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The Structure of Vector Bundles on Non-primary Hopf Manifolds*

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Abstract Let X be a Hopf manifold with non-Abelian fundamental group and E be a holomorphic vector bundle over X, with trivial pull-back to $\mathbb{C}^n - \{0\}$. The authors show that there exists a line bundle L over X such that $E \otimes L$ has a nowhere vanishing section. It is proved that in case $\dim(X) \geq 3$, $\pi^*(E)$ is trivial if and only if E is filtrable by vector bundles. With the structure theorem, the authors get the cohomology dimension of holomorphic bundle E over X with trivial pull-back and the vanishing of Chern class of E.

Keywords Hopf manifolds, Holomorphic vector bundles, Exact sequence, Cohomology, Filtration, Chern class
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1 Introduction

In recent years holomorphic vector bundles on non-algebraic surfaces have received increasing attentions (see [1–3]). Hopf surfaces constitute a simple but important class of compact non-Kählian surfaces. The geometry of Hopf manifolds has been studied by many authors, see [4–8] for instance. For the results of holomorphic vector bundles in Hopf manifolds, the reader is referred to [9–14]. In this paper we investigate the structure of holomorphic vector bundles with trivial pullback on non-primary Hopf manifolds with non-Abelian fundamental group. It generalizes the results of [9] on the primary Hopf manifolds and [10] on the non-primary Hopf manifolds with Abelian fundamental group.

The universal covering of an n-dimensional Hopf manifold X, $n \geq 2$, is biholomorphic to the punctured n-space $W = \mathbb{C}^n - \{0\}$. X can be written as a quotient space W/G with a group G generated by some biholomorphic transformations of W which act on W properly discontinuous and free. When $n \geq 2$, W is simply connected and G is the fundamental group of X. When G is the infinite cyclic group $\mathbb{Z}$, the Hopf manifold is called primary, otherwise it is called non-primary. By a contraction $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $n \geq 2$, we mean an automorphism of $\mathbb{C}^n$ fixing 0 with the property that the eigenvalues $\mu_1, \cdots, \mu_n$ of $f'(0)$, the differential of f at 0, are inside the unit circle. We always assume, without loss of generality, that $0 \leq |\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_n| < 1$. A contraction of the form $f : (z_1, \cdots, z_n) \to (\mu_1 z_1, \cdots, \mu_n z_n)$ is called diagonal. Similar to Kodaira's argument for Hopf surface in [8], the fundamental group G of X has the following properties: G contains a contraction f and the infinite cyclic subgroup $\mathbb{Z}$ generated by f has

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a finite index in G; the subgroup $\mathbb{Z}$ is located in the center of G and then normal in G; the group $G/\mathbb{Z}$ is isomorphic to a finite subgroup H of G. And the fundamental group G of a Hopf manifold can be written as $G = \mathbb{Z} \cdot H$ and $f \cdot h = h \cdot f$, $\forall h \in H$. In general, the fundamental group G is not Abelian.

Every holomorphic vector bundle E on X can be lifted by the covering map $\pi : W \to X$ to a holomorphic vector bundle on $\mathbb{C}^n - \{0\}$, the pullback $\pi^*(E)$ of E. Assume that X is an arbitrary Hopf manifold whose fundamental group is $\pi_1(X) \cong \mathbb{Z} \cdot H$ and let E be a holomorphic vector bundle of rank r on X with a holomorphicly trivial pullback $\pi^*(E) \cong \mathcal{O}_W^r$. Because H is finite we may assume that the order of finite group H is m and $H = \{h_0, \cdots, h_{m-1}\}$. Denote $I = \{0, 1, \cdots, m-1\}$. We can get the representation of the fundamental group $\pi_1(X) \cong \mathbb{Z} \cdot H$ of X as follows:

$$\begin{aligned} \rho_E : \pi_1(X) &\to \mathrm{GL}(r, \Gamma(W, \mathcal{O})), \\ f &\mapsto A(z), \\ h_i &\mapsto B_i(z), \quad i \in I, \end{aligned}$$

where f is the contraction which is the generator of $\mathbb{Z}$ and $h_i \in H$. The fundamental group operates on the $W \times \mathbb{C}^r$ in the following way:

$$\begin{aligned} W \times \mathbb{C}^r &\to W \times \mathbb{C}^r, \\ (z, v) &\mapsto (f(z), A(z)v), \\ (z, v) &\mapsto (h_i(z), B_i(z)v), \quad i \in I. \end{aligned}$$

We denote this bundle E by

$$W \times \mathbb{C}^r / \langle f \times A(z), h_i \times B_i(z), i \in I \rangle,$$

where $A(z) = \rho_E(f)$ and $B_i(z) = \rho_E(h_i)$, $i = 1, \cdots, m$. Denote this bundle E by E_{A,B_I} for clarity.

