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An Alternative Way of Utilizing Fixed Point Theory

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Abstract The author proposes an alternative way of using fixed point theory to get the existence for semilinear equations. As an example, a nonlocal ordinary differential equation is considered. The idea is to solve homogeneous equations in the linearization. One feature of this method is that it does not need the equation to have special structures, for instance, variational structures, maximum principle, etc. Roughly speaking, the existence comes from good properties of the suitably linearized equation. The idea may have wider application.

Keywords Existence, Semi-linear equations, Fixed point theory, Homogeneous linearization
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1 Introduction

The method of studying multiple solutions using fixed point theory has long time history. For instance, Schauder linearization (see [1, p. 593]). Leray [6] also suggested the possible connection between multiple solution and fixed point theory. Comparing with the rich development of fixed point theory, the application in multiple solutions for equations without variational structures and maximum principle seems not very fruitful (see [1, p. 594]). In this paper, we give an idea which might help the application of this point theory.

Roughly speaking, the idea is to solve homogeneous linear equations in the reduction from finding solution to fixed point problem. One thing about this idea is that somehow it is helpful to keep the structures of the original nonlinear equations. We use the following model to introduce the idea.

Theorem 1.1 *Assume that V is a continuous operator from $L^p(0,1)$ to $L^1(0,1)$, $1 \le p < \infty$, and it holds that*

(1) $V(u) \ge 0,\ \forall u \in L^p(0,1)$,

(2) $V(\theta u) = |\theta|^p V(u),\ \theta \in R$,

(3) $C\|u\|_{L^p} \ge \|V(u)\|_{L^1} \ge c_0\|u\|_{L^p}$,

where $c_0 > 0$. Then the problem

$$\begin{cases} u'' - u + V(u)u = 0, \\ u'(0) = u'(1) = 0, \end{cases} \qquad x \in (0,1) \tag{1.1}$$

has non-trivial solution $u \in W^{2,1}(0,1)$ and $u > 0$ for $x \in (0,1)$.

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Theorem 1.2 *Assume that V is a continuous operator from $L^1(0,1)$ to $L^q(0,1)$, $1 < q < 2$, and it holds that*

(1) $V(u) \geq 0, \ \forall u \in L^q(0,1)$,

(2) $V(\theta u) = |\theta|^{\frac{1}{q}} V(u), \ \theta \in R$,

(3) $C_0\|u\|_{L^1} \leq \|V(u)\|_{L^q}^q \leq C\|u\|_{L^1}$.

Then the problem

$$\begin{cases} u'' - u + V(u)u = 0, & x \in (0,1) \\ u'(0) = u'(1) = 0, & \end{cases} \tag{1.2}$$

has non-trivial solution $u \in H^2(0,1)$ and $u > 0$ for $x \in (0,1)$.

Remark 1.1 The nonlinear potential is the generalization of the standard nonlinear term $|u|^p$. For specific nonlinear terms, the improvement of regularity of solution usually is standard and simple. These two results could be known. Our focus is on the method.

One example of potential is $V(u)(x) = |u(x)| + |\int_0^1 k(x,y)u(y)\mathrm{d}y|, \ k \in L^1$. Then the potential V is non-local and satisfies the assumptions in Theorem 1.1, and the solution $u \in W^{2,r}, \ r < \infty$.

Remark 1.2 In [3], it was suggested that the study of multiple solution might help the generation of new methods. It seems that the Neumann boundary condition is more convenient than the Dirichlet boundary condition in this sense.

Remark 1.3 The positiveness of the solution comes from the fact that it is a principal eigenfunction of scalar elliptic equation. The existence of the solution does not depend on the positiveness.

The proofs of Theorems 1.1 and 1.2 are essentially little examples how potentially the idea of homogeneous linearization works. They consist of three steps. The first is the reduction to the nonlinear operator. We will define an operator T from the unit sphere of L^p into the projection space of L^p. Suppose that u is an element of unit sphere in L^p. For suitable α, zero is the first eigenvalue of $-\triangle + 1 - \alpha V(u)$. Therefore, Tu is defined as the projection space of the corresponding eigenfunction space. One may think of this operator as potential to eigenfunction mapping. In general, as long as we have a mapping from a bounded manifold to itself or the quotient manifold of it, there is a chance to do something.

