Chin. Ann. Math. Ser. B
42(1), 2021, 41–68
DOI: 10.1007/s11401-021-0244-y

Chinese Annals of Mathematics, Series B

Distinguished Connections on Finsler Algebroids

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Abstract Considering the prolongation of a Lie algebroid, the authors introduce Finsler algebroids and present important geometric objects on these spaces. Important endomorphisms like conservative and Barthel, Cartan tensor and some distinguished connections like Berwald, Cartan, Chern-Rund and Hashiguchi are introduced and studied.

Keywords Chern-Rund connection, Distinguished connections, Finsler algebroid, Hashiguchi connection, Lie algebroid
2010 MR Subject Classification 53B40, 53B05

1 Introduction

The notion of Lie algebroids which was introduced by Pradines [12] is a vector bundle such that its sections involve a real Lie algebra. Each section is anchored on a vector field, by means of a linear bundle map named as anchor map, which is further supposed to induce a Lie algebra homomorphism. The Lie algebroid is a good extension of tangent bundle since the homomorphism property of the anchor map grants the basic notions of tangent bundle to the vector bundle. Recently, Lie algebroids are important issues in physics and mechanics since the extension of Lagrangian and Hamiltonian systems to their entity (see [2, 5–7, 9–10, 18, 20]) and catching the poisson structure (see [11]).

The aim of this paper is to study some concepts of Finsler geometry on Lie algebroid structures. Of course, there are some discussions on Finsler geometry in [9, 19]. Finsler geometry is a generalization of Riemannian geometry such that interfering of direction and position duplicates the degree of freedom in view of configuration. Variety of tensors in Finsler geometry is more than Riemannian case. A very good reference about Finsler geometry is [1].

The paper is organized as follows. In Section 2, we recall differential, contraction and Lie differential operators, generalized Frölicher-Nijenhuis bracket and vertical and complete lifts on Lie algebroids and we study the relation between these concepts. Also, we present the notion of the prolongation of a Lie algebroid and we recall some concepts on it such as horizontal and vertical endomorphisms, Liouville section, semispray, torsion, tension and almost complex structure. Finally, distinguished connections on the prolongation of a Lie algebroid are introduced and torsion and curvature tensor fields of these connections are considered. In Section 3, we introduce the concept of Finsler algebroid and we study important geometric

Manuscript received December 9, 2019. Revised March 17, 2020.
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subjects on this space. Important endomorphisms like Conservative and Barthel, Cartan tensor and some distinguished connections like Berwald, Cartan, Chern-Rund and Hashiguchi are studied by Szilasi and his collaborators from a special point view based on pullback bundle (see [13–17]). In this section we construct them on Finsler algebroids and obtain some results on these concepts.

2 Basic Concepts on Lie Algebroids

Let E be a vector bundle of rank n over a manifold M of dimension m and $\pi : E \to M$ be the vector bundle projection. Denote by $\Gamma(E)$ the $C^\infty(M)$-module of sections of $\pi : E \to M$. A Lie algebroid over M is the triple $(E, [.,.]_E, \rho)$ where $[\cdot,\cdot]_E$ is a Lie bracket on $\Gamma(E)$ and $\rho : E \to TM$ is a bundle map, called the anchor map, such that if we also denote by $\rho : \Gamma(E) \to \chi(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map, then

$$[X, fY]_E = f[X,Y]_E + \rho(X)(f)Y, \quad \forall\, X, Y \in \Gamma(E),\ \forall\, f \in C^\infty(M).$$

Moreover, we have the relations

$$[\rho(X), \rho(Y)] = \rho([X,Y]_E)$$

and

$$[X,[Y,Z]_E]_E + [Y,[Z,X]_E]_E + [Z,[X,Y]_E]_E = 0.$$

Trivial examples of Lie algebroids are real Lie algebras of finite dimension, the tangent bundle TM of an arbitrary manifold M and an integrable distribution of TM.

On Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$ we define the differential of E, $d^E : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^{k+1} E^*)$, as follows

$$\begin{aligned} d^E\mu(X_0, \cdots, X_k) &= \sum_{i=0}^{k} (-1)^i \rho(X_i)(\mu(X_0, \cdots, \widehat{X}_i, \cdots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j]_E, X_0, \cdots, \widehat{X}_i, \cdots, \widehat{X_j}, \cdots, X_k) \end{aligned}$$

for $\mu \in \Gamma(\wedge^k E^*)$ and $X_0, \cdots, X_k \in \Gamma(E)$. In particular, if $f \in \Gamma(\wedge^0 E^*) = C^\infty(M)$ we have $d^E f(X) = \rho(X) f$. Using the above equation it follows that $(d^E)^2 = 0$.

If we take the local coordinates (x^i) on M and a local basis $\{e_\alpha\}$ of sections of E, then we have the corresponding local coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha)$ on E, where $\mathbf{x}^i = x^i \circ \pi$ and $\mathbf{y}^\alpha(u)$ is the α-th coordinate of $u \in E$ in the given basis. Such coordinates determine local functions ρ^i_α, $L^\gamma_{\alpha\beta}$ on M which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \quad [e_\alpha, e_\beta]_E = L^\gamma_{\alpha\beta} e_\gamma,$$

with conditions

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma L^\gamma_{\alpha\beta}, \qquad \sum_{(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial L^\nu_{\beta\gamma}}{\partial x^i} + L^\nu_{\alpha\mu} L^\mu_{\beta\gamma} \right] = 0.$$

A section ω of E^* also defines a function $\widehat{\omega}$ on E by means of

$$\widehat{\omega}(u) = \langle \omega_m, u \rangle, \quad \forall\, u \in E_m.$$

If $\omega = \omega_\alpha \mathrm{e}^\alpha$, then the linear function $\widehat{\omega}$ is $\widehat{\omega}(x, y) = \omega_\alpha \mathbf{y}^\alpha$.

For $X \in \Gamma(\wedge^k E)$, the contraction $i_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k} E^*)$ is defined in standard way and the Lie differential operator $\pounds^E_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k+1} E^*)$ is defined by

$$\pounds^E_X = i_X \circ d^E - (-1)^k d^E \circ i_X.$$

Note that if $E = TM$ and $X \in \Gamma(E) = \chi(M)$, then d^{TM} and $\pounds^{TM}_X$ are the usual differential and the usual Lie derivative with respect to X, respectively. Also, for $K \in \Gamma(\wedge^k E^* \otimes E)$, the contraction

$$i_K : \Gamma(\wedge^n E^*) \to \Gamma(\wedge^{n+k-1} E^*)$$

is defined in the natural way. In particular, for simple tensor $K = \mu \otimes X$, where $\mu \in \Gamma(\wedge^k E^*)$, $X \in \Gamma(E)$, we set $i_K \nu = \mu \wedge i_X \nu$. The corresponding Lie differential is defined by the formula

$$\pounds^E_K = i_K \circ d^E + (-1)^k d^E \circ i_K,$$

and, in particular

$$\pounds^E_{\mu \otimes X} = \mu \wedge \pounds^E_X + (-1)^k d^E \mu \wedge i_X.$$

The contraction i_K can be extended to an operator

$$i_K : \Gamma(\wedge^n E^* \otimes E) \to \Gamma(\wedge^{n+k-1} E^* \otimes E)$$

by the formula $i_K(\mu \otimes X) = i_K(\mu) \otimes X$. The generalized Frölicher-Nijenhuis bracket is defined for simple tensors $\mu \otimes X \in \Gamma(\wedge^k E^* \otimes E)$ and $\nu \otimes Y \in \Gamma(\wedge^l E^* \otimes E)$ by the formula

$$[\mu \otimes X, \nu \otimes Y]^{F-N} = (\pounds_{\mu \otimes X} \nu) \otimes Y - (-1)^{kl} (\pounds_{\nu \otimes Y} \mu) \otimes X + \mu \wedge \nu \otimes [X, Y]_E.$$

Moreover, for $K \in \Gamma(\wedge^k E^* \otimes E)$, $L \in \Gamma(\wedge^l E^* \otimes E)$, $N \in \Gamma(\wedge^n E^* \otimes E)$ and $X, Y \in \Gamma(E)$ we have (see [3–4])

$$\begin{aligned}
&\pounds^E_{[K,L]^{F-N}} = \pounds^E_K \circ \pounds^E_L - (-1)^{kl} \pounds^E_L \circ \pounds^E_K,\\
&(-1)^{kn}[K, [L, N]^{F-N}]^{F-N} + (-1)^{lk}[L, [N, K]^{F-N}]^{F-N} + (-1)^{nl}[N, [K, L]^{F-N}]^{F-N} = 0,\\
&[K, Y]^{F-N}(X) = [K(X), Y]_E - K[X, Y]_E,\\
&[K, L]^{F-N}(X, Y) = [K(X), L(Y)]_E + [L(X), K(Y)]_E + (K \circ L + L \circ K)[X, Y]_E\\
&\qquad - K[X, L(Y)]_E - K[L(X), Y]_E - L[X, K(Y)]_E - L[K(X), Y]_E.
\end{aligned}$$

For a function f on M, one defines its vertical lift $f^\vee$ on E by $f^\vee(u) = f(\pi(u))$ for $u \in E$. Now, let X be a section of E. Then, we can consider the vertical lift of X as the vector field on E given by $X^\vee(u) = X(\pi(u))^\vee_u$, $u \in E$, where ${}^\vee_u : E_{\pi(u)} \to T_u(E_{\pi(u)})$ is the canonical isomorphism between the vector spaces $E_{\pi(u)}$ and $T_u(E_{\pi(u)})$. Let $\{e_\alpha\}$ be a basis of sections of E. The vertical lift $X^\vee$ of $X = X^\alpha e_\alpha \in \Gamma(E)$ has the locally expression $X^\vee = (X^\alpha \circ \pi) \frac{\partial}{\partial \mathbf{y}^\alpha}$. The complete lift of a smooth function $f \in C^\infty(M)$ into $C^\infty(E)$ is the smooth function $f^{\mathrm{c}} : E \to \mathbb{R}$ defined by $f^{\mathrm{c}}(u) = d^E f(u) = \rho(u) f$. In the local basis, we have

$$f^{\mathrm{c}}|_{\pi^{-1}(U)} = \mathbf{y}^\alpha \Big(\Big(\rho^i_\alpha \frac{\partial f}{\partial x^i} \Big) \circ \pi \Big).$$

Let X be a section on E. Then there exists a unique vector field X^{c} on E, the complete lift of X, satisfying the following conditions:

(i) X^{c} is π-projectable on $\rho(X)$,

(ii) $X^{\mathrm{c}}(\widehat{\alpha}) = \widehat{\pounds_X^E \alpha}$,

where $\alpha \in \Gamma(E^*)$. It is known that X^{c} has the following coordinate expression (see [8]):

$$X^{\mathrm{c}} = \{(X^\alpha \rho^i_\alpha) \circ \pi\} \frac{\partial}{\partial \mathbf{x}^i} + \mathbf{y}^\beta \Big\{ \Big(\rho^j_\beta \frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta} \Big) \circ \pi \Big\} \frac{\partial}{\partial \mathbf{y}^\alpha}.$$

Also we have $X^{\mathrm{c}} f^{\mathrm{c}} = (\rho(X) f)^{\mathrm{c}}$ for all $f \in C^\infty(M)$.

2.1 The prolongation of a Lie algebroid

Let $\pounds^\pi E$ be the subset of $E \times TE$ defined by $\pounds^\pi E = \{(u, z) \in E \times TE \mid \rho(u) = \pi_*(z)\}$ and denote by $\pi_\pounds : \pounds^\pi E \to E$ the mapping given by $\pi_\pounds(u, z) = \pi_E(z)$, where $\pi_E : TE \to E$ is the natural projection. Then, $(\pounds^\pi E, \pi_\pounds, E)$ is a vector bundle over E of rank $2n$. Indeed, the total space of the prolongation is the total space of the pull-back of $\pi_* : TE \to TM$ by the anchor map ρ.

We introduce the vertical subbundle

$$v\pounds^\pi E = \ker \tau_\pounds = \{(u, z) \in \pounds^\pi E \mid \tau_\pounds(u, z) = 0\},$$

where $\tau_\pounds : \pounds^\pi E \to E$ is the projection onto the first factor, i.e., $\tau_\pounds(u, z) = u$. Therefore an element of $v\pounds^\pi E$ is of the form $(0, z) \in E \times TE$ such that $\pi_*(z) = 0$ which is called vertical.