Since $f \cdot h_i = h_i \cdot f$ for $i \in I$, we have

$$A(h_i(z))B_i(z) = \rho_E(f \cdot h_i) = \rho_E(h_i \cdot f) = B_i(f(z))A(z),$$

i.e.,

$$h_i^* \cdot A(z)B_i(z) = f^* \cdot B_i(z)A(z), \tag{1.1}$$

where $h_i^* \cdot A(z) := A(h_i(z))$ and $f^* \cdot B_i(z) := B_i(f(z))$.

If we change the trivialization of the bundle $\pi^*(E)$,

$$\mathbb{C}^r \to \mathbb{C}^r, \quad \xi \mapsto \eta := P\xi$$

with $P \in \mathrm{GL}(r, \Gamma(W, \mathcal{O}))$, then the matrix $A(z)$ and $B_i(Z)$, $i = 1, \cdots, m$ are transformed to $P(f(z)) \cdot A(z) \cdot P^{-1}(z)$ and $P(h_i(z)) \cdot B_i(z) \cdot P^{-1}(z)$, i.e., $\mathrm{GL}(r, \Gamma(W, \mathcal{O})$ acts on itself by a "twist conjugacy".

Similarly, let L be a flat line bundle on X, and

$$\rho_L : \pi_1(X) \to \mathrm{GL}(1, \mathbb{C}^*),$$

$$f \mapsto c,$$
$$h_i \mapsto d_i$$

is a representation of the fundamental group $\pi_1(X) \cong \mathbb{Z} \cdot H$ of X. The fundamental group operates in the following way:

$$W \times \mathbb{C}^r \to W \times \mathbb{C}^r,$$
$$(z, v) \mapsto (h_i(z), B_i(z)v),$$
$$(z, v) \mapsto (f(z), A(z)v).$$

We denote this bundle L by

$$W \times \mathbb{C}/\langle f \times c, h_i \times d_i, i \in I\rangle,$$

where $c = \rho_L(f)$ and $d_i = \rho_L(h_i),\ i \in I$. Denote the bundle L by L_{A,B_I} for clarity.

2 Structure Theorem

Theorem 2.1 *If X is a non-primary Hopf manifold and $E \cong W \times \mathbb{C}^r/\langle f \times A(z), h_i \times B_i(z), i \in I\rangle$ is a holomorphic bundle of rank r on X with a trivial pullback $\pi^*(E)$ on W, then there exists a holomorphic line bundle L on X such that $L \otimes E$ has a nowhere vanishing holomorphic section σ.*

Proof If L is a holomorphic line bundle on X, we know that L is flat and $L \cong W \times \mathbb{C}/\langle f \times c, h_i \times d_i, i \in I\rangle$ by [13]. By the representation of E and L, to any $\sigma \in H^0(X, L \bigotimes E)$, under the global sections frame basis, $\pi^*\sigma$ corresponds to a nowhere vanishing entire holomorphic map $\psi : W \to \mathbb{C}^r$ satisfying the function equation system

$$\begin{cases} cA(z)\psi(z) = \psi(f(z)), \\ d_i B_i(z)\psi(z) = \psi(h_i(z)), \quad i = 0, \cdots, m-1. \end{cases} \tag{2.1}$$

So the existence of a nowhere vanishing holomorphic section of $L \otimes E$ is equivalent to the existence of $c,\ d_i \in \mathbb{C}^*,\ i = 0, \cdots, m-1$ and the existence of a nowhere vanishing entire holomorphic map $\psi : W \to \mathbb{C}^r$ satisfying the function equation system (2.1).

Now we shall construct such a $\psi(z)$.