The second is the compactness of the operator. For Theorem 1.1, the proof is trivial. For Theorem 1.2, we introduce the idea of invariant submanifold of the nonlinear operator.

The third is the existence of fixed point. For Theorem 1.1, we use Lefeschetz fixed point theorem. For Theorem 1.2, we use one version of the Schauder fixed point theorem.

The reduction in the first step is possible due to the help of nonlinear zero order term. Namely, the set $\{u \mid 0$ is the eigenvalue of $-\triangle + 1 - V(u)\}$ contains a submanifold of L^p whose co-dimension is 1. This is also essentially the only prerequisite that the method of homogeneous linearization has a chance to work. Somehow our method provides a possible way of getting existence for equations without a priori estimates.

Remark 1.4 The idea of homogeneous linearization is standard in the existence theory for quasilinear elliptic equation. It did not appear before in the literature for semi-linear equation, perhaps due to the following two reasons. One is that the linearized equation is ill-posed. The other is that somehow the method of homogeneous linearization does not work directly for Dirichlet boundary condition.

This paper is organized as follows. In Section 2, we present the reduction of the operator. In Section 3, we prove Theorem 1.1. The proof of Theorem 1.2 is given in Section 4. In Section 5, we discuss the extensions of the model and Dirichlet boundary condition.

2 Reduction of the Operator

In this section, we present the reduction from solving the equation to the fixed point problem. We will do the reduction under the assumptions of Theorem 1.1. The situation of Theorem 1.2 is almost the same as Theorem 1.1 and we omit it. The motivation is the following. Suppose that u satisfies the equation. Then zero is an eigenvalue of the operator, $L_u = -\triangle + 1 - V(u)$, and u is the corresponding eigenfunction. Clearly, not for each u, zero is an eigenvalue of L_u. Roughly speaking, the lucky thing is that, after some suitable modifications of L_u, zero becomes a principal eigenvalue.

We begin with the definition of the adjusting coefficients. For $u \in L^p(0,1),\ u \geq 0,\ \|u\|_{L^p} > 0$, we define

$$\lambda_1(\alpha, u) = \min_{\substack{\|\varphi\|_{L^2}=1 \\ \varphi \in W^{2,1}(0,1) \\ \varphi'(0)=\varphi'(1)=0}} \int_0^1 [|\varphi'|^2 + \varphi^2 - \alpha V(u)\varphi^2] \mathrm{d}x. \tag{2.1}$$

Note that $g(\alpha, u)$ is the principal eigenvalue of the operator $-\triangle + 1 - \alpha V(u)$ with Neumann boundary condition. The following proposition states that the principal eigenvalue is strictly monotone and continuous with respect to the adjusting factor α.

Proposition 2.1 *The principal eigenvalue* λ_1 *satisfies*

(i) $\lambda_1(0, u) > 0,\ \lambda_1(\alpha, u) < 0,\ \alpha \gg 1$.

(ii) $\lambda_1(\alpha, u)$ *is continuously in* α.

(iii) $\lambda_1(\alpha_2, u) < \lambda_1(\alpha_1, u)$, *if* $\alpha_2 > \alpha_1$.

Remark 2.1 The Proposition 2.1(ii) is a small example of spectra stability. The proof below is elementary from spectral theory point of view. We refer to [5] for more on spectra stability.

Proof of Proposition 2.1 (i) is obvious. For (ii), assume $|\alpha_1 - \alpha_2| < \varepsilon$. Standard variational result implies $\exists v_1$, such that

$$\begin{cases} v_1'' - v_1 + \alpha V(u) v_1 = -\lambda_1(\alpha_1, u) v_1, \\ v_1'(0) = v_1'(1) = 0, \\ \|v_1\|_{L^2} = 1, \quad \lambda_1(\alpha_1, u) = \displaystyle\int_0^1 [|v_1'|^2 + v_1^2 - \alpha V(u) v_1^2] \mathrm{d}x. \end{cases} \tag{2.2}$$