For local basis $\{e_\alpha\}$ of sections of E and coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha)$ on E, we have local coordinates $(\mathbf{x}^i, \mathbf{y}^\alpha, k^\alpha, z^\alpha)$ on $\pounds^\pi E$ given as follows. If (u, z) is an element of $\pounds^\pi E$, then using $\rho(u) = \pi_*(z)$, z has the form

$$z = ((\rho^i_\alpha u^\alpha) \circ \pi) \frac{\partial}{\partial \mathbf{x}^i}\Big|_v + z^\alpha \frac{\partial}{\partial \mathbf{y}^\alpha}\Big|_v, \quad z \in T_v E.$$

The local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ of sections of $\pounds^\pi E$ associated to the coordinate system $(\mathbf{x}^i, \mathbf{y}^\alpha)$ is given by [6],

$$\mathcal{X}_\alpha(v) = \Big(e_\alpha(\pi(v)), (\rho^i_\alpha \circ \pi) \frac{\partial}{\partial \mathbf{x}^i}\Big|_v \Big), \quad \mathcal{V}_\alpha(v) = \Big(0, \frac{\partial}{\partial \mathbf{y}^\alpha}\Big|_v \Big).$$

The vertical lift X^{V} and the complete lift X^{C} of a section $X \in \Gamma(E)$ as the sections of $\pounds^\pi E \to E$ are given by

$$X^{\mathrm{V}}(u) = (0, X^\vee(u)), \quad X^{\mathrm{C}}(u) = (X(\pi(u)), X^{\mathrm{c}}(u)), \quad u \in E$$

with locally coordinate expressions

$$X^{\mathrm{V}} = (X^\alpha \circ \pi) \mathcal{V}_\alpha, \quad X^{\mathrm{C}} = (X^\alpha \circ \pi) \mathcal{X}_\alpha + \mathbf{y}^\beta \Big[\Big(\rho^j_\beta \frac{\partial X^\alpha}{\partial x^j} - X^\gamma L^\alpha_{\gamma\beta} \Big) \circ \pi \Big] \mathcal{V}_\alpha, \tag{2.1}$$

where $X = X^\alpha e_\alpha \in \Gamma(E)$.

Here, we consider the anchor map $\rho_\pounds : \pounds^\pi E \to TE$ defined by $\rho_\pounds(u, z) = z$ and the bracket $[\cdot, \cdot]_\pounds$ satisfying the relations

$$[X^{\mathrm{V}}, Y^{\mathrm{V}}]_\pounds = 0, \quad [X^{\mathrm{V}}, Y^{\mathrm{C}}]_\pounds = [X, Y]^{\mathrm{V}}_E, \quad [X^{\mathrm{C}}, Y^{\mathrm{C}}]_\pounds = [X, Y]^{\mathrm{C}}_E$$

for $X, Y \in \Gamma(E)$. Then, this vector bundle $(\pounds^\pi E, \pi_\pounds, E)$ is a Lie algebroid with structure $([\cdot, \cdot]_\pounds, \rho_\pounds)$. The Lie brackets of basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_\pounds = (L^\gamma_{\alpha\beta} \circ \pi) \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_\pounds = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_\pounds = 0.$$

2.2 A setting for semispray on $\pounds^\pi E$

A section of π along smooth map $f : N \to M$ is a smooth map $\sigma : N \to E$ such that $\pi \circ \sigma = f$. The set of sections of π along f will be denoted by $\Gamma_f(\pi)$. Then, there is a canonical isomorphism between $\Gamma(f^*\pi)$ and $\Gamma_f(\pi)$ (see [15]). Now we consider pullback bundle $\pi^*\pi = (\pi^*E, pr_1, E)$ of vector bundle (E, π, M), where

$$\pi^*E := E \times_M E := \{(u, v) \in E \times E \mid \pi(u) = \pi(v)\}$$

and pr_1 is the projection map onto the first component. The fibres of $\pi^*\pi$ are the n-dimensional real vector spaces $\{u\} \times E_{\pi(u)} \cong E_{\pi(u)}$.

We consider the following sequence:

$$0 \to \pi^*(E) \xrightarrow{i} \pounds^\pi E \xrightarrow{j} \pi^*(E) \to 0$$

with $j(u, z) = (\pi_E(z), \mathrm{Id}(u)) = (v, u)$, $z \in T_vE$ and $i(u, v) = (0, v_u^\vee)$, where $v_u^\vee : C^\infty(E) \to \mathbb{R}$ is defined by $v_u^\vee(F) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} F(u + tv)$. Indeed, we have $v_u^\vee = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(u + tv)$. The function $J = i \circ j : \pounds^\pi E \to \pounds^\pi E$ is called the vertical endomorphism (almost tangent structure) of $\pounds^\pi E$. From the definitions of i, j and J, we get

$$\operatorname{Im} J = \operatorname{Im} i = v\pounds^\pi E, \quad \ker J = \ker j = v\pounds^\pi E, \quad J \circ J = 0.$$

If $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ is the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, we get $J = \mathcal{V}_\alpha \otimes \mathcal{X}^\alpha$.

Let δ be the canonical section along π given by $\delta(u) = (u, u) \in \pi^*E$ for each $u \in E$. The section C given by $C := i \circ \delta$ is called Liouville or Euler section. The Liouville section C has the coordinate expression

$$C = \mathbf{y}^\alpha \mathcal{V}_\alpha \tag{2.2}$$

with respect to $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. Let X be a section of E. Then, we have

$$\text{(i) } [J, C]_\pounds^{F-N} = J, \quad \text{(ii) } [X^\mathrm{V}, C]_\pounds = X^\mathrm{V}, \quad \text{(iii) } JC = 0. \tag{2.3}$$

A section $\widetilde{X}$ of vector bundle $(\pounds^\pi E, \pi_\pounds, E)$ is said to be homogeneous of degree r (r is an integer), if $[C, \widetilde{X}]_\pounds = (r-1)\widetilde{X}$. Moreover, $\widetilde{f} \in C^\infty(E)$ is said to be homogeneous of degree r if $\pounds_C^\pounds \widetilde{f} = \rho_\pounds(C)(\widetilde{f}) = r\widetilde{f}$. It is known that if $\widetilde{X} = \widetilde{X}^\alpha \mathcal{X}_\alpha + \widetilde{Y}^\alpha \mathcal{V}_\alpha$, $\widetilde{X}$ is homogeneous of degree r if and only if

$$\mathbf{y}^\alpha \frac{\partial \widetilde{X}^\beta}{\partial \mathbf{y}^\alpha} = (r-1)\widetilde{X}^\beta, \quad \mathbf{y}^\alpha \frac{\partial \widetilde{Y}^\beta}{\partial \mathbf{y}^\alpha} = r\widetilde{Y}^\beta. \tag{2.4}$$

Also, the real valued smooth function $\widetilde{f}$ on E is homogeneous of degree r if and only if $\mathbf{y}^\alpha \frac{\partial \widetilde{f}}{\partial \mathbf{y}^\alpha} = r\widetilde{f}$.

A section S of the vector bundle $(\pounds^\pi E, \pi_\pounds, E)$ is said to be a semispray if it satisfies the condition $J(S) = C$. Moreover, if S is homogeneous of degree 2, i.e., $[C, S]_\pounds = S$, we call it spray. A semispray S has the coordinate expression $S = \mathbf{y}^\alpha \mathcal{X}_\alpha + S^\alpha \mathcal{V}_\alpha$ and S is a spray if and only if $2S^\beta = \mathbf{y}^\alpha \frac{\partial S^\beta}{\partial \mathbf{y}^\alpha}$.

A function $h : \pounds^\pi E \to \pounds^\pi E$ is called a horizontal endomorphism if $h \circ h = h$, $\ker h = v\pounds^\pi E$ and h is smooth on $\overset{\circ}{\pounds^\pi E} = \pounds^\pi E - \{0\}$. Also, $v := \mathrm{Id} - h$ is called the vertical projector associated to h. Setting $h\pounds^\pi E := \operatorname{Im} h$ we have the following splitting for $\pounds^\pi E$:

$$\pounds^\pi E = v\pounds^\pi E \oplus h\pounds^\pi E. \tag{2.5}$$

Also, from the definition of the horizontal endomorphism we have

$$\ker h = \operatorname{Im} J = \ker J = \operatorname{Im} v = v\pounds^{\pi}E.$$

$$\text{(i) } hJ = hv = Jv = 0, \quad \text{(ii) } v \circ v = v, \quad \text{(iii) } vh = 0, \quad \text{(iv) } Jh = J = vJ. \tag{2.6}$$

It is known that h has the following locally expression:

$$h = (\mathcal{X}_\beta + \mathcal{B}_\beta^\alpha \mathcal{V}_\alpha) \otimes \mathcal{X}^\beta. \tag{2.7}$$

Definition 2.1 *For $k \in \mathbb{N}$, $K \in \Gamma(\wedge^k E^* \otimes E)$ is called semibasic if*

$$JoK = 0, \quad i_{JX}K = 0, \quad \forall\, X \in \Gamma(E).$$

Let h be a horizontal endomorphism on $\pounds^{\pi}E$. Then, $H = [h, C]_{\pounds}^{F-N} : \pounds^{\pi}E \to \pounds^{\pi}E$, $t = [J, h]_{\pounds}^{F-N} \in \Gamma(\pounds^{\pi}E)$ and $T = i_S t + H$ are called the tension, weak torsion and strong torsion of h, respectively, where $[\cdot,\cdot]_{\pounds}^{F-N}$ is the generalized Frölicher-Nijenhuis bracket on $\pounds^{\pi}E$. If $H = 0$, h is called homogeneous. Here, H, t and T have the following coordinate expressions:

$$H = \Big(\mathcal{B}_\beta^\alpha - \mathbf{y}^\gamma \frac{\partial \mathcal{B}_\beta^\alpha}{\partial \mathbf{y}^\gamma}\Big)\mathcal{V}_\alpha \otimes \mathcal{X}^\beta, \tag{2.8}$$

$$t = \frac{1}{2} t_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{2.9}$$

$$T = \Big(\mathcal{B}_\beta^\alpha - \mathbf{y}^\gamma \frac{\partial \mathcal{B}_\gamma^\alpha}{\partial \mathbf{y}^\beta} - \mathbf{y}^\gamma (L_{\gamma\beta}^\alpha \circ \pi)\Big)\mathcal{V}_\alpha \otimes \mathcal{X}^\beta, \tag{2.10}$$

where

$$t_{\alpha\beta}^\gamma := \frac{\partial \mathcal{B}_\beta^\gamma}{\partial \mathbf{y}^\alpha} - \frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{y}^\beta} - (L_{\alpha\beta}^\gamma \circ \pi). \tag{2.11}$$

Theorem 2.1 (see [8]) *If h_1 and h_2 are horizontal endomorphisms with same associated semisprays and strong torsions, then $h_1 = h_2$.*

The curvature of a horizontal endomorphism h is defined by $\Omega = -N_h$, where N_h is the Nijenhuis tensor of h given by

$$N_h(\widetilde{X}, \widetilde{Y}) = [h\widetilde{X}, h\widetilde{Y}] - h[h\widetilde{X}, \widetilde{Y}] - h[\widetilde{X}, h\widetilde{Y}] + h[\widetilde{X}, \widetilde{Y}], \quad \forall\, \widetilde{X}, \widetilde{Y} \in \Gamma(\pounds^{\pi}E).$$

The curvature Ω has the following coordinate expression:

$$\Omega = -\frac{1}{2} R_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{2.12}$$

where

$$R_{\alpha\beta}^\gamma = (\rho_\alpha^i \circ \pi)\frac{\partial \mathcal{B}_\beta^\gamma}{\partial \mathbf{x}^i} - (\rho_\beta^i \circ \pi)\frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{x}^i} + \mathcal{B}_\alpha^\lambda \frac{\partial B_\beta^\gamma}{\partial \mathbf{y}^\lambda} - \mathcal{B}_\beta^\lambda \frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{y}^\lambda} + (L_{\beta\alpha}^\lambda \circ \pi)\mathcal{B}_\lambda^\gamma. \tag{2.13}$$

Let the horizontal endomorphism h be given on $\pounds^{\pi}E$. If S is an arbitrary semispray of $\pounds^{\pi}E$, $\overline{S} = hS$ is also a semispray of $\pounds^{\pi}E$ which does not depend on the choice of S. $\overline{S}$ is called the semispray associated to h. If the horizontal endomorphism h is homogeneous, the semispray associated to h is spray.

Let S be a semispray on $\pounds^{\pi}E$. We consider the map $h_S : \pounds^{\pi}E \to \pounds^{\pi}E$ given by $h_S := \frac{1}{2}(1_{\pounds^{\pi}E} + [J, S]_{\pounds}^{F-N})$. It is known that h_S is a horizontal endomorphism on $\pounds^{\pi}E$ which is called horizontal endomorphism generated by semispray S (see [8] for more details). We have the following theorem.

Theorem 2.2 (see [8]) *Let h be a homogeneous horizontal endomorphism on $\pounds^\pi E$ and S be the semispray associated to h. Then, we have*

$$h_S = h - \frac{1}{2} i_S t,$$

where t is the weak torsion of h and h_S is the horizontal endomorphism generated by S.

Let S be the semispray associated to h. We consider the map $F : \pounds^\pi E \to \pounds^\pi E$ given by $F := h[S,h]_{\pounds}^{F-N} - J$. Then, F is an almost complex structure on $\pounds^\pi E$ ($F^2 = -\mathrm{Id}$) which is called the almost complex structure induced by h. F has the following coordinate expression:

$$F = -(\mathcal{B}_\alpha^\gamma(\mathcal{X}_\gamma + \mathcal{B}_\gamma^\beta \mathcal{V}_\beta) + \mathcal{V}_\alpha) \otimes \mathcal{X}^\alpha + (\mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta) \otimes \mathcal{V}^\alpha. \tag{2.14}$$

The following relations hold in [8]:

$$\text{(i) } F \circ J = h, \quad \text{(ii) } F \circ h = -J, \quad \text{(iii) } J \circ F = v, \quad \text{(iv) } F \circ v = h \circ F. \tag{2.15}$$

The map $\mathcal{H} := F \circ i : E \times_M E \to \pounds^\pi E$ is called the horizontal map for $\pounds^\pi E$ associated to h. Also, the map $\mathcal{V} := j \circ F : \pounds^\pi E \to E \times_M E$ is called the vertical map for $\pounds^\pi E$ associated to h.

