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Reflected Quadratic BSDEs Driven by G-Brownian Motions*

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Abstract In this paper, the authors consider a reflected backward stochastic differential equation driven by a G-Brownian motion (G-BSDE for short), with the generator growing quadratically in the second unknown. The authors obtain the existence by the penalty method, and some a priori estimates which imply the uniqueness, for solutions of the G-BSDE. Moreover, focusing their discussion at the Markovian setting, the authors give a nonlinear Feynman-Kac formula for solutions of a fully nonlinear partial differential equation.

Keywords G-Brownian motion, G-Martingale, Quandratic growth, G-BSDEs, Probabilistic representation
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1 Introduction

A general backward stochastic differential equation (BSDE for short) takes the following form:

$$Y_t = \xi + \int_t^{\mathrm{T}} f(s, Y_s, Z_s)\mathrm{d}s - \int_t^{\mathrm{T}} Z_s \mathrm{d}W_s, \quad t \in [0, T].$$

The function f is conventionally called the generator and the random variable ξ is called the terminal value. Bismut [2–3] initially gave a complete linear theory, where the generator is linear in both unknown variables, and derived the stochastic Riccati equation as a particular nonlinear BSDE where the generator is quadratic in the second unkown variable. Pardoux and Peng [29] established the existence and uniqueness result when the generator f is uniformly Lipschitz continuous in both unknown variables and the terminal value ξ is square integrable. Subsequently, an intensive attention has been given to relax the assumption of the uniformly Lipschitz continuity on the generator. In particular, the one-dimensional BSDE with a quadratic generator (i.e., the so-called quadratic BSDE) was studied by Kobylanski [18] for a bounded terminal value ξ, and by Briand and Hu [5–6] for an unbounded terminal value ξ of some suitable exponential moments. The multi-dimensional quadratic BSDE was discussed by Tang [39] and Hu and Tang [16].

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As a constrained BSDE, a reflected backward stochastic differential equation (RBSDE for short) was formulated and studied by El Karoui et al. [11], where the first unknown Y is required to stay up a given continuous process S and an additional increasing process which satisfies the Skorohod condition, is thus introduced into the equation. Subsequently, much efforts have been made to relax the Lipschitz assumption on the generator. For the quadratic case, see Kobylanski et al. [19] with bounded terminal values, and Lepeltier and Xu [21] with unbounded terminal values.

To incorporate the Knightian uncertainty, Peng [31–34] introduced the notion of G-expectation as a time-consistent sub-linear expectation, and constructed (via a fully non-linear PDE) the so-called G-Brownian motion $\{B_t,\ t \in [0, +\infty)\}$, whose quadratic variation process $\langle B\rangle$—in contrast to the classical Brownian motion—is not deterministic. The stochastic integral with respect to the G-Brownian motion and its quadratic variation were also discussed by Peng [31]. Denis et al. [10] prove that the G-expectation is in fact the upper expectation over a collection of mutually singular martingale measures $\mathcal{P}$. Hu et al. [12] showed that there is a unique triple of processes (Y, Z, K) in a proper Banach space satisfying the following scalar-valued BSDE driven by the G-Brownian motion B:

$$\begin{aligned} Y_t = \xi + \int_t^{\mathrm{T}} g(s, Y_s, Z_s)\mathrm{d}s + \int_t^{\mathrm{T}} f(s, Y_s, Z_s)\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} Z_s \mathrm{d}B_s \\ - \int_t^{\mathrm{T}} \mathrm{d}K_s, \quad t \in [0, T], \end{aligned} \tag{1.1}$$

where f and g are uniformly Lipshchitz in both unknown variables. Hu et al. [15] proved the existence and uniqueness for adapted solutions to the scalar-valued z-quadratic BSDE (1.1) driven by the G-Brownian motion B for a bounded terminal value ξ. Very recently, Li, Peng, and Soumana Hima [24] discussed a reflected BSDE driven by the G-Brownian motion subject to a lower obstacle under the uniformly Lipschitz condition, where a G-martingale condition rather than the conventional Skorohod condition, is used to characterize the unknown bounded variational process which is introduced into the equation to keep the first unknown process stay up the lower obstacle under the G-expectation. More precisely, they showed that there is a unique triple (Y, Z, A) of processes satisfying the following equation:

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^{\mathrm{T}} g(s, Y_s, Z_s)\mathrm{d}s + \int_t^{\mathrm{T}} f(s, Y_s, Z_s)\mathrm{d}\langle B\rangle_s \\ \qquad - \displaystyle\int_t^{\mathrm{T}} Z_s \mathrm{d}B_s + \int_t^{\mathrm{T}} \mathrm{d}A_s, \quad t \in [0, T]; \\ Y_t \geq S_t, \quad 0 \leq t \leq T; \quad \displaystyle\int_0^{\cdot} (S_s - Y_s)\mathrm{d}A_s \text{ is a non-increasing } G\text{-martingale.} \end{cases} \tag{1.2}$$

A subsequent study of Li and Peng [22] reported the following unexpected observation on the upper obstacle problem for the reflected BSDE driven by a G-Brownian motion: The proof of the uniqueness of solutions in the lower obstacle problem turns out to be difficult to be adapted to the upper obstacle problem. Since the preceding two equations hold $\mathbb{P}$-a.s. for each $\mathbb{P} \in \mathcal{P}$, they are also associated to second order BSDEs, which were discussed by Cheridito et al. [8], Soner et al. [37] and Possamaï and Zhou [35]. Moreover, Matoussi, Piozin and Possamaï [26] and Matoussi, Possamaï and Zhou [27, 28] discussed the reflected second order BSDEs. In the context of a G-BSDE, the solution is universally discussed in a "better" space of processes, and its existence naturally requires more regularity of the coefficients.

As a generalized counterpart of the classical reflected quadratic BSDEs, the existence and uniqueness result for reflected quadratic BSDEs driven by G-Brownian motions still remains to be studied. The main objective of this paper is to provide the well-posedness of the reflected G-BSDE (1.2) when the generator has a quadratic growth and the terminal value ξ is bounded. As noted in Li, Peng and Soumana Hima [24] and Possamaï and Zhou [35], the dominated convergence theorem does not hold under the G-framework, and a bounded sequence in $M_G^p(0,T)$ does not necessarily have the weak compactness. These striking differences prevent us from adapting the method of Kobylanski et al. [19] to approximate the quadratic generator with Lipshcitz ones and then to prove the solutions of the approximating reflected BSDEs to converge to that of the original reflected quadratic BSDE. Instead in this paper, we use a penalty method in the spirit of El Karoui et al. [11] (for a BSDE in a Wiener space) and Li, Peng and Soumana Hima [24] (for a G-BSDE). Since our generator is allowed to grow quadratically in the second unknown variable, the terminal value ξ is assumed to be bounded for simplicity of exposition, and then the symmetric martingale part of the underlying BSDE is discussed in the BMO space.

As in Hu et al. [13] and Li, Peng and Soumana Hima [24], the solution of a forward backward differential equation driven by G-Brownian motion (G-FBSDEs for short) can be interpreted as a viscosity solution of a PDE. We first prove the existence of solutions of quadratic G-BSDEs in a Markovian setting. We then give the nonlinear Feynman-Kac formula for a fully nonlinear parabolic variational inequality via the quadratic G-BSDEs and the reflected quadratic G-BSDEs.

The paper is organized as follows. Section 2 is dedicated to preliminaries on the G-framework, the formulation of reflected G-BSDEs, G-BMO martingales and G-Girsanov theorem. In Section 3, we introduce some a priori estimates for quadratic reflected G-BSDEs through the G-Girsanov transformation, which yield the uniqueness in a straightforward way. In Section 4, we establish the approximation method via penalization. We state some convergence properties of the solutions to the penalized G-BSDEs. In Section 5, we prove our main result and a comparison theorem. Finally, in Section 6, we give a nonlinear Feynmann-Kac formula and address the relation between quadratic G-BSDEs and nonlinear parabolic PDEs.

2 Preliminaries

2.1 Notations and results on G-expectation and quadratic G-BSDEs

In this section, we first recall notations and basic results concerning G-expectation, G-Brownian motion and related G-stochastic calculus, and quadratic G-BSDEs. More details can be found in [12, 13, 25, 31–33].

Let Ω be a complete separable metric space, and let $\mathcal{H}$ be a linear space of real-valued functions defined on Ω satisfying $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables.

Definition 2.1 (Sublinear Expectation Space) *A sublinear expectation* $\widehat{\mathbb{E}}[\cdot]$ *is a functional* $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ *satisfying the following properties*: *For all* $X, Y \in \mathcal{H}$,

(1) *monotonicity. If* $X \geq Y$, *then* $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(2) *constant preservation.* $\widehat{\mathbb{E}}[c] = c,\ c \in \mathbb{R}$;

(3) *sub-additivity.* $\widehat{\mathbb{E}}[X+Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$;

(4) *positive homogeneity.* $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$ *for all* $\lambda \geq 0$.

We call the triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *a sublinear expectation space.*

Definition 2.2 (Independence) *In a sublinear expectation space* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, *a random vector* $Y = (Y_1, Y_2, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ *is said to be independent of another random vector* $X = (X_1, X_2, \cdots, X_m)$, $X_i \in \mathcal{H}$, *if* $\widehat{\mathbb{E}}[\phi(X,Y)] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\phi(x,Y)]|_{x=X}]$ *for all* $\phi \in \mathcal{C}_{l,\text{lip}}(\mathbb{R}^{m+n})$, *where* $\mathcal{C}_{l,\text{lip}}(\mathbb{R}^n)$ *is the space of real continuous functions defined on* $\mathbb{R}^n$ *such that*

$$|\phi(x) - \phi(y)| \le C(1 + |x|^k + |y|^k)|x-y|, \quad \forall\, x, y \in \mathbb{R}^n,$$

where k *and* C *depend only on* ϕ.

Definition 2.3 (G-Normal Distribution) *We say that the random vector* $X = (X_1, X_2, \cdots, X_d)$ *is* G*-normally distributed, if for any function* $\phi \in \mathcal{C}_{l,\text{lip}}(\mathbb{R}^d)$, *the function* u *defined by* $u(t,x) := \widehat{\mathbb{E}}[\phi(x + \sqrt{t}X)]$, $(t,x) \in [0,+\infty) \times \mathbb{R}^d$, *is a viscosity of* G*-heat equation*:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0,x) = \phi(x).$$

Here G *denotes the function*

$$G(A) := \frac{1}{2}\widehat{\mathbb{E}}[\langle AX, X\rangle] : \mathbb{S}_d \to \mathbb{R},$$

where $\mathbb{S}_d$ *denotes the collection of* $d \times d$ *symmetric matrices.*

The function $G(\cdot)$ is a monotonic, sublinear mapping on $\mathbb{S}_d$ and

$$G(A) = \frac{1}{2}\widehat{\mathbb{E}}[\langle AX, X\rangle] \le \frac{1}{2}|A|\widehat{\mathbb{E}}[|X|^2] := \frac{1}{2}|A|\overline{\sigma}^2$$

implies that there exists a bounded, convex and closed subset $\Gamma \subseteq \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative elements in $\mathbb{S}_d$.

In this paper, we only consider a non-degenerate G-normal distribution, i.e., there exists some $\underline{\sigma} > 0$ such that $G(A) - G(B) \ge \underline{\sigma}^2 \text{tr}[A-B]$ for any $A \ge B$.

We now fix $\Omega := C_0([0,\infty);\mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous functions $\{\omega_t,\ t \in [0,+\infty)\}$ with $\omega_0 = 0$. Let $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0,+\infty)\}$ be the nature filtration generated by the canonical process $\{B_t,\ t \in [0,+\infty)\}$, i.e., $B_t(\omega) = \omega_t$ for $(t,\omega) \in [0,\infty) \times \Omega$. Set $\Omega_T := C_0([0,T];\mathbb{R}^d)$. Let us consider the function spaces defined by

$$\text{Lip}(\Omega_T) := \{\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) : 0 \le t_1 \le t_2 \le \cdots \le t_n \le T,\ \phi \in \mathcal{C}_{l,\text{lip}}(\mathbb{R}^{d\times n})\}$$

for $T > 0$, and $\text{Lip}(\Omega) = \bigcup_{n=1}^{\infty} \text{Lip}(\Omega_n)$.

Definition 2.4 (G-Brownian Motion and G-Expectation) *On the sublinear expectation space* $(\Omega, \text{Lip}(\Omega), \widehat{\mathbb{E}})$, *the canonical process* $\{B_t,\ t \in [0,+\infty)\}$ *is called* G*-Brownian motion if the following properties are satisfied*:

(1) $B_0 = 0$;

(2) *for each* $t, s > 0$, *the increment* $B_{t+s} - B_t$ *is independent of* $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ *for each* $n \in \mathbb{N}$ *and* $0 \le t_1 \le t_2 \le \cdots \le t_n \le t$;

(3) $B_{t+s} - B_t$ *is* G*-normally distributed.*

Moreover, the sublinear expectation $\widehat{\mathbb{E}}[\cdot]$ *is called* G*-expectation.*

Definition 2.5 (Conditional G-Expectation) *For the random variable $\xi \in \mathrm{Lip}(\Omega_T)$ of the following form*:

$$\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}), \quad \phi \in \mathcal{C}_{l,\mathrm{lip}}(\mathbb{R}^{d\times n}),$$

the conditional G-expectation $\widehat{\mathbb{E}}_{t_i}[\cdot]$, $i = 1, 2, \cdots, n$, is defined as follows:

$$\widehat{\mathbb{E}}_{t_i} = [\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] = \widetilde{\phi}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\widetilde{\phi}(x_1, x_2, \cdots, x_i) = \widehat{\mathbb{E}}[\phi(x_1, x_2, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_n} - B_{t_{n-1}})].$$

If $t \in (t_i, t_{i+1})$, the conditional G-expectation $\widehat{\mathbb{E}}_t[\xi]$ could be defined by reformulating ξ as

$$\xi = \widehat{\phi}(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_t - B_{t_i}, B_{t_{i+1}} - B_t, \cdots, B_{t_n} - B_{t_{n-1}}), \quad \widehat{\phi} \in \mathcal{C}_{l,\mathrm{lip}}(\mathbb{R}^{d\times(n+1)}).$$

For $\xi \in \mathrm{Lip}(\Omega_T)$ and $p \geq 1$, we consider the norm $\|\xi\|_{L_G^p} = (\widehat{\mathbb{E}}[|\xi|^p])^{\frac{1}{p}}$. Denote by $L_G^p(\Omega_T)$ the Banach completion of $\mathrm{Lip}(\Omega_T)$ under $\|\cdot\|_{L_G^p}$. It is easy to check that the conditional G-expectation $\widehat{\mathbb{E}}_t[\cdot]: \mathrm{Lip}(\Omega_T) \to \mathrm{Lip}(\Omega_t)$ is a continuous mapping and thus can be extended to $\widehat{\mathbb{E}}_t: L_G^p(\Omega_T) \to L_G^p(\Omega_t)$.

Definition 2.6 (G-Martingale) *A process $\{M_t,\ t \in [0,T]\}$ is called a G-martingale if*
(i) $M_t \in L_G^1(\Omega_t)$ *for any* $t \in [0,T]$;
(ii) $\widehat{\mathbb{E}}_s[M_t] = M_s$ *for all* $0 \leq s \leq t \leq T$.
The process $\{M_t,\ t \in [0,T]\}$ is called a symmetric G-martingale if $-M$ is also a G-martingale.

The following representation result of G-expectation on $L_G^1(\Omega_T)$ can be found in Denis et al. [10, Propositions 49 and 50, p. 157–158] and Hu and Peng [14, Theorem 3.5, p. 544].

Theorem 2.1 *There exists a weakly compact set $\mathcal{P} \subseteq \mathcal{M}_1(\Omega_T)$ (i.e., the set of all probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$), such that*

$$\widehat{\mathbb{E}}[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\xi], \quad \forall\, \xi \in L_G^1(\Omega_T),$$

where $E^{\mathbb{P}}[\cdot]$ is the expectation operator with respect to probability $\mathbb{P}$. Such $\mathcal{P}$ is called a representative set of $\widehat{\mathbb{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\widehat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity $c(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega_T)$.

Definition 2.7 (Quasi-sure) *A set $A \in \mathcal{B}(\Omega_T)$ is a polar set if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.*

In what follows, two random variables X and Y will not be distinguished if $X = Y$, q.s.

Soner et al. [36, Proposition 3.4, p. 272] gave the following characterization of the conditional G-expectation.

Theorem 2.2 *For any $\xi \in L_G^1(\Omega_T)$, $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}$,*

$$\widehat{\mathbb{E}}_t[\xi] = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} E_t^{\mathbb{P}'}[\xi], \quad \mathbb{P}\text{-}a.s.,$$

where

$$\mathcal{P}(t,\mathbb{P}) := \{\mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\}.$$

In view of Theorem 2.2, it is easy to check the following property for G-martingales.

Proposition 2.1 *Assume that* $\{M_s,\ s\in[0,T]\}$ *is a* G*-Martingale and* $\{\eta_s,\ s\in[0,T]\}$ *is a process satisfying* $\eta_s\in L^1_G(\Omega_s)$ *for any* $s\in[0,T]$. *Then we have for any* $t\in[0,T]$,

$$\widehat{\mathbb{E}}_t[\eta_t+M_T-M_t]=\eta_t.$$

For the terminal value of quadratic G-BSDE, we define the space $L^\infty_G(\Omega_T)$ as the completion of $\mathrm{Lip}(\Omega_T)$ under the norm

$$\|\xi\|_{L^\infty_G}:=\inf\{M\geq 0:|\xi|\leq M,\ \text{q.s.}\}.$$

For $\xi\in\mathrm{Lip}(\Omega_T)$ and $p\geq 1$, define $\|\xi\|_{p,\mathcal{E}}=\widehat{\mathbb{E}}\big[\sup\limits_{t\in[0,T]}\widehat{\mathbb{E}}_t[|\xi|^p]\big]^{\frac{1}{p}}$ and denote by $L^p_{\mathcal{E}}(\Omega_T)$ the completion of $\mathrm{Lip}(\Omega_T)$ under $\|\cdot\|_{p,\mathcal{E}}$. Song [38, Theorem 3.4, p. 293]) gave the following estimate.

Theorem 2.3 *For any* $\alpha\geq 1$ *and* $\delta>0$, $L^{\alpha+\delta}_G(\Omega_T)\subseteq L^\alpha_{\mathcal{E}}(\Omega_T)$. *More precisely, for any* $1<\gamma<\beta:=\frac{\alpha+\delta}{\alpha}$, $\gamma\leq 2$, *we have*

$$\|\xi\|^\alpha_{\alpha,\mathcal{E}}\leq\gamma^*\{\|\xi\|^\alpha_{L^{\alpha+\delta}_G}+14^{\frac{1}{\gamma}}C_{\frac{\beta}{\gamma}}\|\xi\|^{\frac{\alpha+\beta}{\gamma}}_{L^{\alpha+\delta}_G}\},\quad\forall\,\xi\in\mathrm{Lip}(\Omega_T),$$

where $C_{\frac{\beta}{\gamma}}=\sum\limits_{i=1}^{\infty}i^{-\frac{\beta}{\gamma}}$, $\gamma^*=\frac{\gamma}{\gamma-1}$.

Remark 2.1 In view of [12, Remark 2.9], there exists C_1 depending only on α and δ such that

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\widehat{\mathbb{E}}_t[|\xi|^\alpha]\Big]\leq C_1\{\widehat{\mathbb{E}}[|\xi|^{\alpha+\delta}]^{\frac{\alpha}{\alpha+\delta}}+\widehat{\mathbb{E}}[|\xi|^{\alpha+\delta}]\}.$$

Let B be a d-dimensional G-Brownian motion. For each fixed $a\in\mathbb{R}^d$, $B^a_t=\langle a,B_t\rangle$ is a 1-dimensional G_a-Brownian motion, where $G_a(\alpha)=G(aa^{\mathrm{T}})\alpha^+ +G(-aa^{\mathrm{T}})\alpha^-$. The quadratic variation process of B^a is defined by

$$\langle B^a\rangle_t=\lim_{\mu(\pi^N_t)\to 0}\sum_{j=0}^{N-1}(B^a_{t^N_{j+1}}-B^a_{t^N_j})^2,$$

where π^N_t, $N=1,2,\cdots$, are refining partitions of $[0,t]$. By Peng [33, Corollary 3.5.5, Chapter 3, p. 70], for all $t,s>0$, $\langle B^a\rangle_{t+s}-\langle B^a\rangle_t\in[-2G(-aa^{\mathrm{T}})s,2G(aa^{\mathrm{T}})s]$, q.s.

For each fixed $a,\overline{a}\in\mathbb{R}^d$, the mutual variation process of B^a and $B^{\overline{a}}$ is defined by

$$\langle B^a,B^{\overline{a}}\rangle_t=\frac{1}{4}[\langle B^{a+\overline{a}}\rangle_t-\langle B^{a-\overline{a}}\rangle_t].$$

Next we discuss the stochastic integrals with respect to the G-Brownian motion and its quadratic variation.

Definition 2.8 *Let* $M^0_G(0,T)$ *be the collection of processes* η *of the following form: For a given partition* $\{t_1,r_2,\cdots,t_n\}=\pi_T$ *of* $[0,T]$,

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)1_{[t_j,t_{j+1})}(t),$$

where $\xi_i \in \mathrm{Lip}(\Omega_{t_i})$ *for* $i=0,1,2,\cdots,N-1$*. For* $p\geq 1$ *and* $\eta\in M_G^0(0,T)$*, define*

$$\|\eta\|_{H_G^p}:=\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}}|\eta_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}},\quad \|\eta\|_{M_G^p}:=\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}}|\eta_s|^p\mathrm{d}s\Big)\Big]^{\frac{1}{p}}.$$

Denote by $H_G^p(0,T)$ *and* $M_G^p(0,T)$ *the completion of* $M_G^0(0,T)$ *under norms* $\|\cdot\|_{H_G^p}$ *and* $\|\cdot\|_{M_G^p}$*, respectively.*

For both processes $\eta\in M_G^2(0,T)$ and $\xi\in M_G^1(0,T)$, the G-Itô integrals $\big\{\int_0^t\eta_s\mathrm{d}B_s^i,\ 0\leq t\leq T\big\}$ and $\{\int_0^t\xi_s\mathrm{d}\langle B^i,B^j\rangle_s,\ 0\leq t\leq T\}$ are well defined in [25, 33]. Moreover, the following BDG inequality can be found in [38, Proposition 4.3, p. 295].

Proposition 2.2 *For* $\eta\in H_G^\alpha(0,T)$ *with* $\alpha\geq 1$ *and* $p\in(0,\alpha]$*, we have*

$$\underline{\sigma}^p c_p\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|\eta_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\leq\widehat{\mathbb{E}}_t\Big[\sup_{u\in[t,T]}\Big|\int_t^u\eta_s\mathrm{d}B_s\Big|^p\Big]\leq\overline{\sigma}^pC_p\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|\eta_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big],\quad \textit{q.s.}$$

Denote by $\mathcal{C}_{b,\mathrm{lip}}(\mathbb{R}^{1+d\times n})$ the collection of all bounded and Lipschitz functions on $\mathbb{R}^{1+d\times n}$. Define

$$S_G^0(0,T):=\Big\{h(t,B_{t_1\wedge t},B_{t_2\wedge t}-B_{t_1\wedge t},\cdots,B_{t_n\wedge t}-B_{t_{n-1}\wedge t}):\\ h\in\mathcal{C}_{b,\mathrm{lip}}(\mathbb{R}^{1+d\times n})\text{ and }t_1,t_2,\cdots,t_n\in[0,T]\Big\}.$$

For $p\geq1$ and $\eta\in S_G^0(0,T)$, set

$$\|\eta\|_{S_G^p}:=\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\eta_t|^p\Big]^{\frac{1}{p}}.$$

Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$. The following continuity of $Y\in S_G^p(0,T)$ for $p>1$ can be found in Li, Peng and Song [23, Lemma 3.7, p. 12].

Lemma 2.1 *For* $Y\in S_G^p(0,T)$ *with* $p>1$*, we have, by setting* $Y_s=Y_T$ *for* $s>T$*,*

$$F(Y):=\limsup_{\varepsilon\to0}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|Y_t-Y_s|^p\Big]^{\frac{1}{p}}=0.$$

Similar to $S_G^p(0,T)$, we can define the space $S_G^\infty(0,T)$ as the completion of $S_G^0(0,T)$ under the norm $\|\eta\|_{S_G^\infty}:=\big\|\sup_{t\in[0,T]}|\eta_t|\big\|_{L_G^\infty}$.

We now introduce some results on quadratic G-BSDEs in [15]. For simplicity, we assume $d=1$ and consider the following type of equation:

$$\begin{aligned}Y_t=\xi+\int_t^{\mathrm{T}}g(s,\omega_{\cdot\wedge s},Y_s,Z_s)\mathrm{d}s+\int_t^{\mathrm{T}}f(s,\omega_{\cdot\wedge s},Y_s,Z_s)\mathrm{d}\langle B\rangle_s\\ -\int_t^{\mathrm{T}}Z_s\mathrm{d}B_s-\int_t^{\mathrm{T}}\mathrm{d}K_s,\quad\text{q.s.},\end{aligned}\tag{2.1}$$

where the generator $(f,g):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}^2$ and the terminal value ξ are supposed to satisfy the following conditions:

(H1) $\int_0^{\mathrm{T}}|f(t,\omega,0,0)|^2\mathrm{d}t+\int_0^{\mathrm{T}}|g(t,\omega,0,0)|^2\mathrm{d}t+|\xi(\omega)|\leq M_0$, q.s.;

(H2) The generator (f,g) is uniformly continuous in (t,ω), i.e., there is a non-decreasing continuous function $w:[0,+\infty)\to[0,+\infty)$ such that $w(0)=0$ and

$$\sup_{y,z\in\mathbb{R}}|f(t,\omega,y,z)-f(t',\omega',y,z)|\le w(|t-t'|+\|\omega-\omega'\|_\infty),$$
$$\sup_{y,z\in\mathbb{R}}|g(t,\omega,y,z)-g(t',\omega',y,z)|\le w(|t-t'|+\|\omega-\omega'\|_\infty);$$

(H3) There are two positive constants L_y and L_z such that for each $(t,\omega)\in[0,T]\times\Omega$,

$$|f(t,\omega,y,z)-f(t,\omega,y',z')|+|g(t,\omega,y,z)-g(t,\omega,y',z')|\le L_y|y-y'|+L_z(1+|z|+|z'|)|z-z'|.$$

Remark 2.2 In [15], the triple (f,g,ξ) is supposed to satisfy the following condition:

(H1') For each $t\in[0,T]$, $|f(t,\omega,0,0)|+|g(t,\omega,0,0)|+|\xi(\omega)|\le M_0$, q.s.

The results there still hold if **(H1')** is replaced with **(H1)**, by a similar analysis as in [12, 15].

Remark 2.3 **(H3)** implies

$$|h(t,\omega,y,z)|\le|h(t,\omega,0,0)|+L_y|y|+L_z(|z|+|z|^2)\le|h(t,\omega,0,0)|+\frac{1}{2}L_z+L_y|y|+\frac{3}{2}L_z|z|^2$$

with $h=f,g$. So (f,g) are linear in y and quadratic in z.

For simplicity, we denote by $\mathfrak{G}_G^p(0,T)$ the collection of process (Y,Z,K) such that $(Y,Z)\in S_G^p(0,T)\times H_G^p(0,T)$ and K is a decreasing G-martingale with $K_0=0$ and $K_T\in L_G^p(\Omega_T)$. Hu et al. [15, Theorem 5.3, p. 22; (3.2) and (3.3), p. 13] gave the following theorem.

Theorem 2.4 *Assume that $\xi\in L_G^\infty(\Omega_T)$ and the triple (f,g,ξ) satisfies* **(H1)**–**(H3)**. *Then* (2.1) *has a unique solution $(Y,Z,K)\in\mathfrak{G}_G^2(0,T)$ such that*

$$\|Y\|_{S_G^\infty}+\|Z\|_{\mathrm{BMO}_G}\le C(M_0,L_y,L_z)$$

and

$$\widehat{\mathbb{E}}[|K_T|^p]\le C(p,M_0,L_y,L_z),\quad\forall\,p\ge1,$$

where the norm $\|\cdot\|_{\mathrm{BMO}_G}$ will be defined in Subsection 2.3.