We first estimate $\lambda_1(\alpha_1, u)$. Picking $\varphi = 1$ as test function implies

$$\lambda_1(\alpha_1, u) \leq 1 + \alpha_1 C \|u\|_{L^p}.$$

For the lower bound, using the estimate

$$\begin{aligned}\Big|\int_0^1 V(u)v_1^2\mathrm{d}x\Big| &\le \|V(u)\|_{L^1}\|v_1\|_{L^\infty}^2\\ &\le \|V(u)\|_{L^1}(C_\varepsilon\|v_1\|_{L^2}^2+\varepsilon\|v_1\|_{H^1}^2),\end{aligned}$$

we get

$$\begin{aligned}\lambda_1(\alpha_1,u) &= \int_0^1 [|v_1'|^2+v_1^2-\alpha_1 V(u)v_1^2]\mathrm{d}x\\ &\ge \|v_1\|_{H^1}^2-\Big(\frac{1}{2}\|v_1\|_{H^1}^2+C\|v_1\|_{L^2}^2\Big)\\ &> -C\|v_1\|_{L^2}^2=-C,\end{aligned}$$

where $C=C(\alpha_1,\|u\|_{L^p})$.

Combining the lower and upper bounds, we get

$$|\lambda_1(\alpha_1,u)|\le C,\quad C=C(\alpha_1,\|u\|_{L^p}). \tag{2.3}$$

Then we can estimate $\|v_1\|_{H^1}$, which is similar to the lower bound of λ_1. Using (2.2), we have

$$\begin{aligned}\|v_1\|_{H^1}^2 &= \lambda_1(\alpha_1,u)+\int_0^1 \alpha_1 V(u)v_1^2\mathrm{d}x\\ &\le C+\alpha_1\|V(u)\|_{L^1}\|v_1\|_{L^\infty}^2\\ &\le C+\alpha_1\|u\|_{L^p}\Big(C\|v_1\|_{L^2}^2+\frac{1}{2\alpha_1\|u\|_{L^p}}\|v_1\|_{H^1}^2\Big).\end{aligned}$$

So

$$\|v_1\|_{H^1}^2\le C,\quad C=C(\alpha_1,\|u\|_{L^p}). \tag{2.4}$$

Next we estimate $|\lambda_1(\alpha_1,u)-\lambda_1(\alpha_2,u)|$. The idea is to use v_1 as test function. By definition of λ_1, we have

$$\begin{aligned}\lambda_1(\alpha_2,u) &\le \int_0^1 [|v_1'|^2+v_1^2-\alpha_2 V(u)v_1^2]\mathrm{d}x\\ &= \lambda_1(\alpha_1,u)+(\alpha_1-\alpha_2)\int_0^1 V(u)v_1^2\mathrm{d}x\\ &\le \lambda_1(\alpha_1,u)+C\varepsilon.\end{aligned}$$

So

$$\lambda_1(\alpha_2,u)-\lambda_1(\alpha_1,u)\le C\varepsilon.$$

Similarly we have

$$\lambda_1(\alpha_1,u)-\lambda_1(\alpha_2,u)\le C\varepsilon.$$

Therefore

$$|\lambda_1(\alpha_1,u)-\lambda_1(\alpha_2,u)|\le C\varepsilon.$$

Then (ii) is proven. For (iii), still let v_1 be the one in (2.2). Then we have

$$\begin{aligned}\lambda_1(\alpha_2,u) &\le \int_0^1 [|v_1'|^2 + v_1^2 - \alpha_2 V(u)v_1^2]\mathrm{d}x \\ &= \lambda_1(\alpha_1,u) + (\alpha_1-\alpha_2)\int_0^1 V(u)v_1^2\mathrm{d}x \\ &< \lambda_1(\alpha_1,u).\end{aligned}$$

So (iii) is proven. The proof of current proposition is finished.

Rewrite $f_u(\alpha)=\lambda_1(\alpha,u)$. The adjusting coefficient is defined as

$$\alpha(u) = f_u^{-1}(0). \tag{2.5}$$

Proposition 2.1 implies that for $u\in L^p,\ u\ge 0,\ \|u\|_{L^p}>0,\ \alpha(u)$ is well-defined.