Let h be a horizontal endomorphism on $\pounds^\pi E$. We consider the map

$$X \in \Gamma(E) \to X^h := hX^{\mathrm{C}} \in h\pounds^\pi E,$$

and we call it horizontal lift by h. If $X = X^\alpha e_\alpha$, we have

$$X^h = (X^\alpha \circ \pi)(\mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta). \tag{2.16}$$

The following equations hold in [8]:

$$\text{(i) } JX^h = X^{\mathrm{V}}, \quad \text{(ii) } h[X^h, Y^h]_{\pounds} = [X,Y]_E^h, \quad \text{(iii) } [X,Y]_E^{\mathrm{V}} = J[X^h, Y^h]_{\pounds}.$$

Setting $\delta_\alpha = e_\alpha^h = \mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta \mathcal{V}_\beta = h(\mathcal{X}_\alpha)$, it is easy to see that $\{\delta_\alpha\}$ generates a basis of $h\pounds^\pi E$ and the frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\pounds^\pi E$ adapted to splitting (2.5) which is called the adapted basis. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$, where

$$\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{B}_\beta^\alpha \mathcal{X}^\beta.$$

Lie brackets of the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are

$$[\delta_\alpha, \delta_\beta]_{\pounds} = (L_{\alpha\beta}^\gamma \circ \pi)\delta_\gamma + R_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta]_{\pounds} = -\frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{y}^\beta} \mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\pounds} = 0, \tag{2.17}$$

where $R_{\alpha\beta}^\gamma$ is given by (2.13). It is easy to see that h and F have the following coordinate expressions with respect to the adapted basis

$$h = \delta_\alpha \otimes \mathcal{X}^\alpha, \quad F = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta\mathcal{V}^\alpha.$$

2.3 Distinguished connections on Lie algebroids

A linear connection on a Lie algebroid $(E, [,]_E, \rho)$ is a map

$$D : \Gamma(E) \times \Gamma(E) \to \Gamma(E),$$

which satisfies the rules

$$D_{fX+Y}Z = fD_XY + D_YZ,$$
$$D_X(fY+Z) = (\rho(X)f)Y + fD_XY + D_XZ$$

for any function $f \in C^\infty(M)$ and $X, Y, Z \in \Gamma(E)$. Let D be a linear connection on $\pounds^\pi E$ and h be a horizontal endomorphism on $\pounds^\pi E$. Then, (D, h) is called a distinguished connection (or d-connection) on $\pounds^\pi E$, if

(i) D is reducible, i.e., $Dh = 0$,

(ii) D is almost complex, i.e., $DF = 0$,

where F is the almost complex structure associated by h. It is known that a d-connection has the following coordinate expression:

$$D_{\delta_\alpha}\mathcal{V}_\beta = F^\gamma_{\alpha\beta}\mathcal{V}_\gamma, \quad D_{\mathcal{V}_\alpha}\mathcal{V}_\beta = C^\gamma_{\alpha\beta}\mathcal{V}_\gamma, \tag{2.18}$$
$$D_{\delta_\alpha}\delta_\beta = F^\gamma_{\alpha\beta}\delta_\gamma, \quad D_{\mathcal{V}_\alpha}\delta_\beta = C^\gamma_{\alpha\beta}\delta_\gamma. \tag{2.19}$$

Let (D, h) be a d-connection. Then

$$D^h_{\widetilde{X}}\widetilde{Y} := D_{h\widetilde{X}}\widetilde{Y}, \quad D^v_{\widetilde{X}}\widetilde{Y} := D_{v\widetilde{X}}\widetilde{Y}$$

are called h-covariant derivative and v-covariant derivative, respectively. Moreover,

$$h^*(DC)(\widetilde{X}) := D_{h\widetilde{X}}C, \quad v^*(DC)(\widetilde{X}) := D_{v\widetilde{X}}C \tag{2.20}$$

are called h-deflection and v-deflection of (D, h), respectively, where $\widetilde{X}, \widetilde{Y} \in \Gamma(\pounds^\pi E)$. It is easy to see that $h^*(DC)$ has the following coordinate expression:

$$h^*(DC) = (\mathcal{B}^\gamma_\alpha + \mathbf{y}^\beta F^\gamma_{\alpha\beta})\mathcal{V}_\gamma \otimes \mathcal{X}^\alpha. \tag{2.21}$$

Similarly, we can see that $v^*(DC)$ has the following coordinate expression:

$$v^*(DC) = (\delta^\gamma_\alpha + \mathbf{y}^\beta C^\gamma_{\alpha\beta})\mathcal{V}_\gamma \otimes \delta\mathcal{V}^\alpha, \tag{2.22}$$

where δ^γ_α is the Kronecker symbol.

Theorem 2.3 *Let (D, h) be a d-connection on $\pounds^\pi E$. Then, the torsion tensor field T of D is determined by the following, completely*:

$$A(\widetilde{X}, \widetilde{Y}) := hT(h\widetilde{X}, h\widetilde{Y}) = D_{h\widetilde{X}}h\widetilde{Y} - D_{h\widetilde{Y}}h\widetilde{X} - h[h\widetilde{X}, h\widetilde{Y}]_\pounds, \tag{2.23}$$
$$B(\widetilde{X}, \widetilde{Y}) := hT(h\widetilde{X}, J\widetilde{Y}) = -D_{J\widetilde{Y}}h\widetilde{X} - h[h\widetilde{X}, J\widetilde{Y}]_\pounds, \tag{2.24}$$
$$R^1(\widetilde{X}, \widetilde{Y}) := vT(h\widetilde{X}, h\widetilde{Y}) = -v[h\widetilde{X}, h\widetilde{Y}]_\pounds, \tag{2.25}$$
$$P^1(\widetilde{X}, \widetilde{Y}) := vT(h\widetilde{X}, J\widetilde{Y}) = D_{h\widetilde{X}}J\widetilde{Y} - v[h\widetilde{X}, J\widetilde{Y}]_\pounds, \tag{2.26}$$
$$S^1(\widetilde{X}, \widetilde{Y}) := vT(J\widetilde{X}, J\widetilde{Y}) = D_{J\widetilde{X}}J\widetilde{Y} - D_{J\widetilde{Y}}J\widetilde{X} - v[J\widetilde{X}, J\widetilde{Y}]_\pounds, \tag{2.27}$$

where A, B, R^1, P^1 and R^1 are called h-horizontal, h-mixed, v-horizontal, v-mixed and v-vertical torsion, respectively.

It is easy to check that the components of the torsion tensor field have the following coordinate expressions:

$$\begin{cases} A = T^\gamma_{\alpha\beta}\delta_\gamma \otimes \mathcal{X}^\alpha \otimes \mathcal{X}^\beta, \quad B = -C^\gamma_{\alpha\beta}\delta_\gamma \otimes \mathcal{X}^\alpha \otimes \mathcal{X}^\beta, \\ R^1 = -R^\gamma_{\alpha\beta}\mathcal{V}_\gamma \otimes \mathcal{X}^\alpha \otimes \mathcal{X}^\beta, \quad P^1 = P^\gamma_{\alpha\beta}\mathcal{V}_\gamma \otimes \mathcal{X}^\alpha \otimes \mathcal{X}^\beta, \\ Q^1 = S^\gamma_{\alpha\beta}\mathcal{V}_\gamma \otimes \mathcal{X}^\alpha \otimes \mathcal{X}^\beta, \end{cases} \tag{2.28}$$

where

$$\text{(i) } T^{\gamma}_{\alpha\beta} = F^{\gamma}_{\alpha\beta} - F^{\gamma}_{\beta\alpha} - (L^{\gamma}_{\alpha\beta} \circ \pi), \quad \text{(ii) } P^{\gamma}_{\alpha\beta} = F^{\gamma}_{\alpha\beta} + \frac{\partial \mathcal{B}^{\gamma}_{\alpha}}{\partial \mathbf{y}^{\beta}}, \quad \text{(iii) } S^{\gamma}_{\alpha\beta} = C^{\gamma}_{\alpha\beta} - C^{\gamma}_{\beta\alpha}. \tag{2.29}$$

Theorem 2.4 *Let (D, h) be a d-connection on $\pounds^{\pi} E$. The curvature tensor field K of D is determined by the following, completely*:

$$\begin{aligned} \text{(i) } R(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(h\widetilde{X}, h\widetilde{Y})J\widetilde{Z}, \\ \text{(ii) } P(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(h\widetilde{X}, J\widetilde{Y})J\widetilde{Z}, \\ \text{(iii) } Q(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= K(J\widetilde{X}, J\widetilde{Y})J\widetilde{Z}. \end{aligned}$$

Here, R, P and Q are called horizontal, mixed and vertical curvature, respectively.

By a direct calculation, we can see that the horizontal, mixed and vertical curvature, have the following coordinate expressions:

$$\begin{aligned} R &= R_{\alpha\beta\gamma}{}^{\lambda}\, \mathcal{V}_{\lambda} \otimes \mathcal{X}^{\alpha} \otimes \mathcal{X}^{\beta} \otimes \mathcal{X}^{\gamma}, \\ P &= P_{\alpha\beta\gamma}{}^{\lambda}\, \mathcal{V}_{\lambda} \otimes \mathcal{X}^{\alpha} \otimes \mathcal{X}^{\beta} \otimes \mathcal{X}^{\gamma}, \\ Q &= S_{\alpha\beta\gamma}{}^{\lambda}\, \mathcal{V}_{\lambda} \otimes \mathcal{X}^{\alpha} \otimes \mathcal{X}^{\beta} \otimes \mathcal{X}^{\gamma}, \end{aligned}$$

where

$$\begin{aligned} R_{\alpha\beta\gamma}{}^{\lambda} = {} & (\rho^{i}_{\alpha} \circ \pi)\frac{\partial F^{\lambda}_{\beta\gamma}}{\partial \mathbf{x}^{i}} + \mathcal{B}^{\mu}_{\alpha}\frac{\partial F^{\lambda}_{\beta\gamma}}{\partial \mathbf{y}^{\mu}} - (\rho^{i}_{\beta} \circ \pi)\frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial \mathbf{x}^{i}} - \mathcal{B}^{\mu}_{\beta}\frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial \mathbf{y}^{\mu}} + F^{\mu}_{\beta\gamma}F^{\lambda}_{\alpha\mu} \\ & - F^{\mu}_{\alpha\gamma}F^{\lambda}_{\beta\mu} - (L^{\mu}_{\alpha\beta} \circ \pi)F^{\lambda}_{\mu\gamma} - R_{\alpha\beta}{}^{\mu}C^{\lambda}_{\mu\gamma}, \end{aligned} \tag{2.30}$$

$$P_{\alpha\beta\gamma}{}^{\lambda} = (\rho^{i}_{\alpha} \circ \pi)\frac{\partial C^{\lambda}_{\beta\gamma}}{\partial \mathbf{x}^{i}} + \mathcal{B}^{\mu}_{\alpha}\frac{\partial C^{\lambda}_{\beta\gamma}}{\partial \mathbf{y}^{\mu}} + C^{\mu}_{\beta\gamma}F^{\lambda}_{\alpha\mu} - \frac{\partial F^{\lambda}_{\alpha\gamma}}{\partial \mathbf{y}^{\beta}} - F^{\mu}_{\alpha\gamma}C^{\lambda}_{\beta\mu} + \frac{\partial \mathcal{B}^{\mu}_{\alpha}}{\partial \mathbf{y}^{\beta}}C^{\lambda}_{\mu\gamma}, \tag{2.31}$$

$$S_{\alpha\beta\gamma}{}^{\lambda} = \frac{\partial C^{\lambda}_{\beta\gamma}}{\partial \mathbf{y}^{\alpha}} + C^{\mu}_{\beta\gamma}C^{\lambda}_{\alpha\mu} - \frac{\partial C^{\lambda}_{\alpha\gamma}}{\partial \mathbf{y}^{\beta}} - C^{\mu}_{\alpha\gamma}C^{\lambda}_{\beta\mu}. \tag{2.32}$$

3 Finsler Algebroids

In this section, we introduce Finsler algebroids as a generalization of Finsler manifolds and we present some basic objects such as conservative endomorphism (in particular, Barthel endomorphism) and Cartan tensor on these spaces.