2.2 Formulation of the problem

For simplicity, we consider the G-expectation space $(\Omega,L_G^1(\Omega_T),\widehat{\mathbb{E}})$ for the case of $d=1$ and $\overline{\sigma}^2=\widehat{\mathbb{E}}[B_1^2]\ge-\widehat{\mathbb{E}}[-B_1^2]=\underline{\sigma}^2>0$. Consider the following equation:

$$\begin{cases}Y_t=\xi+\displaystyle\int_t^{\mathrm{T}}g(s,\omega_{\cdot\wedge s},Y_s,Z_s)\mathrm{d}s+\int_t^{\mathrm{T}}f(s,\omega_{\cdot\wedge s},Y_s,Z_s)\mathrm{d}\langle B\rangle_s\\ \qquad-\displaystyle\int_t^{\mathrm{T}}Z_s\mathrm{d}B_s+\int_t^{\mathrm{T}}\mathrm{d}A_s,\quad\text{q.s. } t\in[0,T];\\ Y_t\ge S_t,\quad\text{q.s. } t\in[0,T];\\ \text{the process }-\displaystyle\int_0^{\cdot}(Y_s-S_s)\mathrm{d}A_s\text{ is a non-increasing }G\text{-martingale on }[0,T],\end{cases}\tag{2.2}$$

where the generator $(f,g):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}^2$ and the terminal value ξ are assumed to satisfy **(H1)**–**(H3)**. Moreover, the obstacle process $\{S_t,\ t\in[0,T]\}$ is supposed to satisfy the following conditions:

(H4) $S \in \bigcap_{\alpha>1} S_G^\alpha(0,T)$ with $S_T \le \xi$, q.s. Furthermore, there is a positive constant N_0 such that $S_t \le N_0$, q.s. for any $t \in [0,T]$.

(H5) S is uniformly continuous in (t,ω), i.e., there is a non-decreasing continuous function $w : [0,+\infty) \to [0,+\infty)$ with $w(0) = 0$ such that

$$|S_t(\omega) - S_{t'}(\omega')| \le w(|t-t'| + \|\omega - \omega'\|_\infty).$$

Remark 2.4 Like in [15], **(H2)** and **(H5)** are used to ensure the existence of solutions to our subsequent penalized quadratic G-BSDEs.

A solution of reflected G-BSDEs is defined as follows.

Definition 2.9 *A triple of processes (Y,Z,A) belongs to $\mathcal{S}_G^\alpha(0,T)$ for $\alpha > 1$ if $(Y,Z) \in S_G^\alpha(0,T) \times H_G^\alpha(0,T)$ and A is a continuous nondecreasing process such that $A_0 = 0$ and $A_T \in L_G^\alpha(\Omega_T)$. The triple (Y,Z,A) is said to be a solution to the reflected G-BSDE* (2.2) *if $(Y,Z,A) \in \mathcal{S}_G^\alpha(0,T)$, and satisfies* (2.2) *for $t \in [0,T]$.*

Our objective is to establish the existence and uniqueness result for the quadratic G-BSDE (2.2). For simplicity of exposition, we assume $g \equiv 0$ in what follows. Corresponding results still hold for the case of $g \neq 0$.

2.3 *G*-BMO martingales and *G*-Girsanov theorem

We now introduce some results of G-BMO martingale and G-Girsanov theorem in [15, 35].

Definition 2.10 *For $Z \in H_G^2(0,T)$, a symmetric G-martingale $\int_0^\cdot Z_s \mathrm{d}B_s$ on $[0,T]$ is called a G-BMO martingale if*

$$\|Z\|^2_{\mathrm{BMO}_G} := \sup_{\mathbb{P}\in\mathcal{P}} \|Z\|^2_{\mathrm{BMO}(\mathbb{P})} = \sup_{\mathbb{P}\in\mathcal{P}} \Big[\sup_{\tau\in\mathcal{T}_0^{\mathrm{T}}} \Big\| E_\tau^{\mathbb{P}} \Big[\int_\tau^{\mathrm{T}} |Z_t|^2 \mathrm{d}\langle B\rangle_t \Big] \Big\|_{L^\infty(\mathbb{P})} \Big] < +\infty,$$

where $\mathcal{T}_0^{\mathrm{T}}$ denotes the totality of all $\mathcal{F}$-stopping times taking values in $[0,T]$ and $\|Z\|_{\mathrm{BMO}(\mathbb{P})}$ stands for the BMO norm of $\int_0^\cdot Z_s \mathrm{d}B_s$ under probability measure $\mathbb{P}$.

Set

$$\mathrm{BMO}_G := \{Z \in H_G^2(0,T) : \|Z\|_{\mathrm{BMO}_G} < +\infty\}.$$

In a straightforward manner, we have the following important norm estimate for a G-BMO martingale $\int_0^\cdot Z_s \mathrm{d}B_s$.

Lemma 2.2 *For $Z \in \mathrm{BMO}_G$, we have for each $t \in [0,T]$,*

$$\widehat{\mathbb{E}}_t \Big[\Big(\int_t^{\mathrm{T}} |Z_s|^2 \mathrm{d}\langle B\rangle_s \Big)^{\frac{\alpha}{2}} \Big] \le C_\alpha \|Z\|^\alpha_{\mathrm{BMO}_G}, \quad q.s.\ \forall \alpha \ge 1,$$

where C_α is a positive constant depending on α.

Proof Fix some $(t,\mathbb{P}) \in [0,T] \times \mathcal{P}$. In view of [17, Corollary 2.1, p. 28], for each $\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})$ we have

$$E_t^{\mathbb{P}'} \Big[\Big(\int_t^{\mathrm{T}} |Z_s|^2 \mathrm{d}\langle B\rangle_s \Big)^{\frac{\alpha}{2}} \Big] \le C_\alpha \|Z\|^\alpha_{\mathrm{BMO}(\mathbb{P}')} \le C_\alpha \|Z\|^\alpha_{\mathrm{BMO}_G}, \quad \mathbb{P}'\text{-a.s.},$$

where C_α is a positive constant depending only on α. In view of the definition of $\mathcal{P}(t,\mathbb{P})$, we have

$$E_t^{\mathbb{P}'}\Big[\Big(\int_t^{\mathrm{T}}|Z_s|^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big]\le C_\alpha\|Z\|_{\mathrm{BMO}_G}^\alpha,\quad \mathbb{P}\text{-a.s.}$$

In view of Theorem 2.2 and noting that C_α is independent of $\mathbb{P}'$, we have

$$\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|Z_s|^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big]=\operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})}E_t^{\mathbb{P}'}\Big[\Big(\int_0^{\mathrm{T}}|Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^{\frac{\alpha}{2}}\Big]\le C_\alpha\|Z\|_{\mathrm{BMO}_G}^\alpha,\quad \mathbb{P}\text{-a.s.}$$

Notice that C_α is independent of $\mathbb{P}$ and we get the lemma.

Like in the classical stochastic analysis, a G-BMO martingale can be used to define an exponential G-martingale. Hu et al. [15, Lemma 3.2, p. 11] gave the following lemma.

Lemma 2.3 *For $Z\in\mathrm{BMO}_G$, the process*

$$\mathscr{E}(Z)_t:=\exp\Big(\int_0^t Z_s\mathrm{d}B_s-\frac{1}{2}\int_0^t|Z_s|^2\mathrm{d}\langle B\rangle_s\Big),\quad t\ge 0$$

is a symmetric G-martingale.

In a way similar to [35], we have the following lemmas.

Lemma 2.4 (Reverse Hölder Inequality) *Let $\phi(x)=\big\{1+\frac{1}{x^2}\log\frac{2x-1}{2(x-1)}\big\}^{\frac{1}{2}}-1$ and $1<q<+\infty$. If $\|Z\|_{\mathrm{BMO}_G}<\phi(q)$, we have*

$$\sup_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}_0^{\mathrm{T}}}\Big\|E_\tau^{\mathbb{P}}\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_\tau}\Big\}^q\Big]\Big\|_{L^\infty(\mathbb{P})}\le C_q$$

for a constant $C_q>0$ depending only on q.

Proof For each $\mathbb{P}\in\mathcal{P}$,

$$\|Z\|_{\mathrm{BMO}(\mathbb{P})}\le\|Z\|_{\mathrm{BMO}_G}<\phi(q).$$

Then, from [17, Theorem 3.1, p. 54], we have

$$\sup_{\tau\in\mathcal{T}_0^{\mathrm{T}}}\Big\|E_\tau^{\mathbb{P}}\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_\tau}\Big\}^q\Big]\Big\|_{L^\infty(\mathbb{P})}\le C_q,\quad\forall\,\mathbb{P}\in\mathcal{P}$$

for a positive constant C_q which does not depend on $\mathbb{P}$.

Lemma 2.5 *Let $1<r<+\infty$. If $\|Z\|_{\mathrm{BMO}_G}<\frac{\sqrt{2}}{2}(\sqrt{r}-1)$, then*

$$\sup_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}_0^{\mathrm{T}}}\Big\|E_\tau^{\mathbb{P}}\Big[\Big\{\frac{\mathscr{E}(Z)_\tau}{\mathscr{E}(Z)_T}\Big\}^{\frac{1}{r-1}}\Big]\Big\|_{L^\infty(\mathbb{P})}\le C_r$$

holds with a constant $C_r>0$ depending only on r.

Proof In a way similar to the proof of Lemma 2.4, the desired result is an immediate consequence of [17, Theorem 2.4, p. 33] for all $\mathbb{P}\in\mathcal{P}$.

Remark 2.5 Assume $\|Z\|_{\mathrm{BMO}_G} < \phi(q)$ for some $q \in (1,+\infty)$. Fix some $(t,\mathbb{P}) \in [0,T]\times\mathcal{P}$. In view of Lemma 2.4, we have for each $\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})$,

$$E_t^{\mathbb{P}'}\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}\Big\}^q\Big] \le C_q, \quad \mathbb{P}'\text{-a.s.}$$

In view of the definition of $\mathcal{P}(t,\mathbb{P})$, we have

$$E_t^{\mathbb{P}'}\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}\Big\}^q\Big] \le C_q, \quad \mathbb{P}\text{-a.s.}$$

Thus in view of Theorem 2.2, we get

$$\widehat{\mathbb{E}}_t\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}\Big\}^q\Big] = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} E_t^{\mathbb{P}'}\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}\Big\}^q\Big] \le C_q, \quad \mathbb{P}\text{-a.s.}$$

Noting that C_q is independent of $\mathbb{P}$, we have the following reverse Hölder inequality:

$$\widehat{\mathbb{E}}_t\Big[\Big\{\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}\Big\}^q\Big] \le C_q, \quad \text{q.s.}$$

Similarly, in view of Theorem 2.2 and Lemma 2.5, we have

$$\widehat{\mathbb{E}}_t\Big[\Big\{\frac{\mathscr{E}(Z)_t}{\mathscr{E}(Z)_T}\Big\}^{\frac{1}{r-1}}\Big] \le C_r, \quad \text{q.s.},$$

if $\|Z\|_{\mathrm{BMO}_G} < \frac{\sqrt{2}}{2}(\sqrt{r}-1)$ for some $r \in (1,+\infty)$.

Remark 2.6 The reverse Hölder inequality in Remark 2.5 is used in the proof of Hu et al. [15, Lemma 3.4]. We give a proof here for convenience of the reader.

Remark 2.7 Suppose that there exist $\{Z^n\}_{n\in\mathbb{N}} \subseteq H_G^2(0,T)$ such that $\|Z^n\|_{\mathrm{BMO}_G} \le M$ for all $n \in \mathbb{N}$. Taking $t=0$ in Reamrk 2.5, we can know that there exist $q>1$ and $r>1$ which are depending only on M such that

$$\widehat{\mathbb{E}}[\mathscr{E}(Z^n)_T^q] \le C_q, \quad \widehat{\mathbb{E}}[\{\mathscr{E}(Z^n)_T\}^{\frac{1}{1-r}}] \le C_r.$$

With the exponential martingale, we can generalize the Girsanov theorem. In [15], we know that we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ with $\mathscr{E}(Z)$ satisfying

$$\widetilde{\mathbb{E}}[X] = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\mathscr{E}(Z)_T X] = \widehat{\mathbb{E}}[\mathscr{E}(Z)_T X], \quad \forall\, X \in L_G^p(\Omega_T), \tag{2.3}$$

where $p > \frac{q}{q-1}$ and q is the order in the reverse Hölder inequality for $\mathscr{E}(Z)$. Moreover, the conditional expectation $\widetilde{\mathbb{E}}_t[\cdot]$ is well-defined following the procedure introduced in [15, 40]. And we have

$$\widetilde{\mathbb{E}}_t[X] = \widehat{\mathbb{E}}_t\Big[\frac{\mathscr{E}(Z)_T}{\mathscr{E}(Z)_t}X\Big], \quad \text{q.s. } \forall\, X \in L_G^p(\Omega_T). \tag{2.4}$$

The following two lemmas give the Girsanov theorem in the G-framework, and can be found in Hu et al. [15].

Lemma 2.6 *Suppose that $Z \in \mathrm{BMO}_G$. We define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(Z)$. Then the process $B - \int Z\mathrm{d}\langle B\rangle$ is a G-Brownian motion under $\widetilde{\mathbb{E}}[\cdot]$.*

Lemma 2.7 *Suppose that $Z \in \mathrm{BMO}_G$. We define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(Z)$. Suppose that K is a decreasing G-martingale such that $K_0 = 0$ and for some $p > \frac{q}{q-1}$, $K_t \in L_G^p(\Omega_t)$, $0 \le t \le T$, where q is the order in the reverse Hölder inequality for $\mathscr{E}(Z)$. Then K is a decreasing G-martingale under $\widetilde{\mathbb{E}}[\cdot]$.*

3 A Priori Estimates for Solutions of Reflected Quadratic G-BSDEs

With G-BMO martingale and G-Girsanov theorem, we have the following comparison theorem for quadratic G-BSDEs.

Theorem 3.1 *Let the triplet (ξ^i, f^i, g^i) satisfy* **(H1)**–**(H3)** *for $i = 1, 2$. Let $(Y^i, Z^i, K^i) \in \mathfrak{G}_G^2(0,T)$ be the solution to the following G-BSDE:*

$$\begin{aligned}Y_t^i = \xi^i &+ \int_t^T g^i(s, Y_s^i, Z_s^i)\mathrm{d}s + \int_t^T f^i(s, Y_s^i, Z_s^i)\mathrm{d}\langle B\rangle_s + \int_t^T \mathrm{d}V_s^i \\ &- \int_t^T Z_s^i \mathrm{d}B_s - \int_t^T \mathrm{d}K_s^i \quad q.s.\ t \in [0,T],\end{aligned}$$

where V^i is a continuous finite variation process, for $i = 1, 2$. Assume that

$$(Y^i, Z^i, K_T^i, V^i) \in S_G^\infty(0,T) \times \mathrm{BMO}_G \times \bigcap_{p\geq 1} L_G^p(\Omega_T) \times \bigcap_{p\geq 1} S_G^p(0,T),$$

and K^i is a decreasing G-martingale. If $\xi^1 \geq \xi^2$, $g^1 \geq g^2$, $f^1 \geq f^2$, q.s. and $V^1 - V^2$ is an increasing process, then we have $Y_t^1 \geq Y_t^2$, q.s. for any $t \in [0,T]$.

Proof Without loss of generality, we assume that $g^1 = g^2 = 0$.

Define $\widehat{\xi} := \xi^1 - \xi^2$ and for $t \in [0,T]$,

$$\widehat{Y}_t := Y_t^1 - Y_t^2, \quad \widehat{Z}_t := Z_t^1 - Z_t^2, \quad \widehat{K}_t := K_t^1 - K_t^2, \quad \widehat{V}_t := V_t^1 - V_t^2,$$

and $\widehat{f}_t := f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2)$. As in the proof of [15, Proposition 3.5], we use the method of linearization to write

$$\widehat{Y}_t = \widehat{\xi} + \int_t^T (\widehat{f}_s + \widehat{m}_s^\varepsilon + \widehat{a}_s^\varepsilon \widehat{Y}_s + \widehat{b}_s^\varepsilon \widehat{Z}_s)\mathrm{d}\langle B\rangle_s - \int_t^T \widehat{Z}_s \mathrm{d}B_s - \int_t^T \mathrm{d}\widehat{K}_s + \int_t^T d\widehat{V}_s, \quad \text{q.s.},$$

where for $0 \leq s \leq T$,

$$\begin{aligned}\widehat{a}_s^\varepsilon &:= [1 - l(\widehat{Y}_s)]\frac{f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)}{\widehat{Y}_s}\mathbf{1}_{\{|\widehat{Y}_s|>0\}}, \\ \widehat{b}_s^\varepsilon &:= [1 - l(\widehat{Z}_s)]\frac{f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)}{|\widehat{Z}_s|^2}\widehat{Z}_s\mathbf{1}_{\{|\widehat{Z}_s|>0\}}, \\ \widehat{m}_s^\varepsilon &:= l(\widehat{Y}_s)[f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)] + l(\widehat{Z}_s)[f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)]\end{aligned}$$

for a scalar Lipschitz continuous function l such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \leq l(x) \leq \mathbf{1}_{[-2\varepsilon,2\varepsilon]}(x)$ with $x \in (-\infty, +\infty)$. We also have

$$\begin{aligned}|\widehat{a}_s^\varepsilon| \leq L_y, \quad |\widehat{b}_s^\varepsilon| \leq L_z(1 + |Z_s^1| + |Z_s^2|), \\ |\widehat{m}_s^\varepsilon| \leq 2\varepsilon(L_y + L_z(1 + 2\varepsilon + 2|Z_s^1|)).\end{aligned}$$

Define $\widetilde{B}_t := B_t - \int_0^t \widehat{b}_s^\varepsilon \mathrm{d}\langle B\rangle_s$ for $t \in [0,T]$. In view of [15, Lemma 3.6], we know that $\widehat{b}^\varepsilon \in \mathrm{BMO}_G$. Therefore, we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(\widehat{b}^\varepsilon)$, such that $\widetilde{B}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}[\cdot]$. Then the last G-BSDE reads

$$\widehat{Y}_t = \widehat{\xi} + \int_t^T (\widehat{f}_s + \widehat{m}_s^\varepsilon + \widehat{a}_s^\varepsilon \widehat{Y}_s)\mathrm{d}\langle B\rangle_s - \int_t^T \widehat{Z}_s \mathrm{d}\widetilde{B}_s - \int_t^T \mathrm{d}\widehat{K}_s + \int_t^T \mathrm{d}\widehat{V}_s, \quad \text{q.s.}$$

Applying Itô's formula to $\mathrm{e}^{\int_0^t \widehat{a}_s^\varepsilon \mathrm{d}\langle B\rangle_s}\widehat{Y}_t$, we have

$$\begin{aligned}
&\mathrm{e}^{\int_0^t \widehat{a}_s^\varepsilon \mathrm{d}\langle B\rangle_s}\widehat{Y}_t \\
&= \mathrm{e}^{\int_0^{\mathrm{T}} \widehat{a}_s^\varepsilon \mathrm{d}\langle B\rangle_s}\widehat{\xi} + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{f}_s \mathrm{d}\langle B\rangle_s + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{m}_s^\varepsilon \mathrm{d}\langle B\rangle_s \\
&\quad - \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{Z}_s \mathrm{d}\widetilde{B}_s - \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\mathrm{d}\widehat{K}_s + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\mathrm{d}\widehat{V}_s \\
&\geq \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{m}_s^\varepsilon \mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{Z}_s \mathrm{d}\widetilde{B}_s + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\mathrm{d}K_s^2, \quad \text{q.s.}
\end{aligned}$$

So we have

$$-\mathrm{e}^{\int_0^t \widehat{a}_s^\varepsilon \mathrm{d}\langle B\rangle_s}\widehat{Y}_t + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\mathrm{d}K_s^2 \leq -\int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{m}_s^\varepsilon \mathrm{d}\langle B\rangle_s + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{Z}_s \mathrm{d}\widetilde{B}_s, \quad \text{q.s.}$$

In view of Hu et al. [12, Lemma 3.4] and Lemma 2.7, we know $\int_0^\cdot \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\mathrm{d}K_s^2$ is a decreasing G-martingale under both $\widehat{\mathbb{E}}[\cdot]$ and $\widetilde{\mathbb{E}}[\cdot]$. Taking conditional G-expectation on both sides, we have

$$-\mathrm{e}^{\int_0^t \widehat{a}_s^\varepsilon \mathrm{d}\langle B\rangle_s}\widehat{Y}_t \leq \widetilde{\mathbb{E}}_t\Big[-\int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s \widehat{a}_u^\varepsilon \mathrm{d}\langle B\rangle_u}\widehat{m}_s^\varepsilon \mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.}$$

Since $|\widehat{a}_s^\varepsilon| \leq L_y$, we have

$$\widehat{Y}_t \geq -\mathrm{e}^{2L_y\langle B\rangle_T}\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |\widehat{m}_s^\varepsilon| \mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.}$$

Finally, it remains to prove the limit

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |\widehat{m}_s^\varepsilon| \mathrm{d}\langle B\rangle_s\Big] = 0, \quad \text{q.s.}$$

Let $\phi(x) = \big\{1 + \frac{1}{x^2}\log\frac{2x}{2(x-1)}\big\}^{\frac{1}{2}} - 1$. We know that there exists $p > 1$ independent of ε such that

$$\|\widehat{b}_s^\varepsilon\|_{\mathrm{BMO}_G} \leq \|L_z(1 + |Z_1| + |Z_2|)\|_{\mathrm{BMO}_G} < \phi(p'),$$

where $p' = \frac{p}{p-1}$. Then according to Lemma 2.4, for $X \in L_G^p(\Omega_T)$, we have

$$\widetilde{\mathbb{E}}_t[X] = \widehat{\mathbb{E}}_t\Big[\frac{\mathscr{E}(\widehat{b}^\varepsilon)_T}{\mathscr{E}(\widehat{b}^\varepsilon)_t}X\Big] \leq \widehat{\mathbb{E}}_t\Big[\Big(\frac{\mathscr{E}(\widehat{b}^\varepsilon)_T}{\mathscr{E}(\widehat{b}^\varepsilon)_t}\Big)^{p'}\Big]^{\frac{1}{p'}}\widehat{\mathbb{E}}_t[|X|^p]^{\frac{1}{p}} \leq C_p\widehat{\mathbb{E}}_t[|X|^p]^{\frac{1}{p}}, \quad \text{q.s.}$$

In view of Lemma 2.2, we have

$$\begin{aligned}
\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |\widehat{m}_s^\varepsilon| \mathrm{d}\langle B\rangle_s\Big] &\leq 2\varepsilon\overline{\sigma}^2 T(L_y + L_z + 2L_z\varepsilon) + 4\varepsilon\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |Z_s^1| \mathrm{d}\langle B\rangle_s\Big] \\
&\leq 2\varepsilon\overline{\sigma}^2 T(L_y + L_z + 2L_z\varepsilon) + 4\varepsilon\overline{\sigma}^2 T\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |Z_s^1|^2 \mathrm{d}\langle B\rangle_s\Big]^{\frac{1}{2}} \\
&\leq 2\varepsilon\overline{\sigma}^2 T(L_y + L_z + 2L_z\varepsilon) + 4\varepsilon C_p\overline{\sigma}^2 T\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}} |Z_s^1|^2 \mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{2p}} \\
&\leq 2\varepsilon\overline{\sigma}^2 T(L_y + L_z + 2L_z\varepsilon) + 4\varepsilon C_p C_p''\|Z^1\|_{\mathrm{BMO}_G}, \quad \text{q.s.}
\end{aligned}$$

So we get $\lim\limits_{\varepsilon\to 0}\widetilde{\mathbb{E}}_t\big[\int_t^{\mathrm{T}}|\widehat{m}_s^{\varepsilon}|\mathrm{d}\langle B\rangle_s\big]=0$, q.s.

Consider the following type of BSDE:

$$\begin{cases} Y_t=\xi+\displaystyle\int_t^{\mathrm{T}} f(s,\omega_{\cdot\wedge s},Y_s,Z_s)\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}} Z_s\mathrm{d}B_s+\int_t^{\mathrm{T}}\mathrm{d}A_s, \quad \text{q.s. } t\in[0,T]; \\ Y_t\geq S_t, \quad \text{q.s. } t\in[0,T]; \quad \displaystyle\int_0^{\cdot}(S_s-Y_s)\mathrm{d}A_s \text{ is a non-increasing } G\text{-martingale,} \end{cases} \tag{3.1}$$

with A being a continuous nondecreasing process and $A_0=0$.

Proposition 3.1 *Let f satisfy* **(H1)** *and* **(H3)**. *Assume that (Y,Z,A) solves*

$$Y_t=\xi+\int_t^{\mathrm{T}} f(s,Y_s,Z_s)\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}} Z_s\mathrm{d}B_s+\int_t^{\mathrm{T}}\mathrm{d}A_s, \quad q.s.\ t\in[0,T],$$

where

$$(Y,Z)\in S_G^{\infty}(0,T)\times H_G^2(0,T),$$

and A is a continuous nondecreasing process with $A_0=0$.

Then there exists a constant $C_1:=C_1(\|Y\|_{S_G^{\infty}},T,L_z,L_y,M_0,\overline{\sigma})$ such that

$$\|Z\|_{\mathrm{BMO}_G}\leq C_1,$$

and a constant $C_2:=C_2(\|Y\|_{S_G^{\infty}},T,L_z,L_y,M_0,\overline{\sigma},\alpha)$ for any $\alpha\geq 1$, such that

$$\widehat{\mathbb{E}}[|A_T|^{\alpha}]\leq C_2.$$

Proof For each $\mathbb{P}\in\mathcal{P}$, we know

$$Y_t=\xi+\int_t^{\mathrm{T}} f(s,Y_s,Z_s)\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}} Z_s\mathrm{d}B_s+\int_t^{\mathrm{T}}\mathrm{d}A_s, \quad \mathbb{P}\text{-a.s. } t\in[0,T].$$

Then, for some $a>0$, applying Itô's formula under $\mathbb{P}$ to e^{-aY_t}, we have for each $\tau\in\mathcal{T}_0^{\mathrm{T}}$,

$$\begin{aligned}&\frac{a^2}{2}\int_\tau^{\mathrm{T}}\mathrm{e}^{-aY_s}Z_s^2\mathrm{d}\langle B\rangle_s\\&=\mathrm{e}^{-a\xi}-\mathrm{e}^{-aY_\tau}-\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}f(s,Y_s,Z_s)\mathrm{d}\langle B\rangle_s+\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}Z_s\mathrm{d}B_s\\&\quad-\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}\mathrm{d}A_s, \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Since A is a continuous nondecreasing process, noting $a>0$ and Remark 2.3, we have

$$\begin{aligned}&\frac{a^2}{2}\int_\tau^{\mathrm{T}}\mathrm{e}^{-aY_s}|Z_s|^2\mathrm{d}\langle B\rangle_s\\&\leq\mathrm{e}^{-a\xi}-\mathrm{e}^{-aY_\tau}-\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}f(s,Y_s,Z_s)\mathrm{d}\langle B\rangle_s+\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}Z_s\mathrm{d}B_s\\&\leq\mathrm{e}^{-a\xi}-\mathrm{e}^{-aY_\tau}+\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}\Big(|f(s,0,0)|+\frac{1}{2}L_z+L_y|Y_s|\Big)\mathrm{d}\langle B\rangle_s\\&\quad+\frac{3aL_z}{2}\int_\tau^{\mathrm{T}}\mathrm{e}^{-aY_s}|Z_s|^2\mathrm{d}\langle B\rangle_s+\int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}Z_s\mathrm{d}B_s, \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Taking $a = 4L_z$, noting $Y \in S_G^\infty(0,T)$ and taking conditional expectations under $\mathbb{P}$ on both sides, we have

$$
\begin{aligned}
&2L_z^2 E_\tau^{\mathbb{P}}\Big[\int_\tau^{\mathrm{T}} \mathrm{e}^{-aY_s}|Z_s|^2 \mathrm{d}\langle B\rangle_s\Big] \\
&\leq E_\tau^{\mathbb{P}}\Big[\mathrm{e}^{-a\xi} - \mathrm{e}^{-aY_\tau} + \int_\tau^{\mathrm{T}} a\mathrm{e}^{-aY_s}\Big(|f(s,0,0)| + \frac{1}{2}L_z + L_y|Y_s|\Big)\mathrm{d}\langle B\rangle_s\Big] \\
&\leq 2\mathrm{e}^{4L_z\|Y\|_{S_G^\infty}} + 4L_z\overline{\sigma}^2\Big(\sqrt{T}E_\tau^{\mathbb{P}}\Big[\Big(\int_0^{\mathrm{T}} |f(s,0,0)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big] + \frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)\mathrm{e}^{4L_z\|Y\|_{S_G^\infty}} \\
&\leq 2\mathrm{e}^{4L_z\|Y\|_{S_G^\infty}} + 4L_z\overline{\sigma}^2\Big(\sqrt{TM_0} + \frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)\mathrm{e}^{4L_z\|Y\|_{S_G^\infty}}, \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Then with the arbitrariness of τ, we obtain for all $\mathbb{P} \in \mathcal{P}$,

$$
\|Z\|_{\mathrm{BMO}(\mathbb{P})}^2 \leq \frac{1}{L_z^2}\mathrm{e}^{8L_z\|Y\|_{S_G^\infty}} + \frac{2}{L_z}\overline{\sigma}^2\Big(\sqrt{TM_0} + \frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)\mathrm{e}^{8L_z\|Y\|_{S_G^\infty}}.
$$

Finally, with the arbitrariness of $\mathbb{P}$, we get

$$
\|Z\|_{\mathrm{BMO}_G}^2 \leq \frac{1}{L_z^2}\mathrm{e}^{8L_z\|Y\|_{S_G^\infty}} + \frac{2}{L_z}\overline{\sigma}^2\Big(\sqrt{TM_0} + \frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)\mathrm{e}^{8L_z\|Y\|_{S_G^\infty}}.
$$

Now we get the estimate for Z. We have

$$
A_T = Y_0 - \xi - \int_0^{\mathrm{T}} f(s,Y_s,Z_s)\mathrm{d}\langle B\rangle_s + \int_0^{\mathrm{T}} Z_s\mathrm{d}B_s, \quad \text{q.s.}
$$

In view of BDG inequality and Remark 2.3, we have for each $\alpha \geq 1$,

$$
\begin{aligned}
\widehat{\mathbb{E}}[A_T^\alpha] &\leq C_\alpha\widehat{\mathbb{E}}[|Y_0|^\alpha + |\xi|^\alpha] + C_\alpha\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |f(s,Y_s,Z_s)|\mathrm{d}\langle B\rangle_s\Big)^\alpha\Big] \\
&\quad + C_\alpha\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^{\frac{\alpha}{2}}\Big] \\
&\leq 2C_\alpha\|Y\|_{S_G^\infty}^\alpha + \widetilde{C}_\alpha\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |f(s,0,0)|\mathrm{d}\langle B\rangle_s\Big)^\alpha + \Big(\int_0^{\mathrm{T}} \frac{1}{2}L_z + L_y|Y_s|\mathrm{d}\langle B\rangle_s\Big)^\alpha\Big] \\
&\quad + \frac{3\widetilde{C}_\alpha L_z}{2}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^\alpha\Big] + C_\alpha\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^{\frac{\alpha}{2}}\Big] \\
&\leq 2C_\alpha\|Y\|_{S_G^\infty}^\alpha + \widetilde{C}_\alpha\overline{\sigma}^{2\alpha}\Big\{\widehat{\mathbb{E}}\Big[\Big(T\int_0^{\mathrm{T}} |f(s,0,0)|^2\mathrm{d}s\Big)^{\frac{\alpha}{2}}\Big] + \Big(\frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)^\alpha\Big\} \\
&\quad + \frac{3\widetilde{C}_\alpha L_z}{2}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^\alpha\Big] + C_\alpha\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_t|^2\mathrm{d}\langle B\rangle_t\Big)^{\frac{\alpha}{2}}\Big].
\end{aligned}
$$

In view of Lemma 2.2, we have

$$
\begin{aligned}
\widehat{\mathbb{E}}[A_T^\alpha] \leq\ & 2C_\alpha\|Y\|_{S_G^\infty}^\alpha + \widetilde{C}_\alpha\overline{\sigma}^{2\alpha}\Big\{(TM_0)^{\frac{\alpha}{2}} + \Big(\frac{1}{2}L_zT + L_yT\|Y\|_{S_G^\infty}\Big)^\alpha\Big\} \\
& + \frac{3\widetilde{C}_\alpha\overline{C}_{2\alpha}L_z}{2}\|Z\|_{\mathrm{BMO}_G}^{2\alpha} + C_\alpha\overline{C}_\alpha\|Z\|_{\mathrm{BMO}_G}^\alpha.
\end{aligned}
$$

Substituting the estimate for Z, we get the estimate for A.