Define

$$A(0,1)=\{u\in L^p(0,1)\mid u\ge 0,\ x\in(0,1),\ \|u\|_{L^p}>0\}. \tag{2.6}$$

Proposition 2.2 *Suppose $u\in A(0,1)$. Then*

$$\alpha(u)\le \frac{1}{\|V(u)\|_{L^1}}. \tag{2.7}$$

Proof Using $\varphi=1$ as test function, we have

$$\lambda_1\Big(\frac{1}{\|V(u)\|_{L^1}},u\Big)\le \int_0^1 \Big[|\varphi'^2|+\varphi^2-\frac{1}{\|V(u)\|_{L^1}}V(u)\varphi\Big]\mathrm{d}x=0.$$

So $\lambda_1\big(\frac{1}{\|V(u)\|_{L^1}},u\big)\le 0\Rightarrow \alpha(u)\le\frac{1}{\|V(u)\|_{L^1}}$. The proposition is proven.

Remark 2.2 It seems that Proposition 2.2 does not hold true if L^1 is replaced by $L^r, r>1$. Perhaps this is a major trouble in dimensions two and higher. It also looks hard to extend Proposition 2.2 to Dirichlet boundary condition.

Proposition 2.3 *The adjusting coefficient $\alpha(u)$ is continuous in $A(0,1)$.*

Proof The idea is similar to that of Proposition 2.1. Take $u_1,u_2\in A(0,1), \|u_1-u_2\|_{L^p}\le\varepsilon$ where $\varepsilon\ll\|u_1\|_{L^p}$. Let $\alpha_i=\alpha(u_i)$, v_1 be the minimizer of $g(\alpha_1,u_1)$, $\|v_1\|_{L^2}=1$. Then

$$\begin{aligned}\lambda_1(\alpha_1+C\varepsilon,u_2) &\le \int_0^1 [|v_1'|^2+v_1^2-(\alpha_1+C\varepsilon)V(u_2)v_1^2]\mathrm{d}x \\ &= \lambda_1(\alpha_1,u_1)+\int_0^1[\alpha_1 V(u_1)v_1^2-(\alpha_1+C\varepsilon)V(u_2)v_1^2]\mathrm{d}x \\ &= \lambda_1(\alpha_1,u_1)+(\alpha_1+C\varepsilon)\int_0^1(V(u_1)-V(u_2))v_1^2\mathrm{d}x - C\varepsilon\int_0^1 V(u_1)v_1^2\mathrm{d}x \\ &\le -C\varepsilon\int_0^1 V(u_1)v_1^2\mathrm{d}x+\varepsilon(\alpha_1+C\varepsilon)\|v_1\|_{L^\infty}^2.\end{aligned} \tag{2.8}$$

Using (2.4) and Proposition 2.2, we know that

$$(1+\alpha_1)\|v_1\|_{L^\infty}^2\le C_1,\quad C_1=C_1(\|u_1\|_{L^p}). \tag{2.9}$$

Next we estimate the lower bound of $\int_0^1 V(u_1)v_1^2\mathrm{d}x$. The definitions of λ_1, α_1 and v_1 imply

$$\begin{aligned}\int_0^1 V(u_1)v_1^2\mathrm{d}x &= \frac{\|v_1\|_{H^1}^2}{\alpha_1}\\ &\geq \|v_1\|_{H^1}^2\|V(u_1)\|_{L^1} \quad \text{(using Proposition 2.2)}\\ &= (\|v_1'\|_{L^2}^2+\|v_1\|_{L^2}^2)\cdot\|V(u_1)\|_{L^1}\\ &\geq c\|u_1\|_{L^p} \quad (\|v_1\|_{L^2}=1). \end{aligned} \tag{2.10}$$

Plugging (2.9) and (2.10) into (2.8), we get

$$\begin{aligned}\lambda_1(\alpha_1+C\varepsilon,u_2) &\leq \varepsilon[-C\|u_1\|_{L^p}+C_1\varepsilon)+C_1]\\ &= \varepsilon[C(-\|u_1\|_{L^p}+C_1\varepsilon)+C_1]\\ &\leq 0 \quad \text{for } C=\frac{2C_1}{\|u_1\|_{L^p}},\ \varepsilon\leq\frac{\|u_1\|_{L^p}}{2C_1}.\end{aligned}$$

So $\lambda_1(\alpha_1+C\varepsilon,u_2)\leq 0=\lambda_1(\alpha_2,u_2)$ implies $\alpha_2\leq\alpha_1+C\varepsilon$. Similarly we have

$$\alpha_1\leq\alpha_2+C\varepsilon.$$

Therefore $|\alpha_1-\alpha_2|\leq C\varepsilon$. The proof is finished.