Definition 3.1 *Finsler algebroid $(E, \mathcal{F})$ is a Lie algebroid $\pounds^{\pi} E$ provided with a fundamental Finsler function $\mathcal{F} : E \to \mathbb{R}$ satisfying the conditions*:

(i) *$\mathcal{F}$ is a scalar differentiable function on the manifold $\overset{\circ}{E} = E - \{0\}$ and continuous on the null section of $\pi : E \to M$,*

(ii) *$\mathcal{F}$ is a positive function and homogeneous of degree 2, i.e., $\pounds^{\pounds}_{C}\mathcal{F} = 2\mathcal{F}$,*

(iii) *the fundamental form $\omega = d^{\pounds} d^{\pounds}_{J}\mathcal{F}$ is nondegenerate, where*

$$d^{\pounds}_{J}\mathcal{F} = i_{J}d^{\pounds}\mathcal{F} = d^{\pounds}\mathcal{F} \circ J.$$

For the basis $\{\mathcal{X}_{\alpha}, \mathcal{V}_{\alpha}\}$ of $\Gamma(\pounds^{\pi} E)$ and the dual basis $\{\mathcal{X}^{\alpha}, \mathcal{V}^{\alpha}\}$ of it, we get $d^{\pounds}_{J}\mathcal{F}(\mathcal{V}_{\alpha}) = 0$ and $d^{\pounds}_{J}\mathcal{F}(\mathcal{X}_{\alpha}) = \frac{\partial \mathcal{F}}{\partial \mathbf{y}^{\alpha}}$. Therefore, $d^{\pounds}_{J}\mathcal{F}$ has the following coordinate expression:

$$d^{\pounds}_{J}\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \mathbf{y}^{\alpha}}\mathcal{X}^{\alpha}. \tag{3.1}$$

Lemma 3.1 *The fundamental form ω of a Finsler algebroid has the following coordinate expression*:

$$\omega=\Big((\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\boldsymbol{x}^i\partial\boldsymbol{y}^\beta}-\frac{1}{2}\frac{\partial\mathcal{F}}{\partial\boldsymbol{y}^\gamma}(L_{\alpha\beta}^\gamma\circ\pi)\Big)\mathcal{X}^\alpha\wedge\mathcal{X}^\beta-\frac{\partial^2\mathcal{F}}{\partial\boldsymbol{y}^\alpha\partial\boldsymbol{y}^\beta}\mathcal{X}^\alpha\wedge\mathcal{V}^\beta. \tag{3.2}$$

Proof Using (3.1), we have

$$\omega=d^\pounds d_J^\pounds\mathcal{F}=d^\pounds\Big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\Big)\wedge\mathcal{X}^\gamma+\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}d^\pounds\mathcal{X}^\gamma. \tag{3.3}$$

It is easy to see that $(d^\pounds\mathcal{X}^\gamma)(\mathcal{X}_\alpha,\mathcal{X}_\beta)=-(L_{\alpha\beta}^\gamma\circ\pi)$ and $(d^\pounds\mathcal{X}^\gamma)(\mathcal{X}_\alpha,\mathcal{V}_\beta)=(d^\pounds\mathcal{X}^\gamma)(\mathcal{V}_\alpha,\mathcal{V}_\beta)=0$. Thus, we have

$$d^\pounds\mathcal{X}^\gamma=-\frac{1}{2}(L_{\alpha\beta}^\gamma\circ\pi)\mathcal{X}^\alpha\wedge\mathcal{X}^\beta.$$

Also, it is easy to check that $\big(d^\pounds\big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\big)\big)(\mathcal{X}_\beta)=\rho_\beta^i\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma}$ and $\big(d^\pounds\big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\big)\big)(\mathcal{V}_\beta)=\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}$. Hence, we have

$$d^\pounds\Big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\Big)=(\rho_\beta^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma}\mathcal{X}^\beta+\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}\mathcal{V}^\beta.$$

Setting the above two equations in (3.3) impies (3.2).

From (3.2), we deduce that the fundamental form ω is nondegenarate if and only if the symmetric matrix $\big(\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}\big)$ is regular.

Proposition 3.1 *For the fundamental form ω we have the following identities*:

$$\text{(i) } i_J\omega=0,\quad \text{(ii) } \pounds_C^\pounds\omega=\omega,\quad \text{(iii) } i_C\omega=d_J^\pounds\mathcal{F}.$$

Proof We have

$$i_J\omega=i_{\mathcal{X}^\gamma\otimes\mathcal{V}_\gamma}\omega=\mathcal{X}^\gamma\wedge i_{\mathcal{V}_\gamma}\omega.$$

It is easy to check that $i_{\mathcal{V}_\gamma}\mathcal{X}^\alpha=0$ and $i_{\mathcal{V}_\gamma}\mathcal{V}^\alpha=\delta_\gamma^\alpha$. Therefore, from (3.2), we get $i_{\mathcal{V}_\gamma}\omega=\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\gamma}\mathcal{X}^\alpha$ and consequently

$$i_J\omega=\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\gamma}\mathcal{X}^\gamma\wedge\mathcal{X}^\alpha.$$

It is easy to see that $\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\gamma}\mathcal{X}^\gamma\wedge\mathcal{X}^\alpha=-\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\gamma}\mathcal{X}^\gamma\wedge\mathcal{X}^\alpha$. Thus, we deduce $i_J\omega=0$. Now we prove (ii). Since $[C,\mathcal{X}_\alpha]=0$, using (3.2) we derive that

$$\begin{aligned}(\pounds_C^\pounds\omega)(\mathcal{X}_\alpha,\mathcal{X}_\beta)&=\rho_\pounds(C)\Big((\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}-(\rho_\beta^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha}-\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(L_{\alpha\beta}^\gamma\circ\pi)\Big)\\&=\mathbf{y}^\lambda\Big((\rho_\alpha^i\circ\pi)\frac{\partial^3\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta\partial\mathbf{y}^\lambda}-(\rho_\beta^i\circ\pi)\frac{\partial^3\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha\partial\mathbf{y}^\lambda}\\&\quad-\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\lambda}(L_{\alpha\beta}^\gamma\circ\pi)\Big).\end{aligned} \tag{3.4}$$

Since $\mathcal{F}$ is homogeneous of degree 2, we can obtain

$$\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}=\mathbf{y}^\lambda\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\lambda}. \tag{3.5}$$

Using the above equation in (3.4), we get

$$(\pounds_C^{\pounds}\omega)(\mathcal{X}_\alpha,\mathcal{X}_\beta)=\Big(\rho^i_\alpha\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}-\rho^i_\beta\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha}-\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}L^\gamma_{\alpha\beta}\Big)=\omega(\mathcal{X}_\alpha,\mathcal{X}_\beta).$$

Similarly, we can obtain

$$(\pounds_C^{\pounds}\omega)(\mathcal{X}_\alpha,\mathcal{V}_\beta)=-\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}=\omega(\mathcal{X}_\alpha,\mathcal{V}_\beta),\quad(\pounds_C^{\pounds}\omega)(\mathcal{V}_\alpha,\mathcal{V}_\beta)=0=\omega(\mathcal{V}_\alpha,\mathcal{V}_\beta).$$

Hence, we have (ii). It is easy to check that $i_C\mathcal{X}^\gamma=0$ and $i_C\mathcal{V}^\gamma=\mathbf{y}^\gamma$. Using (3.2) and (3.5), we get

$$i_C\omega=\mathbf{y}^\gamma\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\gamma}\mathcal{X}^\alpha=\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\alpha}\mathcal{X}^\alpha=d_J^{\pounds}\mathcal{F}.$$

Definition 3.2 *Let $(E,\mathcal{F})$ be a Finsler algebroid with fundamental form ω. The map*

$$\mathcal{G}:\Gamma(v\pounds^\pi\overset{\circ}{E})\times\Gamma(v\pounds^\pi\overset{\circ}{E})\to C^\infty(\pounds^\pi\overset{\circ}{E})$$

defined by $\mathcal{G}(J\widetilde{X},J\widetilde{Y}):=\omega(J\widetilde{X},\widetilde{Y})$ is called the vertical metric of Finsle algebroid $(E,\mathcal{F})$.

It is easy to check that $\mathcal{G}$ is bilinear, symmetric and nondegenerate on $v\pounds^\pi\overset{\circ}{E}$. So, we can deduce the following.

Proposition 3.2 *Let h be a horizontal endomorphism and $\mathcal{G}$ be the vertical metric of Finsler manifold $(E,\mathcal{F})$. The function $\widetilde{\mathcal{G}}:\Gamma(\pounds^\pi\overset{\circ}{E})\times\Gamma(\pounds^\pi\overset{\circ}{E})\to C^\infty(\pounds^\pi\overset{\circ}{E})$ given by*

$$\widetilde{\mathcal{G}}(\widetilde{X},\widetilde{Y}):=\mathcal{G}(J\widetilde{X},J\widetilde{Y})+\mathcal{G}(v\widetilde{X},v\widetilde{Y}),\quad\forall\,\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi\overset{\circ}{E})\tag{3.6}$$

is a pseudo-Riemannian metric on $\pounds^\pi\overset{\circ}{E}$.

The pseudo-Riemannian metric $\widetilde{\mathcal{G}}$ introduced in the above proposition, is called the prolongation of $\mathcal{G}$ along h.

Using (3.2), we obtain the following coordinate expression:

$$\mathcal{G}_{\alpha\beta}:=\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=\omega(\mathcal{V}_\alpha,\mathcal{X}_\beta)=\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}.\tag{3.7}$$

Also, using (3.6) we can obtain

$$\widetilde{\mathcal{G}}(\delta_\alpha,\delta_\beta)=\mathcal{G}_{\alpha\beta},\quad\widetilde{\mathcal{G}}(\delta_\alpha,\mathcal{V}_\beta)=0,\quad\widetilde{\mathcal{G}}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=\mathcal{G}_{\alpha\beta}$$

and consequently

$$\widetilde{\mathcal{G}}=\mathcal{G}_{\alpha\beta}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta+\mathcal{G}_{\alpha\beta}\delta\mathcal{V}^\alpha\otimes\delta\mathcal{V}^\beta.\tag{3.8}$$

Proposition 3.3 *For the metrics $\mathcal{G}$, $\widetilde{\mathcal{G}}$ and sections X, Y of $\overset{\circ}{E}$, we have*

$$\widetilde{\mathcal{G}}(X^{\mathrm{V}},Y^{\mathrm{V}})=\mathcal{G}(X^{\mathrm{V}},Y^{\mathrm{V}})=\rho_\pounds(X^{\mathrm{V}})(\rho_\pounds(Y^{\mathrm{V}})\mathcal{F}),\tag{3.9}$$

$$\widetilde{\mathcal{G}}(C,C)=\mathcal{G}(C,C)=2\mathcal{F}.\tag{3.10}$$

Proof Using (3.7), we get

$$\mathcal{G}(C,C)=\mathbf{y}^\alpha\mathbf{y}^\beta\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}.$$

Since $\mathcal{F}$ is homogeneous of degree 2, we can obtain $\mathbf{y}^\alpha\mathbf{y}^\beta\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}=2\mathcal{F}$. Thus, we deduce $\mathcal{G}(C,C)=2\mathcal{F}$. Using (iii) of (2.3) and (3.6), we get

$$\widetilde{\mathcal{G}}(C,C)=\mathcal{G}(JC,JC)+\mathcal{G}(vC,vC)=\mathcal{G}(C,C)=2\mathcal{F}.$$

Now, let $X=X^\alpha e_\alpha$ and $Y=Y^\beta e_\beta$ be sections of $\overset{\circ}{E}$. Then, we have

$$\begin{aligned}\mathcal{G}(X^{\mathrm{V}},Y^{\mathrm{V}})&=\mathcal{G}((X^\alpha\circ\pi)\mathcal{V}_\alpha,(Y^\beta\circ\pi)\mathcal{V}_\beta)=(X^\alpha\circ\pi)(Y^\beta\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta}\\&=\rho_\pounds(X^{\mathrm{V}})(\rho_\pounds(Y^{\mathrm{V}})\mathcal{F}).\end{aligned}$$

Using (3.6) and the above equation, we can obtain

$$\widetilde{\mathcal{G}}(X^{\mathrm{V}},Y^{\mathrm{V}})=\rho_\pounds(X^{\mathrm{V}})(\rho_\pounds(Y^{\mathrm{V}})\mathcal{F}).$$

Let h be a horizontal endomorphism on $\pounds^\pi E$ and $\widetilde{\mathcal{G}}$ be a pseudo-Riemannian metric given by (3.6). We consider

$$\mathcal{K}_h(\widetilde{X},\widetilde{Y})=\widetilde{\mathcal{G}}(\widetilde{X},J\widetilde{Y})-\widetilde{\mathcal{G}}(J\widetilde{X},\widetilde{Y}),\quad\forall\,\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi E)$$

and we call it the Kähler form with respect to $\widetilde{\mathcal{G}}$.

Proposition 3.4 *We have* $\mathcal{K}_h=i_v\omega$.