Proposition 3.2 *Let* (ξ, f, S) *satisfy* **(H1)** *and* **(H3)**–**(H4)**. *Assume that the triplet* $(Y, Z, A) \in \mathcal{S}_G^{2p}(0,T)$, *with some* $p > 1$, *is a solution to the reflected G-BSDE with data* (ξ, f, S). *Moreover, we suppose*

$$\|L_z(1+|Z|)\|_{\mathrm{BMO}_G} < \phi(q) := \Big\{1 + \frac{1}{q^2}\log\frac{2q-1}{2(q-1)}\Big\}^{\frac{1}{2}} - 1$$

with q *satisfying* $p > \frac{q}{q-1}$.

Then there exists a constant $C := C(T, L_z, L_y, \overline{\sigma}, N_0)$ *such that*

$$\|Y_t\|_{L_G^\infty} \le C\Big(1 + \|\xi\|_{L_G^\infty} + \Big\|\int_0^{\mathrm{T}} |f(s,0,0)|^2 \mathrm{d}s\Big\|_{L_G^\infty}^{\frac{1}{2}}\Big), \quad \forall\, t \in [0,T].$$

Proof For some $r > 0$, applying Itô's formula to $\mathrm{e}^{rt}|Y_t - N_0|^2$, we have for each $t \in [0,T]$,

$$\begin{aligned}
&\mathrm{e}^{rt}|Y_t - N_0|^2 + r\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s - N_0|^2\mathrm{d}s + \int_t^{\mathrm{T}} \mathrm{e}^{rs}|Z_s|^2\mathrm{d}\langle B\rangle_s\\
&= \mathrm{e}^{rT}|\xi - N_0|^2 + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)f(s, Y_s, Z_s)\mathrm{d}\langle B\rangle_s\\
&\quad - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)Z_s\mathrm{d}B_s + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)\mathrm{d}A_s, \quad \text{q.s.}
\end{aligned}$$

We have

$$f(s, Y_s, Z_s) = f(s,0,0) + m_s^\varepsilon + a_s^\varepsilon Y_s + b_s^\varepsilon Z_s,$$

where

$$\begin{aligned}
a_s^\varepsilon &:= [1 - l(Y_s)]\frac{f(s,Y_s,Z_s) - f(s,0,Z_s)}{Y_s}\mathbf{1}_{\{|Y_s|>0\}},\\
b_s^\varepsilon &:= [1 - l(Z_s)]\frac{f(s,0,Z_s) - f(s,0,0)}{|Z_s|^2}Z_s\mathbf{1}_{\{|Z_s|>0\}},\\
m_s^\varepsilon &:= l(Y_s)[f(s,Y_s,Z_s) - f(s,0,Z_s)] + l(Z_s)[f(s,0,Z_s) - f(s,0,0)]
\end{aligned}$$

with $s \in [0,T]$ and the function l being Lipschitz continuous such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \le l(x) \le \mathbf{1}_{[-2\varepsilon,2\varepsilon]}(x)$ for $x \in (-\infty, +\infty)$. Moreover,

$$|a_s^\varepsilon| \le L_y, \quad |b_s^\varepsilon| \le L_z(1+|Z_s|), \quad |m_s^\varepsilon| \le 2\varepsilon(L_y + L_z(1+2\varepsilon)).$$

In view of [15, Lemma 3.6], we know that $b^\varepsilon \in \mathrm{BMO}_G$. Set $\widetilde{B}_t := B_t - \int_0^t b_s^\varepsilon \mathrm{d}\langle B\rangle_s$ for $t \in [0,T]$. Thus we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(b_s^\varepsilon)$, such that $\widetilde{B}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}[\cdot]$. Then we have for each $t \in [0,T]$,

$$\begin{aligned}
&\mathrm{e}^{rt}|Y_t - N_0|^2 + r\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s - N_0|^2\mathrm{d}s + \int_t^{\mathrm{T}} \mathrm{e}^{rs}|Z_s|^2\mathrm{d}\langle B\rangle_s\\
&\le \mathrm{e}^{rT}|\xi - N_0|^2 + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)(f(s,0,0) + m_s^\varepsilon + a_s^\varepsilon Y_s + b_s^\varepsilon Z_s)\mathrm{d}\langle B\rangle_s\\
&\quad - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)Z_s\mathrm{d}B_s + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)\mathrm{d}A_s\\
&\le \mathrm{e}^{rT}|\xi - N_0|^2 + (1 + 2L_y)\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s - N_0|^2\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s - N_0)Z_s\mathrm{d}\widetilde{B}_s
\end{aligned}$$

$$+\int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0)+|m_s^{\varepsilon}|+N_0L_y)^2\mathrm{d}\langle B\rangle_s+\int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s-S_s)\mathrm{d}A_s, \quad \text{q.s.}$$

Setting $r>\overline{\sigma}^2(1+2L_y)$ and taking conditional expectations on both sides, we have

$$\begin{aligned}&\mathrm{e}^{rt}|Y_t-N_0|^2+\widetilde{\mathbb{E}}_t\Big[-\int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s-S_s)\mathrm{d}A_s\Big]\\ \le\ &\widetilde{\mathbb{E}}_t[\mathrm{e}^{rT}|\xi-N_0|^2]+\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0)+|m_s^{\varepsilon}|+N_0L_y)^2\mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.}\end{aligned}$$

From (3.1), we know that $\{-\int_0^t(Y_s-S_s)\mathrm{d}A_s\}_{t\in[0,T]}$ is a non-increasing G-martingale under $\widehat{\mathbb{E}}[\cdot]$. Moreover,

$$\widehat{\mathbb{E}}\Big[\Big(\int_0^t(Y_s-S_s)\mathrm{d}A_s\Big)^p\Big]\le\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|Y_s-S_s|^p\Big(\int_0^{\mathrm{T}}\mathrm{d}A_s\Big)^p\Big]\le\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|Y_s-S_s|^{2p}\Big]^{\frac12}\widehat{\mathbb{E}}[|A_T|^{2p}]^{\frac12}.$$

Note $p>\frac{q}{q-1}$ and

$$\|b_s^{\varepsilon}\|_{\mathrm{BMO}_G}\le\|L_z(1+|Z|)\|_{\mathrm{BMO}_G}<\phi(q).$$

In view of Lemma 2.7, we know that $\big\{-\int_0^t(Y_s-S_s)\mathrm{d}A_s\big\}_{t\in[0,T]}$ is a non-increasing G-martingale under $\widetilde{\mathbb{E}}[\cdot]$. Then for each $t\in[0,T]$,

$$\begin{aligned}\mathrm{e}^{rt}|Y_t-N_0|^2&\le\widetilde{\mathbb{E}}_t[\mathrm{e}^{rT}|\xi-N_0|^2]+\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0)+|m_s^{\varepsilon}|+N_0L_y)^2\mathrm{d}\langle B\rangle_s\Big]\\&\le 2\mathrm{e}^{rT}(\|\xi\|^2_{L_G^{\infty}}+N_0^2)+2\mathrm{e}^{rT}\overline{\sigma}^2\Big\{(2\varepsilon(L_y+L_z(1+2\varepsilon))+N_0L_y)^2T\\&\quad+\Big\|\int_0^{\mathrm{T}}|f(t,0,0)|^2\mathrm{d}t\Big\|_{L_G^{\infty}}\Big\}, \quad \text{q.s.}\end{aligned}$$

Letting $\varepsilon\to0$, we have

$$\mathrm{e}^{rt}|Y_t-N_0|^2\le 2\mathrm{e}^{rT}(\|\xi\|^2_{L_G^{\infty}}+N_0^2)+2\mathrm{e}^{rT}\overline{\sigma}^2\Big(N_0^2L_y^2T+\Big\|\int_0^{\mathrm{T}}|f(t,0,0)|^2\mathrm{d}t\Big\|_{L_G^{\infty}}\Big), \quad \text{q.s.}$$

So we get the estimate for Y.

Proposition 3.3 *Let (ξ^1,f^1,S^1) and (ξ^2,f^2,S^2) be two sets of data, each one satisfying* **(H1)** *and* **(H3)**–**(H4)***. Assume that the triplet $(Y^i,Z^i,A^i)\in\mathcal{S}_G^{2p}(0,T)$, with some $p>1$, is a solution of the reflected G-BSDE with data (ξ^i,f^i,S^i), $i=1,2$. Moreover, we suppose*

$$\|L_z(1+|Z^1|+|Z^2|)\|_{\mathrm{BMO}_G}<\phi(q):=\Big\{1+\frac{1}{q^2}\log\frac{2q-1}{2(q-1)}\Big\}^{\frac12}-1$$

with q satisfying $p>\frac{q}{q-1}$. Then there exists a constant $C_1:=C_1(q,T,L_z,L_y,N_0)$ such that for each $t\in[0,T]$,

$$\begin{aligned}|Y_t^1-Y_t^2|^2&\le C_1\|\xi^1-\xi^2\|^2_{L_G^{\infty}}+C_1`\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|\widehat{\lambda}_s|^2\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac1p}\\&\quad+C_1\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|S_t^1-S_t^2|^{2p}\Big]^{\frac{1}{2p}}\widehat{\mathbb{E}}_t[|A_T^1-A_t^1|^{2p}+|A_T^2-A_t^2|^{2p}]^{\frac{1}{2p}}, \quad q.s.,\end{aligned}$$

where

$$\widehat{\lambda}_s := f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2).$$

Moreover, there exists a constant $C_2 := C_2(T, L_z, L_y, \overline{\sigma}, N_0, M_0)$ *such that*

$$\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} |Z_s^1 - Z_s^2|^2 \mathrm{d}\langle B\rangle_s\Big] \le C_2 \|\widehat{Y}\|_{S_G^\infty}(1 + \widehat{\mathbb{E}}_t[|A_T^1 - A_t^1| + |A_T^2 - A_t^2|]), \quad q.s.\ \forall\, t \in [0, T].$$

Proof First, with Propositions 3.1 and 3.2, we know that there exists a constant $C := C(T, L_z,$
$L_y, \overline{\sigma}, N_0, M_0)$ such that

$$\sum_{i=1}^{2} (\|Y^i\|_{S_G^\infty} + \|Z^i\|_{\mathrm{BMO}_G}) \le C. \tag{3.2}$$

Define

$$\widehat{Y}_t := Y_t^1 - Y_t^2, \quad \widehat{Z}_t := Z_t^1 - Z_t^2, \quad \widehat{S}_t := S_t^1 - S_t^2, \quad \widehat{\xi} := \xi^1 - \xi^2.$$

With the condition of f^1 and f^2, we see that $\widehat{\lambda} \in H_G^{2p}(0, T)$. As in the proof of Proposition 3.2, define for $0 \le s \le T$,

$$\begin{aligned}
\widehat{a}_s^\varepsilon &:= [1 - l(\widehat{Y}_s)] \frac{f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)}{\widehat{Y}_s} \mathbf{1}_{\{|\widehat{Y}_s| > 0\}}, \\
\widehat{b}_s^\varepsilon &:= [1 - l(\widehat{Z}_s)] \frac{f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)}{|\widehat{Z}_s|^2} \widehat{Z}_s \mathbf{1}_{\{|\widehat{Z}_s| > 0\}}, \\
\widehat{m}_s^\varepsilon &:= l(\widehat{Y}_s)[f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)] + l(\widehat{Z}_s)[f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)],
\end{aligned}$$

where l is a Lipschitz continuous function such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \le l(x) \le \mathbf{1}_{[-2\varepsilon,2\varepsilon]}(x)$. Also define $\widehat{A} := A^1 - A^2$. We have

$$\widehat{Y}_t = \widehat{\xi} + \int_t^{\mathrm{T}} (\widehat{\lambda}_s + \widehat{m}_s^\varepsilon + \widehat{a}_s^\varepsilon \widehat{Y}_s + \widehat{b}_s^\varepsilon \widehat{Z}_s) \mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} \widehat{Z}_s \mathrm{d}B_s + \int_t^{\mathrm{T}} \mathrm{d}\widehat{A}_s, \quad \text{q.s. } t \in [0, T],$$

and for each $s \in [0, T]$,

$$\begin{aligned}
|\widehat{a}_s^\varepsilon| \le L_y, &\quad |\widehat{b}_s^\varepsilon| \le L_z(1 + |Z_s^1| + |Z_s^2|), \\
|\widehat{m}_s^\varepsilon| &\le 2\varepsilon(L_y + L_z(1 + 2\varepsilon + 2|Z_s^1|)).
\end{aligned}$$

Then we have

$$\widehat{Y}_t = \widehat{\xi} + \int_t^{\mathrm{T}} (\widehat{\lambda}_s + \widehat{m}_s^\varepsilon + \widehat{a}_s^\varepsilon \widehat{Y}_s) \mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} \widehat{Z}_s \mathrm{d}\widetilde{B}_s + \int_t^{\mathrm{T}} \mathrm{d}\widehat{A}_s, \quad \text{q.s. } t \in [0, T],$$

where $\mathrm{d}\widetilde{B}_s = \mathrm{d}B_s - \widehat{b}_s^\varepsilon \mathrm{d}\langle B\rangle_s$. In a way similar to the proof of Proposition 3.2, we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(\widehat{b}^\varepsilon)$, such that $\widetilde{B}$ is a G-Brownian Motion under $\widetilde{\mathbb{E}}[\cdot]$.

For some $r > 0$, applying Itô's formula to $\mathrm{e}^{rt}|\widehat{Y}_t|^2$, we have for each $t \in [0, T]$,

$$\begin{aligned}
&\mathrm{e}^{rt}|\widehat{Y}_t|^2 + r\int_t^{\mathrm{T}} \mathrm{e}^{rs}|\widehat{Y}_s|^2 \mathrm{d}s + \int_t^{\mathrm{T}} \mathrm{e}^{rs}|\widehat{Z}_s|^2 \mathrm{d}\langle B\rangle_s \\
&= \mathrm{e}^{rT}|\widehat{\xi}|^2 + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}\widehat{Y}_s(\widehat{\lambda}_s + \widehat{m}_s^\varepsilon + \widehat{a}_s^\varepsilon \widehat{Y}_s) \mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}\widehat{Y}_s \mathrm{d}\widetilde{B}_s + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}\widehat{Y}_s \mathrm{d}\widehat{A}_s
\end{aligned}$$

$$
\begin{aligned}
&\le \mathrm{e}^{rT}|\widehat{\xi}|^2+\int_t^{\mathrm{T}}\mathrm{e}^{rs}(|\widehat{\lambda}_s|^2+|\widehat{m}_s^{\varepsilon}|^2)\mathrm{d}\langle B\rangle_s+(2+2L_y)\int_t^{\mathrm{T}}\mathrm{e}^{rs}|\widehat{Y}_s|^2\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}}2\mathrm{e}^{rs}\widehat{Y}_s\mathrm{d}\widetilde{B}_s\\
&\quad+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}\widehat{S}_s\mathrm{d}\widehat{A}_s+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}(\widehat{Y}_s-\widehat{S}_s)\mathrm{d}\widehat{A}_s\\
&\le \mathrm{e}^{rT}|\widehat{\xi}|^2+\int_t^{\mathrm{T}}\mathrm{e}^{rs}(|\widehat{\lambda}_s|^2+|\widehat{m}_s^{\varepsilon}|^2)\mathrm{d}\langle B\rangle_s+(2+2L_y)\int_t^{\mathrm{T}}\mathrm{e}^{rs}|\widehat{Y}_s|^2\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}}2\mathrm{e}^{rs}\widehat{Y}_s\mathrm{d}\widetilde{B}_s\\
&\quad+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}\widehat{S}_sd\widehat{A}_s+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}(Y_s^1-S_s^1)\mathrm{d}A_s^1+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}(Y_s^2-S_s^2)\mathrm{d}A_s^2,\quad \text{q.s.}
\end{aligned}
$$

In view of a similar argument as in the proof of Proposition 3.2, we see that $\int_0^{\cdot}(S_s^i-Y_s^i)\,\mathrm{d}A_s^i$ is a non-increasing G-martingale on $[0,T]$ under $\widetilde{\mathbb{E}}[\cdot]$ for $i=1,2$.

Setting $r>\overline{\sigma}^2(2+2L_y)$ and taking conditional expectations on both sides, we have

$$
\begin{aligned}
\mathrm{e}^{rt}|\widehat{Y}_t|^2 &\le \widetilde{\mathbb{E}}_t[\mathrm{e}^{rT}|\widehat{\xi}|^2]+\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}\mathrm{e}^{rs}(|\widehat{\lambda}_s|^2+|\widehat{m}_s^{\varepsilon}|^2)\mathrm{d}\langle B\rangle_s+\int_t^{\mathrm{T}}2\mathrm{e}^{rs}\widehat{S}_s\mathrm{d}\widehat{A}_s\Big]\\
&\le \mathrm{e}^{rT}\Big\{\|\widehat{\xi}\|^2_{L_G^\infty}+\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}(|\widehat{\lambda}_s|^2+|\widehat{m}_s^{\varepsilon}|^2)\mathrm{d}\langle B\rangle_s\Big]+2\widetilde{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|\widehat{S}_s||\widehat{A}_T-\widehat{A}_t|\Big]\Big\},\quad \text{q.s.}
\end{aligned}
$$

Note that $\|\widehat{b_s^{\varepsilon}}\|_{\mathrm{BMO}_G}\le\|L_z(1+|Z_1|+|Z_2|)\|_{\mathrm{BMO}_G}<\phi(q)<\phi(p')$ where $p'=\frac{p}{p-1}$. Then according to Lemma 2.4, $\forall\, X\in L_G^p(\Omega_T)$, we have for each $t\in[0,T]$,

$$
\widetilde{\mathbb{E}}_t[X]=\widehat{\mathbb{E}}_t\Big[\frac{\mathscr{E}(\widehat{b^{\varepsilon}})_T}{\mathscr{E}(\widehat{b^{\varepsilon}})_t}X\Big]\le\widehat{\mathbb{E}}_t\Big[\Big(\frac{\mathscr{E}(\widehat{b^{\varepsilon}})_T}{\mathscr{E}(\widehat{b^{\varepsilon}})_t}\Big)^{p'}\Big]^{\frac{1}{p'}}\widehat{\mathbb{E}}_t[X^p]^{\frac{1}{p}}\le C_p\widehat{\mathbb{E}}_t[X^p]^{\frac{1}{p}},\quad \text{q.s.}
$$

So by the Hölder inequality, we have for each $t\in[0,T]$,

$$
\begin{aligned}
\widetilde{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|\widehat{S}_s||\widehat{A}_T-\widehat{A}_t|\Big]&\le C_p\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|\widehat{S}_s|^p|\widehat{A}_T-\widehat{A}_t|^p\Big]^{\frac{1}{p}}\\
&\le C_pC_p'\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|\widehat{S}_s|^p\Big]^{\frac{1}{2p}}\widehat{\mathbb{E}}_t\Big[|A_T^1-A_t^1|^{2p}+|A_T^2-A_t^2|^{2p}\Big]^{\frac{1}{2p}},\quad \text{q.s.}
\end{aligned}
$$

Then there exists a constant $C_1:=C_1(q,T,L_z,L_y,N_0)$ such that for each $t\in[0,T]$,

$$
\begin{aligned}
|\widehat{Y}_t^2|&\le C_1\Big\{\|\widehat{\xi}\|^2_{L_G^\infty}+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|\widehat{\lambda}_s|^2\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{p}}+\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|\widehat{m}_s^{\varepsilon}|^2\mathrm{d}\langle B\rangle_s\Big]\\
&\quad+\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]}|\widehat{S}_t|^{2p}\Big]^{\frac{1}{2p}}\widehat{\mathbb{E}}_t[|A_T^1-A_t^1|^{2p}+|A_T^2-A_t^2|^{2p}]^{\frac{1}{2p}}\Big\},\quad \text{q.s.}
\end{aligned}
$$

Finally, we just need to prove

$$
\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|\widehat{m}_s^{\varepsilon}|^2\mathrm{d}\langle B\rangle_s\Big]=0,\quad \text{q.s.}
$$

In view of Lemma 2.2, we have

$$
\begin{aligned}
\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|\widehat{m}_s^{\varepsilon}|^2\mathrm{d}\langle B\rangle_s\Big]&\le 8\varepsilon^2\overline{\sigma}^2T(L_y+L_z+2L_z\varepsilon)^2+32\varepsilon^2\widetilde{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|Z_s^1|^2\mathrm{d}\langle B\rangle_s\Big]\\
&\le 8\varepsilon^2\overline{\sigma}^2T(L_y+L_z+2L_z\varepsilon)^2+32\varepsilon^2C_p\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|Z_s^1|^2\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{p}}
\end{aligned}
$$

$$\leq 8\varepsilon^2\overline{\sigma}^2 T(L_y + L_z + 2L_z\varepsilon)^2 + 32\varepsilon^2 C_p C_p''\|Z^1\|^2_{\mathrm{BMO}_G}, \quad \text{q.s.}$$

So we get $\lim_{\varepsilon\to 0}\widetilde{\mathbb{E}}_t\big[\int_t^{\mathrm{T}}|\widehat{m}_s^{\varepsilon}|^2\mathrm{d}\langle B\rangle_s\big] = 0$. And we get the estimate for $\widehat{Y}$.

Then we consider the estimate for $\widehat{Z}$. Applying Itô's formula to $\mathrm{e}^{rt}|\widehat{Y}_t|^2$, we have for each $t\in[0,T]$,

$$\begin{aligned}&|\widehat{Y}_t|^2 + \int_t^{\mathrm{T}}|\widehat{Z}_s|^2\mathrm{d}\langle B\rangle_s\\ &= |\widehat{\xi}|^2 + \int_t^{\mathrm{T}} 2\widehat{Y}_s(f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2))\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}}2\widehat{Y}_s\mathrm{d}B_s + \int_t^{\mathrm{T}}2\widehat{Y}_s\mathrm{d}\widehat{A}_s, \quad \text{q.s.}\end{aligned}$$

Taking conditional expectations on both sides, we have

$$\begin{aligned}&\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|\widehat{Z}_s|^2\mathrm{d}\langle B\rangle_s\Big]\\ &\leq \|\widehat{Y}\|_{S_G^\infty}\Big(M_0 + 2\sum_{i=1}^2\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|f^i(s,Y_s^i,Z_s^i)|\mathrm{d}\langle B\rangle_s\Big] + \widehat{\mathbb{E}}_t[|\widehat{A}_T - \widehat{A}_t|]\Big), \quad \text{q.s.}\end{aligned}$$

Note that $\forall\, i = 1,2$,

$$\begin{aligned}&\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|f^i(s,Y_s^i,Z_s^i)|\mathrm{d}\langle B\rangle_s\Big]\\ &\leq \widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}\Big(|f^i(s,0,0)| + \frac{L_z}{2} + L_y|Y_s^i| + \frac{3L_z}{2}|Z_s^i|^2\Big)\mathrm{d}\langle B\rangle_s\Big]\\ &\leq \Big(\sqrt{M_0T} + \frac{L_zT}{2}\Big)\overline{\sigma}^2 + L_y\overline{\sigma}^2T\|Y^i\|_{S_G^\infty} + \frac{3L_z}{2}\|Z^i\|^2_{\mathrm{BMO}_G}, \quad \text{q.s.}\end{aligned}$$

With (3.2), we get the estimate for $\widehat{Z}$.

Remark 3.1 The uniqueness for solutions to the reflected quadratic G-BSDE is an immediate consequence of Proposition 3.3.

4 Penalized *G*-BSDEs and Their Limit

Similar to [22, 24], we use a penalized method. In this section, we first prove some convergence properties of solutions to the penalized G-BSDEs. For (f,ξ,S) satisfying (**H1**)–(**H5**) and $n\in\mathbb{N}$, we consider the following penalized G-BSDE:

$$\begin{aligned}Y_t^n = \xi + \int_t^{\mathrm{T}} f(s,Y_s^n,Z_s^n)\mathrm{d}\langle B\rangle_s + n\int_t^{\mathrm{T}}(Y_s^n - S_s)^-\mathrm{d}s\\ - \int_t^{\mathrm{T}} Z_s^n\mathrm{d}B_s - \int_t^{\mathrm{T}}\mathrm{d}K_s^n, \quad \text{q.s. } t\in[0,T].\end{aligned} \tag{4.1}$$

Define $L_t^n := n\int_0^t(Y_s^n - S_s)^-\mathrm{d}s$ for $t\in[0,T]$. The penalized G-BSDE reads:

$$Y_t^n = \xi + \int_t^{\mathrm{T}} f(s,Y_s^n,Z_s^n)\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} Z_s^n\mathrm{d}B_s - \int_t^{\mathrm{T}}\mathrm{d}K_s^n + \int_t^{\mathrm{T}}\mathrm{d}L_s^n, \quad \text{q.s. } t\in[0,T]. \tag{4.2}$$

From (**H4**), we have

$$\int_0^T |(-S_s)^-|^2 \mathrm{d}s = \int_0^T |S_s^+|^2 \mathrm{d}s \le N_0^2 T.$$

Thus we can check that the generators of the penalized BSDE (4.1) or (4.2) satisfy (**H1**)–(**H3**). In view of Theorem 2.4, the penalized BSDE (4.1) or (4.2) has a unique solution $(Y^n, Z^n, K^n) \in \mathfrak{G}_G^2(0,T)$ such that

$$\|Y^n\|_{S_G^\infty} + \|Z^n\|_{\mathrm{BMO}_G} \le C(M_0, L_y, L_z, n)$$

and

$$\widehat{\mathbb{E}}[|K_T^n|^p] \le C(p, M_0, L_y, L_z, n), \quad \forall\, p \ge 1.$$

Both estimates depend on n. In fact, (Y^n, Z^n, K^n, L^n) is uniformly bounded in n.

Lemma 4.1 *There exist two positive constants C and C_p which are independent of n, such that*

$$\|Y^n\|_{S_G^\infty} + \|Z^n\|_{\mathrm{BMO}_G} \le C$$

and

$$\widehat{\mathbb{E}}[|K_T^n|^p] + \widehat{\mathbb{E}}[|L_T^n|^p] \le C_p, \quad \forall\, p \ge 1.$$

Proof First we consider the estimate for Y^n. The proof is very similar to that of Proposition 3.2.