Now we are in a position to define the nonlinear operator T. Let

$$S=\{u \mid \|u\|_{L^p}=1\}$$

be the unit space in $L^p(0,1)$. Similar to the finite dimensional case, let

$$PX:=\{\{t\varphi,\ t\in R\}\mid\varphi\in X,\ \varphi\neq 0\} \tag{2.11}$$

be the projection space of linear space X.

Define

$$\widetilde{L}_u\varphi=-\varphi''+\varphi-\alpha(u)V(u)\varphi. \tag{2.12}$$

Also define

$$E(u)=\{v\in W^{2,1}\mid v'(0)=v'(1)=0,\ \widetilde{L}_uv=0\}. \tag{2.13}$$

Then the operator T is defined as

$$Tu:=PE(u). \tag{2.14}$$

From spectral theory (see [4]), we get

(i)

$$\dim(E(u))=1, \tag{2.15}$$

(ii)

$$E(u)=\{tv\mid t\in R,\ v(x)>0,\ x\in(0,1)\}. \tag{2.16}$$

Remark 2.3 The property (2.15) implies that the operator T defined in the current paper is single-valued. In general, the operator would be set-valued. Set-valued is also called multivalued.

Proposition 2.4 *The operator T from S to PL^p is continuous.*

Proof We will use the following elementary result in mathematical analysis. Suppose that $\{y_m\}$ is a sequence. If for any subsequence, there exists a convergent subsequent, and all the subsequences converge to the same limit, then the original sequence converges. The assumptions in Theorem 1.1 and embedding theorem imply that the operator T is well-defined and $Tu \in PL^p$ for $u \in S$. For continuity, take $\{u_m\} \subset L^p$, $\alpha_m = \alpha(u_m)$. Then the continuity of α (see Proposition 2.3) implies

$$\lim_{m\to\infty} \alpha_m = \alpha(u).$$

So

$$\sup_m \alpha_m, \quad \alpha(u) \le \frac{C}{\|V(u)\|_{L^1}}. \tag{2.17}$$

Let $v_m \in E(u_m),\ v \in E(u)$, $\|v_m\|_{L^2} = \|v\|_{L^2} = 1$. Then (2.17) and (2.4) give

$$\sup \|v_m\|_{H^1} + \|v_m\|_{W^{2,1}} \le C. \tag{2.18}$$

Now take any subsequence of $\{u_m\}$ denoted as $\{u_{m_k}\}$. Sobolev compact embedding theorem and (2.18) imply that $\exists v_{m_{k_i}}$, $\overline{v}$ such that

$$v_{m_{k_i}} \to \overline{v} \quad \text{in } H^1. \tag{2.19}$$

Passing to the limit, we see that $\overline{v} \in E(u)$. Therefore

$$\lim_{i\to\infty} Tu_{m_{k_i}} = Tu. \tag{2.20}$$

This means that for any subsequence $\{u_{m_k}\}$ of $\{u_m\}$, we can find a subsequence of $\{u_{m_k}\}$ such that $Tu_{m_{k_i}}$ converges to the same limit Tu. Therefore Tu_m converges to Tu. The proposition is proven.

Remark 2.4 The procedure presented in this section also works for high dimensional and system cases. The keys are Proposition 2.1 and 2.3. Certainly, we need to assume more regularity on the nonlinear potential.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We will use the following fixed point theorem, which looks like textbook things in the literature of fixed point theory.

Lemma 3.1 *Assume that X is a seperable Banach space. Then any compact continuous mapping from the unit sphere in X to PX has fixed point, i.e., $\exists u \in X,\ \|u\|_X = 1,\ Fu = [u]$.*

Proof The idea is standard: Use finite dimensional case to approximate infinite dimensional case. It is well-known that any continuous mapping from even dimensional sphere to even dimensional projection space has fixed point. The tool is Lefschetz fixed point theorem (see [2]).

Let $\{e_i\}$ be the basis for X. Define $J_m = \text{span}\{e_i\}_{i=1}^{2m+1}$. Fix a small ε. The compactness of $F\{\|u\|_X = 1\}$ implies $\text{dist}(PJ_m, F\{\|u\|_X = 1\}) \leq \varepsilon,\ m \gg 1$.