First we assume that f is a diagonal contraction given by

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \quad (z_1, \cdots, z_n) \mapsto (\mu_1 z_1, \cdots, \mu_n z_n),$$

where $0 \le |\mu_1| \le |\mu_2| \le \cdots \le |\mu_n| < 1$. We use the method of Mall [9] to prove that there exists a $c \in \mathbb{C}^*$ such that the functional equation $cA(z)\psi(z) = \psi(f(z))$ has a nowhere vanishing entire holomorphic vector-valued solution. We develop $A(z)$ in a Taylor series about the origin

$$A(z) = \sum_{\alpha \in N_0^n} A_\alpha z^\alpha, \quad A_\alpha \in M(r, \mathbb{C}),$$

and $A_0 \in \mathrm{GL}(r, \mathbb{C})$.

Without loss of generality, we may assume that A_0 is in Jordan normal form with eigenvalues λ_i, $i = 1, \cdots, r$. Let $u := \min_{1 \le i \le r} \{|\lambda_i|\}$ and choose a j such that $|\lambda_j| = u$. Let $c := \lambda_j^{-1}$. Then at least one eigenvalue of cA_0 is equal to 1 and the modulus of the others is greater or equal to 1. So for convention, in the following, we instead $cA(z)$ by $A(z)$ and cA_0 by A_0 in the functional equation (2.1). Then the first equation (2.1) reduces to $A(z)\psi(z) = \psi(f(z))$.

We use the following partial order on $\mathbb{N}^n : \alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for all i. We write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Assume $\phi(z) = \sum \phi_\gamma z^\gamma$, where $\phi_\gamma \in \mathbb{C}^n$ is a solution of $A(z)\psi(z) = \psi(f(z))$. This means that $\phi_\gamma,\ \gamma = 1, 2, \cdots$ satisfy the equations

$$\sum_{\alpha+\beta=\gamma} A_\alpha \phi_\beta = \mu^\gamma \phi_\gamma, \quad \gamma \in N_0^n. \tag{2.2}$$

We rewrite (2.2) in the form

$$(A_0 - \mu^\gamma \mathrm{Id})\phi_r\gamma = -\sum_{\substack{\alpha+\beta=\gamma \\ \beta<\gamma}} A_\alpha \phi_\beta, \tag{2.3}$$

and solve this system of equations inductively: For $\gamma = (0, \cdots, 0)$, (2.3) implies $(A_0 - \mathrm{Id})\phi_0 = 0$. Hence either $\phi_0 = 0$ which implies that $\phi_\gamma = 0$ for all γ or ϕ_0 is an eigenvector with eigenvalue 1 (in which case $\phi_0 \neq 0$).

Assume that all ϕ_β with $\beta < \gamma$ are already determined. And ϕ_γ satisfies (2.3) if and only if

$$\sum_{\substack{\alpha+\beta=\gamma \\ \beta<\gamma}} A_\alpha \phi_\beta \in \mathrm{Im}(A_0 - \mu^\gamma \mathrm{Id}). \tag{2.4}$$

Since $|\mu_i| < 1$ and $|\lambda_i| \geq 1$ (the λ_i being the eigenvalues of A_0) for all i, the μ^γ for $\gamma > (0, \cdots, 0)$ cannot be eigenvalues of A_0 and (2.4) is satisfied. Proceeding in this manner we gain a solution of $A(z)\psi(z) = \psi(f(z))$. By the deformation argument as in [9], we can prove that ϕ is convergent in a neighborhood of 0.

Now we used $\phi(z)$ to construct a solution of functional equations (2.1). Let $d_i \in \mathbb{C}^*$, $i = 0, 1, \cdots, m-1$, such that

$$\rho_L : H \to \mathbb{C}^*,$$
$$h_i \mapsto d_i$$

is a representation of the finite group $H = \{h_0, \cdots,\ h_{m-1}\}$. Using the method of group action, we construct

$$\psi(z) := \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i(z)^{-1} \cdot h_i^* \cdot \phi(z), \tag{2.5}$$

where $h_i^* \cdot \phi(z) := \phi(h_i(z))$, and $h_i^* \cdot B_i(z) := B(h_i(z))$.