For some $r > 0$, applying Itô's formula to $\mathrm{e}^{rt}|Y_t - N_0|^2$, we have for each $t \in [0,T]$,

$$\begin{aligned}
&\mathrm{e}^{rt}|Y_t^n - N_0|^2 + r\int_t^T \mathrm{e}^{rs}|Y_s^n - N_0|^2 \mathrm{d}s + \int_t^T \mathrm{e}^{rs}|Z_s^n|^2 \mathrm{d}\langle B\rangle_s \\
&= \mathrm{e}^{rT}|\xi - N_0|^2 + \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0) f(s, Y_s^n, Z_s^n)\mathrm{d}\langle B\rangle_s - \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0)Z_s^n \mathrm{d}B_s \\
&\quad + \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0)\mathrm{d}(L_s^n - K_s^n), \quad \text{q.s.}
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_t^T \mathrm{e}^{rs}(Y_s^n - N_0)\mathrm{d}L_s^n \\
&= n\int_t^T \mathrm{e}^{rs}(Y_s^n - N_0)(Y_s^n - S_s)^- \mathrm{d}s \le n\int_t^T \mathrm{e}^{rs}(Y_s^n - S_s)(Y_s^n - S_s)^- \mathrm{d}s \le 0, \quad \text{q.s.},
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{e}^{rt}|Y_t^n - N_0|^2 + r\int_t^T \mathrm{e}^{rs}|Y_s^n - N_0|^2 \mathrm{d}s \\
&\le \mathrm{e}^{rT}|\xi - N_0|^2 + \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0) f(s, Y_s^n, Z_s^n)\mathrm{d}\langle B\rangle_s - \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0)Z_s^n \mathrm{d}B_s \\
&\quad - \int_t^T 2\mathrm{e}^{rs}(Y_s^n - N_0)\mathrm{d}K_s^n, \quad \text{q.s.}
\end{aligned}$$

In a way similar to the proof of Proposition 3.2, we have for each $s \in [0,T]$,

$$f(s, Y_s^n, Z_s^n) = f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon} Y_s^n + b_s^{n,\varepsilon} Z_s^n,$$

where

$$|a_s^{n,\varepsilon}| \le L_y, \quad |b_s^{n,\varepsilon}| \le L_z(1+|Z_s^n|),$$
$$|m_s^{n,\varepsilon}| \le 2\varepsilon(L_y + L_z(1+2\varepsilon)).$$

So we have

$$\begin{aligned}
&\mathrm{e}^{rt}|Y_t^n - N_0|^2 + r\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s^n - N_0|^2\mathrm{d}s + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s^n - N_0)^+\mathrm{d}K_s^n \\
&\le \mathrm{e}^{rt}|Y_t^n - N_0|^2 + r\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s^n - N_0|^2\mathrm{d}s + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s^n - N_0)\mathrm{d}K_s^n \\
&\le \mathrm{e}^{rT}|\xi - N_0|^2 + \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s^n - N_0)(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n + b_s^{n,\varepsilon}Z_s^n)\mathrm{d}\langle B\rangle_s \\
&\quad - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s^n - N_0)Z_s^n\mathrm{d}B_s \\
&\le \mathrm{e}^{rT}|\xi - N_0|^2 + (1+2L_y)\int_t^{\mathrm{T}} \mathrm{e}^{rs}|Y_s^n - N_0|^2\mathrm{d}\langle B\rangle_s \\
&\quad + \int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0) + |m_s^{n,\varepsilon}| + N_0L_y)^2\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} 2\mathrm{e}^{rs}(Y_s^n - N_0)Z_s^n\mathrm{d}\widetilde{B}_s, \quad \text{q.s.,}
\end{aligned}$$

where $\mathrm{d}\widetilde{B}_s^{n,\varepsilon} = \mathrm{d}B_s - b_s^{n,\varepsilon}\mathrm{d}\langle B\rangle_s$. In view of [15, Lemma 3.6], we know that $b^{n,\varepsilon} \in \mathrm{BMO}_G$. Thus we can define a new G-expectation $\widetilde{\mathbb{E}}^{n,\varepsilon}[\cdot]$ by $\mathscr{E}(b_s^{n,\varepsilon})$, such that $\widetilde{B}^{n,\varepsilon}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}^{n,\varepsilon}[\cdot]$.

In view of Hu et al. [12, Lemma 3.4] and Lemma 2.7, we know that the process

$$\int_0^{\cdot} 2\mathrm{e}^{rs}(Y_s^n - N_0)^+\mathrm{d}K_s^n$$

is a decreasing G-martingale under both $\widehat{\mathbb{E}}[\cdot]$ and $\widetilde{\mathbb{E}}^{n,\varepsilon}[\cdot]$. Setting $r > \overline{\sigma}^2(1+2L_y)$ and taking conditional expectations in the last inequality, we have for each $t \in [0,T]$,

$$\mathrm{e}^{rt}|Y_t^n - N_0|^2 \le \widetilde{\mathbb{E}}_t^{n,\varepsilon}[\mathrm{e}^{rT}|\xi - N_0|^2] + \widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[\int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0) + |m_s^{n,\varepsilon}| + N_0L_y)^2\mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.}$$

Then

$$\begin{aligned}
\mathrm{e}^{rt}|Y_t^n - N_0|^2 &\le \widetilde{\mathbb{E}}_t^{n,\varepsilon}[\mathrm{e}^{rT}|\xi - N_0|^2] + \widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[\int_t^{\mathrm{T}} \mathrm{e}^{rs}(f(s,0,0) + |m_s^{n,\varepsilon}| + N_0L_y)^2\mathrm{d}\langle B\rangle_s\Big] \\
&\le 2\mathrm{e}^{rT}(\|\xi\|_{L_G^\infty}^2 + N_0^2) + 2\mathrm{e}^{rT}\overline{\sigma}^2\Big\{\Big\|\int_0^{\mathrm{T}} |f(s,0,0)|^2\mathrm{d}s\Big\|_{L_G^\infty} \\
&\quad + (2\varepsilon(L_y + L_z + 2L_z\varepsilon) + N_0L_y)^2T\Big\}, \quad \text{q.s.}
\end{aligned}$$

Setting $\varepsilon \to 0$, we have

$$\mathrm{e}^{rt}|Y_t^n - N_0|^2 \le 2\mathrm{e}^{rT}(\|\xi\|_{L_G^\infty}^2 + N_0^2) + 2\mathrm{e}^{rT}\overline{\sigma}^2\Big\{\Big\|\int_0^{\mathrm{T}} |f(s,0,0)|^2\mathrm{d}s\Big\|_{L_G^\infty} + N_0^2L_y^2T\Big\}, \quad \text{q.s.}$$

So we know that there exists a constant C' independent of n such that $\|Y^n\|_{S_G^\infty} \le C'$.

Then by Proposition 3.1, we know that there exist two constants C'' and C'_p which are independent of n, such that

$$\|Z^n\|_{\mathrm{BMO}_G} \le C''$$

and

$$\widehat{\mathbb{E}}[|L_T^n - K_T^n|^p] \le C'_p, \quad \forall\, p \ge 1.$$

We have

$$\|Y^n\|_{S_G^\infty} + \|Z^n\|_{\mathrm{BMO}_G} \le C$$

with $C = 2(C' + C'')$, and

$$\widehat{\mathbb{E}}[|L_T^n|^p] + \widehat{\mathbb{E}}[|K_T^n|^p] \le 2\widehat{\mathbb{E}}[|L_T^n - K_T^n|^p] \le C_p$$

with $C_p = 2C'_p$.

The following lemma plays a key role in the proof of the convergence of $\{Y^n\}$. It gives the convergence of $(Y^n - S)^-$ in $S_G^\alpha(0,T)$.

Lemma 4.2 *For each $\alpha > 1$, we have*

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |(Y_t^n - S_t)^-|^\alpha\Big] = 0.$$

Proof The lemma was proved by Li, Peng and Soumana Hima [24, Lemma 4.3] when the generator (f, g) is uniformly Lipschitz continuous. Their arguments can be adapted to our general case.

First, we sketch the main ideas. Under our z-quadratic generator, we will still use the method of linearization. By the G-Girsanov theorem, we can rewrite the G-BSDE (4.1) so that the generator is independent of z under a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$. Similarly as in [24, Lemma 4.3], the following holds true:

$$\lim_{n\to\infty} \widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T]} |(Y_t^n - S_t)^-|^\alpha\Big] = 0, \quad \forall\, \alpha > 1.$$

Then from Lemmas 2.4 and 2.5, we see that $\widetilde{\mathbb{E}}[\cdot]$ can be replaced with $\widehat{\mathbb{E}}[\cdot]$ in the last limit, which completes the proof.

Now we begin our proof. Similar to the proof of Proposition 3.2 and Lemma 4.1, we first rewrite the G-BSDE (4.1) by linearization into the form:

$$f(s, Y_s^n, Z_s^n) = f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon} Y_s^n + b_s^{n,\varepsilon} Z_s^n, \quad s \in [0,T],$$

with

$$|a_s^{n,\varepsilon}| \le L_y, \quad |b_s^{n,\varepsilon}| \le L_z(1 + |Z_s^n|), \quad |m_s^{n,\varepsilon}| \le 2\varepsilon(L_y + L_z(1+2\varepsilon)), \quad s \in [0,T].$$

So the G-BSDE (4.1) reads

$$\begin{aligned} Y_t^n = \xi &+ \int_t^{\mathrm{T}} (f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon} Y_s^n) \mathrm{d}\langle B\rangle_s + n\int_t^{\mathrm{T}} (Y_s^n - S_s)^- \mathrm{d}s \\ &- \int_t^{\mathrm{T}} Z_s^n d\widetilde{B}_s^{n,\varepsilon} - \int_t^{\mathrm{T}} \mathrm{d}K_s^n, \quad \text{q.s.}, \end{aligned}$$

where $\mathrm{d}\widetilde{B}_s^{n,\varepsilon} = \mathrm{d}B_s - b_s^{n,\varepsilon}\mathrm{d}\langle B\rangle_s$. In view of [15, Lemma 3.6], we know that $b^{n,\varepsilon} \in \mathrm{BMO}_G$. Thus we can define a new G-expectation $\widetilde{\mathbb{E}}^{n,\varepsilon}[\cdot]$ by $\mathscr{E}(b^{n,\varepsilon})$, such that $\widetilde{B}^{n,\varepsilon}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}^{n,\varepsilon}[\cdot]$.

We now prove

$$\lim_{n\to\infty} \widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]} |(Y_t^n - S_t)^-|^\alpha\Big] = 0. \tag{4.3}$$

Set

$$\begin{aligned} y_t^n = \xi + \int_t^{\mathrm{T}} f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon} Y_s^n \mathrm{d}\langle B\rangle_s + n\int_t^{\mathrm{T}} (S_s - y_s^n)\mathrm{d}s \\ - \int_t^{\mathrm{T}} z_s^n \mathrm{d}\widetilde{B}_s^{n,\varepsilon} - \int_t^{\mathrm{T}} \mathrm{d}k_s^n, \quad \text{q.s. } t\in[0,T]. \end{aligned}$$

Then we have for each $t\in[0,T]$,

$$y_t^n = \mathrm{e}^{nt}\widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[\mathrm{e}^{-nT}\xi + \int_t^{\mathrm{T}} n\mathrm{e}^{-ns}\mathrm{d}s + \int_t^{\mathrm{T}} \mathrm{e}^{-ns}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)\mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.}$$

In view of [13, Theorem 3.6], we have for each $t\in[0,T]$,

$$Y_t^n - S_t \geq y_t^n - S_t = \widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[\widetilde{S}_t^n + \int_t^{\mathrm{T}} \mathrm{e}^{n(t-s)}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)\mathrm{d}\langle B\rangle_s\Big], \quad \text{q.s.,}$$

where

$$\widetilde{S}_t^n := \mathrm{e}^{n(t-T)}(\xi - S_t) + \int_t^{\mathrm{T}} n\mathrm{e}^{n(t-s)}(S_s - S_t)\mathrm{d}s, \quad t\in[0,T].$$

It follows that

$$(Y_t^n - S_t)^- \leq (y_t^n - S_t)^- \leq \widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[|\widetilde{S}_t^n| + \Big|\int_t^{\mathrm{T}} \mathrm{e}^{n(t-s)}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)\mathrm{d}\langle B\rangle_s\Big|\Big], \quad \text{q.s.}$$

We have for any $\alpha > 1$,

$$\begin{aligned} &\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]}\Big|\int_t^{\mathrm{T}} \mathrm{e}^{n(t-s)}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)\mathrm{d}\langle B\rangle_s\Big|^\alpha\Big] \\ &\leq \overline{\sigma}^{2\alpha}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}} (f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)^2\mathrm{d}s\Big)^{\frac{\alpha}{2}} \sup_{t\in[0,T]}\Big(\int_t^{\mathrm{T}} \mathrm{e}^{2n(t-s)}\mathrm{d}s\Big)^{\frac{\alpha}{2}}\Big] \\ &\leq \Big(\frac{1-\mathrm{e}^{-2nT}}{n}\Big)^{\frac{\alpha}{2}}\overline{\sigma}^{2\alpha}\Big\{\Big\|\int_0^{\mathrm{T}} |f(s,0,0)|^2\mathrm{d}s\Big\|_{L_G^\infty} \\ &\quad + T[L_y\|Y^n\|_{S_G^\infty} + 2\varepsilon(L_y + L_z + 2L_z\varepsilon)]^2\Big\}^{\frac{\alpha}{2}}. \end{aligned}$$

In view of Lemma 4.1, we have

$$\lim_{n\to\infty} \widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]}\Big|\int_t^{\mathrm{T}} \mathrm{e}^{n(t-s)}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon}Y_s^n)\mathrm{d}\langle B\rangle_s\Big|^\alpha\Big] = 0. \tag{4.4}$$

For $\epsilon > 0$, it is straightforward to show for each $t\in[0,T]$,

$$|\widetilde{S}_t^n| = \Big|\mathrm{e}^{n(t-T)}(\xi - S_t) + \int_{t+\epsilon}^{\mathrm{T}} n\mathrm{e}^{n(t-s)}(S_s - S_t)\,\mathrm{d}s + \int_t^{t+\epsilon} n\mathrm{e}^{n(t-s)}(S_s - S_t)\mathrm{d}s\Big|$$

$$\leq \mathrm{e}^{n(t-T)}|\xi - S_t| + \mathrm{e}^{-n\epsilon} \sup_{s\in[t+\epsilon,T]} |S_t - S_s| + \sup_{s\in[t,t+\epsilon]} |S_t - S_s|, \quad \text{q.s.}$$

For $\delta \in (0, T)$, we have

$$\begin{aligned}
&\sup_{t\in[0,T-\delta]} |\widetilde{S}_t^n| \\
&\leq \mathrm{e}^{-n\delta} \sup_{t\in[0,T-\delta]} |\xi - S_t| + \mathrm{e}^{-n\epsilon} \sup_{t\in[0,T-\delta]} \sup_{s\in[t+\epsilon,T]} |S_t - S_s| \\
&\quad + \sup_{t\in[0,T-\delta]} \sup_{s\in[t,t+\epsilon]} |S_t - S_s| \\
&\leq \mathrm{e}^{-n\delta}\Big(\sup_{t\in[0,T]} |S_t| + |\xi|\Big) + 2\mathrm{e}^{-n\epsilon} \sup_{t\in[0,T]} |S_t| + \sup_{t\in[0,T]} \sup_{s\in[t,t+\epsilon]} |S_t - S_s|, \quad \text{q.s.} \qquad (4.5)
\end{aligned}$$

Define the function

$$\phi(x) := \Big(1 + \frac{1}{x^2} \log \frac{2x-1}{2(x-1)}\Big)^{\frac{1}{2}} - 1, \quad x \in (1, \infty).$$

In view of Lemma 4.1, we can choose $p > 1$ independent of n and ε, such that

$$\|b^{n,\varepsilon}\|_{\mathrm{BMO}_G} \leq L_z(1 + \|Z^n\|_{\mathrm{BMO}_G}) < \phi(p).$$

Set $q = \frac{p}{p-1}$. Then in view of Lemma 2.4, we have for each $\alpha > 1$ and $X \in L_G^q(\Omega_T)$,

$$\widetilde{\mathbb{E}}^{n,\varepsilon}[X] = \widehat{\mathbb{E}}\big[\mathscr{E}(b^{n,\varepsilon})_T X\big] \leq \widehat{\mathbb{E}}\big[\mathscr{E}(b^{n,\varepsilon})_T^p\big]^{\frac{1}{p}} \widehat{\mathbb{E}}[|X|^q]^{\frac{1}{q}} \leq C_p \widehat{\mathbb{E}}[|X|^q]^{\frac{1}{q}}, \qquad (4.6)$$

where C_p depends only on p.

In view of **(H4)** on S, we know

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |S_t|^\alpha\Big] < +\infty, \quad \forall\, \alpha > 1.$$

So we have for all $\alpha > 1$,

$$\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]} |S_t|^\alpha\Big] \leq C_p \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |S_t|^{\alpha q}\Big]^{\frac{1}{q}}$$

and

$$\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]} \sup_{s\in[t,t+\epsilon]} |S_t - S_s|^\alpha\Big] \leq C_p \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \sup_{s\in[t,t+\epsilon]} |S_t - S_s|^{\alpha q}\Big]^{\frac{1}{q}}.$$

From (4.5), we know

$$\limsup_{n\to\infty} \widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]} |\widetilde{S}_t^n|^\alpha\Big] \leq C_\alpha C_p \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \sup_{s\in[t,t+\epsilon]} |S_t - S_s|^{\alpha q}\Big]^{\frac{1}{q}}. \qquad (4.7)$$

Then, in view of (4.4), (4.7) and Remark 2.1, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]} |(Y_t^n - S_t)^-|^\alpha\Big] \\
&\leq \limsup_{n\to\infty} \widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]} \widetilde{\mathbb{E}}_t^{n,\varepsilon}\Big[|\widetilde{S}_t^n| + \Big|\int_t^{\mathrm{T}} \mathrm{e}^{n(t-s)}(f(s,0,0) + m_s^{n,\varepsilon} + a_s^{n,\varepsilon} Y_s^n)\mathrm{d}\langle B\rangle_s\Big|\Big]^\alpha\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le C\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]}\widetilde{\mathbb{E}}^{n,\varepsilon}_t\Big[\sup_{u\in[0,T-\delta]}|\widetilde{S}^n_u|^\alpha\Big]\Big]\\
&\quad+C\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]}\widetilde{\mathbb{E}}^{n,\varepsilon}_t\Big[\sup_{u\in[0,T-\delta]}\Big|\int_u^{\mathrm{T}}\mathrm{e}^{n(t-s)}(f(s,0,0)+m^{n,\varepsilon}_s+a^{n,\varepsilon}_sY^n_s)\mathrm{d}\langle B\rangle_s\Big|^\alpha\Big]\Big]\\
&=C\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]}\widetilde{\mathbb{E}}^{n,\varepsilon}_t\Big[\sup_{u\in[0,T-\delta]}|\widetilde{S}^n_u|^\alpha\Big]\Big]\\
&\le C'\limsup_{n\to\infty}\Big\{\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{u\in[0,T-\delta]}|\widetilde{S}^n_u|^{2\alpha}\Big]+\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{u\in[0,T-\delta]}|\widetilde{S}^n_u|^{2\alpha}\Big]^{\frac12}\Big\}\\
&\le C''\Big\{\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\epsilon]}|S_t-S_s|^{2\alpha q}\Big]^{\frac1q}+\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\sup_{s\in[t,t+\epsilon]}|S_t-S_s|^{2\alpha q}\Big]^{\frac{1}{2q}}\Big\},
\end{aligned}
$$

where C'' is independent of n, δ and ϵ. Therefore, in view of Lemma 2.1, setting $\epsilon\to 0$, we have

$$\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]}|(Y^n_t-S_t)^-|^\alpha\Big]=0.$$

In view of Theorem 3.1, we get $Y^n_t\ge Y^1_t$ and then obtain

$$
\begin{aligned}
&\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]}|(Y^n_t-S_t)^-|^\alpha\Big]\\
&\le\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T-\delta]}|(Y^n_t-S_t)^-|^\alpha\Big]+\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[T-\delta,T]}|(Y^n_t-S_t)^-|^\alpha\Big]\\
&\le\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[T-\delta,T]}|(Y^1_t-S_t)^-|^\alpha\Big].
\end{aligned}
$$

By Lemma 2.1 again and noting that $(Y^1_T-S_T)^-=0$, we obtain

$$\lim_{\delta\to0}\widehat{\mathbb{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y^1_t-S_t)^-|^\alpha\Big]=0,\quad\forall\,\alpha>1.$$

Finally, with (4.6), we derive that

$$\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[T-\delta,T]}|(Y^1_t-S_t)^-|^\alpha\Big]\le C_p\widehat{\mathbb{E}}\Big[\sup_{t\in[T-\delta,T]}|(Y^1_t-S_t)^-|^{q\alpha}\Big]^{\frac1q}.$$

Let $\delta\to0$ and we know

$$\limsup_{n\to\infty}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[T-\delta,T]}|(Y^1_t-S_t)^-|^\alpha\Big]=0.$$

Therefore, we have (4.3).

Next we want to change the G-expectation in the last equality. Actually, in view of Lemma 2.5 and Remark 2.7, there exists $r>1$ which is independent of n and ε, such that

$$\widehat{\mathbb{E}}[\{\mathscr{E}(b^{n,\varepsilon})_T\}^{\frac{1}{1-r}}]\le C_r$$

for some positive constant C_r which depends only on r. Thus, for each $\alpha>1$, we have

$$
\begin{aligned}
\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|(Y^n_t-S_t)^-|^\alpha\Big]&=\widehat{\mathbb{E}}\Big[\mathscr{E}(b^{n,\varepsilon})^{\frac1r}_T\mathscr{E}(b^{n,\varepsilon})^{-\frac1r}_T\sup_{t\in[0,T]}|(Y^n_t-S_t)^-|^\alpha\Big]\\
&\le\widehat{\mathbb{E}}\Big[\mathscr{E}(b^{n,\varepsilon})_T\sup_{t\in[0,T]}|(Y^n_t-S_t)^-|^{\alpha r}\Big]^{\frac1r}\widehat{\mathbb{E}}[\{\mathscr{E}(b^{n,\varepsilon})_T\}^{\frac{1}{1-r}}]^{\frac{r-1}{r}}
\end{aligned}
$$

$$\leq C_r^{\frac{r-1}{r}}\widetilde{\mathbb{E}}^{n,\varepsilon}\Big[\sup_{t\in[0,T]}|(Y_t^n-S_t)^-|^{\alpha r}\Big]^{\frac{1}{r}}.$$

So

$$\limsup_{n\to\infty}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|(Y_t^n-S_t)^-|^{\alpha}\Big]=0.$$

Now we show the convergence of the sequence $\{Y^n\}_{n=1}^{\infty}$.

Lemma 4.3 *The sequence $\{Y^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $S_G^{\alpha}(0,T)$ for any $\alpha\geq 2$.*

Proof For $m,n\in\mathbb{N}$ and each $t\in[0,T]$, set

$$\widehat{Y}_t^{n,m}=Y_t^n-Y_t^m,\quad \widehat{Z}_t^{n,m}=Z_t^n-Z_t^m,\quad \widehat{K}_t^{n,m}=K_t^n-K_t^m,\quad \widehat{L}_t^{n,m}=L_t^n-L_t^m.$$

We use the method of linearization. Similarly as the proof of Proposition 3.2 and Lemma 4.1, $\forall\,\varepsilon>0$, we write for each $s\in[0,T]$,

$$f(s,Y_s^n,Z_s^n)-f(s,Y_s^m,Z_s^m)=m_s^{n,m,\varepsilon}+a_s^{n,m,\varepsilon}\widehat{Y}_s^{n,m}+b_s^{n,m,\varepsilon}\widehat{Z}_s^{n,m}$$

with

$$|a_s^{n,m,\varepsilon}|\leq L_y,\quad |b_s^{n,m,\varepsilon}|\leq L_z(1+|Z_s^n|+|Z_s^m|),$$
$$|m_s^{n,m,\varepsilon}|\leq 2\varepsilon(L_y+L_z(1+2\varepsilon+2|Z_s^n|)).$$

So we have for each $t\in[0,T]$,

$$\begin{aligned}\widehat{Y}_t^{n,m}&=\int_t^{\mathrm{T}}(m_s^{n,m,\varepsilon}+a_s^{n,m,\varepsilon}\widehat{Y}_s^{n,m}+b_s^{n,m,\varepsilon}\widehat{Z}_s^{n,m})\mathrm{d}\langle B\rangle_s\\&\quad-\int_t^{\mathrm{T}}\widehat{Z}_s^{n,m}\mathrm{d}B_s-\int_t^{\mathrm{T}}\mathrm{d}\widehat{K}_s^{n,m}+\int_t^{\mathrm{T}}\mathrm{d}\widehat{L}_s^{n,m}\\&=\int_t^{\mathrm{T}}(m_s^{n,m,\varepsilon}+a_s^{n,m,\varepsilon}\widehat{Y}_s^{n,m})\mathrm{d}\langle B\rangle_s-\int_t^{\mathrm{T}}\widehat{Z}_s^{n,m}\mathrm{d}\widetilde{B}_s^{n,m,\varepsilon}-\int_t^{\mathrm{T}}\mathrm{d}\widehat{K}_s^{n,m}+\int_t^{\mathrm{T}}\mathrm{d}\widehat{L}_s^{n,m},\quad \text{q.s.},\end{aligned}$$

where $\mathrm{d}\widetilde{B}_s^{n,m,\varepsilon}=\mathrm{d}B_s-b_s^{n,m,\varepsilon}\widehat{Z}_s^{n,m}\mathrm{d}\langle B\rangle_s$. In view of [15, Lemma 3.6], we know that $b^{n,m,\varepsilon}\in \mathrm{BMO}_G$ and we define a new G-expectation $\widetilde{\mathbb{E}}^{n,m,\varepsilon}[\cdot]$ by $\mathscr{E}(b^{n,m,\varepsilon})$, such that $\widetilde{B}^{n,m,\varepsilon}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}^{n,m,\varepsilon}[\cdot]$.