Suppose $[u] \in F\{\|u\|_X = 1\}$. Let $h_m([u])$ be the projection of $[u]$ on J_m, i.e.,

$$| h_m([u]) - [u] |= \text{dist}([u],\ PJ_m).$$

Note that h_m is well-defined and continuous since ε is small and PJ_m is compact. Also note that

$$\sup | h_m([u]) - h_k[u] |\leq \varepsilon, \quad [u] \in F(\{\|u\|_X = 1\}),\ m, k \gg 1.$$

Therefore we can show

$$h_m[v_m] \to [v], \quad \text{ if } [v_m] \to v. \tag{3.1}$$

Clearly $J_m \bigcap\{\|u\|_X = 1\}$ is isomorphic to S^{2m} and PJ_m is isomorphic to RP^{2m}. So there exists u_m such that

$$(h_m \circ F)(u_m) = [u_m], \quad u_m \in J_m \cap \{\|u\|_X = 1\}. \tag{3.2}$$

The compactness of F implies that $\{F(u_m)\}$ has a convergent subsequence, still denoted by $\{F(u_m)\}$. The property (3.1) gives that $(h_m \circ F)(u_m)$ also converges. Using (3.2), we know that $\{[u_m]\}$ converges, denote by $[u_m]$.

Since $[u_m] = (u_m,\ -u_m)$, $\|u_m\| = 1$, so $\exists\{u_{m_i}\}, \|u_{m_i}\| = 1$, $\{u_{m_i}\}$ converges. We denote the limit by u. Hence $\|u\| = 1$, $u_{m_i} \to u$ in X. Passing limit in (3.2), we prove the lemma.

Proposition 3.1 *The operator T is compact from $\{\|u\|_{L^p} = 1\}$ to PL^p.*

Proof It follows from Proposition 2.2 and the proof of Proposition 2.4.

Remark 3.1 The compactness for Theorem 1.1 is so easy to prove mainly since L^1 is the admissible space in 1D and $V(u) \in L^1, u \in L^p$. In 2D and higher, the compactness seems to be a major issue. The difficulty seems to be the control of the adjusting coefficients.

Proof of Theorem 1.1 Lemma 3.1 and Proposition 3.1 imply $\exists u \in S$, s.t $F(u) = [u]$. So $u'(0) = u'(1) = 0$,

$$u'' - u + \alpha(u)V(u)u = 0. \tag{3.3}$$

Set

$$v = \alpha^{\frac{1}{p}}(u)u. \tag{3.4}$$

Then v satisfies (1.1). Using (2.16), we can pick $v > 0$, $x \in (0, 1)$. The theorem is proven.

Here we want to emphasize that the solution to (1.1) may not generate the fixed point of the operator defined in Section 2. Certainly if u solves the equation, then zero is the eigenvalue of $-\triangle + 1 - V(u)$, and u is the corresponding eigenfunction. But zero might not be the principal eigenvalue. Therefore $\frac{u}{\|u\|_{L^p}}$ is not the fixed point of the operator defined in Section 2.

4 Proof of Theorem 1.2

In this section we will prove Theorem 1.2. The main difference with Theorem 1.1 is that the L^1 norm of $V(u)$ cannot be controlled from below. So we have to deal with the compactness in another way. We present Theorem 1.2 mainly because the proof of it contains the idea of invariant submanifold of the reduced operator, which may be useful in the study of high dimensional case.

Lemma 4.1 *Suppose*

$$\begin{cases} v'' + wv = 0, \quad x \in (0,1), \\ v'(0) = v'(1) = 0. \end{cases} \tag{4.1}$$

Let $K = \|w\|_{L^q} + 1,\ q > 1$. *Then*

$$\|v\|_{L^2} \le CK^{\frac{1}{2}\frac{q}{2q-1}} \|v\|_{L^1}, \quad \textit{where } C \textit{ is absolute constant.} \tag{4.2}$$

Proof Energy integration gives

$$\|v\|_{H^1}^2 = \int_0^1 (|v'|^2 + wv^2)\mathrm{d}x \le CK\|v\|_{L^{2q'}}^2, \quad \frac{1}{q'} + \frac{1}{q} = 1.$$

So

$$\|v\|_{H^1} \le CK^{\frac{1}{2}}\|v\|_{L^{2q'}} \le CK^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}(1+\frac{1}{q'})} \cdot \|v\|_{H^1}^{\frac{1}{2}(1-\frac{1}{q'})}.$$