For $h \in H,$ let $\rho_L(h) = d$, where $d \in \{d_0, \cdots, d_{m-1}\}$. Since $H = \{h_0, \cdots, h_{m-1}\}$ is a finite group, we have $h_i \cdot h = h_{\lambda(i)}$, where $\lambda : \{0, 1, \cdots, m-1\} \to \{0, 1, \cdots, m-1\}$ is a permutation of $\lambda : \{0, 1, \cdots, m-1\}$. Hence we have $d_i \cdot d = \rho_L(h_i \cdot h) = \rho_L(h_{\lambda(i)}) = d_{\lambda(i)}$.

Similarly $\forall h \in H,$ let $\rho_E(h) = B(z)$. Since $H = \{h_0, \cdots, h_{m-1}\}$ is a finite group, we have $h_i \cdot h = h_{\lambda(i)}$, where $\lambda : \{0, 1, \cdots, m-1\} \to \{0, 1, \cdots, m-1\}$ is a permutation of $\lambda : \{0, 1, \cdots, m-1\}$. Hence we have $B_i(h(z)) \cdot B(z) = \rho_E(h_i \cdot h) = \rho_E(h_{\lambda(i)}) = B_{\lambda(i)}(z)$.

Now we prove that the $\psi(z)$ just constructed satisfies (2.1). Firstly

$$\psi(h(z)) = h^* \cdot \psi(z)$$

$$
\begin{aligned}
&= \sum_{i=0}^{m-1} d_i^{-1} \cdot (h^* \cdot B_i^{-1}(z)) \cdot h^* \cdot (h_i^* \cdot \phi(z)) \\
&= \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i^{-1}(h(z)) \cdot (h_i \cdot h)^* \cdot \phi(z) \\
&= dB(z) \sum_{i=0}^{m-1} (d^{-1} d_i^{-1}) \cdot (B^{-1}(z) B_i^{-1}(h(z))) \cdot (h \cdot h_i)^* \cdot \phi(z) \\
&= dB(z) \sum_{i=0}^{m-1} d_{\lambda(i)}^{-1} \cdot B_{\lambda(i)}^{-1}(z) h_{\lambda(i)}^* \cdot \phi(z) \\
&= dB(z) \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i(z)^{-1} \cdot h_i^* \cdot \phi(z) \\
&= dB(z)\psi(z).
\end{aligned}
$$

Secondly

$$
\begin{aligned}
\psi(f(z)) &= f^* \cdot \psi(z) \\
&= \sum_{i=0}^{m-1} d_i^{-1} \cdot (f^* \cdot B_i^{-1}(z)) \cdot f^* \cdot \phi(h_i(z)) \\
&= \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i^{-1}(f(z)) \cdot (cA(h_i(z))\phi(h_i(z))) \\
&= c \sum_{i=0}^{m-1} d_i^{-1} \cdot (B_i^{-1}(f(z)) \cdot A(h_i(z)))\phi(h_i(z)) \\
&= c \sum_{i=0}^{m-1} d_i^{-1} \cdot (A(z) \cdot B_i^{-1}(z)) h_i^* \cdot \phi(z) \quad \text{(by (1.1))} \\
&= cA(z) \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i(z)^{-1} \cdot h_i^* \cdot \phi(z) \\
&= cA(z)\psi(z).
\end{aligned}
$$

Therefore $\psi(z)$ is a solution of (2.1).

We claim that we can find $d_0, \cdots, d_{m-1}$ such that $\psi(0) \neq 0$. Otherwise, we assume that

$$
\psi(0) = \sum_{i=0}^{m-1} d_i^{-1} \cdot B_i(0)^{-1} \cdot \phi(0) = 0
$$

for all $d_0, \cdots, d_{m-1}$, where $d_i \in \mathbb{C}^*,\ i = 0, 1, \cdots, m-1$, are given by a representation of the finite group $H = \{h_0, \cdots, h_{m-1}\}$,

$$
\begin{aligned}
\rho_L : H &\to \mathbb{C}^*, \\
h_i &\mapsto d_i.
\end{aligned}
$$

We shall prove that it will lead to a contradiction.