For all $\alpha\geq 2$, by applying Itô's formula to $|\widehat{Y}_t^{n,m}|^{\alpha}\mathrm{e}^{rt}$, we get for each $t\in[0,T]$,

$$\begin{aligned}&|\widehat{Y}_t^{n,m}|^{\alpha}\mathrm{e}^{rt}+\int_t^{\mathrm{T}}r\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha}\mathrm{d}s+\frac{1}{2}\alpha(\alpha-1)\int_t^{\mathrm{T}}\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}|\widehat{Z}_s^{n,m}|^2\mathrm{d}\langle B\rangle_s\\&=\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}m_s^{n,m,\varepsilon}\mathrm{d}\langle B\rangle_s+\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha}a_s^{n,m,\varepsilon}\mathrm{d}\langle B\rangle_s\\&\quad-\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}\widetilde{B}_s^{n,m,\varepsilon}-\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{K}_s^{n,m}\\&\quad+\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{L}_s^{n,m},\quad \text{q.s.}\end{aligned}$$

Let $r>L_y\alpha\overline{\sigma}^2$. Noting that $|a_s^{n,m,\varepsilon}|\leq L_y$, we get

$$|\widehat{Y}_t^{n,m}|^{\alpha}\mathrm{e}^{rt}\leq\int_t^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}m_s^{n,m,\varepsilon}\mathrm{d}\langle B\rangle_s$$

$$
\begin{aligned}
&-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}\widetilde{B}_s^{n,m,\varepsilon}\\
&-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{K}_s^{n,m}+\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{L}_s^{n,m}, \quad \text{q.s.}
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
&\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{L}_s^{n,m}\\
&=-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^n-S_s)\mathrm{d}L_s^m-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^n\\
&\quad+\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^n-S_s)\mathrm{d}L_s^n+\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^m\\
&\le-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^n-S_s)\mathrm{d}L_s^m-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^n, \quad \text{q.s.}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}\mathrm{d}\widehat{K}_s^{n,m} \ge& \int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(\widehat{Y}_s^{n,m})^+\mathrm{d}K_s^m\\
&+\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(\widehat{Y}_s^{n,m})^-\mathrm{d}K_s^n, \quad \text{q.s.,}
\end{aligned}
$$

we have

$$
\begin{aligned}
|\widehat{Y}_t^{n,m}|^\alpha \mathrm{e}^{rt}+M_T-M_t \le& \int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}m_s^{n,m,\varepsilon}\mathrm{d}\langle B\rangle_s\\
&-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^n-S_s)\mathrm{d}L_s^m\\
&-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^n, \quad \text{q.s.,}
\end{aligned}
$$

where

$$
M_t:=\int_0^t \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}[(\widehat{Y}_s^{n,m})^+\mathrm{d}K_s^m+(\widehat{Y}_s^{n,m})^-\mathrm{d}K_s^n+\widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}\widetilde{B}_s^{n,m,\varepsilon}].
$$

In view of [12, Lemma 3.3] and [15, Lemma 3.4], we conclude that M is a G-martingale under $\widetilde{\mathbb{E}}^{n,m,\varepsilon}[\cdot]$. Thus we obtain

$$
\begin{aligned}
&|\widehat{Y}_t^{n,m}|^\alpha \mathrm{e}^{rt}-\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}\widehat{Y}_s^{n,m}m_s^{n,m,\varepsilon}\mathrm{d}\langle B\rangle_s\Big]\\
&\le \widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^n-S_s)\,\mathrm{d}L_s^m\\
&\quad-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^n\Big], \quad \text{q.s.}
\end{aligned}
\tag{4.8}
$$

Noting the following estimate

$$
\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[-\int_t^{\mathrm{T}} \alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m-S_s)\mathrm{d}L_s^n\Big]
$$

$$
\begin{aligned}
&= \widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[-\int_t^{\mathrm{T}} n\alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-2}(Y_s^m - S_s)(Y_s^n - S_s)^-\mathrm{d}s\Big]\\
&\le \alpha \mathrm{e}^{rT}\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_t^{\mathrm{T}} n|(Y_s^n - S_s)-(Y_s^m - S_s)|^{\alpha-2}(Y_s^m - S_s)^-(Y_s^n - S_s)^-\mathrm{d}s\Big]\\
&\le C\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_t^{\mathrm{T}} n|(Y_s^n - S_s)^-|^{\alpha-1}(Y_s^m - S_s)^-\mathrm{d}s\Big]\\
&\quad + C\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_t^{\mathrm{T}} n|(Y_s^m - S_s)^-|^{\alpha-1}(Y_s^n - S_s)^-\mathrm{d}s\Big], \quad \text{q.s.,}
\end{aligned}
$$

where C is independent of n, m and ε, we deduce from (4.8) that

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha}\Big] - \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_0^{\mathrm{T}}\alpha \mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big]\Big]\\
&\le C\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_0^{\mathrm{T}}(m+n)|(Y_s^n - S_s)^-|^{\alpha-1}(Y_s^m - S_s)^-\mathrm{d}s\Big]\Big]\\
&\quad + C\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}\Big[\int_0^{\mathrm{T}}(m+n)|(Y_s^m - S_s)^-|^{\alpha-1}(Y_s^n - S_s)^-\mathrm{d}s\Big]\Big]. \qquad (4.9)
\end{aligned}
$$

Recall that

$$\phi(x) = \Big(1 + \frac{1}{x^2}\log\frac{2x-1}{2(x-1)}\Big)^{\frac{1}{2}} - 1, \quad x > 1.$$

In view of Lemma 4.1, we can choose $p > 1$ independent of n, m and ε, such that

$$\|b^{n,m,\varepsilon}\|_{\mathrm{BMO}_G} \le L_z(1 + \|Z^n\|_{\mathrm{BMO}_G} + \|Z^m\|_{\mathrm{BMO}_G}) < \phi(p).$$

Set $q = \frac{p}{p-1}$. Then in view of Lemma 2.4, we have for each $\alpha > 1$ and $X \in L_G^q(\Omega_T)$,

$$
\begin{aligned}
\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}[X] &= \widehat{\mathbb{E}}_t\Big[\frac{\mathscr{E}(b^{n,m,\varepsilon})_T}{\mathscr{E}(b^{n,m,\varepsilon})_t}X\Big] \le \widehat{\mathbb{E}}_t\Big[\Big(\frac{\mathscr{E}(b^{n,m,\varepsilon})_T}{\mathscr{E}(b^{n,m,\varepsilon})_t}\Big)^p\Big]^{\frac{1}{p}}\widehat{\mathbb{E}}_t[|X|^q]^{\frac{1}{q}}\\
&\le C_p\widehat{\mathbb{E}}_t[|X|^q]^{\frac{1}{q}}, \quad \text{q.s.,} \qquad (4.10)
\end{aligned}
$$

where C_p depends only on p.

Then we have for some $\beta > 1$,

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_t^{\mathrm{T}} n|(Y_s^n - S_s)^-|^{\alpha-1}(Y_s^m - S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\\
&\le \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{s\in[0,T]}\{|(Y_s^n - S_s)^-|^{(\alpha-2)\beta}|(Y_s^m - S_s)^-|^{\beta}\}\Big(\int_t^{\mathrm{T}} n(Y_s^n - S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\\
&\le \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{s\in[0,T]}|(Y_s^n - S_s)^-|^{4(\alpha-2)\beta}\Big]^{\frac{1}{4}}\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{s\in[0,T]}|(Y_s^m - S_s)^-|^{4\beta}\Big]^{\frac{1}{4}}\\
&\quad \times \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_t^{\mathrm{T}} n(Y_s^n - S_s)^-\mathrm{d}s\Big)^{2\beta}\Big]^{\frac{1}{2}}\\
&\le C_p^3\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^n - S_s)^-|^{4(\alpha-2)\beta q}\Big]^{\frac{1}{4q}}\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^m - S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q}}\\
&\quad \times \widehat{\mathbb{E}}\Big[\Big(\int_t^{\mathrm{T}} n(Y_s^n - S_s)^-\mathrm{d}s\Big)^{2\beta q}\Big]^{\frac{1}{2q}}
\end{aligned}
$$

$$\leq C_1\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^m-S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q}}, \tag{4.11}$$

and

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_t^{\mathrm{T}} m|(Y_s^n-S_s)^-|^{\alpha-1}(Y_s^m-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\\
&\leq \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{s\in[0,T]}|(Y_s^n-S_s)^-|^{(\alpha-1)\beta}\Big(\int_t^{\mathrm{T}} m(Y_s^m-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\\
&\leq \widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{s\in[0,T]}|(Y_s^n-S_s)^-|^{2(\alpha-1)\beta}\Big]^{\frac{1}{2}}\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_t^{\mathrm{T}} m(Y_s^m-S_s)^-\mathrm{d}s\Big)^{2\beta}\Big]^{\frac{1}{2}}\\
&\leq C_p^2\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^n-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q}}\widehat{\mathbb{E}}\Big[\Big(\int_t^{\mathrm{T}} m(Y_s^m-S_s)^-\mathrm{d}s\Big)^{2\beta q}\Big]^{\frac{1}{2q}}\\
&\leq C_2\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^n-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q}}.
\end{aligned} \tag{4.12}$$

Moreover, in view of the assumption on S and Lemma 4.1, we know that C_1 and C_2 are independent of n, m and ε. From Remark 2.1, there exists a constant C' independent of n, m and ε, such that

$$\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}\widetilde{\mathbb{E}}_t^{n,m,\varepsilon}[|X|]\Big]\leq C'(\widetilde{\mathbb{E}}^{n,m,\varepsilon}[|X|^{\beta}]^{\frac{1}{\beta}}+\widetilde{\mathbb{E}}^{n,m,\varepsilon}[|X|^{\beta}]).$$

Then with (4.9) and (4.11)–(4.12), we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha}\Big]-C'\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta}\Big]^{\frac{1}{\beta}}\\
&-C'\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}\alpha\mathrm{e}^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta}\Big]\\
&\leq CC'\Big\{\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}(m+n)|(Y_s^n-S_s)^-|^{\alpha-1}(Y_s^m-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]^{\frac{1}{\beta}}\\
&+\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}(m+n)|(Y_s^n-S_s)^-|^{\alpha-1}(Y_s^m-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\\
&+\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}(m+n)|(Y_s^m-S_s)^-|^{\alpha-1}(Y_s^n-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]^{\frac{1}{\beta}}\\
&+\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}(m+n)|(Y_s^m-S_s)^-|^{\alpha-1}(Y_s^n-S_s)^-\mathrm{d}s\Big)^{\beta}\Big]\Big\}\\
&\leq \overline{C}\sum_{j=m,n}\Big(\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q}}+\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q}}\\
&+\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q\beta}}+\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q\beta}}\Big),
\end{aligned} \tag{4.13}$$

where $\overline{C}$ is independent of n, m and ε.

In view of Lemma 2.5 and Remark 2.7, there is $r>1$ which is independent of n, m and ε, such that

$$\widehat{\mathbb{E}}[\{\mathscr{E}(b^{n,m,\varepsilon})_T\}^{\frac{1}{1-r}}]\leq C_r,$$

where C_r depends only on r. Thus, for each $\alpha' \geq 2$, we have

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha'}\Big] &= \widehat{\mathbb{E}}\Big[\mathscr{E}(b^{n,m,\varepsilon})_T^{\frac{1}{r}}\mathscr{E}(b^{n,m,\varepsilon})_T^{-\frac{1}{r}}\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha'}\Big] \\ &\leq \widehat{\mathbb{E}}\Big[\mathscr{E}(b^{n,m,\varepsilon})_T\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha' r}\Big]^{\frac{1}{r}}\widehat{\mathbb{E}}[\{\mathscr{E}(b^{n,m,\varepsilon})_T\}^{\frac{1}{1-r}}]^{\frac{r-1}{r}} \\ &\leq C_r^{\frac{r-1}{r}}\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha' r}\Big]^{\frac{1}{r}}.\end{aligned}$$

Setting $\alpha = \alpha' r > 2$ in (4.13), we have

$$\begin{aligned}&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\widehat{Y}_t^{n,m}|^{\alpha'}\Big]^r - C_r^{r-1}C'\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}\alpha e^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta}\Big]^{\frac{1}{\beta}} \\ &- C_r^{r-1}C'\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}\alpha e^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta}\Big] \\ \leq{}& C_r^{r-1}\overline{C}\sum_{j=m,n}\Big(\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q}} + \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q}} \\ &+ \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{4\beta q}\Big]^{\frac{1}{4q\beta}} + \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|(Y_s^j-S_s)^-|^{2(\alpha-1)\beta q}\Big]^{\frac{1}{2q\beta}}\Big).\end{aligned} \tag{4.14}$$

On the other hand,

$$\begin{aligned}&\widetilde{\mathbb{E}}^{n,m,\varepsilon}\Big[\Big(\int_0^{\mathrm{T}}\alpha e^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta}\Big] \\ &\leq C_p\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}}\alpha e^{rs}|\widehat{Y}_s^{n,m}|^{\alpha-1}|m_s^{n,m,\varepsilon}|\mathrm{d}\langle B\rangle_s\Big)^{\beta q}\Big]^{\frac{1}{q}} \\ &\leq 2\varepsilon C_p\alpha e^{rT}\|\widehat{Y}^{n,m}\|_{S_G^\infty}^{\beta(\alpha-1)}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}}(L_y+L_z(1+2\varepsilon+2|Z_s^n|))\mathrm{d}\langle B\rangle_s\Big)^{\beta q}\Big]^{\frac{1}{q}} \\ &\to 0 \quad \text{as} \quad \varepsilon\to 0.\end{aligned}$$

Let $\varepsilon \to 0$ in (4.14). Then in view of Lemma 4.2, we conclude that $(Y^n)_{n=1}^\infty$ is a Cauchy sequence in $S_G^{\alpha'}(0,T)$.

5 Existence and Uniqueness Result on Reflected Quadratic *G*-BSDEs

Our main result is stated as follows.

Theorem 5.1 *Let the triple* (ξ, f, S) *satisfy* **(H1)**–**(H5)**. *Then, the reflected G-BSDE* (3.1) *has a unique solution* (Y, Z, A) *such that* $(Y, Z) \in S_G^\infty(0,T) \times \mathrm{BMO}_G$ *and* $A \in \bigcap_{\alpha\geq 2} S_G^\alpha(0,T)$.

Proof The uniqueness of the solution is referred to Remark 3.1. We now prove the existence. Recalling the penalized G-BSDE (4.2), for $m, n \in \mathbb{N}$ and each $t \in [0,T]$, define

$$\widehat{Y}_t^{n,m} := Y_t^n - Y_t^m, \quad \widehat{Z}_t^{n,m} := Z_t^n - Z_t^m, \quad \widehat{K}_t^{n,m} := K_t^n - K_t^m, \quad \widehat{L}_t^{n,m} := L_t^n - L_t^m,$$

and

$$\widehat{f}_t^{n,m} := f(t, Y_t^n, Z_t^n) - f(t, Y_t^m, Z_t^m).$$

In view of Lemma 4.3, there exists $Y \in S_G^\alpha(0,T)$ satisfying

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |Y_t - Y_t^n|^\alpha\Big] = 0, \quad \forall\, \alpha \geq 2.$$

Note that there is $L := L(L_y, L_z)$ such that for each $t \in [0,T]$,

$$|\widehat{f}_t^{n,m}| \leq L_y|\widehat{Y}_t^{n,m}| + L_z(1 + |Z_t^m| + |Z_t^n|)|\widehat{Z}_t^{n,m}| \leq L(1 + |\widehat{Y}_t^{n,m}| + |Z_t^m|^2 + |Z_t^n|^2).$$

Applying Itô's formula to $|\widehat{Y}_t^{n,m}|^2$, we get for each $t \in [0,T]$,

$$\begin{aligned}
&|\widehat{Y}_t^{n,m}|^2 + \int_t^{\mathrm{T}} |\widehat{Z}_s^{n,m}|^2 \mathrm{d}\langle B\rangle_s \\
&= 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\widehat{f}_s^{n,m}\mathrm{d}\langle B\rangle_s - 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\mathrm{d}\widehat{K}_s^{n,m} + 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\mathrm{d}\widehat{L}_s^{n,m} - 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}B_s \\
&\leq 2L\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}(1 + |\widehat{Y}_s^{n,m}| + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}\langle B\rangle_s - 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\mathrm{d}\widehat{K}_s^{n,m} \\
&\quad + 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\mathrm{d}\widehat{L}_s^{n,m} - 2\int_t^{\mathrm{T}} \widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}B_s, \quad \text{q.s.}
\end{aligned}$$

Setting $t = 0$, we have

$$\begin{aligned}
\int_0^{\mathrm{T}} |\widehat{Z}_s^{n,m}|^2\mathrm{d}\langle B\rangle_s &\leq 2L\overline{\sigma}^2 T \sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^2 + 2L\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}| \int_0^{\mathrm{T}} (1 + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}\langle B\rangle_s \\
&\quad + 2\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}| \sum_{j=m,n} (|K_T^j| + |L_T^j|) - 2\int_0^{\mathrm{T}} \widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}\mathrm{d}B_s, \quad \text{q.s.}
\end{aligned}$$

With the B-D-G inequality and Hölder's inequality, we have

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{Z}_s^{n,m}|^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big] \\
&\leq C_\alpha\Big\{2L\overline{\sigma}^2 T\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^\alpha\Big] + 2L\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^{\frac{\alpha}{2}}\Big(\int_0^{\mathrm{T}} (1 + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big] \\
&\quad + 2\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^{\frac{\alpha}{2}}\Big(\sum_{j=m,n} (|K_T^j| + |L_T^j|)\Big)^{\frac{\alpha}{2}}\Big] + 2\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{Y}_s^{n,m}\widehat{Z}_s^{n,m}|^2\mathrm{d}s\Big)^{\frac{\alpha}{4}}\Big]\Big\} \\
&\leq C'_\alpha\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^\alpha\Big] + C'_\alpha\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^\alpha\Big]^{\frac{1}{2}}\Big\{\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} (1 + |Z_s^m|^2 + |Z_s^n|^2)\mathrm{d}\langle B\rangle_s\Big)^\alpha\Big]^{\frac{1}{2}} \\
&\quad + \widehat{\mathbb{E}}\Big[\sum_{j=m,n} (|K_T^j|^\alpha + |L_T^j|^\alpha)\Big]^{\frac{1}{2}} + \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_s^n|^2 + |Z_s^m|^2\mathrm{d}s\Big)^{\frac{\alpha}{2}}\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

In view of Lemmas 4.1 and 2.2, there exists a constant C_1 independent of m and n, such that

$$\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{Z}_s^{n,m}|^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big] \leq C_1\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^\alpha\Big] + C_1\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\widehat{Y}_s^{n,m}|^\alpha\Big]^{\frac{1}{2}}.$$

In view of Lemma 4.3, we know that $\{Z^n\}_{n=1}^\infty$ is a Cauchy sequence in $H_G^\alpha(0,T)$ for $\alpha \geq 2$. Thus there exists $Z \in H_G^\alpha(0,T)$ satisfying

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |Z_s - Z_s^n|\mathrm{d}s\Big)^{\frac{\alpha}{2}}\Big] = 0, \quad \forall\, \alpha \geq 2.$$

Now set $A^n := L^n - K^n$. It is easy to check that $(A^n_t)_{t\in[0,T]}$ is a nondecreasing process and

$$A^n_t - A^m_t = \widehat{Y}^{n,m}_0 - \widehat{Y}^{n,m}_t - \int_0^t \widehat{f}^{n,m}_s \mathrm{d}\langle B\rangle_s + \int_0^t \widehat{Z}^{n,m}_s \mathrm{d}B_s, \quad \text{q.s.}$$

So we get

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |A^n_t - A^m_t|^\alpha\Big] \\
&\le C_2\Big\{\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\widehat{Y}^{n,m}_t|^\alpha\Big] + \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{f}^{n,m}_s|\mathrm{d}\langle B\rangle_s\Big)^\alpha\Big] + \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{Z}^{n,m}_s|^2\mathrm{d}s\Big)^{\frac{\alpha}{2}}\Big]\Big\}. \qquad (5.1)
\end{aligned}$$

From the assumption on f, we have

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} \widehat{f}^{n,m}_s \mathrm{d}\langle B\rangle_s\Big)^\alpha\Big] \\
&\le \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} (L_y|\widehat{Y}^{n,m}_s| + L_z(1+|Z^m_s|+|Z^n_s|)|\widehat{Z}^{n,m}_t|)\mathrm{d}\langle B\rangle_s\Big)^\alpha\Big] \\
&\le C_3\Big\{\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\widehat{Y}^{n,m}_t|^\alpha\Big] + \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} (1+|Z^m_s|+|Z^n_s|)^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big(\int_0^{\mathrm{T}} |\widehat{Z}^{n,m}_t|^2\mathrm{d}\langle B\rangle_s\Big)^{\frac{\alpha}{2}}\Big]\Big\} \\
&\le C_3\Big\{\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|\widehat{Y}^{n,m}_t|^\alpha\Big] + \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} (1+|Z^m_s|+|Z^n_s|)^2\mathrm{d}\langle B\rangle_s\Big)^{\alpha}\Big]^{\frac{1}{2}} \\
&\quad \times \widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{Z}^{n,m}_t|^2\mathrm{d}\langle B\rangle_s\Big)^{\alpha}\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Then in view of Lemmas 4.1, 2.2 and (5.1), we know that $\{A^n\}_{n=1}^\infty$ is a Cauchy sequence in $S^\alpha_G(0,T)$ for each $\alpha \ge 2$. There exists a nondecreasing process $(A_t)_{t\in[0,T]}$ such that

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |A_t - A^n_t|^\alpha\Big] = 0.$$

Now, we prove $Y \in S^\infty_G(0,T)$. In view of Lemma 4.1, we know that that there exists a constant $C > 0$ such that $\|Y^n\|_{S^\infty_G} \le C$. Recall that

$$\widehat{\mathbb{E}}[X] = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[X], \quad \forall\, X \in L^1_G(\Omega_T).$$

From

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |Y_t - Y^n_t|^2\Big] = 0,$$

we see that for each $\mathbb{P}\in\mathcal{P}$, $\Big\{\sup_{t\in[0,T]}|Y^n_t|,\ n=1,2,\cdots\Big\}$ converges in probability $\mathbb{P}$ to $\sup_{t\in[0,T]}|Y_t|$. Then, there exists a sub-sequence of $\Big\{\sup_{t\in[0,T]}|Y^n_t|\Big\}$ such that $\mathbb{P}$-a.s.,

$$\lim_{k\to\infty}\sup_{t\in[0,T]}|Y^{n_k}_t| = \sup_{t\in[0,T]}|Y_t|.$$

Since $\sup_{t\in[0,T]}|Y^{n_k}_t| \le C$ for a positive constant C independent of $\mathbb{P}$, we have $\sup_{t\in[0,T]}|Y_t| \le C$ $\mathbb{P}$-a.s. for each $\mathbb{P}\in\mathcal{P}$, and then $\sup_{t\in[0,T]}|Y_t| \le C$, q.s., which yields the inequality $\|Y\|_{S^\infty_G} \le C$. In view of Proposition 3.1, we have $Z \in \mathrm{BMO}_G$.

From Lemma 4.2, we have $Y_t \geq S_t$ for $t \in [0,T]$. We claim that $\int_0^\cdot (S_s - Y_s)\,\mathrm{d}A_s$ is a non-increasing G-martingale on $[0,T]$. Set $\widetilde{K}_t^n = \int_0^t (Y_s - S_s)\mathrm{d}K_s^n$. Since $Y_t \geq S_t$ for $t \in [0,T]$ and K^n is a decreasing G-martingale, $\widetilde{K}^n$ is a decreasing G-martingale.

We have

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big| -\int_0^t (Y_s - S_s)\mathrm{d}A_s - \widetilde{K}_t^n \Big| \\
&\leq \sup_{t\in[0,T]} \Big\{ \Big| -\int_0^t (Y_s - S_s)\mathrm{d}A_s + \int_0^t (Y_s - S_s)\mathrm{d}A_s^n \Big| + \Big| \int_0^t (Y_s^n - Y_s)\mathrm{d}A_s^n \Big| \\
&\quad + \Big| \int_0^t (Y_s^n - Y_s)\mathrm{d}K_s^n \Big| + \Big| \int_0^t -(Y_s^n - S_s)\mathrm{d}L_s^n \Big| \Big\} \\
&\leq \sup_{t\in[0,T]} \Big\{ \Big| \int_0^t (\widetilde{Y}_s^m - \widetilde{S}_s^m)\mathrm{d}(A_s^n - A_s) \Big| + \Big| \int_0^t \{Y_s - S_s - (\widetilde{Y}_s^m - \widetilde{S}_s^m)\}\mathrm{d}(A_s^n - A_s) \Big| \Big\} \\
&\quad + \sup_{t\in[0,T]} |Y_t - Y_t^n| [|A_T^n| + |K_T^n|] + \sup_{t\in[0,T]} (Y_s^n - S_s)^- |L_T^n|, \quad \text{q.s.,}
\end{aligned}$$

with

$$\widetilde{Y}_t^m := \sum_{i=0}^{m-1} Y_{t_i^m} \mathbf{1}_{[t_i^m, t_{i+1}^m)}(t), \quad \widetilde{S}_t^m := \sum_{i=0}^{m-1} S_{t_i^m} \mathbf{1}_{[t_i^m, t_{i+1}^m)}(t)$$

and

$$t_i^m := \frac{iT}{m}, \quad i = 0, 1, \cdots, m.$$

In view of Lemma 4.1 and identically as in the proof of [24, Theorem 5.1], we have

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \Big| -\int_0^t (Y_s - S_s)\mathrm{d}A_s - \widetilde{K}_t^n \Big| \Big] = 0,$$

which implies that $\int_0^\cdot (S_s - Y_s)\,\mathrm{d}A_s$ is a non-increasing G-martingale on $[0,T]$.

In an identical way, we have the following theorem.

Theorem 5.2 *Suppose that ξ, f, g, and S satisfy* **(H1)**–**(H5)**. *Then the reflected G-BSDE (2.2) has a unique solution (Y,Z,A) such that $(Y,Z) \in L_G^\infty[0,T] \times \mathrm{BMO}_G$ and $A \in \bigcap_{\alpha\geq 2} S_G^\alpha[0,T]$.*

We have the following comparison theorem for reflected quadratic G-BSDEs.

Theorem 5.3 *Let the set (ξ^i, f^i, g^i, S^i) satisfy* **(H1)**–**(H5)**, *and $(Y^i, Z^i, A^i) \in \mathcal{S}_G^2(0,T)$ be the solution to the following reflected G-BSDE:*

$$\begin{cases} Y_t^i = \xi^i + \displaystyle\int_t^T g^i(s, \omega_{\cdot\wedge s}, Y_s^i, Z_s^i)\mathrm{d}s + \int_t^T f^i(s, \omega_{\cdot\wedge s}, Y_s^i, Z_s^i)\mathrm{d}\langle B\rangle_s - \int_t^T Z_s^i \mathrm{d}B_s + \int_t^T \mathrm{d}A_s^i, \quad q.s.; \\ Y_t^i \geq S_t^i, \quad q.s.\ 0 \leq t \leq T; \quad \displaystyle\int_0^\cdot (S_s^i - Y_s^i)\,\mathrm{d}A_s^i \text{ is a non-increasing } G\text{-martingale} \end{cases}$$

with $i = 1, 2$. Assume that $(Y^i, Z^i) \in S_G^\infty(0,T) \times \mathrm{BMO}_G$ and $A_T^i \in \bigcap_{p\geq 1} L_G^p(\Omega_T)$ for $i = 1, 2$. If $\xi^1 \geq \xi^2$, $g^1 \geq g^2$, $f^1 \geq f^2$, and $S^1 \geq S^2$, q.s., then $Y_t^1 \geq Y_t^2$, q.s. for any $t \in [0,T]$.

Proof The proof is identical to that of [24, Theorem 5.3].

We first consider the following G-BSDE:

$$\begin{aligned} y_t^n = \xi^2 + \int_t^{\mathrm{T}} g^2(s, y_s^n, z_s^n)\mathrm{d}s + \int_t^{\mathrm{T}} f^2(s, y_s^n, z_s^n)\mathrm{d}\langle B\rangle_s + \int_t^{\mathrm{T}} n(y_s^n - S_s^2)^-\mathrm{d}s \\ - \int_t^{\mathrm{T}} z_s^n \mathrm{d}B_s - \int_t^{\mathrm{T}} \mathrm{d}K_s^n, \quad \forall\, t \in [0, T] \end{aligned}$$

for $n = 1, 2, \cdots$. As before, we have

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |Y_t^2 - y_t^n|^\alpha\Big] = 0, \quad \forall\, \alpha \geq 2.$$

Noting that $Y_t^1 \geq S_t^1$, we can rewrite the equation for (Y^1, Z^1, A^1) as

$$\begin{aligned} Y_t^1 = \xi^1 + \int_t^{\mathrm{T}} g^1(s, Y_s^1, Z_s^1)\mathrm{d}s + \int_t^{\mathrm{T}} f^1(s, Y_s^1, Z_s^1)\mathrm{d}\langle B\rangle_s \\ + \int_t^{\mathrm{T}} n(Y_s^1 - S_s^1)^-\mathrm{d}s - \int_t^{\mathrm{T}} Z_s^1 \mathrm{d}B_s + \int_t^{\mathrm{T}} \mathrm{d}A_s^1, \quad \text{q.s. } t \in [0, T]. \end{aligned}$$

Using Theorem 3.1, we have $Y_t^1 \geq y_t^n$, q.s. for all $n \in \mathbb{N}$. Letting $n \to \infty$, we conclude that $Y_t^1 \geq Y_t^2$, q.s.

6 Relation Between Quadratic *G*-BSDEs and Nonlinear Parabolic PDEs

Consider the following PDE:

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n; \\ u(T, x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

where

$$\begin{aligned} F(A, p, y, x, t) := G(\sigma^{\mathrm{T}}(t, x)A\sigma(t, x) + 2f(t, x, y, \sigma^{\mathrm{T}}(t, x)p) + 2h^{\mathrm{T}}(t, x)p) \\ + b^{\mathrm{T}}(t, x)p + g(t, x, y, \sigma^{\mathrm{T}}(t, x)p) \end{aligned}$$

for each $(A, p, y, x, t) \in \mathbb{S}_n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0, T]$.

We will give a nonlinear Feynman-Kac formula for the fully nonlinear PDE (6.1) when the functions f and g are quadratic in the last argument. Similarly as [24, Section 6], we give the relationship between solutions of the obstacle problem for nonlinear parabolic PDEs and the related reflected quadratic G-BSDEs.

In what follows, we consider the G-expectation space $(\Omega, L_G^1(\Omega_T), \widehat{\mathbb{E}})$ for the case of $d = 1$ and $\overline{\sigma}^2 = \widehat{\mathbb{E}}[B_1^2] \geq -\widehat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$.

6.1 Nonlinear Feynman-Kac formula

Our main assumptions of this subsection are formulated as follows.

For deterministic functions $b, h, \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{R}$, and $f, g : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we make the following assumptions.

(A1) The functions b, h, σ, f, g are uniformly continuous in t, i.e., there is a non-decreasing continuous function $w : [0, +\infty) \to [0, +\infty)$ such that $w(0) = 0$ and

$$\sup_{x,y,z\in\mathbb{R}} |l_1(t, x, y, z) - l_1(t', x, y, z)| \leq w(|t - t'|), \quad l_1 = f, g,$$

$$\sup_{x\in\mathbb{R}} |l_2(t,x) - l_2(t',x)| \le w(|t-t'|), \quad l_2 = b, h, \sigma.$$

(A2) There exists a positive integer m and a constant $L > 0$ such that for each $(t, x, x', y, y', z, z') \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}
&|b(t,x) - b(t,x')| + |h(t,x) - h(t,x')| + |\sigma(t,x) - \sigma(t,x')| \le L|x-x'|,\\
&|\phi(x) - \phi(x')| \le L(1 + |x|^m + |x'|^m)|x-x'|,\\
&|f(t,x,y,z) - f(t,x',y',z')| + |g(t,x,y,z) - g(t,x',y',z')|\\
\le\ & L[(1 + |x|^m + |x'|^m)|x-x'| + |y-y'| + (1 + |z| + |z'|)|z-z'|].
\end{aligned}$$

(A3) There is a positive constant M_0 such that

$$\int_0^{\mathrm{T}} \sup_{x\in\mathbb{R}^n} [|f(t,x,0,0)|^2 + |g(t,x,0,0)|^2]\,\mathrm{d}t + \sup_{x\in\mathbb{R}^n} |\phi(x)| \le M_0.$$

(A4) There are two constants $\varepsilon > 0$ and $K > 0$ such that for each $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\varepsilon I \le \sigma\sigma^{\mathrm{T}}(t,x) \le KI.$$

Remark 6.1 **(A4)** implies that σ is bounded on $[0,T] \times \mathbb{R}^n$.