Therefore

$$\|v\|_{H^1}^{\frac{1}{2}(1+\frac{1}{q'})} \le CK^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}(1+\frac{1}{q'})}.$$

This means

$$\|v\|_{H^1} \le CK^{\frac{1}{1+q'}}\|v\|_{H^1}. \tag{4.3}$$

Using Nirenberg inequality, we have

$$\begin{aligned} \|v\|_{L^2} &\le \|v\|_{L^1}^{\frac{2}{3}} \cdot \|v\|_{H^1}^{\frac{1}{3}} \\ &\le \|v\|_{L^1}^{\frac{2}{3}} \cdot CK^{\frac{1}{3}\cdot\frac{1}{1+q'}}\|v\|_{L^2}^{\frac{1}{3}}. \end{aligned}$$

So

$$\|v\|_{L^2}^{\frac{2}{3}} \le CK^{\frac{1}{3}\cdot\frac{q}{2q-1}}\|v\|_{L^1}^{\frac{2}{3}}.$$

Therefore

$$\|v\|_{L^2} \le CK^{\frac{1}{2}\cdot\frac{q}{2q-1}}\|v\|_{L^1}.$$

The lemma is proven.

Lemma 4.2 *Under the assumptions of Theorem* 1.2, *the set* $\{u \in L^2 \mid \|u\|_{L^2} = 1, \|u\|_{L^1} \ge \frac{1}{M}\}, M \gg 1$ *is an invariant submanifold of* T, i.e.,

$$T\Big\{u \in L^2 \Big| \|u\|_{L^2} = 1, \|u\|_{L^1} \ge \frac{1}{M}\Big\} = \Big\{(u, -u) \Big| \|u\|_{L^2} = 1, \|u\|_{L^1} \ge \frac{1}{M}\Big\}.$$

Proof Clearly it is a submanifold. For invariance, assume $u \in L^2$, $\|u\|_{L^2} = 1$, $\|u\|_{L^1} \geq \frac{1}{M}$, then

$$\|V(u)\|_{L^q}, \|V(u)\|_{L^{2q}} \leq C, \quad \|V(u)\|_{L^q} \geq \frac{C}{M}.$$

Hölder inequality implies

$$\|V(u)\|_{L^q} \leq \|V(u)\|_{L^1}^{\frac{1}{2q-2}} \cdot \|V(u)\|_{L^{2q}}^{\frac{2q-1}{2q-2}}.$$

So

$$\|V(u)\|_{L^1} \geq \Big(\frac{1}{C} \cdot \|V(u)\|_{L^q}\Big)^{2q-1} \geq C\Big(\frac{1}{M}\Big)^{2q-1}.$$

Using Proposition 2.2, we have

$$\alpha(V(u)) \leq \frac{1}{\|V(u)\|_{L^1}} \leq C \cdot M^{2q-1}.$$

By definition,

$$Tu = P\{v \mid v'' - v + \alpha(V(u))V(u)v = 0,\ v'(0) = v'(1) = 0\}.$$

Lemma 4.1 implies

$$\begin{aligned}\|v\|_{L^2} &\leq C\|\alpha(V(u))V(u) - 1\|_{L^q}^{\frac{1}{2}\cdot\frac{q}{2q-1}} \cdot \|v\|_{L^1} \\ &\leq CM^{(2q-1)\frac{1}{2}\cdot\frac{q}{2q-1}} \cdot \|V(u)\|_{L^q}^{\frac{1}{2}\cdot\frac{q}{2q-1}} \cdot \|v\|_{L^1} \\ &\leq CM^{\frac{q}{2}} \cdot \|v\|_{L^1} \\ &\leq M\|v\|_{L^1}, \quad M \gg 1.\end{aligned}$$

The lemma is proven.

Define

$$D = \Big\{u \in L^2(0,1) \Big| \|u\|_{L^2} = 1,\ u \geq 0,\ x \in (0,1),\ \|u\|_{L^1} \geq \frac{1}{M}\Big\}. \tag{4.4}$$

Note that

$$D \cap -D = \phi. \tag{4.5}$$

Then Lemma 4.2 and (2.16) imply that T is an operator from D to D.

Now we recall the following definition of convex in manifold by Berger.