Proof Let $\widetilde{X},\widetilde{Y}\in\Gamma(\pounds^\pi E)$. Then, we have

$$\begin{aligned}(i_v\omega)(\widetilde{X},\widetilde{Y})&=\omega(v\widetilde{X},\widetilde{Y})+\omega(\widetilde{X},v\widetilde{Y})=\omega(v\widetilde{X},\widetilde{Y})-\omega(v\widetilde{Y},\widetilde{X})\\&=\widetilde{\mathcal{G}}(\widetilde{X},J\widetilde{Y})-\widetilde{\mathcal{G}}(J\widetilde{X},\widetilde{Y})\\&=\mathcal{K}_h(\widetilde{X},\widetilde{Y}).\end{aligned}$$

Using (3.8), the Kähler form $\mathcal{K}_h$ has the following coordinate expression with respect to $\{\delta_\alpha,\mathcal{V}_\alpha\}$:

$$\mathcal{K}_h=\mathcal{G}_{\alpha\beta}\delta\mathcal{V}^\alpha\wedge\mathcal{X}^\beta.$$

Definition 3.3 *Let $(E,\mathcal{F})$ be a Finsler algebroid with fundamental form ω. If $\phi:E\to\mathbb{R}$ is a smooth function, then the section* $\operatorname{grad}\phi\in\Gamma(\pounds^\pi E)$ *characterized by*

$$d^\pounds\phi=i_{\operatorname{grad}\phi}\omega\tag{3.11}$$

is called the gradient of ϕ.

In the above definition, the nondegeneracy of ω guarantees the existence and unicity of the gradient section.

If β is a nonzero 1-form on $\pounds^\pi E$, we denote by $\beta^\sharp$ the section corresponding to ω, i.e., $i_{\beta^\sharp}\omega=\beta$. Also, we can introduce the gradient of ϕ by $\operatorname{grad}\phi=(d^\pounds\phi)^\sharp$. Since $\operatorname{grad}\phi\in\Gamma(\pounds^\pi E)$, we can write it as follows

$$\operatorname{grad}\phi=(\operatorname{grad}\phi)^\alpha\mathcal{X}_\alpha+(\operatorname{grad}\phi)^{\overline{\alpha}}\mathcal{V}_\alpha.\tag{3.12}$$

Using (3.2) and (3.11), we get

$$\frac{\partial \phi}{\partial \mathbf{y}^\beta} = (d^\pounds \phi)(\mathcal{V}_\beta) = (i_{\mathrm{grad}\,\phi}\omega)(\mathcal{V}_\beta) = -(\mathrm{grad}\,\phi)^\alpha \frac{\partial^2 \mathcal{F}}{\partial \mathbf{y}^\alpha \partial \mathbf{y}^\beta} = -(\mathrm{grad}\,\phi)^\alpha \mathcal{G}_{\alpha\beta}$$

which yields

$$(\mathrm{grad}\,\phi)^\alpha = -\mathcal{G}^{\alpha\beta} \frac{\partial \phi}{\partial \mathbf{y}^\beta}, \tag{3.13}$$

where $(\mathcal{G}^{\alpha\beta})$ is the inverse matric of $(\mathcal{G}_{\alpha\beta})$. Similarly, using (3.2), (3.11) and the above equation we obtain

$$\begin{aligned}(\rho^i_\beta \circ \pi)\frac{\partial \phi}{\partial \mathbf{x}^i} &= (d^\pounds \phi)(\mathcal{X}_\beta) = (i_{\mathrm{grad}\,\phi}\omega)(\mathcal{X}_\beta) \\ &= -\mathcal{G}^{\alpha\gamma}\frac{\partial \phi}{\partial \mathbf{y}^\gamma}\Big((\rho^i_\alpha \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\beta} - (\rho^i_\beta \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\alpha} - \frac{\partial \mathcal{F}}{\partial \mathbf{y}^\gamma}(L^\gamma_{\alpha\beta} \circ \pi)\Big) + (\mathrm{grad}\,\phi)^{\overline{\alpha}}\mathcal{G}_{\alpha\beta},\end{aligned}$$

which gives us

$$\begin{aligned}(\mathrm{grad}\,\phi)^{\overline{\alpha}} = \mathcal{G}^{\alpha\beta}\Big\{(\rho^i_\beta \circ \pi)\frac{\partial \phi}{\partial \mathbf{x}^i} + \mathcal{G}^{\lambda\gamma}\frac{\partial \phi}{\partial \mathbf{y}^\gamma}\Big((\rho^i_\lambda \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\beta} - (\rho^i_\beta \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\lambda} \\ - \frac{\partial \mathcal{F}}{\partial \mathbf{y}^\gamma}(L^\gamma_{\lambda\beta} \circ \pi)\Big)\Big\}.\end{aligned} \tag{3.14}$$

Plugging (3.13) and (3.14) into (3.12) implies the following local expression for gradient:

$$\begin{aligned}\mathrm{grad}\,\phi = -\mathcal{G}^{\alpha\beta}\frac{\partial \phi}{\partial \mathbf{y}^\beta}\mathcal{X}_\alpha + \mathcal{G}^{\alpha\beta}\Big\{(\rho^i_\beta \circ \pi)\frac{\partial \phi}{\partial \mathbf{x}^i} + \mathcal{G}^{\lambda\gamma}\frac{\partial \phi}{\partial \mathbf{y}^\gamma}\Big((\rho^i_\lambda \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\beta} \\ - (\rho^i_\beta \circ \pi)\frac{\partial^2 \mathcal{F}}{\partial \mathbf{x}^i \partial \mathbf{y}^\lambda} - \frac{\partial \mathcal{F}}{\partial \mathbf{y}^\gamma}(L^\gamma_{\lambda\beta} \circ \pi)\Big)\Big\}\mathcal{V}_\alpha.\end{aligned} \tag{3.15}$$

Proposition 3.5 *Let $(E, \mathcal{F})$ be a Finsler algebroid and $f \in C^\infty(M)$. We have*

(i) $\mathrm{grad}\,f^\vee \in \Gamma(v\pounds^\pi E)$, (ii) $[C, \mathrm{grad}\,f^\vee]_\pounds = -\mathrm{grad}\,f^\vee$, (iii) $\rho_\pounds(\mathrm{grad}\,f^\vee)(\mathcal{F}) = f^{\mathrm{c}}$.

Proof Since $f^\vee = f \circ \pi$ is a function with respect to $(\mathbf{x}^i)$, we have $\frac{\partial f^\vee}{\partial \mathbf{y}^\beta} = 0$. From (3.15), we deduce that $\mathrm{grad}\,f^\vee$ has the following coordinate expression:

$$\mathrm{grad}\,f^\vee = \mathcal{G}^{\alpha\beta}(\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\mathcal{V}_\alpha. \tag{3.16}$$

Thus we have (i). The above equation and (2.2) give us

$$[C, \mathrm{grad}\,f^\vee]_\pounds = \Big(\mathbf{y}^\alpha \frac{\partial \mathcal{G}^{\beta\gamma}}{\partial \mathbf{y}^\alpha}(\rho^i_\gamma \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i} - \mathcal{G}^{\beta\gamma}(\rho^i_\gamma \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\Big)\mathcal{V}_\beta.$$

But using (3.5), we can deduce $\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\alpha} = 0$ and consequently $\frac{\partial \mathcal{G}^{\beta\gamma}}{\partial \mathbf{y}^\alpha} = 0$. Setting this equation in the above equation implies

$$[C, \mathrm{grad}\,f^\vee]_\pounds = -\mathcal{G}^{\beta\gamma}(\rho^i_\gamma \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\mathcal{V}_\beta = -\mathrm{grad}\,f^\vee.$$

Therefore, we have (ii). To prove (iii), we use (3.5) and (3.16) as follows

$$\begin{aligned}\rho_\pounds(\mathrm{grad}\,f^\vee)(\mathcal{F}) &= \mathcal{G}^{\alpha\beta}(\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\rho_\pounds(\mathcal{V}_\alpha)(\mathcal{F}) = \mathcal{G}^{\alpha\beta}(\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\frac{\partial \mathcal{F}}{\partial \mathbf{y}^\alpha} \\ &= \mathcal{G}^{\alpha\beta}(\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i}\mathbf{y}^\lambda \mathcal{G}_{\alpha\lambda} = \mathbf{y}^\beta(\rho^i_\beta \circ \pi)\frac{\partial (f \circ \pi)}{\partial \mathbf{x}^i} = f^{\mathrm{c}}.\end{aligned}$$

3.1 Conservative endomorphism on Finsler algebroids

Definition 3.4 *A horizontal endomorphism h on Finsler algebroid $(E,\mathcal{F})$ is called conservative if $d_h^{\pounds}\mathcal{F}=0$.*

Using (2.7), it is easy to check that h is conservative if and only if

$$(\rho_\alpha^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\mathcal{B}_\alpha^\beta\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0. \tag{3.17}$$

Proposition 3.6 *Let h be a conservative horizontal endomorphism on Finsler algebroid $(E,\mathcal{F})$. We have $d_H^{\pounds}\mathcal{F}=0$, where H is the tension of h.*

Proof Using (2.8), we can obtain $d_H^{\pounds}\mathcal{F}(\mathcal{V}_\alpha)=0$ and

$$d_H^{\pounds}\mathcal{F}(\mathcal{X}_\alpha)=\Big(\mathcal{B}_\alpha^\beta-\mathbf{y}^\gamma\frac{\partial\mathcal{B}_\alpha^\beta}{\partial\mathbf{y}^\gamma}\Big)\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}. \tag{3.18}$$

Since h is conservative, differentiating (3.17) with respect to $\mathbf{y}^\gamma$ we obtain

$$(\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma}+\frac{\partial\mathcal{B}_\alpha^\beta}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}+\mathcal{B}_\alpha^\beta\frac{\partial^2 F}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}=0. \tag{3.19}$$

Contracting the above equation by $\mathbf{y}^\gamma$ and using homogeneity of $\mathcal{F}$ we get

$$(\rho_\alpha^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\mathbf{y}^\gamma\frac{\partial\mathcal{B}_\alpha^\beta}{\partial\mathbf{y}^\gamma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0. \tag{3.20}$$

Setting the above equation in (3.18) and using (3.17) we deduce $d_H^{\pounds}\mathcal{F}(\mathcal{X}_\alpha)=0$. Therefore $d_H^{\pounds}\mathcal{F}=0$.

Lemma 3.2 *If ω is the fundamental two-form of Finsler algebroid $(E,\mathcal{F})$ and h is a conservative horizontal endomorphism on $\pounds^\pi E$, then*

$$i_h\omega=\omega+i_t d^{\pounds}\mathcal{F}.$$

Proof Since h is conservative, we have (3.19). Then, using (3.2) and (3.19) we get

$$\begin{aligned}(i_h\omega)(\mathcal{X}_\alpha,\mathcal{X}_\beta)=(\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}-(\rho_\beta^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha}-2\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(L_{\alpha\beta}^\gamma\circ\pi)\\ -\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\beta}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}+\frac{\partial\mathcal{B}_\beta^\lambda}{\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}.\end{aligned}$$

Also, (2.9) and (2.11) give us

$$(i_t d^{\pounds}\mathcal{F})(\mathcal{X}_\alpha,\mathcal{X}_\beta)=\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\Big(\frac{\partial\mathcal{B}_\beta^\gamma}{\partial\mathbf{y}^\alpha}-\frac{\partial\mathcal{B}_\alpha^\gamma}{\partial\mathbf{y}^\beta}-(L_{\alpha\beta}^\gamma\circ\pi)\Big).$$

Two above equations yield

$$\begin{aligned}(i_h\omega-i_t d^{\pounds}\mathcal{F})(\mathcal{X}_\alpha,\mathcal{X}_\beta)&=(\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}-(\rho_\beta^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha}-\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(L_{\alpha\beta}^\gamma\circ\pi)\\ &=\omega(\mathcal{X}_\alpha,\mathcal{X}_\beta).\end{aligned}$$

Similarly we get

$$(i_h\omega - i_t d^{\pounds}\mathcal{F})(\mathcal{X}_\alpha, \mathcal{V}_\beta) = (i_h\omega)(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \omega(h\mathcal{X}_\alpha, \mathcal{V}_\beta) = \omega(\mathcal{X}_\alpha, \mathcal{V}_\beta),$$

and

$$(i_h\omega - i_t d^{\pounds}\mathcal{F})(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0 = \omega(\mathcal{V}_\alpha, \mathcal{V}_\beta).$$

Corollary 3.1 *If ω is the fundamental two-form of Finsler algebroid $(E, \mathcal{F})$ and h is a torsion free conservative horizontal endomorphism on $\pounds^\pi E$, then*

$$i_h\omega = \omega.$$

On any Finsler algebroid there is a spray $S_\circ : E \to \pounds^\pi E$, which is uniquely determined on $\overset{\circ}{\pounds^\pi} E$ by the formula

$$i_{S_\circ}\omega = -d^{\pounds}\mathcal{F}. \tag{3.21}$$

This spray is called the canonical spray of the Finsler algebroid. Using (3.8) and the above equation, the canonical spray $S_\circ$ has the coordinate expression $S_\circ = y^\alpha\mathcal{X}_\alpha + S_\circ^\alpha\mathcal{V}_\alpha$, where

$$S_\circ^\alpha = \mathcal{G}^{\alpha\beta}\Big((\rho_\beta^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i} + \mathbf{y}^\gamma\Big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L_{\gamma\beta}^\lambda\circ\pi) - (\rho_\gamma^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}\Big)\Big) \tag{3.22}$$

and $(\mathcal{G}^{\alpha\beta})$ is the inverse matrix of $(\mathcal{G}_{\alpha\beta})$.

Proposition 3.7 *Let $S_\circ$ be the canonical spray and h be a conservative horizontal endomorphism on Finsler algebroid $(E, \mathcal{F})$ with the associated semispray S. We have*

$$S - S_\circ = (d^{\pounds}_{i_S t}\mathcal{F})^\sharp,$$

where $i_{(d^{\pounds}_{i_S t}\mathcal{F})^\sharp}\omega = d^{\pounds}_{i_S t}\mathcal{F}$.