For each $(t,\xi) \in [0,T] \times \bigcap_{p\ge 2} L_G^p(\Omega_t;\mathbb{R}^n)$, we consider the following G-SDE:

$$X_s = \xi + \int_t^s b(u,X_u)\mathrm{d}u + \int_t^s h(u,X_u)\mathrm{d}\langle B\rangle_u + \int_t^s \sigma(u,X_u)\mathrm{d}B_u, \quad \text{q.s. } s \in [t,T]. \tag{6.2}$$

Denote by $X^{t,\xi}$ the solution to G-SDE (6.2). Then, we have the following proposition.

Proposition 6.1 (see [33, Exercise 5.4.8, Chapter 5, p. 111]) *Let $\xi, \xi' \in L_G^p(\Omega_t;\mathbb{R}^n)$ with $p \ge 2$. Then we have, for each $\delta \in [0, T-t]$,*

$$\begin{aligned}
&\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi} - X_s^{t,\xi'}|^p\Big] \le C|\xi-\xi'|^p,\\
&\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi}|^p\Big] \le C(1+|\xi|^p),\\
&\widehat{\mathbb{E}}_t\Big[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi} - \xi|^p\Big] \le C(1+|\xi|^p)\delta^{\frac{p}{2}},
\end{aligned}$$

where the constant C depends on L, G, p, n and T.

Proposition 6.2 *Let the triplet (b^i, h^i, σ^i) satisfy* **(A1)**–**(A2)** *for $i = 1, 2$. For each $(t,\xi) \in [0,T] \times L_G^p(\Omega_t;\mathbb{R}^n)$, $p \ge 2$, let $X^{t,\xi,i}$ be the solution to the following G-SDE:*

$$X_s^{t,\xi,i} = \xi + \int_t^s b^i(s,X_u^{t,\xi,i})\mathrm{d}u + \int_t^s h^i(u,X_u^{t,\xi,i})\mathrm{d}\langle B\rangle_u + \int_t^s \sigma^i(u,X_u^{t,\xi,i})\mathrm{d}B_u, \quad \text{q.s. } s \in [t,T].$$

Then for each $\delta \in [0, T-t]$, there exists a constant C depending only on L, G, p and T, such that

$$\widehat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi,1} - X_{t+\delta}^{t,\xi,2}|^p]$$

$$\leq C\Big(\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{b}_u|\mathrm{d}u\Big)^p\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{h}_u|\mathrm{d}u\Big)^p\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{\sigma}_u|^2\mathrm{d}u\Big)^{\frac{p}{2}}\Big]\Big),\quad q.s.,$$

where for each $u\in[t,T]$*,*

$$\widehat{l}_u:=l^1(u,X_u^{t,\xi,2})-l^2(u,X_u^{t,\xi,2}),\quad l=b,h,\sigma.$$

Proof For simplicity, we assume $h^i=0$. Then we have

$$X_{t+\delta}^{t,\xi,1}-X_{t+\delta}^{t,\xi,2}=\int_t^{t+\delta}(\widehat{b}_u+b^1(t,X_{t+\delta}^{t,\xi,1})-b^1(t,X_{t+\delta}^{t,\xi,2}))\mathrm{d}u$$
$$+\int_t^{t+\delta}(\widehat{\sigma}_u+\sigma^1(t,X_{t+\delta}^{t,\xi,1})-\sigma^1(t,X_{t+\delta}^{t,\xi,2}))\mathrm{d}B_u.$$

In view of BDG inequality, we have

$$\widehat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi,1}-X_{t+\delta}^{t,\xi,2}|^p]$$
$$\leq C_1\Big(\widehat{\mathbb{E}}_t\Big[\int_t^{t+\delta}|X_u^{t,\xi,1}-X_u^{t,\xi,2}|^p\mathrm{d}u\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{b}_u|\mathrm{d}u\Big)^p\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{\sigma}_u|^2\mathrm{d}u\Big)^{\frac{p}{2}}\Big]\Big)$$
$$\leq C_1\Big(\int_t^{t+\delta}\widehat{\mathbb{E}}_t[|X_u^{t,\xi,1}-X_u^{t,\xi,2}|^p]\mathrm{d}u+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{b}_u|\mathrm{d}u\Big)^p\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{\sigma}_u|^2\mathrm{d}u\Big)^{\frac{p}{2}}\Big]\Big).$$

By the Gronwall's inequality, we obtain

$$\widehat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi,1}-X_{t+\delta}^{t,\xi,2}|^p]\leq C_1\mathrm{e}^{C_1T}\Big(\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{b}_u|\mathrm{d}u\Big)^p\Big]+\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{t+\delta}|\widehat{\sigma}_u|^2\mathrm{d}u\Big)^{\frac{p}{2}}\Big]\Big).$$

We now consider the following G-BSDE:

$$Y_s=\phi(X_T^{t,\xi})+\int_s^{\mathrm{T}}g(u,X_u^{t,\xi},Y_u,Z_u)\mathrm{d}u+\int_s^{\mathrm{T}}f(u,X_u^{t,\xi},Y_u,Z_u)\mathrm{d}\langle B\rangle_u$$
$$-\int_s^{\mathrm{T}}Z_u\mathrm{d}B_u-\int_s^{\mathrm{T}}\mathrm{d}K_u,\quad \text{q.s. } s\in[t,T]. \tag{6.3}$$

We should point out that [15, Theorem 5.3] can not be used to our case directly, for it is hard to check **(H2)** directly in our Markovian case. We now give the existence of the solution to G-BSDE (6.3) in the spirit of the method in Hu et al. [12] and Hu et al. [15].

Without loss of generality, we assume $h=0$, $g=0$ and $t=0$. For each $x_0\in\mathbb{R}^n$, we consider the following forward and backward differential equations in the G-framework (G-FBSDE for short):

$$X_t=x_0+\int_0^t b(u,X_u)\mathrm{d}u+\int_0^t\sigma(u,X_u)\mathrm{d}B_u,\quad \text{q.s. } t\in[0,T], \tag{6.4}$$

$$Y_t=\phi(X_T)+\int_t^{\mathrm{T}}f(u,X_u,Y_u,Z_u)\mathrm{d}\langle B\rangle_u-\int_t^{\mathrm{T}}Z_u\mathrm{d}B_u-\int_t^{\mathrm{T}}\mathrm{d}K_u,\quad \text{q.s. } t\in[0,T], \tag{6.5}$$

where b,σ,f and ϕ satisfy **(A1)**–**(A4)**.

First, we introduce the following fully nonlinear PDE on $[0,T]$:

$$\begin{cases}\partial_t u+G(\sigma^{\mathrm{T}}(t,x)D_x^2u\sigma(t,x)+2f(t,x,u,\sigma^{\mathrm{T}}(t,x)D_xu))+b^{\mathrm{T}}(t,x)D_xu=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,T)\times\mathbb{R}^n;\\ u(T,x)=\phi(x),\quad x\in\mathbb{R}^n.\end{cases} \tag{6.6}$$

We make the following assumptions on the coefficients of the PDE (6.6).

(A5) The function $f(t,x,y,z)$ is continuously differentiable in (x,y,z), differentiable in t, and twice differentiable in (x,y,z), where the first-order time derivative of f and the second-order derivatives of f in (x,y,z) are bounded on the set $[0,T]\times\mathbb{R}^n\times[-M_y,M_y]\times[-M_z,M_z]$, for any $M_y,M_z>0$.

(A6) Both functions b and σ are differentiable in t and twice differentiable in x, where the first-order time derivative of (b,σ) and the second-order spatial derivatives of (b,σ) are bounded on the set $[0,T]\times\mathbb{R}^n$.

(A7) The functions b is bounded on the set $[0,T]\times\mathbb{R}^n$. The function f is bounded on the set $[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$.

(A8) There exists a constant $L>0$ such that for each $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$,

$$|\phi(x)-\phi(x')|+|f(t,x,y,z)-f(t,x',y,z)|\le L|x-x'|,\quad \forall\, x,x'\in\mathbb{R}^n.$$

Note that Peng [33, Appendix C] used Krylov [20, Theorem 6.4.3] to prove that there is a classical solution to PDE (6.6) when $b=0$, $f=0$, and $\sigma=1$. In a similar way, we prove that there is a classical solution to PDE (6.6) and further that $u(t,\cdot)$ is uniformly Lipschitz continuous.

Proposition 6.3 *Assume that b,σ,f and ϕ satisfy* **(A1)**–**(A8)**. *Then the* PDE (6.6) *admits a classical solution $u\in C([0,T]\times\mathbb{R}^n)$ bounded by $M:=M(M_0,L)$, and there exists a constant $\alpha\in(0,1)$ such that for each $k\in(0,T)$,*

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T-k]\times\mathbb{R}^n)}<\infty.$$

Moreover, there exists a constant $C>0$ such that for all $t\in[0,T]$,

$$|u(t,x)-u(t,x')|\le C|x-x'|,\quad \forall\, x,x'\in\mathbb{R}^n.$$

Proof First, we introduce the truncation function. For each integer N, let $\rho_N:\mathbb{R}\to\mathbb{R}$ be a smooth modification of the projection on $[-N,N]$ such that $|\rho_N|\le N$, $|\rho_N'|\le 1$ and $\rho_N(z)=z$ when $|z|\le N-1$. We consider the following PDE:

$$\begin{cases}\partial_t u+G(\sigma^{\mathrm T}(t,x)D_x^2u\sigma(t,x)+2f^N(t,x,u,\sigma^{\mathrm T}(t,x)D_xu))+b^{\mathrm T}(t,x)D_xu=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,T)\times\mathbb{R}^n;\\ u(T,x)=\phi(x),\quad x\in\mathbb{R}^n,\end{cases}\tag{6.7}$$

where f^N is defined as

$$f^N(t,x,y,z):=f(t,x,y,\rho_N(z)),\quad \forall\,(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}.$$

It is easy to check that f^N is uniformly Lipschitz in z.

Considering the PDE for the quantity $\mathrm{e}^{(L\overline{\sigma}^2+1)(t-T)}u(t,x)$ as in [33, Appendix C], in view of [20, Theorem 6.4.3], we can prove that the PDE (6.7) admits a classical solution $u^N\in C([0,T]\times\mathbb{R}^n)$ dominated by a constant $M:=M(M_0,L)$, such that for some constant $\alpha\in(0,1)$, the related restriction of u^N belongs to $C^{1+\frac{\alpha}{2},2+\alpha}([0,k]\times\mathbb{R}^n)$ with any $k\in(0,T)$.

We now rewrite PDE (6.7) into a HJB equation, and then estimate the gradient D_xu^N. Since

$$G(a)=\frac12(\overline{\sigma}^2a^+-\underline{\sigma}^2a^-)=\sup_{v\in[\underline{\sigma},\overline{\sigma}]}\frac12 v^2a,$$

the PDE (6.7) is the following HJB equation:

$$\begin{cases}\partial_t u + \sup\limits_{v\in[\underline{\sigma},\overline{\sigma}]} H^N(t,x,u(t,x),D_xu(t,x),D_x^2u(t,x),v) = 0,\\ u(T,x)=\phi(x),\end{cases} \tag{6.8}$$

where the Hamiltonian H^N is defined as follows: For $(t,x,y,p,A,v)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}_n\times[\underline{\sigma},\overline{\sigma}]$,

$$H^N(t,x,y,p,A,v) := \frac{1}{2}\widehat{\sigma}^{\mathrm{T}}(t,x,v)A\widehat{\sigma}(t,x,v) + F^N(t,x,y,\widehat{\sigma}^{\mathrm{T}}(t,x)p,v) + b^{\mathrm{T}}(t,x)p$$

with

$$\widehat{\sigma}(t,x,v) := v\sigma(t,x), \quad F^N(t,x,y,z,v) := v^2 f^N\Big(t,x,y,\frac{z}{v}\Big) \quad \text{for } z\in\mathbb{R}.$$

This shows that u^N is in fact a value function of a control problem.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be the classical Wiener space. Let W be a one-dimension standard Brownian motion under Probability $\mathbb{P}$. For each $(t,x)\in[0,T]\times\mathbb{R}^n$, we consider the FBSDE:

$$\begin{aligned}\mathcal{X}_s^{t,x,v} &= x + \int_t^s b(r,\mathcal{X}_r^{t,x,v})\,\mathrm{d}r + \int_t^s \widehat{\sigma}(r,\mathcal{X}_r^{t,x,v},v_r)\,\mathrm{d}W_r, \quad \mathbb{P}\text{-a.s. } s\in[t,T],\\ \mathcal{Y}_s^{t,x,v,N} &= \phi(\mathcal{X}_T^{t,x,v}) + \int_s^{\mathrm{T}} F^N(r,\mathcal{X}_r^{t,x,v},\mathcal{Y}_r^{t,x,v,N},\mathcal{Z}_r^{t,x,v,N},v_r)\,\mathrm{d}r\\ &\quad - \int_s^{\mathrm{T}} \mathcal{Z}_r^{t,x,v,N}\mathrm{d}W_r, \quad \mathbb{P}\text{-a.s. } s\in[t,T].\end{aligned}$$

Let $\mathcal{F}^W$ be the filtration generated by W and augmented by all $\mathbb{P}$-null sets. Let $\mathcal{V}$ be the set of all $\mathcal{F}_t^W$-progressively measurable processes valued in $[\underline{\sigma},\overline{\sigma}]$. In view of [30, Theorem 4.2] or [7, Theorems 4.2 and 5.3] and noting that u^N is a viscosity solution of the PDE (6.8), we have

$$u^N(t,x) = \sup_{v\in\mathcal{V}} \mathcal{Y}_t^{t,x,v,N}.$$

Note that for each $z\in\mathbb{R}$,

$$|\rho_N(z)|\le z, \quad |\rho_N'(z)|\le 1.$$

We have for each $(t,x,y,v)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times[\underline{\sigma},\overline{\sigma}]$,

$$\begin{aligned}&|F^N(t,x,y,z,v)-F^N(t,x,y',z',v)|\\ &= \Big|v^2 f^N\Big(t,x,y,\frac{z}{v}\Big) - v^2 f^N\Big(t,x,y',\frac{z'}{v}\Big)\Big|\\ &\le Lv^2\Big(1+\Big|\rho_N\Big(\frac{z}{v}\Big)\Big| + \Big|\rho_N\Big(\frac{z'}{v}\Big)\Big|\Big)\Big|\rho_N\Big(\frac{z}{v}\Big) - \rho_N\Big(\frac{z'}{v}\Big)\Big|\\ &\le L(\overline{\sigma}+|z|+|z'|)|z-z'|.\end{aligned}$$

In view of [5, Lemma 1], there exists a constant C_1 independent of v and N such that for each $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$\sup_{s\in[t,T]} |\mathcal{Y}_s^{t,x,v,N}| \le C_1.$$

In view of [4, Proposition 2.1], there exists a constant C_2 independent of v and N such that for each $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$\Big\|\int_t^{\cdot} \mathcal{Z}_s^{t,x,v,N}\mathrm{d}W_s\Big\|_{\mathrm{BMO}(\mathbb{P})} \le C_2.$$

By a similar stability result as in [1, Theorem 5.1], there exists a constant C_3 and some $p > 2$ which are independent of v and N such that for each $(t, x, x') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|\mathcal{Y}_t^{t,x,v,N} - \mathcal{Y}_t^{t,x',v,N}| \leq C_3\Big\{E^{\mathbb{P}}[|\phi(\mathcal{X}_T^{t,x,v}) - \phi(\mathcal{X}_T^{t,x',v})|^p]^{\frac{1}{p}} + E^{\mathbb{P}}\Big[\Big(\int_t^{\mathrm{T}} |\delta F_s^N|\mathrm{d}s\Big)^p\Big]^{\frac{1}{p}}\Big\},$$

where

$$\delta F_s^N := F^N(s, \mathcal{X}_s^{t,x,v}, \mathcal{Y}_s^{t,x,v,N}, \mathcal{Z}_s^{t,x,v,N}, v_s) - F^N(s, \mathcal{X}_s^{t,x',v}, \mathcal{Y}_s^{t,x,v,N}, \mathcal{Z}_s^{t,x,v,N}, v_s).$$

Thus we get

$$|\mathcal{Y}_t^{t,x,v,N} - \mathcal{Y}_t^{t,x',v,N}| \leq LC_3\Big\{E^{\mathbb{P}}[|\mathcal{X}_T^{t,x,v} - \mathcal{X}_T^{t,x',v}|^p]^{\frac{1}{p}} + E^{\mathbb{P}}\Big[\Big(\int_t^{\mathrm{T}} |\mathcal{X}_s^{t,x,v} - \mathcal{X}_s^{t,x',v}|\mathrm{d}s\Big)^p\Big]^{\frac{1}{p}}\Big\}.$$

By [7, (3.3)], there exists a constant C_4 independent of v and N such that for each $(t, x, x') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$E^{\mathbb{P}}\Big[\sup_{s\in[t,T]} |\mathcal{X}_s^{t,x,v} - \mathcal{X}_s^{t,x',v}|^p\Big] \leq C_4|x - x'|^p.$$

Thus there exists a constant C independent of v and N such that for each $(t, x, x') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|\mathcal{Y}_t^{t,x,v,N} - \mathcal{Y}_t^{t,x',v,N}| \leq C|x - x'|,$$

which means $|u^N(t, x) - u^N(t, x')| \leq C|x - x'|$. So we get

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |D_x u^N(t, x)| \leq C.$$

In view of Remark 6.1, we know that σ is bounded. Let

$$N > 1 + C \cdot \max_{(t,x)\in[0,T]\times\mathbb{R}^n} |\sigma(t, x)|.$$

It is easy to check that for each $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$f^N(t, x, u^N, \sigma^{\mathrm{T}}(t, x)D_x u^N) = f(t, x, u^N, \sigma^{\mathrm{T}}(t, x)D_x u^N).$$

Set $u := u^N$ and we know that u is the solution of PDE (6.6).

For each $t \in [0, T]$, we set $Y_t := u(t, X_t)$, $Z_t := \sigma^{\mathrm{T}}(t, X_t)D_x u(t, X_t)$ and

$$\begin{aligned} K_t = &\int_0^t \Big(\frac{1}{2}D_x^2 u(s, X_s) + f(s, X_s, u(s, X_s), \sigma^{\mathrm{T}}(s, X_s)D_x u(s, X_s))\Big)\mathrm{d}\langle B\rangle_s \\ &- \int_0^t G(D_x^2 u(s, X_s) + f(s, X_s, u(s, X_s), \sigma^{\mathrm{T}}(s, X_s)D_x u(s, X_s)))\mathrm{d}s. \end{aligned}$$

For any $0 < k < T$, applying Itô's formula to $u(t, X_t)$ for $t \in [0, k]$, we get

$$Y_t = u(k, X_k) + \int_t^k f(s, X_s, Y_s, Z_s)\mathrm{d}\langle B\rangle_s - \int_t^k Z_s \mathrm{d}B_s - \int_t^k \mathrm{d}K_s, \quad \text{q.s.} \tag{6.9}$$

Similarly as [12, (4.3)], in view of Proposition 6.3 and (6.9), we could obtain that there exists a constant $C > 0$, for $t_1, t_2 \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^n$,

$$|u(t_1, x_1) - u(t_2, x_2)| \leq C((1 + |x_1|)\sqrt{|t_1 - t_2|} + |x_1 - x_2|).$$

Then we can deduce that $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{S}_G^p(0, T)$ and (Y, Z, K) is a solution to (6.5). So we have the following lemma.

Lemma 6.1 *Assume that b, σ, f and ϕ satisfy* **(A1)**–**(A8)**. *Then G-BSDE (6.5) has a solution $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{S}_G^p(0, T)$.*

As an immediate consequence of both proofs of [15, Proposition 3.5] and Proposition 3.3, we have the following stability property for quadratic G-BSDEs.

Proposition 6.4 *Let the triplet (ξ^i, f^i, g^i) satisfy* **(H1)** *and* **(H3)** *for $i = 1, 2$. Let $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^2(0, T)$ be the solution to the following G-BSDE:*

$$Y_t^i = \xi^i + \int_t^{\mathrm{T}} f^i(s, Y_s^i, Z_s^i)\,\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} Z_s^i\,\mathrm{d}B_s - \int_t^{\mathrm{T}} \mathrm{d}K_s^i, \quad q.s.\ t \in [0, T].$$

Moreover, we suppose

$$\|L_z(1 + |Z_1| + |Z_2|)\|_{\mathrm{BMO}_G} < \phi(q) := \Big\{1 + \frac{1}{q^2}\log\frac{2q-1}{2(q-1)}\Big\}^{\frac{1}{2}} - 1.$$

Then for each $p > \frac{q}{q-1}$, there exists a constant $C := C(p, T, L, M_0)$ such that for any $t \in [0, T]$,

$$|Y_t^1 - Y_t^2| \leq C\Big\{\widehat{\mathbb{E}}_t[|\xi^1 - \xi^2|^p]^{\frac{1}{p}} + \widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}} |(f^1 - f^2)(s, Y_s^2, Z_s^2)|\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{p}}\Big\}, \quad q.s.$$

Remark 6.2 We still have [15, (3.2)], which means that the constants C_1 and p in Proposition 6.4 depend only on T, L and M_0.

The main result of this section is stated as follows.

Theorem 6.1 *Assume that b, σ, f and ϕ satisfy* **(A1)**–**(A4)**. *Then G-BSDE (6.5) has a unique solution $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{S}_G^p(0, T)$.*

Proof The uniqueness result directly comes from Proposition 6.4. Now we focus on the existence result. We borrow the idea of Hu et al. [15] to mollify the coefficients of the G-FBSDE.

Step 1 We assume that f satisfies the following condition.

(A5') The first-order time derivative of f in t, and the spatial derivatives of f up to the second-order are bounded on the set $[0, T] \times \mathbb{R}^n \times [-M_y, M_y] \times [-M_z, M_z]$, for any $M_y, M_z > 0$.

We replace **(A5)** with **(A5')**. Assume that b, σ, f and ϕ satisfy **(A1)**–**(A4)**, **(A5')** and **(A6)**–**(A8)**. Then we can obtain that G-BSDE (6.5) has one solution $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{S}_G^p(0, T)$ with exactly the same method of Step 1 in Hu et al. [15, Section 5].

Step 2 We assume that b, σ, f and ϕ satisfy **(A1)**–**(A4)** and **(A6)**–**(A8)**. For each $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, we define

$$f^k(t, x, y, z) := \int_{\mathbb{R}^{n+1}} f(t - \widetilde{t}, x - \widetilde{x}, y, z)\rho_k(\widetilde{t}, \widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x},$$

where ρ_k is a positive smooth function such that its support is contained in a $\frac{1}{k}$-ball in $\mathbb{R}^{n+1}$ and $\int_{\mathbb{R}^{n+1}} \rho_k = 1$. In addition, we define the extension of f on $\mathbb{R}$, i.e., $f(t, \cdot, \cdot, \cdot) = f(t^+ \wedge T, \cdot, \cdot, \cdot)$. We can check that f^k satisfies **(A5')**. Therefore, in view of the result in Step 1, we obtain that the G-BSDE (6.5) with the coefficients (ϕ, f^k) admits a solution $(Y^k, Z^k, K^k) \in \bigcap_{p\geq 2} \mathfrak{S}_G^p(0, T)$.

Noting that for each $k_1 \geq k_2$ and $t \in [0, T]$,

$$|(f^{k_1} - f^{k_2})(t, X_t, Y_t^{k_2}, Z_t^{k_2})|$$

$$
\begin{aligned}
&\le |(f^{k_1}-f)(t,X_t,Y_t^{k_2},Z_t^{k_2})| + |(f^{k_2}-f)(t,X_t,Y_t^{k_2},Z_t^{k_2})| \\
&\le \int_{\mathbb{R}^{n+1}} |f(t-\widetilde{t},X_t-\widetilde{x},Y_t^{k_2},Z_t^{k_2}) - f(t,X_t,Y_t^{k_2},Z_t^{k_2})|\rho_{k_1}(\widetilde{t},\widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x} \\
&\quad + \int_{\mathbb{R}^{n+1}} |f(t-\widetilde{t},X_t-\widetilde{x},Y_t^{k_2},Z_t^{k_2}) - f(t,X_t,Y_t^{k_2},Z_t^{k_2})|\rho_{k_2}(\widetilde{t},\widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x} \\
&\le w\Big(\frac{1}{k_1}\Big) + w\Big(\frac{1}{k_2}\Big) + L\Big(\frac{1}{k_1}+\frac{1}{k_2}\Big),
\end{aligned}
$$

we could deduce that the sequence $\{Y^k\}_{k=1}^\infty$ is a Cauchy sequence in $S_G^p(0,T)$ for any $p \ge 2$ by Proposition 6.4 and Remark 6.2. Thus we could conclude that G-BSDE (6.5) has a solution $(Y,Z,K) \in \bigcap_{p\ge 2} \mathfrak{G}_G^p(0,T)$ in a similar way as in Step 1 in Hu et al. [15, Section 5].

Step 3 We assume that b,σ,f and ϕ satisfy **(A1)**–**(A4)** and **(A7)**–**(A8)**. For each $(t,x) \in [0,T]\times\mathbb{R}^n$, we define

$$
l^k(t,x) := \int_{\mathbb{R}^{n+1}} l(t-\widetilde{t},x-\widetilde{x})\rho_k(\widetilde{t},\widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x}, \quad l=b,\sigma,
$$

where ρ_k is a positive smooth function and is supported in a $\frac{1}{k}$-ball in $\mathbb{R}^{n+1}$ and $\int_{\mathbb{R}^{n+1}}\rho_k = 1$. We can check that b^k and σ^k satisfy **(A6)**. Moreover, **(A4)** still holds here for σ^k when k is large enough. Actually, if we assume that σ satisfies **(A4)** with constants ε and K, we can check that for k large enough and for each $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$
\frac{\varepsilon}{2}I \le \sigma^k(t,x)(\sigma^k)^{\mathrm{T}}(t,x) \le 2KI.
$$

Let X^k be the solution of the following G-SDE:

$$
X_t^k = x_0 + \int_0^t b^k(u,X_u^k)\mathrm{d}u + \int_0^t \sigma^k(u,X_u^k)\mathrm{d}B_u, \quad \text{q.s. } t\in[0,T].
$$

From Step 2, we can let (Y^k,Z^k,K^k) be the solution to the following G-BSDE:

$$
Y_t^k = \phi(X_T^k) + \int_t^{\mathrm{T}} f(s,X_s^k,Y_s^k,Z_s^k)\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} Z_s^k\mathrm{d}B_s - \int_t^{\mathrm{T}} \mathrm{d}K_s^k, \quad \text{q.s. } t\in[0,T].
$$

For each $k_1,k_2\in\mathbb{N}$ and $t\in[0,T]$, set

$$
\widehat{l}_t := l^{k_1}(t,X_t^{k_2}) - l^{k_2}(t,X_t^{k_2}), \quad l=b,\sigma.
$$

It is easy to check that for $l=b,\sigma$ and each $t\in[0,T]$,

$$
\begin{aligned}
|\widehat{l}_t| &\le |l^{k_1}(t,X_t^{k_2}) - l(t,X_t^{k_2})| + |l^{k_2}(t,X_t^{k_2}) - l(t,X_t^{k_2})| \\
&\le \int_{\mathbb{R}^2} |l(t-\widetilde{t},X_t^{k_2}-\widetilde{x}) - l(t,X_t^{k_2})|\rho_{k_1}(\widetilde{t},\widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x} \\
&\quad + \int_{\mathbb{R}^2} |l(t-\widetilde{t},X_t^{k_2}-\widetilde{x}) - l(t,X_t^{k_2})|\rho_{k_2}(\widetilde{t},\widetilde{x})\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{x} \\
&\le w\Big(\frac{1}{k_1}\Big) + w\Big(\frac{1}{k_2}\Big) + L\Big(\frac{1}{k_1}+\frac{1}{k_2}\Big).
\end{aligned}
$$

In view of Proposition 6.2, we obtain for each $p\ge 2$ and $t\in[0,T]$,

$$
\widehat{\mathbb{E}}[|X_t^{k_1}-X_t^{k_2}|^p] \le C\Big(\widehat{\mathbb{E}}\Big[\Big(\int_0^t |\widehat{b}_s|\mathrm{d}s\Big)^p\Big] + \widehat{\mathbb{E}}\Big[\Big(\int_0^t |\widehat{\sigma}_s|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big)
$$

$$\leq C_p\Big\{w\Big(\frac{1}{k_1\wedge k_2}\Big)+\Big(\frac{1}{k_1\wedge k_2}\Big)\Big\}^p. \tag{6.10}$$

On the other hand, in view of Proposition 6.4 and [15, (3.2)], we obtain that for some $p>1$ and each $t\in[0,T]$,

$$\begin{aligned}|Y_t^{k_1}-Y_t^{k_2}| &\leq C\widehat{\mathbb{E}}_t[|\phi(X_T^{k_1})-\phi(X_T^{k_2})|^p]^{\frac{1}{p}}\\ &\quad + C\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}}|f(s,X_s^{k_1},Y_s^{k_2},Z_s^{k_2})-f(s,X_s^{k_2},Y_s^{k_2},Z_s^{k_2})|\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{p}}\\ &\leq C_1\widehat{\mathbb{E}}_t[|X_T^{k_1}-X_T^{k_2}|^p]^{\frac{1}{p}}+C_2\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}}|X_s^{k_1}-X_s^{k_2}|^p\mathrm{d}s\Big]^{\frac{1}{p}} \quad \text{q.s.}\end{aligned}$$

Therefore,

$$\begin{aligned}&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|Y_t^{k_1}-Y_t^{k_2}|\Big]\\ &\leq C_3\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\widehat{\mathbb{E}}_t[|X_T^{k_1}-X_T^{k_2}|^p]\Big]^{\frac{1}{p}}+C_4\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\widehat{\mathbb{E}}_t\Big[\int_0^{\mathrm{T}}|X_s^{k_1}-X_s^{k_2}|^p\mathrm{d}s\Big]\Big]^{\frac{1}{p}}.\end{aligned}$$

In view of (6.10), we obtain that for each $\delta>0$,

$$\begin{aligned}&\widehat{\mathbb{E}}\Big[\int_0^{\mathrm{T}}|X_s^{k_1}-X_s^{k_2}|^{p+\delta}\mathrm{d}s\Big]\\ &\leq \int_0^{\mathrm{T}}\widehat{\mathbb{E}}[|X_s^{k_1}-X_s^{k_2}|^{p+\delta}]\mathrm{d}s\leq C_{p,\delta}T\Big\{w\Big(\frac{1}{k_1\wedge k_2}\Big)+\Big(\frac{1}{k_1\wedge k_2}\Big)\Big\}^{p+\delta}.\end{aligned}$$

Then in view of Remark 2.1, we know that $\{Y^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $S_G^p(0,T)$ for any $p\geq 2$. Thus we could conclude that G-BSDE (6.5) has a solution $(Y,Z,K)\in\bigcap_{p\geq 2}\mathfrak{S}_G^p(0,T)$.