Definition 4.1 *Suppose that D is a subset of a manifold. If for any $a, b \in D$, there exists a unique geodesic in D connecting a and b, then we say D is convex.*

Proposition 4.1 *The set D is a closed convex bounded set in $\{\|u\|_{L^2} = 1\}$.*

Proof Clearly D is closed and bounded. For convexity, taking $u_1, u_2 \in D$, there are two geodesic connecting u_1, u_2 in $\{\|u\|_{L^2} = 1\}$,

$$\gamma_1 = \Big\{\frac{\lambda u_1 + (1-\lambda)u_2}{\|\lambda u_1 + (1-\lambda)u_2\|_{L^2}},\ \lambda \in [0,1]\Big\}$$

$$\gamma_2 = \Big\{\frac{-\lambda u_1 + (1-\lambda)u_2}{\|-\lambda u_1 + (1-\lambda)u_2\|_{L^2}},\ \lambda \in [0,1]\Big\}$$
$$\bigcup \Big\{\frac{-\lambda u_1 - (1-\lambda)u_2}{\|\lambda u_1 + (1-\lambda)u_2\|_{L^2}},\ \lambda \in (0,1)\Big\}$$
$$\bigcup \Big\{\frac{\lambda u_1 - (1-\lambda)u_2}{\|\lambda u_1 - (1-\lambda)u_2\|_{L^2}},\ \lambda \in [0,1]\Big\}.$$

Clearly γ_2 does not completely belong to D. For γ_1, the key is that $u_i \geq 0$.

Note that

$$Q = \|\lambda u_1 + (1-\lambda)u_2\|_{L^2} \leq 1.$$

So

$$\begin{aligned}\frac{\|\lambda u_1 + (1-\lambda)u_2\|_{L^1}}{Q} &= \frac{1}{Q}\int_0^1 [\lambda u_1 + (1-\lambda)u_2]\mathrm{d}x \\ &= \frac{1}{Q}\lambda\|u_1\|_{L^1} + (1-\lambda)\|u_1\|_{L^1} \\ &\geq \frac{1}{Q}\Big[\lambda\frac{1}{M} + (1-\lambda)\frac{1}{M}\Big] \\ &\geq \frac{1}{M}.\end{aligned}$$

So $\gamma_1 \subset D$. The proposition is proven.

Proof of Theorem 1.2 Lemma 4.2 and Proposition 4.1 imply that the operator T is compact continuous operator from D to itself. The existence of fixed point comes from the following version of Schauder fixed point theorem. Any compact, continuous mapping from a closed convex bounded set of a manifold to the set itself has fixed point. The rest is essentially the same as Theorem 1.1.

Probably the property (2.16) would only hold true for the principle eigenfunction of second order equation. The other way of getting existence of fixed point is to use homotopic invariance of Nielsen number. Such number is not easy to estimate in multivalued cases, and the construction of the homotopy may be hard. But both things look doable.

5 Further Discussions

Extension of the model From the very simple model (1.1), there are various ways of improvement, for instance, the removing of artificial helping term $-u$, system case, non-self-adjoint linearized operator, compactness of the nonlinear operator in higher dimensions, etc. Among all the generalizations, the compactness issue looks a lot harder than others. The latter could be difficult or very complicated, but still looks to be within the reach of current method. But for compactness in dimension two and higher, the method seems not clear. Somehow the role of compactness is like a priority estimates in the existence theory based on Leray-Schauder fixed point theorem.

Dirichlet boundary condition It is perhaps hard to extend the method to Dirichlet boundary condition directly. At first glance, this seems to be a drawback of the method. But after further thinking, this even maybe a little supporting evidence for the method. As suggested in [3], one major goal of studying multiple solutions in bounded domain is to induce method for studying singluar solutions. But for singular solutions, usually the first typical case is the whole space case. From this point of view, Dirichlet or Neumann does not matter.

More importantly, consider the following standard semilinear elliptic equation:

$$\triangle u + |u|^\sigma u = 0, \quad x \in \Omega,$$

where Ω is a bounded domain in R^n. It is well-known that if $\sigma \geq \frac{4}{n-2}$, $n \geq 3$ and the domain is convex, then the equation above does not have nontrivial solution for zero Dirichlet boundary condition. This property suggests that Dirichlet boundary condition might not be a first choice in the search for method of utilizing the help of nonlinear terms.

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