Proof Let $h = (\mathcal{X}_\alpha + \mathcal{B}_\alpha^\beta\mathcal{V}_\beta)\otimes\mathcal{X}^\alpha$, $S = \mathbf{y}^\alpha\mathcal{X}_\alpha + S^\alpha\mathcal{V}_\alpha$ and $S_\circ = \mathbf{y}^\alpha\mathcal{X}_\alpha + S_\circ^\alpha\mathcal{V}_\alpha$, where $S_\circ^\alpha$ are given by (3.22). Since $(i_{\mathcal{V}_\alpha}\omega)(\mathcal{X}_\beta) = \frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta} = \mathcal{G}_{\alpha\beta}$ and $(i_{\mathcal{V}_\alpha}\omega)(\mathcal{V}_\beta) = 0$, we have $i_{\mathcal{V}_\alpha}\omega = \mathcal{G}_{\alpha\beta}\mathcal{X}^\beta$. Therefore, using (3.22) we get

$$\begin{aligned} i_{S-S_\circ}\omega = (S - S_\circ)i_{\mathcal{V}_\alpha}\omega = \Big(&S^\alpha\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta} - (\rho_\beta^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i} - \mathbf{y}^\gamma\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L_{\gamma\beta}^\lambda\circ\pi)\\ &+ (\rho_\gamma^i\circ\pi)\mathbf{y}^\gamma\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}\Big)\mathcal{X}^\beta. \end{aligned}$$

From $S = hS_\circ$, we deduce $S^\alpha = \mathbf{y}^\gamma\mathcal{B}_\gamma^\alpha$. Setting this in the above equation gives us

$$\begin{aligned} i_{S-S_\circ}\omega = \Big(&\mathbf{y}^\gamma\mathcal{B}_\gamma^\alpha\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta} - (\rho_\beta^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i} - \mathbf{y}^\gamma\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L_{\gamma\beta}^\lambda\circ\pi)\\ &+ (\rho_\gamma^i\circ\pi)\mathbf{y}^\gamma\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\beta}\Big)\mathcal{X}^\beta. \end{aligned}$$

Since h is conservative, we have (3.17) and (3.19)–(3.20). Using these equations in the above equation and using (2.11) we get

$$\begin{aligned} i_{S-S_\circ}\omega &= \mathbf{y}^\alpha\Big(\frac{\partial\mathcal{B}_\beta^\gamma}{\partial\mathbf{y}^\alpha} - \frac{\partial\mathcal{B}_\alpha^\gamma}{\partial\mathbf{y}^\beta} - (L_{\alpha\beta}^\gamma\circ\pi)\Big)\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\mathcal{X}^\beta = \mathbf{y}^\alpha t_{\alpha\beta}^\gamma\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}\mathcal{X}^\beta\\ &= \frac{1}{2}t_{\alpha\beta}^\gamma\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}(\mathbf{y}^\alpha\mathcal{X}^\beta - \mathbf{y}^\beta\mathcal{X}^\alpha) = \frac{1}{2}t_{\alpha\beta}^\gamma(\mathbf{y}^\alpha\mathcal{X}^\beta - \mathbf{y}^\beta\mathcal{X}^\alpha)\rho_{\pounds}(\mathcal{V}_\gamma)(\mathcal{F})\\ &= \frac{1}{2}t_{\alpha\beta}^\gamma(\mathbf{y}^\alpha\mathcal{X}^\beta - \mathbf{y}^\beta\mathcal{X}^\alpha)i_{\mathcal{V}_\gamma}d^{\pounds}\mathcal{F} = i_{i_S t}d^{\pounds}\mathcal{F} = d^{\pounds}_{i_S t}\mathcal{F} = i_{(d^{\pounds}_{i_S t}\mathcal{F})^\sharp}\omega. \end{aligned}$$

3.2 Barthel endomorphism on Finsler algebroids

Let $S_\circ$ be the canonical spray on Finsler algebroid $(E,\mathcal{F})$. We consider

$$h_\circ=\frac{1}{2}(1_{\Gamma(\pounds^\pi E)}+[J,S_\circ]_\pounds^{F-N}).$$

In the coordinate expression, we can obtain

$$h_\circ=\Big(\mathcal{X}_\alpha+\frac{1}{2}\Big(\frac{\partial S_\circ^\beta}{\partial\mathbf{y}^\alpha}-\mathbf{y}^\gamma(L_{\alpha\gamma}^\beta\circ\pi)\Big)\mathcal{V}_\beta\Big)\otimes\mathcal{X}^\alpha. \tag{3.23}$$

From the above equation, we deduce $h_\circ^2=h_\circ$ and $\ker h_\circ=v\pounds^\pi E$. Thus, $h_\circ$ is a horizontal endomorphism on $\pounds^\pi E$ which is called Barthel endomorphism. Since S_0 is a spray on $(E,\mathcal{F})$, we can deduce that the Barthel endomorphism is homogeneous.

Proposition 3.8 *Let h be a conservative and homogeneous horizontal endomorphism and $h_\circ$ be the Barthel endomorphism on Finsle algebroid $(E,\mathcal{F})$. We have*

$$h=h_\circ+\frac{1}{2}i_St+\frac{1}{2}[J,(d_{i_St}^\pounds\mathcal{F})^\sharp]_\pounds.$$

Proof Let S be the semispray associated to h and h' be the horizontal endomorphism generated by S. Using Theorem 2.2 we get

$$\begin{aligned}h_\circ&=\frac{1}{2}(1_{\Gamma(\pounds^\pi E)}+[J,S_\circ]_\pounds)=\frac{1}{2}(1_{\Gamma(\pounds^\pi E)}+[J,S]_\pounds-[J,(d_{i_St}^\pounds\mathcal{F})^\sharp]_\pounds)\\&=h'-\frac{1}{2}[J,(d_{i_St}^\pounds\mathcal{F})^\sharp]_\pounds=h-\frac{1}{2}i_St-\frac{1}{2}[J,(d_{i_St}^\pounds\mathcal{F})^\sharp]_\pounds.\end{aligned}$$

Theorem 3.1 *Barthel endomorphism of Finsler algebroid $(E,\mathcal{F})$ is conservative.*

Proof Using (3.17), it is sufficient to show that

$$(\rho_\alpha^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\mathcal{B}_\alpha^\beta\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=0, \tag{3.24}$$

where $\mathcal{B}_\alpha^\beta=\frac{1}{2}\big(\frac{\partial S_\circ^\beta}{\partial\mathbf{y}^\alpha}-\mathbf{y}^\gamma(L_{\alpha\gamma}^\beta\circ\pi)\big)$ and $S_\circ^\beta$ are given by (3.22). Using (3.5) we deduce

$$\text{(i)}\ \frac{\partial\mathcal{F}}{\partial\mathbf{y}^\gamma}=\mathbf{y}^\lambda\mathcal{G}_{\gamma\lambda},\quad\text{(ii)}\ \mathbf{y}^\mu\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\lambda\partial\mathbf{y}^\mu}=0. \tag{3.25}$$

From (i) of (3.25) we derive that

$$\mathcal{B}_\alpha^\beta\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\beta}=\frac{1}{2}\Big(\frac{\partial S_\circ^\beta}{\partial\mathbf{y}^\alpha}-\mathbf{y}^\gamma(L_{\alpha\gamma}^\beta\circ\pi)\Big)\mathbf{y}^\mu\mathcal{G}_{\mu\beta}. \tag{3.26}$$

Using (3.22) we obtain

$$\begin{aligned}\frac{\partial S_\circ^\beta}{\partial\mathbf{y}^\alpha}\mathbf{y}^\mu\mathcal{G}_{\mu\beta}=\mathbf{y}^\mu\mathcal{G}_{\mu\beta}\Big(\frac{\partial\mathcal{G}^{\beta\sigma}}{\partial\mathbf{y}^\alpha}\Big)\Big((\rho_\sigma^i\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}+\mathbf{y}^\gamma\Big(\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L_{\gamma\sigma}^\lambda\circ\pi)\\-(\rho_\gamma^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\sigma}\Big)\Big)+\mathbf{y}^\sigma\Big((\rho_\sigma^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\alpha}\\+\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}(L_{\alpha\sigma}^\lambda\circ\pi)-(\rho_\alpha^i\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\sigma}+\mathbf{y}^\gamma\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\lambda}(L_{\gamma\sigma}^\lambda\circ\pi)\end{aligned}$$

$$-\mathbf{y}^{\gamma}(\rho^i_{\gamma}\circ\pi)\frac{\partial^3\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^{\alpha}\partial\mathbf{y}^{\sigma}}\Big). \tag{3.27}$$

But (ii) of (3.25) implies

$$\mathbf{y}^{\mu}\mathcal{G}_{\mu\beta}\frac{\partial\mathcal{G}^{\beta\sigma}}{\partial\mathbf{y}^{\alpha}}=-\mathbf{y}^{\mu}\mathcal{G}^{\beta\sigma}\frac{\partial\mathcal{G}_{\mu\beta}}{\partial\mathbf{y}^{\alpha}}=-\mathbf{y}^{\mu}\mathcal{G}^{\beta\sigma}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^{\alpha}\partial\mathbf{y}^{\mu}\partial\mathbf{y}^{\beta}}=0.$$

Moreover, we have $\mathbf{y}^{\gamma}\mathbf{y}^{\sigma}(L^{\lambda}_{\gamma\sigma}\circ\pi)=0$, $\mathbf{y}^{\sigma}\frac{\partial^3\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^{\alpha}\partial\mathbf{y}^{\sigma}}=\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^{\alpha}}$ and $\mathbf{y}^{\sigma}\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^{\sigma}}=2\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i}$, because $\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^{\alpha}}$ and $\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}$ are homogeneous of degree 1 and 2, respectively. Therefore, (3.27) reduces to

$$\frac{\partial S^{\beta}_{\circ}}{\partial\mathbf{y}^{\alpha}}\mathbf{y}^{\mu}\mathcal{G}_{\mu\beta}=\mathbf{y}^{\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^{\lambda}}(L^{\lambda}_{\alpha\sigma}\circ\pi)-2(\rho^i_{\alpha}\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}.$$

Setting the above equation in (3.26), we deduce $\mathcal{B}^{\beta}_{\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^{\beta}}=-(\rho^i_{\alpha}\circ\pi)\frac{\partial\mathcal{F}}{\partial\mathbf{x}^i}$ from which we have (3.24).

Theorem 3.2 *Let h_1 and h_2 be conservative horizontal endomorphisms on Finsler algebroid $(E,\mathcal{F})$. If h_1 and h_2 have common strong torsions, then $h_1=h_2$.*

Proof We denote by S_1 and S_2 the associated semisprays of h_1 and h_2, respectively and let T_1 and T_2 be the strong torsions of h_1 and h_2, respectively. From hypothesis we have $d^{\pounds}_{h_1}\mathcal{F}=d^{\pounds}_{h_2}\mathcal{F}=0$ and $T_1=T_2$. Also, from the last equation in the proof of Proposition 3.7, we deduce $i_{S_1-S_0}\omega=d^{\pounds}_{i_{S_1}t_1}\mathcal{F}$, $i_{S_2-S_0}\omega=d^{\pounds}_{i_{S_2}t_2}\mathcal{F}$ and consequently

$$i_{S_1-S_2}\omega=d^{\pounds}_{i_{S_1}t_1}\mathcal{F}-d^{\pounds}_{i_{S_2}t_2}\mathcal{F}, \tag{3.28}$$

where t_1 and t_2 are weak torsions of h_1 and h_2, respectively. From the definition of strong torsion we have

$$d^{\pounds}_{i_{S_1}t_1}\mathcal{F}=d^{\pounds}_{T_1-H_1}\mathcal{F}=d^{\pounds}_{T_1}\mathcal{F},$$

because $d^{\pounds}_{H_1}\mathcal{F}=0$, where H_1 is the tension of h_1. Similarly we obtain $d^{\pounds}_{i_{S_1}t_1}\mathcal{F}=d^{\pounds}_{T_2}\mathcal{F}$. Setting this equation together with the above equation in (3.28), we deduce $i_{S_1-S_2}\omega=d^{\pounds}_{T_1}\mathcal{F}-d^{\pounds}_{T_2}\mathcal{F}=0$. Since ω is nondegenerate, this equation gives us $S_1=S_2$ and consequently using Theorem 2.1, we deduce $h_1=h_2$.

From the above results, we understand that Barthel endomorphism is homogeneous, conservative and torsion free. Moreover, since Barthel endomorphism is homogeneous and torsion free, we deduce that its strong torsion is zero. Also, from the above theorem we derive that if h is a homogeneous, conservative and torsion free horizontal endomorphism, then it is coincide with Barthel endomorphism. Hence, we have the following theorem.