From the above assumption, we know that $\forall k \in \mathbb{Z}$,

$$
\rho_L : H \to \mathbb{C}^*,
$$

$$h_i \mapsto d_i^k$$

is another representation of the finite group $H = \{h_0, \cdots, h_{m-1}\}$. By the assumption, we have

$$\psi(0) = \sum_{i=0}^{m-1} d_i^k \cdot B_i(0)^{-1} \cdot \phi(0) = 0.$$

We rewrite it into

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_1 & d_2 & \cdots & d_m \\ \vdots & \vdots & & \vdots \\ d_1^{m-1} & d_2^{m-1} & \cdots & d_m^{m-1} \end{pmatrix} \cdot \begin{pmatrix} B_0^{-1} \\ B_1^{-1} \\ \vdots \\ B_{m-1}^{-1} \end{pmatrix} \cdot \phi(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{mr}.$$

It follows that

$$\begin{pmatrix} I & I & \cdots & I \\ d_1 I & d_2 I & \cdots & d_m I \\ \vdots & \vdots & & \vdots \\ d_1^{m-1} I & d_2^{m-1} I & \cdots & d_m^{m-1} I \end{pmatrix} \cdot \begin{pmatrix} B_0^{-1} \\ B_1^{-1} \\ \vdots \\ B_{m-1}^{-1} \end{pmatrix} \cdot \phi(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{mr}.$$

It is clear that

$$\begin{pmatrix} B_0^{-1} \\ B_1^{-1} \\ \vdots \\ B_{m-1}^{-1} \end{pmatrix} \cdot \phi(0) \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{mr}.$$

Now we need to prove only

$$V := \begin{vmatrix} I & I & \cdots & I \\ d_1 I & d_2 I & \cdots & d_m I \\ \vdots & \vdots & & \vdots \\ d_1^{m-1} I & d_2^{m-1} I & \cdots & d_m^{m-1} I \end{vmatrix} \neq 0.$$

This is a generalized Van der Mond determinant. By the computation of determinant, we get the formula

$$V = (d_2 - d_1)^r (d_3 - d_1)^r \cdots (d_m - d_1)^r (d_3 - d_2)^r (d_4 - d_2)^r \cdots (d_m - d_2)^r \cdots (d_m - d_{m-1})^r.$$

Now we take $d_1, d_2, \cdots, d_m$ to m different unit roots, then it is clear that $V \neq 0$. Hence it leads to a contradiction.

We have $\psi(a) \neq 0$, $\forall a \in \mathbb{C}^n$. In fact, since $\psi(0) \neq 0$ there exists a neighborhood U of 0 such that $\psi|_U \neq 0$. By the first equation of (2.1), we have $\psi(f^n(a)) = c^n A(f^{n-1}(a)) \cdots A(a)\psi(a) = 0$. But since f is a contraction, when $n \to \infty$, $f^n(a) \to 0$. Because $\psi|_U \neq 0$, so when $n \to \infty$, $\psi(f^n(a)) \neq 0$. So it leads to a contradiction.

Therefore, we proved Theorem 2.1 for a diagonal contraction f. If f is an arbitary contraction, applying the upper-semicontinuity theorem of Grauert [15] and the deformation method as in [9], Theorem 2.1 is proved.

Theorem 2.2 *Let X be an arbitrary non-primary Hopf manifold and E be a holomorphic vector bundle of rand r on X with $\pi^*(E) \cong \mathcal{O}_W^r$. Then there exist upper triangular matrixes $A, B_i \in \mathrm{GL}(r, \Gamma(W, \mathcal{O}))$, $i = 1, \cdots, m-1$ such that $E \cong W \times \mathbb{C}^r / \langle f \times A, h_i \times B_i, i \in I \rangle$.*

Proof For this, we follow the proof in [9] and [10]. It suffices to show that if E is a vector bundle with $\pi^*(E) \cong \mathcal{O}_W^r$, then there is a trivialisation of $\pi^*(E)$ such that the matrixes A, B_i, $i = 1, \cdots, m-1$ representing E is of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

The theorem then follows by induction over the rank of E.

We show in Theorem 2.1 that there exists a line bundle L such that $L \otimes E$ has a nowhere vanishing section s. Hence there is an injection of vector bundles $\mathcal{O}_X \xrightarrow{(\cdot s)} L \otimes E$ induced by multiplication with the section s. This proves the existence of an exact sequence $0 \to \mathcal{O}_X \xrightarrow{(\cdot s)} L \otimes E \to F \to 0$ with F being a vector bundle of rank $r-1$. Applying the functor π^* on the above sequence we obtain

$$0 \to \pi^*\mathcal{O}_X \xrightarrow{\pi^*(\cdot s)} \pi^*(L \otimes E) \to \pi^* F \to 0. \tag{2.6}$$