Step 4 We now consider the situation that $\phi(\cdot)$ and $f(t,\cdot,y,z)$ can be locally Lipschitz. We assume that b,σ,f and ϕ satisfy **(A1)**–**(A4)** and **(A7)**. For each $(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$, we define

$$\begin{aligned}\phi^k(x)&:=\int_{\mathbb{R}^n}\phi(x-\widetilde{x})\rho_k(\widetilde{x})\mathrm{d}\widetilde{x},\\ f^k(t,x,y,z)&:=\int_{\mathbb{R}^n}f(t,x-\widetilde{x},y,z)\rho_k(\widetilde{x})\mathrm{d}\widetilde{x},\end{aligned}$$

where ρ_k is a positive smooth function such that its support is contained in a $\frac{1}{k}$-ball in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n}\rho_k=1$. Noting that

$$\Big|\frac{\partial\phi^k}{\partial x}\Big|\leq\int_{\mathbb{R}}\Big|\phi(\widetilde{x})\frac{\partial\rho_k}{\partial x}(x-\widetilde{x})\Big|\mathrm{d}\widetilde{x}\leq C(k)M_0\Big\|\frac{\partial\rho_k}{\partial x}\Big\|_{\infty},$$

we obtain that ϕ^k is Lipschitz in x. Similarly, it is easy to check that $f^k(t,\cdot,y,z)$ is uniformly Lipschitz. Therefore, in view of the result in Step 3, we obtain that the G-BSDE (6.5) with the coefficients (ϕ^k,f^k) admits a solution $(Y^k,Z^k,K^k)\in\bigcap_{p\geq 2}\mathfrak{S}_G^p(0,T)$. It is easy to check that

$$\begin{aligned}&|\phi^{k_1}(X_T)-\phi^{k_2}(X_T)|\\ &\leq|\phi^{k_1}(X_T)-\phi(X_T)|+|\phi^{k_2}(X_T)-\phi(X_T)|\end{aligned}$$

$$\leq \int_{\mathbb{R}} |\phi(X_T - \widetilde{x}) - \phi(X_T)|\rho_{k_1}(\widetilde{x})d\widetilde{x} + \int_{\mathbb{R}} |\phi(X_T - \widetilde{x}) - \phi(X_T)|\rho_{k_2}(\widetilde{x})\mathrm{d}\widetilde{x}$$
$$\leq 2L\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)(1 + |X_T|^m).$$

Similarly, for each $t \in [0, T]$,

$$|(f^{k_1} - f^{k_2})(t, X_t, Y_t^{k_2}, Z_t^{k_2})| \leq 2L\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)(1 + |X_t|^m).$$

In view of Proposition 6.1, we obtain for each $p \geq 2$,

$$\widehat{\mathbb{E}}[|\phi^{k_1}(X_T) - \phi^{k_2}(X_T)|^p] \leq C\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)^p(1 + \widehat{\mathbb{E}}[|X_T|^{mp}]) \leq C'(1 + |x_0|^{mp})\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)^p$$

and

$$\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |(f^{k_1} - f^{k_2})(t, X_t, Y_t^{k_2}, Z_t^{k_2})|\mathrm{d}t\Big)^p\Big] \leq C''(1 + |x_0|^{mp})\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)^p.$$

Then again in view of Remark 2.1, Proposition 6.4 and Remark 6.2, we know that $\{Y^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $S_G^p(0, T)$ for any $p \geq 2$. Thus we could conclude that G-BSDE (6.5) has a solution $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{G}_G^p(0, T)$.

Step 5 Finally, we remove the boundedness condition on b and f. We assume that b, σ, f and ϕ satisfy **(A1)**–**(A4)**. Set $b^k := [b \vee (-k)] \wedge k$ and $f^k := [f \vee (-k)] \wedge f$. It is easy to check that b^k, σ, f^k and ϕ satisfy **(A1)**–**(A4)** and **(A7)**. Let (X^k, Y^k, Z^k, K^k) be the solution of the following G-FBSDE:

$$\begin{cases} X_t^k = x_0 + \int_0^t b^k(u, X_u^k)\mathrm{d}u + \int_0^t \sigma(u, X_u^k)\mathrm{d}B_u, \quad \text{q.s. } t \in [0, T], \\ Y_t^k = \phi(X_T^k) + \int_t^{\mathrm{T}} f^k(s, X_s^k, Y_s^k, Z_s^k)\mathrm{d}\langle B\rangle_s - \int_t^{\mathrm{T}} Z_s^k\mathrm{d}B_s - \int_t^{\mathrm{T}} \mathrm{d}K_s^k, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{q.s. } t \in [0, T]. \end{cases} \tag{6.11}$$

For each $k_1, k_2 \in \mathbb{N}$ and $t \in [0, T]$, set

$$\widehat{b}_t := b^{k_1}(t, X_t^{k_2}) - b^{k_2}(t, X_t^{k_2})$$

and

$$\widehat{f}_t := f^{k_1}(t, X_t^{k_1}, Y_t^{k_2}, Z_t^{k_2}) - f^{k_2}(t, X_t^{k_2}, Y_t^{k_2}, Z_t^{k_2}).$$

Assume $k_1 < k_2$. Then for each $\delta > 0$ and $t \in [0, T]$, we have

$$|\widehat{b}_t| \leq |b(t, X_t^{k_2})|\mathbf{1}_{\{|b(t,X_t^{k_2})|>k_1\}} \leq \frac{1}{k_1^{\delta}}|b(t, X_t^{k_2})|^{1+\delta}$$

and

$$|\widehat{f}_t| \leq |f^{k_1}(t, X_t^{k_1}, Y_t^{k_2}, Z_t^{k_2}) - f^{k_1}(t, X_t^{k_2}, Y_t^{k_2}, Z_t^{k_2})|$$
$$+ |(f^{k_1} - f^{k_2})(t, X_t^{k_2}, Y_t^{k_2}, Z_t^{k_2})|$$
$$\leq L(1 + |X_t^{k_1}|^m + |X_t^{k_2}|^m)|X_t^{k_1} - X_t^{k_2}| + \frac{1}{k_1^{\delta}}|f(t, X_t^{k_2}, Y_t^{k_2}, Z_t^{k_2})|^{1+\delta}.$$

In view of Proposition 6.2, we obtain for each $p \geq 2$ and $t \in [0,T]$,

$$\begin{aligned}\widehat{\mathbb{E}}[|X_t^{k_1} - X_t^{k_2}|^p] &\leq C\widehat{\mathbb{E}}\Big[\Big(\int_0^t |\widehat{b}_s|\mathrm{d}s\Big)^p\Big] \\ &\leq \frac{C_1}{k_1^p} + \frac{C_1}{k_1^{p\delta}}\widehat{\mathbb{E}}\Big[\int_0^t |b(s,X_s^{k_2})|^{p(1+\delta)}\mathrm{d}s\Big] \\ &\leq \frac{C_1}{k_1^p} + \frac{C_2}{k_1^{p\delta}}\Big(\lambda_T + \Big[\int_0^t \widehat{\mathbb{E}}|X_s^{k_2}|^{p(1+\delta)}\mathrm{d}s\Big]\Big) \\ &\leq \frac{C_1}{k_1^p} + \frac{C_3}{k_1^{p\delta}}(1+\lambda_T + |x_0|^{p(1+\delta)}), \end{aligned} \tag{6.12}$$

where $\lambda_T := \int_0^T |b(s,0)|^{p(1+\delta)}\mathrm{d}s$. In view of Proposition 6.1, we obtain for each $p \geq 2$,

$$\begin{aligned}\widehat{\mathbb{E}}[|\phi(X_T^{k_1}) - \phi(X_T^{k_2})|^p] &\leq C\widehat{\mathbb{E}}[(1+|X_T^{k_1}|^{mp} + |X_T^{k_2}|^{mp})|X_T^{k_1} - X_T^{k_2}|^p] \\ &\leq C_1\widehat{\mathbb{E}}[1+|X_T^{k_1}|^{2mp} + |X_T^{k_2}|^{2mp}]^{\frac{1}{2}}\widehat{\mathbb{E}}[|X_T^{k_1} - X_T^{k_2}|^{2p}]^{\frac{1}{2}} \\ &\leq C_2(1+|x_0|^{mp})\widehat{\mathbb{E}}[|X_T^{k_1} - X_T^{k_2}|^{2p}]^{\frac{1}{2}}. \end{aligned} \tag{6.13}$$

On the other hand, for each $p \geq 2$ and $0 < \delta \leq 1$,

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{f}_t|\mathrm{d}t\Big)^p\Big] &\leq C\int_0^{\mathrm{T}} \widehat{\mathbb{E}}[(1+|X_t^{k_1}|^{mp} + |X_t^{k_2}|^{mp})|X_t^{k_1} - X_t^{k_2}|^p]\mathrm{d}t \\ &\quad + \frac{C}{k_1^{\delta}}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |f(t,X_t^{k_2},Y_t^{k_2},Z_t^{k_2})|^{1+\delta}\mathrm{d}t\Big)^p\Big] \\ &\leq C_1(1+|x_0|^{mp})\int_0^{\mathrm{T}} \widehat{\mathbb{E}}[|X_t^{k_1} - X_t^{k_2}|^{2p}]^{\frac{1}{2}}\mathrm{d}t \\ &\quad + \frac{C_1}{k_1^{\delta}}\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} (|f(t,X_t^{k_2},0,0)|^2 + |Y_t^{k_2}|^2 + |Z_t^{k_2}|^2)\mathrm{d}t\Big)^p\Big]. \end{aligned}$$

Note that [15, (3.2)] still holds here, which means that there exists a constant $C = C(M_0, L, G, T)$, such that

$$\|Y^{k_2}\|_{S_G^\infty} + \|Z^{k_2}\|_{\mathrm{BMO}_G} \leq C.$$

In view of Lemma 2.2 and **(A3)**, we obtain

$$\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\widehat{f}_t|\mathrm{d}t\Big)^p\Big] \leq C_1(1+|x_0|^{mp})\int_0^{\mathrm{T}} \widehat{\mathbb{E}}[|X_t^{k_1} - X_t^{k_2}|^{2p}]^{\frac{1}{2}}\mathrm{d}t + \frac{C_2}{k_1^{\delta}} \tag{6.14}$$

In view of Remarks 2.1 and 6.2, inequalities (6.12)–(6.14) and Proposition 6.4, we see that $\{Y^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $S_G^p(0,T)$ for any $p \geq 2$. Thus we could conclude that G-BSDE (6.5) has a solution $(Y,Z,K) \in \bigcap_{p\geq 2} \mathfrak{G}_G^p(0,T)$.

Moreover, we have the following result with a similar argument before.

Theorem 6.2 *Assume that* $\xi \in \bigcap_{p\geq 2} L_G^p(\Omega_t;\mathbb{R}^n)$ *and* b,h,σ,g,f *and* ϕ *satisfy* **(A1)**–**(A4)** . *Then* G-BSDE (6.3) *has a unique solution* $(Y,Z,K) \in \bigcap_{p\geq 2} \mathfrak{G}_G^p(0,T)$.

Remark 6.3 Once we have a solution $(Y, Z, K) \in \bigcap_{p\geq 2} \mathfrak{G}_G^p(0,T)$ to G-BSDE (6.3). In view of $[15, (3.2) - (3.3)]$, there are two constants $C1 = C1(M_0, L, G, T)$ and $C2 = C2(p, M_0, L, G, T)$ such that for all $p \geq 2$,

$$\|Y\|_{S_G^\infty} + \|Z\|_{\mathrm{BMO}_G} \leq C_1, \quad \widehat{\mathbb{E}}[|K_T|^p] \leq C_2.$$

Now we can give the relationship between quadratic G-FBSDEs and parabolic PDEs. For $(t,\xi) \in [0,T] \times \bigcap_{p\geq 2} L_G^p(\Omega_t;\mathbb{R}^n)$, denote by $(X^{t,\xi}, Y^{t,\xi}, Z^{t,\xi}, K^{t,\xi})$ the solution to the G-FBSDE (6.2)–(6.3).

Proposition 6.5 *For each $t \in [0,T]$ and $\xi, \xi' \in \bigcap_{p\geq 2} L_G^p(\Omega_t;\mathbb{R}^n)$, we have*

$$\begin{aligned}
&|Y_t^{t,\xi} - Y_t^{t,\xi'}| \leq C(1 + |\xi|^m + |\xi'|^m)|\xi - \xi'|, \quad q.s.,\\
&|Y_t^{t,\xi}| \leq C, \quad q.s.,
\end{aligned}$$

where the constant C depends on L, G and T.

Proof For simplicity, we assume $g = 0$ and $h = 0$. In view of Propositions 6.1, 6.4 and Remark 6.3, we obtain

$$\begin{aligned}
|Y_t^{t,\xi} - Y_t^{t,\xi'}| &\leq C\widehat{\mathbb{E}}_t[|\phi(X_T^{t,\xi}) - \phi(X_T^{t,\xi'})|^p]^{\frac{1}{p}}\\
&\quad + C\widehat{\mathbb{E}}_t\Big[\Big(\int_t^{\mathrm{T}} |f(s, X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi}) - f(s, X_s^{t,\xi'}, Y_s^{t,\xi}, Z_s^{t,\xi})|\mathrm{d}\langle B\rangle_s\Big)^p\Big]^{\frac{1}{p}}\\
&\leq C_1\widehat{\mathbb{E}}_t[(1 + |X_T^{t,\xi}|^{mp} + |X_T^{t,\xi'}|^{mp})|X_T^{t,\xi} - X_T^{t,\xi'}|^p]^{\frac{1}{p}}\\
&\quad + C_1\widehat{\mathbb{E}}_t\Big[\int_t^{\mathrm{T}} (1 + |X_s^{t,\xi}|^{mp} + |X_s^{t,\xi'}|^{mp})|X_s^{t,\xi} - X_s^{t,\xi'}|^p \mathrm{d}s\Big]^{\frac{1}{p}}\\
&\leq C_2(1 + |\xi|^m + |\xi'|^m)\widehat{\mathbb{E}}_t[|X_T^{t,\xi} - X_T^{t,\xi'}|^{2p}]^{\frac{1}{2p}}\\
&\quad + C_2(1 + |\xi|^m + |\xi'|^m)\Big(\int_t^{\mathrm{T}} \widehat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^{2p}\mathrm{d}s]^{\frac{1}{2}}\Big)^{\frac{1}{p}}\\
&\leq C_2(1 + |\xi|^m + |\xi'|^m)|\xi - \xi'| \quad \text{q.s.}
\end{aligned}$$

On the other hand, we get $|Y_t^{t,\xi}| \leq C$, q.s., directly from Remark 6.3.

Now for each $(t,x) \in [0,T] \times \mathbb{R}^n$, we define $u(t,x) := Y_t^{t,x}$. Identically as in [13, Reamrk 4.3], we deduce that u is a deterministic function. In view of Proposition 6.5, we immediately have the following estimates:

$$\begin{aligned}
&|u(t,x) - u(t,x')| \leq C(1 + |x|^m + |x'|^m)|x - x'|,\\
&|u(t,x)| \leq C,
\end{aligned}$$

where the constant C depends on L, G and T. Moreover, with the same proof of [13, Theorem 4.4], we have the following proposition.

Proposition 6.6 *For each $\xi \in \bigcap_{p\geq 2} L_G^p(\Omega_t;\mathbb{R}^n)$, we have $u(t,\xi) = Y_t^{t,\xi}$.*

Now we give the main result of this section.

Theorem 6.3 *Let $u(t,x) := Y_t^{t,x}$ for $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, the function u is a viscosity solution to the* PDE (6.1).

Proof Without loss of genearlity, we still assume that $h = 0$ and $g = 0$. First, we show that u is a continuous function. Fix some $(t,x) \in [0,T] \times \mathbb{R}^n$. In view of $Y_{t+\delta}^{t,x} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}}$ and Proposition 6.6, we obtain $Y_{t+\delta}^{t,x} = u(t+\delta, X_{t+\delta}^{t,x})$ for $\delta \in [0, T-t]$. Thus we obtain

$$Y_t^{t,x} = u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \mathrm{d}\langle B\rangle_s - \int_t^{t+\delta} Z_s^{t,x} \mathrm{d}B_s - \int_t^{t+\delta} \mathrm{d}K_s^{t,x} \quad \text{q.s.}$$

The generator can be written as Proposition 3.2 in the following form: For each $s \in [0,T]$,

$$f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) = f(s, X_s^{t,x}, 0, 0) + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x} + b_s^\varepsilon Z_s^{t,x},$$

with

$$\begin{aligned} a_s^\varepsilon &:= (1 - l(Y_s^{t,x})) \frac{f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, 0, Z_s^{t,x})}{Y_s^{t,x}} \mathbf{1}_{\{|Y_s^{t,x}|>0\}}, \\ b_s^\varepsilon &:= (1 - l(Z_s^{t,x})) \frac{f(s, X_s^{t,x}, 0, Z_s^{t,x}) - f(s, X_s^{t,x}, 0, 0)}{|Z_s^{t,x}|^2} Z_s^{t,x} \mathbf{1}_{\{|Z_s^{t,x}|>0\}}, \\ m_s^\varepsilon &:= l(Y_s^{t,x})(f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) - f(s, X_s^{t,x}, 0, Z_s^{t,x})) \\ &\quad + l(Z_s^{t,x})(f(s, X_s^{t,x}, 0, Z_s^{t,x}) - f(s, X_s^{t,x}, 0, 0)) \end{aligned}$$

for a Lipschitz continuous function l such that $\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) \le l(x) \le \mathbf{1}_{[-2\varepsilon,2\varepsilon]}(x)$ at each $x \in (-\infty, +\infty)$. Moreover,

$$|a_s^\varepsilon| \le L, \quad |b_s^\varepsilon| \le L(1 + |Z_s^{t,x}|), \quad |m_s^\varepsilon| \le 4L\varepsilon(1+\varepsilon).$$

In view of [15, Lemma 3.6] and Remark 6.3, we know that $b^\varepsilon \in \mathrm{BMO}_G$. Set $\widetilde{B}_t := B_t - \int_0^t b_s^\varepsilon \mathrm{d}\langle B\rangle_s$ for $t \in [0,T]$. Thus we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(b^\varepsilon)$, such that $\widetilde{B}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}[\cdot]$. Thus we have

$$Y_t^{t,x} = u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} (f_s + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x}) \mathrm{d}\langle B\rangle_s - \int_t^{t+\delta} Z_s^{t,x} \mathrm{d}\widetilde{B}_s - \int_t^{t+\delta} \mathrm{d}K_s^{t,x}, \quad \text{q.s.},$$

where $f_s := f(s, X_s^{t,x}, 0, 0)$. Taking G-expectation $\widetilde{\mathbb{E}}[\cdot]$, we get

$$u(t,x) = \widetilde{\mathbb{E}}\Big[u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} (f_s + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x}) \mathrm{d}\langle B\rangle_s\Big].$$

In view of Proposition 6.5, we obtain

$$\begin{aligned} |u(t,x) - u(t+\delta, x)| &\le \widetilde{\mathbb{E}}\Big[|u(t+\delta, X_{t+\delta}^{t,x}) - u(t+\delta, x)| + \int_t^{t+\delta} |f_s + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x}| \mathrm{d}\langle B\rangle_s\Big] \\ &\le \widetilde{\mathbb{E}}[(1 + |x|^m + |X_{t+\delta}^{t,x}|^m)|X_{t+\delta}^{t,x} - x|] \\ &\quad + \sqrt{\delta}\overline{\sigma}^2 \widetilde{\mathbb{E}}\Big[\int_t^{t+\delta} |f_s + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x}|^2 \mathrm{d}s\Big]^{\frac{1}{2}}. \end{aligned}$$

Let $\varepsilon < 1$. In view of Remark 6.3 and **(A3)**, there exists a constant C depending on M_0, L, G and T, such that

$$\overline{\sigma}^2 \widetilde{\mathbb{E}}\Big[\int_t^{t+\delta} \Big|f_s + m_s^\varepsilon + a_s^\varepsilon Y_s^{t,x}\Big|^2 \mathrm{d}s\Big]^{\frac{1}{2}} \le C.$$

Thus we know

$$|u(t,x)-u(t+\delta,x)|\le\widetilde{\mathbb{E}}[(1+|x|^m+|X^{t,x}_{t+\delta}|^m)^2]^{\frac12}\widetilde{\mathbb{E}}[|X^{t,x}_{t+\delta}-x|^2]^{\frac12}+C\sqrt{\delta}. \tag{6.15}$$

Note that $\|b^\varepsilon_s\|_{\mathrm{BMO}_G}\le\|L(1+|Z^{t,x}|)\|_{\mathrm{BMO}_G}$. Then according to Lemma 2.4 and Remark 6.3, there exists $p>1$ depending on M_0,L,G and T, such that, for each $(s,X)\in[0,T]\times L^p_G(\Omega_T)$,

$$\widetilde{\mathbb{E}}_s[|X|]=\widehat{\mathbb{E}}_s\Big[\frac{\mathscr{E}(b^\varepsilon)_T}{\mathscr{E}(b^\varepsilon)_s}|X|\Big]\le\widehat{\mathbb{E}}_s\Big[\Big(\frac{\mathscr{E}(tb^\varepsilon)_T}{\mathscr{E}(b^\varepsilon)_s}\Big)^{p'}\Big]^{\frac{1}{p'}}\widehat{\mathbb{E}}_s[|X|^p]^{\frac1p}\le C_p\widehat{\mathbb{E}}_s[|X|^p]^{\frac1p},\quad \text{q.s.},$$

where C_p depends only on p. In view of Proposition 6.1, we get

$$\widetilde{\mathbb{E}}[|X^{t,x}_{t+\delta}|^{2m}]\le C_p\widehat{\mathbb{E}}[|X^{t,x}_{t+\delta}|^{2mp}]^{\frac1p}\le C(1+|x|^{2m})$$

and

$$\widetilde{\mathbb{E}}[|X^{t,x}_{t+\delta}-x|^2]\le C(1+|x|^2)\delta.$$

Then from (6.15), we have for each $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$|u(t,x)-u(t+\delta,x)|\le C(1+x^{m+1})\sqrt{\delta},$$

where C depends on M_0,L,G and T. On the other hand, we get from Proposition 6.5 that for each $(t,x,x')\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$,

$$|u(t,x)-u(t,x')|\le C(1+|x|^m+|x'|^m)|x-x'|.$$

It follows that u is continuous.

For any fixed $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, let $\psi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that for each $(t,x)\in([0,T]\times\mathbb{R}^n)\backslash\{(t_0,x_0)\}$,

$$\psi(t,x)-u(t,x)>\psi(t_0,x_0)-u(t_0,x_0)=0. \tag{6.16}$$

Without loss of generality, we may assume that there exists some $m_1>0$ such that for each $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$|\psi(t,x)|+|D^2_x\psi(t,x)|\le C(1+|x|^{m_1}),\quad |D_x\psi(t,x)|\le C. \tag{6.17}$$

We want to prove that

$$\partial_t\psi+G(\sigma^{\mathrm{T}}(t_0,x_0)D^2_x\psi\sigma(t_0,x_0)+2f(t_0,x_0,\psi,\sigma^{\mathrm{T}}(t_0,x_0)D_x\psi))+b^{\mathrm{T}}(t_0,x_0)D_x\psi\ge0.$$

Let us assume that the inequality above does not hold. Let $O_\delta(t_0,x_0)$ be an open ball centered at (t_0,x_0), with radius δ. By continuity, there exists some $\delta\in(0,T-t_0)$ such that for each $(t,x)\in O_\delta(t_0,x_0)$,

$$\partial_t\psi+G(\sigma^{\mathrm{T}}(t,x)D^2_x\psi\sigma(t,x)+2f(t,x,\psi,\sigma^{\mathrm{T}}(t,x)D_x\psi))+b^{\mathrm{T}}(t,x)D_x\psi\le0.$$

Setting $\widetilde{Y}_t:=\psi(t,X^{t_0,x_0}_t)$ and $\widetilde{Z}_t:=\sigma^{\mathrm{T}}(t,X^{t_0,x_0}_t)D_x\psi(t,X^{t_0,x_0}_t)$, it is easy to check that for each $t\in[t_0,T]$,

$$\widetilde{Y}_t=\psi(T,X^{t_0,x_0}_T)-\int_t^T\{\partial_t\psi(s,X^{t_0,x_0}_s)+b^{\mathrm{T}}(s,X^{t_0,x_0}_s)D_x\psi(s,X^{t_0,x_0}_s)\}\mathrm{d}s$$

$$-\int_t^T \frac{1}{2}\sigma^{\mathrm{T}}(s,X_s^{t_0,x_0})D_x^2\psi(s,X_s^{t_0,x_0})\sigma(s,X_s^{t_0,x_0})\mathrm{d}\langle B\rangle_s-\int_t^T \widetilde{Z}_s\mathrm{d}B_s, \quad \text{q.s.}$$

For each $t\in[t_0,T]$, set $\widetilde{K}_t:=\int_{t_0}^t F_s\mathrm{d}\langle B\rangle_s-\int_{t_0}^t G(2F_s)\mathrm{d}s$, where

$$F_s:=\frac{1}{2}\sigma^{\mathrm{T}}(s,X_s^{t_0,x_0})D_x^2\psi(s,X_s^{t_0,x_0})\sigma(s,X_s^{t_0,x_0})+f(s,X_s^{t_0,x_0},\widetilde{Y}_s,\widetilde{Z}_s), \quad s\in[t_0,T].$$

We can check that $\widetilde{K}$ is a decreasing G-martingale. Noting that σ is bounded here, by Proposition 6.1 and (6.17), we deduce that $\widetilde{K}_T\in\bigcap\limits_{p\geq 1}L_G^p(\Omega_T)$. Now we have for each $t\in[t_0,T]$,

$$\begin{aligned}\widetilde{Y}_t&=\psi(T,X_T^{t_0,x_0})-\int_t^T(\partial_t\psi(s,X_s^{t_0,x_0})+b^{\mathrm{T}}(s,X_s^{t_0,x_0})D_x\psi(s,X_s^{t_0,x_0})+G(2F_s))\mathrm{d}s\\&\quad+\int_t^T f(s,X_s^{t_0,x_0},\widetilde{Y}_s,\widetilde{Z}_s)\mathrm{d}\langle B\rangle_s-\int_t^T\widetilde{Z}_s\mathrm{d}B_s-\int_t^T\mathrm{d}\widetilde{K}_s, \quad \text{q.s.}\end{aligned}$$

Now we set $\delta Y:=\widetilde{Y}-Y^{t_0,x_0}$, $\delta Z:=\widetilde{Z}-Z^{t_0,x_0}$ and for each $s\in[t_0,T]$,

$$\widetilde{F}_s:=\partial_t\psi(s,X_s^{t_0,x_0})+b^{\mathrm{T}}(s,X_s^{t_0,x_0})D_x\psi(s,X_s^{t_0,x_0})+G(2F_s).$$