Theorem 3.3 *There exists a unique horizontal endomorphism on Finsler algebroid $(E,\mathcal{F})$ such that it is homogeneous, conservative and torsion free.*

3.3 Cartan tensor on Finsler algebroids

Here, we consider the tensor

$$\begin{cases}\mathcal{C}:\Gamma(\pounds^{\pi}\overset{\circ}{E})\times\Gamma(\pounds^{\pi}\overset{\circ}{E})\to\Gamma(\pounds^{\pi}\overset{\circ}{E}),\\ \qquad\qquad(\widetilde{X},\widetilde{Y})\to\mathcal{C}(\widetilde{X},\widetilde{Y})\end{cases} \tag{3.29}$$

on Finsler algebroid $(E,\mathcal{F})$ which satisfies

$$J\circ\mathcal{C}=0, \tag{3.30}$$

and

$$\mathcal{G}(\mathcal{C}(\widetilde{X},\widetilde{Y}),J\widetilde{Z})=\frac{1}{2}(\pounds^{\pounds}_{J\widetilde{X}}J^{*}\mathcal{G})(\widetilde{Y},\widetilde{Z}), \tag{3.31}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(\overset{\circ}{\pounds^{\pi}}E)$, and we call it the first Cartan tensor. Also, the lowered tensor $\mathcal{C}_{\flat}$ of $\mathcal{C}$ is defined by

$$\mathcal{C}_{\flat}(\widetilde{X},\widetilde{Y},\widetilde{Z})=\mathcal{G}(\mathcal{C}(\widetilde{X},\widetilde{Y}),J\widetilde{Z}),\quad\forall\,\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(\overset{\circ}{\pounds^{\pi}}E). \tag{3.32}$$

(3.30) tells us that $\mathcal{C}(\widetilde{X},\widetilde{Y})$ belongs to $\Gamma(v\overset{\circ}{\pounds^{\pi}}E)$. Also, from (3.31) we deduce that $\mathcal{C}(\mathcal{X}_\alpha,\mathcal{V}_\beta)=\mathcal{C}(\mathcal{V}_\alpha,\mathcal{V}_\beta)=0$ and

$$\mathcal{C}(\mathcal{X}_\alpha,\mathcal{X}_\beta)=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\lambda}\mathcal{V}_\lambda=\frac{1}{2}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}\mathcal{G}^{\gamma\lambda}\mathcal{V}_\lambda.$$

Therefore, the first Cartan tensor has the following coordinate expression:

$$\mathcal{C}=\mathcal{C}^{\gamma}_{\alpha\beta}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta\otimes\mathcal{V}_\gamma, \tag{3.33}$$

where

$$\mathcal{C}^{\gamma}_{\alpha\beta}=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\lambda}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\lambda}=\frac{1}{2}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\lambda}\mathcal{G}^{\gamma\lambda}.$$

From (3.33) and the above equation, we can deduce the following proposition.

Proposition 3.9 *The first Cartan tensor is semibasic. Moreover, it and its the lowered tensor are symmetric tensors.*

Using (3.32)–(3.33), we can obtain the following coordinate expression for the lowered tensor:

$$\mathcal{C}_{\flat}=\mathcal{C}_{\alpha\beta_\gamma}\mathcal{X}^\alpha\otimes\mathcal{X}^\beta\otimes\mathcal{X}^\gamma,$$

where

$$\mathcal{C}_{\alpha\beta\gamma}=\mathcal{C}^{\lambda}_{\alpha\beta}\mathcal{G}_{\gamma\lambda}=\frac{1}{2}\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}.$$

Proposition 3.10 *If S is a semispray on $\pounds^{\pi}E$, we have $i_S\mathcal{C}=i_S\mathcal{C}_{\flat}=0$.*

Proof Let $\widetilde{Y}=\widetilde{Y}^\beta\mathcal{X}_\beta+\widetilde{Y}^{\overline{\beta}}\mathcal{V}_\beta$ and $\widetilde{Z}=\widetilde{Z}^\gamma\mathcal{X}_\gamma+\widetilde{Z}^{\overline{\gamma}}\mathcal{V}_\gamma$ be sections of $\overset{\circ}{\pounds^{\pi}}E$. Using (3.25), we have

$$(i_S\mathcal{C}_{\flat})(\widetilde{Y},\widetilde{Z})=\mathcal{C}_{\flat}(S,\widetilde{Y},\widetilde{Z})=\frac{1}{2}\mathbf{y}^\alpha\widetilde{Y}^\beta\widetilde{Z}^\gamma\frac{\partial^3\mathcal{F}}{\partial\mathbf{y}^\alpha\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}=0.$$

Similarly, we can prove $i_S\mathcal{C}=0$.

Now we consider a horizontal endomorphism h on $\pounds^{\pi}E$ and the prolongation $\widetilde{\mathcal{G}}$ of the vertical metric $\mathcal{G}$ along h. The second Cartan tensor

$$\begin{cases}\widetilde{\mathcal{C}}:\Gamma(\overset{\circ}{\pounds^{\pi}}E)\times\Gamma(\overset{\circ}{\pounds^{\pi}}E)\to\Gamma(\overset{\circ}{\pounds^{\pi}}E),\\ \qquad\qquad(\widetilde{X},\widetilde{Y})\to\widetilde{\mathcal{C}}(\widetilde{X},\widetilde{Y})\end{cases} \tag{3.34}$$

(belonging to h) is defined by the rules

$$J \circ \widetilde{\mathcal{C}} = 0, \tag{3.35}$$

$$\widetilde{\mathcal{G}}(\widetilde{\mathcal{C}}(\widetilde{X}, \widetilde{Y}), J\widetilde{Z}) = \frac{1}{2}(\pounds_{h\widetilde{X}}\widetilde{\mathcal{G}})(J\widetilde{Y}, J\widetilde{Z}), \tag{3.36}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\pounds^{\pi}\overset{\circ}{E})$. Also, the lowered tensor $\widetilde{\mathcal{C}}_\flat$ of $\widetilde{\mathcal{C}}$ is defined by

$$\widetilde{\mathcal{C}}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = \widetilde{\mathcal{G}}(\widetilde{\mathcal{C}}(\widetilde{X}, \widetilde{Y}), J\widetilde{Z}), \quad \forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(\pounds^{\pi}\overset{\circ}{E}). \tag{3.37}$$

In a way similar to the first Cartan tensor, using (3.35)–(3.36) we can deduce that the second Cartan tensor has the following coordinate expression:

$$\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_{\alpha\beta}^{\gamma}\mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{3.38}$$

where

$$\widetilde{\mathcal{C}}_{\alpha\beta}^{\gamma} = \frac{1}{2}\Big((\rho_\alpha^i \circ \pi)\frac{\partial \mathcal{G}_{\beta\mu}}{\partial \mathbf{x}^i}\mathcal{G}^{\gamma\mu} + \mathcal{B}_\alpha^\lambda \frac{\partial \mathcal{G}_{\beta\mu}}{\partial \mathbf{y}^\lambda}\mathcal{G}^{\gamma\mu} + \frac{\partial \mathcal{B}_\alpha^\gamma}{\partial \mathbf{y}^\beta} + \frac{\partial \mathcal{B}_\alpha^\lambda}{\partial \mathbf{y}^\mu}\mathcal{G}^{\gamma\mu}\mathcal{G}_{\beta\lambda}\Big). \tag{3.39}$$

From (3.38), it is easy to see that the second Cartan tensor is semibasic. Moreover, (3.37)–(3.38) give us

$$\widetilde{\mathcal{C}}_\flat = \widetilde{\mathcal{C}}_{\alpha\beta\gamma}\mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{X}^\gamma, \tag{3.40}$$

where

$$\widetilde{\mathcal{C}}_{\alpha\beta\gamma} = \widetilde{\mathcal{C}}_{\alpha\beta}^{\lambda}\mathcal{G}_{\lambda\gamma} = \frac{1}{2}\Big((\rho_\alpha^i \circ \pi)\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{x}^i} + \mathcal{B}_\alpha^\lambda \frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\lambda} + \frac{\partial \mathcal{B}_\alpha^\lambda}{\partial \mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma} + \frac{\partial \mathcal{B}_\alpha^\lambda}{\partial \mathbf{y}^\gamma}\mathcal{G}_{\beta\lambda}\Big). \tag{3.41}$$

Proposition 3.11 *Let $(E, \mathcal{F})$ be a Finsler algebroid. We have*

$$2\mathcal{C}_\flat(X^{\mathrm{C}}, Y^{\mathrm{C}}, Z^{\mathrm{C}}) = \rho_\pounds(X^{\mathrm{V}})(\rho_\pounds(Y^{\mathrm{V}})(\rho_\pounds(Z^{\mathrm{V}})\mathcal{F})), \tag{3.42}$$

$$2\widetilde{\mathcal{C}}_\flat(X^{\mathrm{C}}, Y^{\mathrm{C}}, Z^{\mathrm{C}}) = [Y^{\mathrm{V}}, [X^h, Z^{\mathrm{V}}]_\pounds]_\pounds + \rho_\pounds(Y^{\mathrm{V}})(\rho_\pounds(Z^{\mathrm{V}})(\rho_\pounds(X^h)\mathcal{F})). \tag{3.43}$$

Proof Let X, Y and Z be sections of E. Using the second part of (2.1) we get

$$\begin{aligned}
2\mathcal{C}_\flat(X^{\mathrm{C}}, Y^{\mathrm{C}}, Z^{\mathrm{C}}) &= 2(X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\mathcal{C}_\flat(\mathcal{X}_\alpha, \mathcal{X}_\beta, \mathcal{X}_\gamma) \\
&= (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\frac{\partial^3 \mathcal{F}}{\partial \mathbf{y}^\alpha \partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma} \\
&= (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\rho_\pounds(\mathcal{V}_\alpha)(\rho_\pounds(\mathcal{V}_\beta)(\rho_\pounds(\mathcal{V}_\gamma)\mathcal{F})) \\
&= \rho_\pounds(X^{\mathrm{V}})(\rho_\pounds(Y^{\mathrm{V}})(\rho_\pounds(Z^{\mathrm{V}})\mathcal{F})).
\end{aligned}$$

Now we prove (3.43). Direct calculation gives us

$$\begin{aligned}
&[Y^{\mathrm{V}}, [X^h, Z^{\mathrm{V}}]_\pounds]_\pounds + \rho_\pounds(Y^{\mathrm{V}})(\rho_\pounds(Z^{\mathrm{V}})(\rho_\pounds(X^h)\mathcal{F})) \\
&= (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\Big((\rho_\alpha^i \circ \pi)\frac{\partial^3 \mathcal{F}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma \partial \mathbf{x}^i} + \frac{\partial \mathcal{B}_\alpha^\lambda}{\partial \mathbf{y}^\gamma}\frac{\partial^2 \mathcal{F}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\lambda} \\
&\quad + \frac{\partial \mathcal{B}_\alpha^\lambda}{\partial \mathbf{y}^\beta}\frac{\partial^2 \mathcal{F}}{\partial \mathbf{y}^\gamma \partial \mathbf{y}^\lambda} + \mathcal{B}_\alpha^\lambda \frac{\partial^3 \mathcal{F}}{\partial \mathbf{y}^\beta \partial \mathbf{y}^\gamma \partial \mathbf{y}^\lambda}\Big).
\end{aligned}$$

But, using (3.41) we can see that the above equation is equal to $2\widetilde{\mathcal{C}}_\flat(X^{\mathrm{C}}, Y^{\mathrm{C}}, Z^{\mathrm{C}})$. Thus, we have (3.43).

Proposition 3.12 *Let $(E,\mathcal{F})$ be a Finsler algebroid. If h is a torsion free and conservative horizontal endomorphism on $\pounds^{\pi}E$, the lowered second Cartan tensor is symmetric.*

Proof (3.41) implies that $\widetilde{C}_{\alpha\beta\gamma}$ is symmetric with respect to last two variables. Then, it is sufficient to prove that $\widetilde{C}_{\alpha\beta\gamma}$ is symmetric with respect to first two variables. Since h is conservative, using (3.24) and (i) of (3.25) in (3.41) we obtain

$$\widetilde{C}_{\alpha\beta\gamma}=-\frac{1}{2}\mathbf{y}^{\mu}\frac{\partial^2\mathcal{B}^{\lambda}_{\alpha}}{\partial\mathbf{y}^{\beta}\partial\mathbf{y}^{\gamma}}\mathcal{G}_{\lambda\mu}.$$

Since h is torsion free, using (2.11) we have $\frac{\partial^2\mathcal{B}^{\lambda}_{\alpha}}{\partial\mathbf{y}^{\beta}\partial\mathbf{y}^{\gamma}}=\frac{\partial^2\mathcal{B}^{\lambda}_{\beta}}{\partial\mathbf{y}^{\alpha}\partial\mathbf{y}^{\gamma}}$. Setting this equation in the above equation implies $\widetilde{C}_{\alpha\beta\gamma}=\widetilde{C}_{\beta\alpha\gamma}$.

3.4 Distinguished connections on Finsler algebroids

In this section, we study the existence and uniqueness of the distinguished connections Berwald, Cartan, Chern-Rund and Hashiguchi and we present some properties of them.