Via the isomorphisms $\pi^*\mathcal{O}_X \cong \mathcal{O}_W$, and $\pi^*(L \otimes E) = \pi^* L \otimes \pi^* E \cong \mathcal{O}_W^r$, the map $\pi^*(\cdot s)$ is given by a vector of r holomorphic function $\psi = (s_1, \cdots, s_r)$ on W satisfying the system of function equation (2.1). This means that (2.6) is the restriction of a vector bundle extension

$$0 \to \mathcal{O}_W \xrightarrow{i} \mathcal{O}_W^r \xrightarrow{p} \mathcal{O}_W^r/\mathcal{O}_W \to 0 \tag{2.7}$$

over $\mathbb{C}^n$ but $\mathcal{O}_W^r/\mathcal{O}_W$ is topologically trivial and $\mathbb{C}^n$ is a Stein space, then $\mathcal{O}_W^r/\mathcal{O}_W$ is holomorphically trivial by Grauert's theorem; hence $\mathcal{O}_W^r/\mathcal{O}_W \cong \mathcal{O}_W^{r-1}$. The sequence (2.7) splits, i.e., there exists a holomorphic map $j : \mathcal{O}_W^{r-1} \to \mathcal{O}_W^r$ such that $p \circ j = \mathrm{id}$. This implies that $\mathcal{O}_W \oplus \mathcal{O}_W^{r-1} \xrightarrow{(i,j)} \mathcal{O}_W^r \cong \pi^*(E)$ is an isomorphism and in this new trivialisation the matrixes A and B_i, $i = 1, \cdots, m-1$ have the required form.

Theorem 2.3 (Structure Theorem) *Let X be an arbitrary Hopf manifold of dimension $n \geq 3$. If E is a holomorphic vector bundle of rank r on X, then the following statements are equivalent*:

(1) $\pi^*(E) \cong \mathcal{O}_W^r$,

(2) *E possesses a filtration by vector bundles $0 = E^0 \subset E^1 \cdots \subset E^r = E$ with E^i of rank i.*

Proof (1) $\Rightarrow$ (2) It follows from Theorem 2.2 that we get a holomorphic line subbundle E^1 of E. Now we consider the quotient bundle E/E^1. The pullback of the quotient bundle E/E^1 to the univerisal cover is again trivial following from the Grauert's theorem (see [15]). Then, there exists another holomorphic line subbundle of E/E^1 and this gives a rank two holomorphic vector subbundle E^2 of E. Now one sees inductively that E is filtrable.

(2) $\Rightarrow$ (1) For a line bundle, since all the holomorphic line bundles are flat (see [13]), the statement (2) is trivial and statement (1) is always correct. We proceed by induction. Let E be a holomorphic vector bundle of rank r with a filtration $0 = E^0 \subset E^1 \cdots \subset E^r = E$ and $\pi^*(E^{r-1}) = \mathcal{O}_W^{r-1}, \cdots, \pi^*(E^i) \cong \mathcal{O}_W^i, \cdots, \pi^*(E^1) \cong \mathcal{O}_W$. There is an exact sequence $0 \to E^{r-1} \to E \to L \to 0$ with a line bundle L. We apply the functor π^* and obtain

$$0 \to \pi^* E^{r-1} \to \pi^* E \to \pi^* L \to 0. \tag{2.8}$$

By induction we may assume that $\pi^* E^{r-1} \cong \mathcal{O}_W^{r-1}$. Thus (2.8) is reduced to

$$0 \to \mathcal{O}_W^{r-1} \to \pi^* E \to \mathcal{O}_W \to 0,$$

i.e., (2.8) is an extension of $\mathcal{O}_W$ by $\mathcal{O}_W^{r-1}$. These extensions are classified by the elements of

$$H^1(W, \mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_W^r)) = \bigoplus_{i=1}^{r-1} H^1(W, \mathcal{O}_W).$$

However $H^1(W, \mathcal{O}_W) = 0$ for $n \geq 3$ (see [16]). Hence $\pi^*(E) \cong \mathcal{O}_W^r$.

Remark 2.1 When X is an arbitrary Hopf surface, by Theorem 2.2, we know that "(1) $\Rightarrow$ (2)" in the Structure theorem is also true.