Then for each $t\in[t_0,T]$,

$$\begin{aligned}\delta Y_t&=(\psi-u)(T,X_T^{t_0,x_0})+\int_t^T(f(s,X_s^{t_0,x_0},\widetilde{Y}_s,\widetilde{Z}_s)-f(s,X_s^{t_0,x_0},Y_s^{t_0,x_0},\widetilde{Z}_s^{t_0,x_0}))\mathrm{d}\langle B\rangle_s\\&\quad-\int_t^T\widetilde{F}_s\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}B_s-\int_t^T\mathrm{d}\widetilde{K}_s+\int_t^T\mathrm{d}K_s^{t_0,x_0}, \quad \text{q.s.}\end{aligned}\tag{6.18}$$

As what we do in Proposition 3.3, we have for each $s\in[t_0,T]$,

$$f(s,X_s^{t_0,x_0},\widetilde{Y}_s,\widetilde{Z}_s)-f(s,X_s^{t_0,x_0},Y_s^{t_0,x_0},\widetilde{Z}_s^{t_0,x_0})=m_s^\varepsilon+a_s^\varepsilon\delta Y_s+b_s^\varepsilon\delta Z_s,$$

where

$$|a_s^\varepsilon|\leq L, \quad |b_s^\varepsilon|\leq L(1+|Z_s^{t_0,x_0}|+|\widetilde{Z}_s|), \quad |m_s^\varepsilon|\leq 4L\varepsilon(1+\varepsilon+|\widetilde{Z}_s|).$$

In view of [15, Lemma 3.6], Remark 6.3 and (6.17), we know that $b^\varepsilon\in\mathrm{BMO}_G$. Set $\widetilde{B}_t:=B_t-\int_0^t b_s^\varepsilon\mathrm{d}\langle B\rangle_s$ for $t\in[0,T]$. Thus we can define a new G-expectation $\widetilde{\mathbb{E}}[\cdot]$ by $\mathscr{E}(b^\varepsilon)$, such that $\widetilde{B}$ is a G-Brownian motion under $\widetilde{\mathbb{E}}[\cdot]$. Thus (6.18) can be written as

$$\begin{aligned}\delta Y_t&=(\psi-u)(T,X_T^{t_0,x_0})+\int_t^T m_s^\varepsilon+a_s^\varepsilon\delta Y_s\mathrm{d}\langle B\rangle_s\\&\quad-\int_t^T\widetilde{F}_s\mathrm{d}s-\int_t^T\delta Z_s\mathrm{d}\widetilde{B}_s-\int_t^T\mathrm{d}\widetilde{K}_s+\int_t^T\mathrm{d}K_s^{t_0,x_0}, \quad \text{q.s.}\end{aligned}$$

Applying Itô's formula to $\mathrm{e}^{\int_0^t a_s^\varepsilon\mathrm{d}\langle B\rangle_s}\delta Y_t$, we have

$$\begin{aligned}&\mathrm{e}^{\int_0^t a_s^\varepsilon\mathrm{d}\langle B\rangle_s}\delta Y_t\\&=\mathrm{e}^{\int_0^{\mathrm{T}} a_s^\varepsilon\mathrm{d}\langle B\rangle_s}(\psi-u)(T,X_T^{t_0,x_0})-\int_t^{\mathrm{T}}\mathrm{e}^{\int_0^s a_u^\varepsilon\mathrm{d}\langle B\rangle_u}\widetilde{F}_s\mathrm{d}\langle B\rangle_s+\int_t^{\mathrm{T}}\mathrm{e}^{\int_0^s a_u^\varepsilon\mathrm{d}\langle B\rangle_u}m_s^\varepsilon\mathrm{d}\langle B\rangle_s\end{aligned}$$

$$-\int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \delta Z_s \mathrm{d}\widetilde{B}_s - \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \mathrm{d}\widetilde{K}_s + \int_t^{\mathrm{T}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \mathrm{d}K_s^{t_0,x_0}, \quad \text{q.s.} \qquad (6.19)$$

Let $\mathcal{P}$ be the weakly compact set that represents $\widehat{\mathbb{E}}$. For each $\mathbb{P}\in\mathcal{P}$, let $\tau^{\mathbb{P}}$ be the following stopping time under $\mathbb{P}$:

$$\tau^{\mathbb{P}} := \inf\{s \geq t_0 : (s, X_s^{t_0,x_0}) \notin O_\delta(t_0,x_0)\}.$$

By the strict minimum property (6.16), we notice that

$$\eta := \min_{(t,x)\in\partial O_\delta(t_0,x_0)} (\psi - u)(t,x) > 0.$$

It is easy to check that $\tau^{\mathbb{P}} < T$, $\mathbb{P}$-a.s. and $(\psi-u)(\tau^{\mathbb{P}}, X_{\tau^{\mathbb{P}}}^{t_0,x_0}) \geq \eta$, $\mathbb{P}$-a.s.. From (6.19), we have for each $t\in[t_0,T]$,

$$\begin{aligned}
&\mathrm{e}^{\int_0^{t\wedge\tau^{\mathbb{P}}} a_s^\varepsilon \mathrm{d}\langle B\rangle_s} \delta Y_{t\wedge\tau^{\mathbb{P}}} \\
&= \mathrm{e}^{\int_0^{\tau^{\mathbb{P}}} a_s^\varepsilon \mathrm{d}\langle B\rangle_s} (\psi-u)(\tau^{\mathbb{P}}, X_{\tau^{\mathbb{P}}}^{t_0,x_0}) - \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \widetilde{F}_s \mathrm{d}\langle B\rangle_s + \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} m_s^\varepsilon \mathrm{d}\langle B\rangle_s \\
&\quad - \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \delta Z_s \mathrm{d}\widetilde{B}_s - \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \mathrm{d}\widetilde{K}_s + \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \mathrm{d}K_s^{t_0,x_0}, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Note that for each $(s,\omega)\in[t_0,T]\times\Omega_T$ satisfying $t_0 \leq s \leq \tau^{\mathbb{P}}(\omega)$, $\widetilde{F}_s \leq 0$. Thus we have

$$\begin{aligned}
\mathrm{e}^{\int_0^{t\wedge\tau^{\mathbb{P}}} a_s^\varepsilon \mathrm{d}\langle B\rangle_s} \delta Y_{t\wedge\tau^{\mathbb{P}}} &\geq \mathrm{e}^{-LT\overline{\sigma}^2}\eta + \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} m_s^\varepsilon \mathrm{d}\langle B\rangle_s \\
&\quad - \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \delta Z_s \mathrm{d}\widetilde{B}_s + \int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{e}^{\int_0^s a_u^\varepsilon \mathrm{d}\langle B\rangle_u} \mathrm{d}K_s^{t_0,x_0}, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Since $\widetilde{B}$ is a martingale under the new probability $\mathbb{Q}$ with $\mathrm{d}\mathbb{Q} := \mathscr{E}(b^\varepsilon)_T \mathrm{d}\mathbb{P}$, we have in particular

$$E^{\mathbb{Q}}[\delta Y_{t_0}] \geq \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} E^{\mathbb{Q}}\Big[\int_{t_0}^{\tau^{\mathbb{P}}} |m_s^\varepsilon| \mathrm{d}\langle B\rangle_s\Big] + \mathrm{e}^{2LT\overline{\sigma}^2} E^{\mathbb{Q}}\Big[\int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{d}K_s^{t_0,x_0}\Big].$$

While $\delta Y_{t_0} = (\psi-u)(t_0,x_0)$ and $|m_s^\varepsilon| \leq \rho(\varepsilon)$ for a nonnegative continuous function ρ defined on $\mathbb{R}^+$ with $\rho(0)=0$, we have

$$\begin{aligned}
(\psi-u)(t_0,x_0) &\geq \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} T\overline{\sigma}^2\rho(\varepsilon) + \mathrm{e}^{2LT\overline{\sigma}^2} E^{\mathbb{Q}}\Big[\int_{t\wedge\tau^{\mathbb{P}}}^{\tau^{\mathbb{P}}} \mathrm{d}K_s^{t_0,x_0}\Big] \\
&\geq \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} T\overline{\sigma}^2\rho(\varepsilon) + \mathrm{e}^{2LT\overline{\sigma}^2} E^{\mathbb{Q}}[K_T^{t_0,x_0}] \\
&= \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} T\overline{\sigma}^2\rho(\varepsilon) + \mathrm{e}^{2LT\overline{\sigma}^2} E^{\mathbb{P}}[\mathscr{E}(b^\varepsilon)_T K_T^{t_0,x_0}]
\end{aligned}$$

for each $\mathbb{P}\in\mathcal{P}$. Consequently, we have

$$\begin{aligned}
(\psi-u)(t_0,x_0) &\geq \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} T\overline{\sigma}^2\rho(\varepsilon) + \mathrm{e}^{2LT\overline{\sigma}^2} \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\mathscr{E}(b^\varepsilon)_T K_T^{t_0,x_0}] \\
&= \mathrm{e}^{-2LT\overline{\sigma}^2}\eta - \mathrm{e}^{2LT\overline{\sigma}^2} T\overline{\sigma}^2\rho(\varepsilon) + \mathrm{e}^{2LT\overline{\sigma}^2} \widehat{\mathbb{E}}[\mathscr{E}(b^\varepsilon)_T K_T^{t_0,x_0}]. \qquad (6.20)
\end{aligned}$$

In view of Lemma 2.7 and Remark 6.3, the process K^{t_0,x_0} is a G-martingale under $\widetilde{\mathbb{E}}[\cdot]$, and

$$\widehat{\mathbb{E}}[\mathscr{E}(b^\varepsilon)_T K_T^{t_0,x_0}] = \widetilde{\mathbb{E}}[K_T^{t_0,x_0}] = 0.$$

Letting $\varepsilon \to 0$ in the last inequality, we have

$$0 = (\psi - u)(t_0, x_0) \geq \mathrm{e}^{-2LT\overline{\sigma}^2}\eta > 0,$$

which is a contradiction. Hence, u is a viscosity subsolution.

In a similar way, u can be shown to be a viscosity supersolution.

Remark 6.4 When the functions f and g do not depend on y, one can get the uniqueness of viscosity solution to PDE by the uniqueness result in Da Lio and Ley [9] concerning Bellman-Isaacs equation.

6.2 Relation between reflected quadratic G-BSDEs and obstacle problems for nonlinear parabolic PDEs

With the preceding nonlinear Feynman-Kac formula, we can give the relationship between solutions of the obstacle problem for nonlinear parabolic PDEs and the related reflected quadratic G-BSDEs. For each $(t, \xi) \in [0, T] \times \bigcap_{p \geq 2} L_G^p(\Omega_t; \mathbb{R}^n)$, we consider the following G-SDE:

$$X_s^{t,\xi} = \xi + \int_t^s b(u, X_u^{t,\xi})\mathrm{d}u + \int_t^s h(u, X_u^{t,\xi})\mathrm{d}\langle B\rangle_u + \int_t^s \sigma(u, X_u^{t,\xi})\mathrm{d}B_u, \quad s \in [t, T], \tag{6.21}$$

and the following type of reflected G-BSDE:

$$\begin{cases} Y_s^{t,\xi} = \phi(X_T^{t,\xi}) + \displaystyle\int_s^{\mathrm{T}} g(u, X_u^{t,\xi}, Y_u^{t,\xi}, Z_u^{t,\xi})\mathrm{d}u + \int_s^{\mathrm{T}} f(u, X_u^{t,\xi}, Y_u^{t,\xi}, Z_u^{t,\xi})\mathrm{d}\langle B\rangle_u \\ \qquad\qquad - \displaystyle\int_s^{\mathrm{T}} Z_u^{t,\xi}\mathrm{d}B_u + \int_s^{\mathrm{T}} \mathrm{d}A_u^{t,\xi}, \quad \text{q.s. } s \in [t, T]; \\ Y_s^{t,\xi} \geq l(s, X_s^{t,\xi}), \quad \text{q.s. } s \in [t, T]; \\ \displaystyle\int_t^{\cdot} (l(u, X_u^{t,\xi}) - Y_u^{t,\xi})\mathrm{d}A_u^{t,\xi} \text{ is a non-increasing } G\text{-martingale on } [s, T], \end{cases} \tag{6.22}$$

where $b, h, \sigma, l : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $\phi : \mathbb{R}^n \to \mathbb{R}$, $f, g : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are deterministic functions and satisfy **(A1)**–**(A4)**. Moreover, we have the following assumption on l:

(A9) The function $l(t, \cdot)$ is uniformly Lipschitz and $l(T, x) \leq \phi(x)$ for any $x \in \mathbb{R}^n$. Furthermore, there exists a constant N_0 such that $l(t, x) \leq N_0$ for any $t \in [0, T]$.

(A10) The function $l(\cdot, x)$ is uniformly continuous, i.e., there is a non-decreasing continuous function $w : [0, +\infty) \to [0, +\infty)$ such that $w(0) = 0$ and

$$\sup_{x \in \mathbb{R}^n} |l(t, x) - l(t', x)| \leq w(|t - t'|).$$

Remark 6.5 In the Markovian case, **(H2)** and **(H5)** may not hold directly. However, in view of Remark 2.4, one can still get the results under **(H1)** and **(H3)**–**(H4)** as long as the penalized quadratic G-BSDE has a solution. The reflected G-BSDE (6.22) has one solution in the sense of Definition 2.9 and all results in Sections 3–5 still hold here under **(A1)**–**(A4)** and **(A9)**–**(A10)**.

Consider the following obstacle problem for a parabolic PDE:

$$\begin{cases} \min\{-\partial_t u - F(D_x^2 u, D_x u, u, x, t), u(t, x) - l(t, x)\} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\ u(T, x) = \phi(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{6.23}$$

where

$$F(A,p,y,x,t) := G(\sigma^{\mathrm{T}}(t,x)A\sigma(t,x) + 2f(t,x,y,\sigma^{\mathrm{T}}(t,x)p) + 2h^{\mathrm{T}}(t,x)p) \\ + b^{\mathrm{T}}(t,x)p + g(t,x,y,\sigma^{\mathrm{T}}(t,x)p)$$

for each $(A,p,y,x,t) \in \mathbb{S}_n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0,T]$.

We need to recall the equivalent definition of the viscosity solution of the obstacle problem (6.23) as in [24] or [33].

Definition 6.1 *Let $u \in C([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T] \times \mathbb{R}^n$. We denote by $\mathcal{P}^{2,+}u(t,x)$ the set of triples $(p,q,A) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ satisfying*

$$u(s,y) \le u(t,y) + p(s-t) + q^{\mathrm{T}}(y-x) + \frac{1}{2}A(y-x)^2 + o(|s-t| + |y-x|^2).$$

Similarly, we define $\mathcal{P}^{2,-}u(t,x) := -\mathcal{P}^{2,+}(-u)(t,x)$.

Definition 6.2 *The function $u \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity subsolution of* (6.23) *if $u(T,x) \le \phi(x)$, $x \in \mathbb{R}^n$, and for each $(t,x) \in [0,T) \times \mathbb{R}^n$ and $(p,q,A) \in \mathcal{P}^{2,+}u(t,x)$,*

$$\min\{-p - F(A,q,u(t,x),x,t), u(t,x) - l(t,x)\} \le 0.$$

The function $u \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity supersolution of (6.23) *if $u(T,x) \ge \phi(x)$, $x \in \mathbb{R}^n$, and for each $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(p,q,X) \in \mathcal{P}^{2,-}u(t,x)$,*

$$\min\{-p - F(A,q,u(t,x),x,t), u(t,x) - l(t,x)\} \ge 0.$$

$u \in C([0,T] \times \mathbb{R}^n)$ is said to be a viscosity solution of (6.23) *if it is both a viscosity subsolution and supersolution.*

We now define $u(t,x) := Y_t^{t,x}$. Similarly as before, we can note that u is a deterministic function. We now should prove that $u \in C([0,T] \times \mathbb{R}^n)$.

Lemma 6.2 *Let* **(A1)**–**(A4)** *and* **(A9)**–**(A10)** *hold. For each $t \in [0,T]$, $x_1, x_2 \in \mathbb{R}^n$, we have*

$$|u(t,x_1) - u(t,x_2)|^2 \le C(1 + |x_1|^{2m} + |x_2|^{2m})|x_1 - x_2|^2 + C|x_1 - x_2|.$$

Proof Without loss of generality, we assume $h = 0$ and $g = 0$. In view of Propositions 3.1 and 3.2, we deduce that there exists a constant $C_1 := C_1(T,L,G,M_0,N_0)$ such that

$$\|Z^{t,x_1}\|_{\mathrm{BMO}_G} + \|Z^{t,x_2}\|_{\mathrm{BMO}_G} \le C_1,$$

and a constant $C_2 := C_2(T,L,G,M_0,N_0,\alpha)$, for any $\alpha \ge 1$, such that

$$\widehat{\mathbb{E}}[|A_T^{t,x_1}|^\alpha + |A_T^{t,x_2}|^\alpha] \le C_2.$$

In view of Proposition 3.3 and its proof and noting that u is deterministic, we obtain that there exists a constant $C := C(T,L,G,M_0,N_0)$ and $p \ge 2$ such that for each $t \in [0,T]$,

$$|u(t,x_1) - u(t,x_2)|^2 \\ \le C\Big\{\widehat{\mathbb{E}}[|\phi(X_T^{t,x_1}) - \phi(X_T^{t,x_2})|^{2p}]^{\frac{1}{p}} + \widehat{\mathbb{E}}\Big[\sup_{s\in[t,T]} |l(s,X_s^{t,x_1}) - l(s,X_s^{t,x_2})|^{2p}\Big]^{\frac{1}{2p}}\Big\}$$

$$
\begin{aligned}
&+ C\widehat{\mathbb{E}}\Big[\Big(\int_t^{\mathrm{T}} |f(s, X_s^{t,x_1}, Y_s^{t,x_2}, Z_s^{t,x_2}) - f(s, X_s^{t,x_2}, Y_s^{t,x_2}, Z_s^{t,x_2})|^2 \mathrm{d}s\Big)^p\Big]^{\frac{1}{p}} \\
&\leq C'\widehat{\mathbb{E}}[(1 + |X_T^{t,x_1}|^{2pm} + |X_T^{t,x_2}|^{2pm})^2]^{\frac{1}{2p}} \widehat{\mathbb{E}}[|X_T^{t,x_1} - X_T^{t,x_2}|^{4p}]^{\frac{1}{2p}} \\
&+ C'\widehat{\mathbb{E}}\Big[\sup_{s\in[t,T]} |X_s^{t,x_1} - X_s^{t,x_2}|^{2p}\Big]^{\frac{1}{2p}} \\
&+ C'\widehat{\mathbb{E}}\Big[\int_t^{\mathrm{T}} (1 + |X_s^{t,x_1}|^{2pm} + |X_s^{t,x_2}|^{2pm})^2 \mathrm{d}s\Big]^{\frac{1}{2p}} \widehat{\mathbb{E}}\Big[\int_t^{\mathrm{T}} |X_s^{t,x_1} - X_s^{t,x_2}|^{4p} \mathrm{d}s\Big]^{\frac{1}{2p}} \\
&\leq C''(1 + |x_1|^{2m} + |x_2|^{2m})|x_1 - x_2|^2 + C''|x_1 - x_2|.
\end{aligned}
$$

Lemma 6.3 *Let* **(A1)**–**(A4)** *and* **(A9)**–**(A10)** *hold. The function* $u(t,x) := Y_t^{t,x}$ *is continuous in* t.

Proof For simplicity, we assume $h = 0$ and $g = 0$. We define $X_s^{t,x} := x$, $Y_s^{t,x} := Y_t^{t,x}$, $Z_s^{t,x} := 0$ and $A_s^{t,x} := 0$ for $s \in [0,t]$. It is easy to check that $(Y^{t,x}, Z^{t,x}, A^{t,x})$ is a solution to the following G-BSDE on $[0,T]$:

$$
\begin{cases}
Y_s^{t,x} = \phi(X_T^{t,x}) + \displaystyle\int_s^{\mathrm{T}} \mathbf{1}_{[t,T]}(s) f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \mathrm{d}\langle B\rangle_r - \int_s^{\mathrm{T}} Z_r^{t,x} \mathrm{d}B_r \\
\qquad + \displaystyle\int_s^{\mathrm{T}} \mathrm{d}A_r^{t,\xi}, \quad s \in [0,T], \\
Y_s^{t,\xi} \geq S_s^{t,x}, \quad \text{q.s. } s \in [0,T]; \\
\Big\{ -\displaystyle\int_0^s (Y_r^{t,x} - S_r^{t,x}) \mathrm{d}A_r^{t,x},\ s \in [0,T]\Big\} \text{ is a non-increasing } G\text{-martingale,}
\end{cases}
\tag{6.24}
$$

where

$$
S_s^{t,x} = \begin{cases} l(s, X_s^{t,\xi}), & s \in [t,T], \\ l(s,x), & s \in [0,t]. \end{cases}
$$

Fix $x \in \mathbb{R}^n$. As before, in view of Propositions 3.1–3.3, we have for $0 \leq t_1 \leq t_2 \leq T$ and some $p \geq 2$,

$$
\begin{aligned}
&|u(t_1,x) - u(t_2,x)|^2 = |Y_0^{t_1,x} - Y_0^{t_2,x}|^2 \\
&\leq C\Big\{\widehat{\mathbb{E}}[|\phi(X_T^{t_1,x}) - \phi(X_T^{t_2,x})|^{2p}]^{\frac{1}{p}} + \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |l(s, X_s^{t_1,x}) - l(s, X_s^{t_2,x})|^{2p}\Big]^{\frac{1}{2p}}\Big\} \\
&+ C\widehat{\mathbb{E}}\Big[\Big(\int_0^{\mathrm{T}} |\mathbf{1}_{[t_1,T]}(s) f(s, X_s^{t_1,x}, Y_s^{t_2,x}, Z_s^{t_2,x}) - \mathbf{1}_{[t_2,T]}(s) f(s, X_s^{t_2,x}, Y_s^{t_2,x}, Z_s^{t_2,x})|^2 \mathrm{d}s\Big)^p\Big]^{\frac{1}{p}} \\
&\leq C'\widehat{\mathbb{E}}[(1 + |X_T^{t_1,x}|^{2pm} + |X_T^{t_2,x}|^{2pm})^2]^{\frac{1}{2p}} \widehat{\mathbb{E}}[|X_T^{t_1,x} - X_T^{t_2,x}|^{4p}]^{\frac{1}{2p}} \\
&+ C'\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X_s^{t_1,x} - X_s^{t_2,x}|^{2p}\Big]^{\frac{1}{2p}} + C'\widehat{\mathbb{E}}\Big[\Big(\int_{t_1}^{t_2} |f(s, X_s^{t_1,x}, Y_s^{t_2,x}, Z_s^{t_2,x})|^2 \mathrm{d}s\Big)^p\Big]^{\frac{1}{p}} \\
&+ C'\widehat{\mathbb{E}}\Big[\int_{t_2}^{\mathrm{T}} (1 + |X_s^{t_1,x}|^{2pm} + |X_s^{t_2,x}|^{2pm})^2 \mathrm{d}s\Big]^{\frac{1}{2p}} \widehat{\mathbb{E}}\Big[\int_{t_2}^{\mathrm{T}} |X_s^{t_1,x} - X_s^{t_2,x}|^{4p} \mathrm{d}s\Big]^{\frac{1}{2p}} \\
&\leq C''(1 + |x|^{2m})\widehat{\mathbb{E}}\Big[\sup_{s\in[t_2,T]} |X_s^{t_1,x} - X_s^{t_2,x}|^{4p}\Big]^{\frac{1}{2p}} + C''\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X_s^{t_1,x} - X_s^{t_2,x}|^{2p}\Big]^{\frac{1}{2p}} \\
&+ C''\widehat{\mathbb{E}}\Big[\Big(\int_{t_1}^{t_2} |f(s, X_s^{t_1,x}, Y_s^{t_2,x}, 0)|^2 \mathrm{d}s\Big)^p\Big]^{\frac{1}{p}}.
\end{aligned}
\tag{6.25}
$$

For each $\alpha \geq 2$, we have

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]}|X_s^{t_1,x}-X_s^{t_2,x}|^\alpha\Big]\\
&\leq \widehat{\mathbb{E}}\Big[\sup_{s\in[t_1,t_2]}|X_s^{t_1,x}-x|^\alpha\Big]+\widehat{\mathbb{E}}\Big[\sup_{s\in[t_2,T]}|X_s^{t_2,X_{t_2}^{t_1,x}}-X_s^{t_2,x}|^\alpha\Big]\\
&\leq C_1(1+|x|^\alpha)|t_1-t_2|^{\frac{\alpha}{2}}+C_1\widehat{\mathbb{E}}[|X_{t_2}^{t_1,x}-x|^\alpha]\\
&\leq C_2(1+|x|^\alpha)|t_1-t_2|^{\frac{\alpha}{2}}.
\end{aligned}$$

On the other hand, in view of Proposition 3.2, for each $\alpha \geq 2$,

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\Big(\int_{t_1}^{t_2}|f(s,X_s^{t_1,x},Y_s^{t_2,x},0)|^2\mathrm{d}s\Big)^\alpha\Big]\\
&\leq C\widehat{\mathbb{E}}\Big[\int_{t_1}^{t_2}(|f(s,0,0,0)|^{2\alpha}+|X_s^{t_1,x}|^{2\alpha}+|Y_s^{t_2,x}|^\alpha)\mathrm{d}s\Big]\\
&\leq \widetilde{C}\int_{t_1}^{t_2}(|f(s,0,0,0)|^{2\alpha}+1+\widehat{\mathbb{E}}[|X_s^{t_1,x}|^{2\alpha}])\mathrm{d}s\\
&\leq \widetilde{C}'\int_{t_1}^{t_2}(|f(s,0,0,0)|^{2\alpha}+1+|x|^{2\alpha})\mathrm{d}s.
\end{aligned}$$

Then from (6.25), we know that u is continuous in t.

Now we consider the penalized G-BSDEs:

$$\begin{aligned}
Y_s^{t,x,n}&=\phi(X_T^{t,x})+\int_s^{\mathrm{T}}g(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\mathrm{d}r+\int_s^{\mathrm{T}}f(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\mathrm{d}\langle B\rangle_r\\
&\quad+n\int_s^{\mathrm{T}}(Y_r^{t,x,n}-l(t,X_r^{t,x}))^-\mathrm{d}r-\int_s^{\mathrm{T}}Z_r^{t,x,n}\mathrm{d}B_r-\int_s^{\mathrm{T}}\mathrm{d}K_r^{t,x,n},\quad \text{q.s. } s\in[t,T].
\end{aligned}$$

We define $u_n(t,x) := Y_t^{t,x,n}$, $(t,x) \in [0,T] \times \mathbb{R}^n$. In view of Theorem 6.3, u_n is the viscosity solution to the following PDE:

$$\begin{cases}\partial_t u_n+F_n(D_x^2u_n,D_xu_n,u_n,x,t)=0, & (t,x)\in[0,T)\times\mathbb{R}^n,\\ u_n(T,x)=\phi(x), & x\in\mathbb{R}^n,\end{cases}\tag{6.26}$$

where

$$F_n(DA,p,y,x,t):=F(A,p,y,x,t)+n(y-l(t,x))^-$$

for each $(A,p,y,x,t) \in \mathbb{S}_n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times [0,T]$.

Theorem 6.4 *Let* **(A1)**–**(A4)** *and* **(A9)**–**(A10)** *hold. The function* $u(t,x) := Y_t^{t,x}$ *is a viscosity solution of the obstacle problem* (6.23).

Proof We follow the procedure of [24, Theorem 6.7], and only sketch the main ideas. From the previous results, we have for each $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\lim_{n\to\infty}u_n(t,x)=u(t,x),\quad u_{n+1}(t,x)\geq u_n(t,x),\quad \forall\, n\in\mathbb{Z}^+.$$

Moreover, functions u and u_n are continuous. Then in view of Dini's theorem, u_n uniformly converges to u on any compact subset.

We now show that u is a viscosity subsolution to (6.23). For each fixed $(t,x) \in [0,T] \times \mathbb{R}^n$, let $(p,q,A) \in \mathcal{P}^{2,+}u(t,x)$. We may assume $u(t,x) > l(t,x)$. Similarly as in the proof of [24, Theorem 6.7], we deduce that there exist sequences

$$n_j \to \infty, \quad (t_j, x_j) \to (t,x), \quad (p_j, q_j, A_j) \to (p,q,A),$$

where $(p_j, q_j, X_j) \in \mathcal{P}^{2,+}u_{n_j}(t_j, x_j)$. Since u_n is the viscosity solution to (6.26), it follows that for any j,

$$\min\{-p_j - F_{n_j}(A_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j), u(t_j, x_j) - l(t_j, x_j)\} \le 0.$$

Noting that $u(t,x) > l(t,x)$, by the uniform convergence of u_n, we deduce that $u_j(t_j, x_j) > l(t_j, x_j)$ for sufficiently large integer j. Thus letting $j \to \infty$, we have

$$-p - F(A, q, u(t,x), x, t) \le 0,$$

which means that u is a viscosity subsolution to (6.23).

In a similar way, u is proved to be a viscosity supersolution to (6.23).

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