3 The Application of Structure Theorem

Theorem 3.1 *Let X be an arbitrary Hopf manifold of dimension $n \geq 3$. If E is a holomorphic vector bundle of rank r on X with $\pi^*(E) \cong \mathcal{O}_W^r$, then we have*

(a) $h^0(X, E) = h^1(X, E)$,

(b) $h^i(X, E) = 0$, *if* $2 \leq i \leq n-2$,

(c) $h^{n-1}(X, E) = h^n(X, E)$,

where $h^q(X, \Omega^p(E)) := \dim H^q(X, \Omega^p(E))$.

Proof By the Structure theorem, E possesses a filtration by vector bundles : $0 = E^0 \subset E^1 \subset \cdots \subset E^r = E$. From this filtration, we have the following short exact sequences:

$$\begin{array}{l} 0 \to E^{r-1} \to E \to L_1 \to 0, \\ 0 \to E^{r-2} \to E^{r-1} \to L_2 \to 0, \\ \quad \vdots \\ 0 \to E^2 \to E^3 \to L_{r-2} \to 0, \\ 0 \to E^1 \to E^2 \to L_{r-1} \to 0, \end{array}$$

where $L_i \in H^1(X, \mathcal{O}^*)$, i.e., L_i, $i = 1, \cdots, r-1$ are holomorphic line bundles on X.

Since $\Omega^p(E) = \Omega^p \otimes E$, and by

$$0 \to E^{r-1} \to E \to L_1 \to 0,$$

we have

$$0 \to \Omega^p(E^{r-1}) \to \Omega^p(E) \to \Omega^p(L_1) \to \mathrm{Tor}_1^{\mathcal{O}}(\Omega^p, E^{r-1}) \to \cdots.$$

But because E^{r-1} is local free $\mathcal{O}$ module, we get $\mathrm{Tor}_1^{\mathcal{O}}(\Omega^p, E^{r-1}) = 0$.

Hence from the above short exact sequences, we have

$$\begin{array}{l} 0 \to \Omega^p(E^{r-1}) \to \Omega^p(E) \to \Omega^p(L_1) \to 0, \\ 0 \to \Omega^p(E^{r-2}) \to \Omega^p(E^{r-1}) \to \Omega^p(L_2) \to 0, \\ \quad \vdots \\ 0 \to \Omega^p(E^2) \to \Omega^p(E^3) \to \Omega^p(L_{r-2}) \to 0, \\ 0 \to \Omega^p(E^1) \to \Omega^p(E^2) \to \Omega^p(L_{r-1}) \to 0. \end{array}$$

From the above, we get the following long exact sequences:

$$\cdots \to H^{i-1}(X, \Omega^p(L_1)) \to H^i(X, \Omega^p(E^{r-1})) \to H^i(X, \Omega^p(E)) \to H^i(X, \Omega^p(L_1)) \to \cdots$$

$$\cdots \to H^{i-1}(X,\Omega^p(L_2)) \to H^i(X,\Omega^p(E^{r-2})) \to H^i(X,\Omega^p(E^{r-1})) \to H^i(X,\Omega^p(L_2)) \to \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots \to H^{i-1}(X,\Omega^p(L_{r-2})) \to H^i(X,\Omega^p(E^2)) \to H^i(X,\Omega^p(E^3)) \to H^i(X,\Omega^p(L_{r-2})) \to \cdots$$
$$\cdots \to H^{i-1}(X,\Omega^p(L_{r-1})) \to H^i(X,\Omega^p(E^1)) \to H^i(X,\Omega^p(E^2)) \to H^i(X,\Omega^p(L_{r-1})) \to \cdots .$$

By [13], for holomorphic line bundle L on Hopf manifold, we have

$$\begin{aligned} &h^0(X,\Omega^p(L)) = h^1(X,\Omega^p(L)),\\ &h^i(X,\Omega^p(L)) = 0, \quad \text{if } 2 \le i \le n-2,\\ &h^{n-1}(X,\Omega^p(L)) = h^n(X,\Omega^p(L)). \end{aligned}$$

Hence from the above long exact sequences, for $i = 2, \cdots, \ n-2$, we obtain

$$h^i(X,\Omega^p(E)) = h^i(X,\Omega^p(E^{r-1})) = h^i(X,\Omega^p(E^{r-2})) = \cdots = h^i(X,\Omega^p(E^1)) = 0.$$

On the other hand, by $0 \to \Omega^p(E^1) \to \Omega^p(E^2) \to \Omega^p(L_{r-1}) \to 0$, we have

$$0 \to H^0(E^1) \xrightarrow{f_1} H^0(E^2) \xrightarrow{f_2} H^0(L_{r-1}) \xrightarrow{f_3} H^1(E^1) \xrightarrow{f_4} H^1(E^2) \xrightarrow{f_5} H^1(L_{r-1}) \to 0,$$

where we denote $H^i(X,\Omega^p(E))$ and $h^i(X,\Omega^p(E))$ by $H^i(E)$ and $h^i(E)$ respectively for simplicity. Since E^1 and L are line bundles, it implies $h^0(E^1) = h^1(E^1)$ and $h^0(L_{r-1}) = h^1(L_{r-1})$. Thus we get

$$\begin{aligned} h^0(E^2) &= \dim \operatorname{Ker} f_2 + \dim \operatorname{Im} f_2\\ &= \dim \operatorname{Im} f_1 + \dim \operatorname{Ker} f_3\\ &= h^0(E^1) + h^0(L) - \dim \operatorname{Im} f_3\\ &= h^0(E^1) + h^0(L) - \dim \operatorname{Ker} f_4\\ &= h^0(E^1) + h^0(L) - h^1(E^1) + \dim \operatorname{Im} f_4\\ &= h^0(L) + \dim \operatorname{Ker} f_5\\ &= h^0(L) + h^1(E^2) - \dim \operatorname{Im} f_5\\ &= h^0(L) + h^1(E^2) - h^0(L)\\ &= h^1(E^2). \end{aligned}$$

Proceeding it by induction, we obtain $h^0(E^3) = h^1(E^3), \ \cdots, \ h^0(E) = h^1(E)$.

Similarly we have $h^{n-1}(E) = h^n(E)$.

Remark 3.1 In [14], we have another proof of Theorem 3.1 by the method of Douady sequence. We refer the readers to [11] for more details of the Douady sequence on non-primary Hopf manifolds.

Theorem 3.2 *Let X be a non-primary Hopf manifold of dimension $n \ge 3$. If E is a holomorphic vector bundle of rank r on X with $\pi^*(E) \cong \mathcal{O}_W^r$, then the Chern classes of E $c_i(E) = 0, \ 1 \le i \le n$.*

Proof By the Structure theorem, E possesses a filtration by vector bundles: $0 = E^0 \subset E^1 \cdots \subset E^r = E$. From this filtration, we have the following short exact sequences:

$$\begin{aligned} &0 \to E^{r-1} \to E \to L_1 \to 0,\\ &0 \to E^{r-2} \to E^{r-1} \to L_2 \to 0,\\ &\quad\vdots\\ &0 \to E^2 \to E^3 \to L_{r-2} \to 0,\\ &0 \to E^1 \to E^2 \to L_{r-1} \to 0, \end{aligned}$$

where $L_i,\ i = 1, \cdots, r-1$ are holomorphic line bundles on X. By

$$0 \ \to \ E^1 \ \to \ E^2 \ \to L_{r-1} \ \to \ 0,$$

we have

$$c(E^2) = c(E^1) \ \cdot \ c(L_{r-1}),$$

i.e.,

$$1 + c_1(E^2) + c_2(E^2) = (1 + c_1(E^1)) \cdot (1 + c_1(L_{r-1})).$$

Because all holomorphic line bundles on Hopf manifold are flat, we have $c_1(E^1) = 0$ and $c_1(L_{r-1}) = 0$. So $c_1(E^2) = c_2(E^2) = 0$. Again by

$$0 \ \to \ E^2 \ \to \ E^3 \ \to L_{r-2} \ \to \ 0,$$

we have

$$1 + c_1(E^3) + c_2(E^3) + c_3(E^3) = (1 + c_1(E^2) + c_2(E^2)) \cdot (1 + c_1(L_{r-2})).$$

For $c_1(L_{r-2}) = c_1(E^2) = c_2(E^2) = 0$, we get $c_1(E^3) = c_2(E^3) = c_3(E^3) = 0$.

Proceeding it by induction, we get $c_i(E) = 0,\ i \geq 1$.

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