Theorem 3.4 *Let $(E,\mathcal{F})$ be a Finsler algebroid and h be a conservative horizontal endomorphism on $\pounds^{\pi}E$. There exists a unique d-connection $\overset{\text{BF}}{D}$ on $(E,\mathcal{F})$ such that the v-mixed and h-mixed torsions of $\overset{\text{BF}}{D}$ are zero.*

Proof There exists a d-connection $\overset{\text{BF}}{D}$ on $(E,\mathcal{F})$ such that the v-mixed and h-mixed torsions of it are zero. If we denote by $\overset{\text{BF}}{P^1}$ the v-mixed torsion of $\overset{\text{BF}}{D}$, then we have

$$\begin{aligned}0=&\overset{\text{BF}}{P^1}(\widetilde{X},F\widetilde{Y})=v\overset{\text{BF}}{T}(h\widetilde{X},v\widetilde{Y})=v(\overset{\text{BF}}{D}_{h\widetilde{X}}v\widetilde{Y}-\overset{\text{BF}}{D}_{v\widetilde{Y}}h\widetilde{X}-[h\widetilde{X},v\widetilde{Y}]_{\pounds})\\=&\overset{\text{BF}}{D}_{h\widetilde{X}}v\widetilde{Y}-v[h\widetilde{X},v\widetilde{Y}]_{\pounds},\end{aligned}$$

where $\overset{\text{BF}}{T}$ is the torsion of $\overset{\text{BF}}{D}$. The above equation gives us

$$\overset{\text{BF}}{D}_{h\widetilde{X}}v\widetilde{Y}=v[h\widetilde{X},v\widetilde{Y}]_{\pounds}. \tag{3.44}$$

Since the h-mixed torsion of $\overset{\text{BF}}{D}$ is zero, we have

$$\begin{aligned}0=&\overset{\text{BF}}{B}(\widetilde{Y},F\widetilde{X})=h\overset{\text{BF}}{T}(h\widetilde{Y},v\widetilde{X})=h(\overset{\text{BF}}{D}_{h\widetilde{Y}}v\widetilde{X}-\overset{\text{BF}}{D}_{v\widetilde{X}}h\widetilde{Y}-[h\widetilde{Y},v\widetilde{X}]_{\pounds})\\=&-\overset{\text{BF}}{D}_{v\widetilde{X}}h\widetilde{Y}-h[h\widetilde{Y},v\widetilde{X}]_{\pounds}=-\overset{\text{BF}}{D}_{v\widetilde{X}}h\widetilde{Y}-h[\widetilde{Y},v\widetilde{X}]_{\pounds},\end{aligned}$$

where $\overset{\text{BF}}{B}$ is the h-mixed torsion of $\overset{\text{BF}}{D}$. The above equation gives us

$$\overset{\text{BF}}{D}_{v\widetilde{X}}h\widetilde{Y}=h[v\widetilde{X},\widetilde{Y}]_{\pounds}. \tag{3.45}$$

Since $\overset{\text{BF}}{D}$ is d-connection, using (3.44), (iv) of (2.6) and (i), (iv) of (2.15) we get

$$\begin{aligned}\overset{\text{BF}}{D}_{h\widetilde{X}}h\widetilde{Y}&=F\overset{\text{BF}}{D}_{h\widetilde{X}}J\widetilde{Y}=F\overset{\text{BF}}{D}_{h\widetilde{X}}vJ\widetilde{Y}=Fv[h\widetilde{X},vJ\widetilde{Y}]_{\pounds}\\&=hF[h\widetilde{X},J\widetilde{Y}]_{\pounds}.\end{aligned} \tag{3.46}$$

Since $\overset{\text{BF}}{D}$ is d-connection, (iii)–(iv) of (2.15), (ii), (iv) of (2.6) and (3.45) give us

$$\overset{\text{BF}}{D}_{v\widetilde{X}}v\widetilde{Y}=\overset{\text{BF}}{D}_{v\widetilde{X}}v(v\widetilde{Y})=\overset{\text{BF}}{D}_{v\widetilde{X}}J(Fv\widetilde{Y})=J\overset{\text{BF}}{D}_{v\widetilde{X}}hF\widetilde{Y}=Jh[v\widetilde{X},F\widetilde{Y}]_{\pounds}$$

$$= J[v\widetilde{X}, F\widetilde{Y}]_{\pounds}. \tag{3.47}$$

Relations (3.44)–(3.47) prove the existence and uniqueness of $\overset{\text{BF}}{D}$.

Using (3.44)–(3.47), the d-connection $\overset{\text{BF}}{D}$ has the following coordinate expression:

$$\begin{cases} \overset{\text{BF}}{D}_{\delta_\alpha}\delta_\beta = -\dfrac{\partial \mathcal{B}^\gamma_\alpha}{\partial y^\beta}\delta_\gamma, & \overset{\text{BF}}{D}_{v_\alpha}\mathcal{V}_\beta = 0, \\ \overset{\text{BF}}{D}_{\delta_\alpha}\mathcal{V}_\beta = -\dfrac{\partial \mathcal{B}^\gamma_\alpha}{\partial y^\beta}\mathcal{V}_\gamma, & \overset{\text{BF}}{D}_{\mathcal{V}_\alpha}\delta_\beta = 0. \end{cases} \tag{3.48}$$

Proposition 3.13 *Let $(E, \mathcal{F})$ be a Finsler algebroid, h be a conservative horizontal endomorphism on $\pounds^\pi E$ and $\overset{\text{BF}}{D}$ be the d-connection given by* (3.48)*. If h-deflection and h-horizontal torsions of $\overset{\text{BF}}{D}$ are zero, h is the Barthel endomorphism.*

Proof It is sufficient to show that h is homogeneous and torsion free. Since h-deflection of $(\overset{\text{BF}}{D}, h)$ is zero, we have

$$0 = h^*(\overset{\text{BF}}{D}C)(\delta_\alpha) = \overset{\text{BF}}{D}_{h\delta_\alpha}(C) = \overset{\text{BF}}{D}_{h\delta_\alpha}(\mathbf{y}^\beta\mathcal{V}_\beta) = \Big(\mathcal{B}^\beta_\alpha - \mathbf{y}^\lambda\frac{\partial \mathcal{B}^\beta_\alpha}{\partial \mathbf{y}^\lambda}\Big)\mathcal{V}_\beta.$$

The above equation shows that h is homogeneous. Also, since the h-horizontal torsion of $\overset{\text{BF}}{D}$ is zero, we get

$$\begin{aligned} 0 = hT(\delta_\alpha, \delta_\beta) &= h(\overset{\text{BF}}{D}_{\delta_\alpha}\delta_\beta - \overset{\text{BF}}{D}_{\delta_\beta}\delta_\alpha - [\delta_\alpha, \delta_\beta]_{\pounds}) \\ &= h\Big(\Big(\frac{\partial \mathcal{B}^\gamma_\beta}{\partial \mathbf{y}^\alpha} - \frac{\partial \mathcal{B}^\gamma_\alpha}{\partial \mathbf{y}^\beta} - (L^\gamma_{\alpha\beta}\circ\pi)\Big)\delta_\gamma - R^\gamma_{\alpha\beta}\mathcal{V}\gamma\Big) \\ &= t^\gamma_{\alpha\beta}\delta_\gamma. \end{aligned}$$

From the above equation, we deduce that the weak torsion of h is zero.

If h is the Barthel endomorphism of Finsler algebroid $(E, \mathcal{F})$, the d-connection $\overset{\text{BF}}{D}$ given in (3.48) is called the Berwald connection of $(E, \mathcal{F})$.

Theorem 3.5 *Let $(E, \mathcal{F})$ be a Finsler algebroid, h be a torsion free and conservative horizontal endomorphism on $\pounds^\pi E$ and $\widetilde{\mathcal{G}}$ be the prolongation of $\mathcal{G}$ along h. There exists a unique d-connection $\overset{\text{C}}{D}$ on $(E, \mathcal{F})$ such that $\overset{\text{C}}{D}$ is metrical, i.e., $\overset{\text{C}}{D}\,\widetilde{\mathcal{G}} = 0$ and the v-vertical and h-horizontal torsions of $\overset{\text{C}}{D}$ are zero.*

Proof There exists a d-connection $\overset{\text{C}}{D}$ such that $\overset{\text{C}}{D}$ is metrical and the v-vertical and h-horizontal torsions of $\overset{\text{C}}{D}$ are zero. Since $\overset{\text{C}}{D}$ is metrical, we have

$$\rho_{\pounds}(\delta_\alpha)\widetilde{\mathcal{G}}(\delta_\beta, \delta_\gamma) = \widetilde{\mathcal{G}}(\overset{\text{C}}{D}_{\delta_\alpha}\delta_\beta, \delta_\gamma) + \widetilde{\mathcal{G}}(\delta_\beta, \overset{\text{C}}{D}_{\delta_\alpha}\delta_\gamma), \tag{3.49}$$

$$\rho_{\pounds}(\delta_\beta)\widetilde{\mathcal{G}}(\delta_\gamma, \delta_\alpha) = \widetilde{\mathcal{G}}(\overset{\text{C}}{D}_{\delta_\beta}\delta_\gamma, \delta_\alpha) + \mathcal{G}(\delta_\gamma, \overset{\text{C}}{D}_{\delta_\beta}\delta_\alpha), \tag{3.50}$$

$$-\rho_{\pounds}(\delta_\gamma)\widetilde{\mathcal{G}}(\delta_\alpha, \delta_\beta) = -\widetilde{\mathcal{G}}(\overset{\text{C}}{D}_{\delta_\gamma}\delta_\alpha, \delta_\beta) - \widetilde{\mathcal{G}}(\delta_\alpha, \overset{\text{C}}{D}_{\delta_\gamma}\delta_\beta). \tag{3.51}$$

Since the h-horizontal torsion of $\overset{\text{C}}{D}$ is zero, we have

$$\overset{\text{C}}{D}_{\delta_\alpha}\delta_\beta - \overset{\text{C}}{D}_{\delta_\beta}\delta_\alpha = [\delta_\alpha, \delta_\beta]_{\pounds} = (L^\gamma_{\alpha\beta}\circ\pi)\delta_\gamma + R^\gamma_{\alpha\beta}\mathcal{V}_\gamma.$$

Summing (3.49)–(3.51) and using the above equation give us

$$\begin{aligned}\widetilde{\mathcal{G}}(\overset{\mathrm{C}}{D}_{\delta_\alpha}\delta_\beta,\delta_\gamma)=\frac{1}{2}\Big(&(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\alpha\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}+(\rho^i_\beta\circ\pi)\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\beta\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{y}^\lambda}\\&-(\rho^i_\gamma\circ\pi)\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{x}^i}-\mathcal{B}^\lambda_\gamma\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{y}^\lambda}-(L^\lambda_{\beta\alpha}\circ\pi)\mathcal{G}_{\lambda\gamma}-(L^\lambda_{\alpha\gamma}\circ\pi)\mathcal{G}_{\lambda\beta}\\&-(L^\lambda_{\beta\gamma}\circ\pi)\mathcal{G}_{\alpha\lambda}\Big).\end{aligned}$$

Since h is torsion free, using (2.11) in the above equation we get

$$\begin{aligned}\overset{\mathrm{C}}{D}_{\delta_\alpha}\delta_\beta=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big(&(\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\alpha\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}+(\rho^i_\beta\circ\pi)\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\beta\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{y}^\lambda}\\&-(\rho^i_\gamma\circ\pi)\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{x}^i}-\mathcal{B}^\lambda_\gamma\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\gamma}-\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\beta}\\&+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}-\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\beta}\mathcal{G}_{\alpha\lambda}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\alpha\lambda}\Big)\delta_\mu.\end{aligned}\tag{3.52}$$

Since h is conservative, we have (3.19). Differentiation of this equation with respect to y gives us

$$(\rho^i_\beta\circ\pi)\frac{\partial\mathcal{G}_{\gamma\alpha}}{\partial\mathbf{x}^i}+\frac{\partial^2\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\alpha}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\gamma}+\mathcal{B}^\lambda_\beta\frac{\partial\mathcal{G}_{\gamma\alpha}}{\partial\mathbf{y}^\lambda}=0,\tag{3.53}$$

$$(\rho^i_\gamma\circ\pi)\frac{\partial\mathcal{G}_{\beta\alpha}}{\partial\mathbf{x}^i}+\frac{\partial^2\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\lambda}+\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\alpha}+\frac{\partial\mathcal{B}^\lambda_\gamma}{\partial\mathbf{y}^\alpha}\mathcal{G}_{\lambda\beta}+\mathcal{B}^\lambda_\gamma\frac{\partial\mathcal{G}_{\beta\alpha}}{\partial\mathbf{y}^\lambda}=0.\tag{3.54}$$

Setting two above equations in (3.52), we obtain

$$\overset{\mathrm{C}}{D}_{\delta_\alpha}\delta_\beta=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\alpha\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}\Big)\delta_\mu.\tag{3.55}$$

Since the v-horizontal torsion of $\overset{\mathrm{C}}{D}$ is zero, we have

$$\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta-\overset{\mathrm{C}}{D}_{\mathcal{V}_\beta}\mathcal{V}_\alpha=[\mathcal{V}_\alpha,\mathcal{V}_\beta]_{\pounds}=0.$$

If we replace δ_α, δ_β, δ_γ by $\mathcal{V}_\alpha$, $\mathcal{V}_\beta$, $\mathcal{V}_\gamma$ in (3.49)–(3.51), summing these equations and using the above equation we get

$$\widetilde{\mathcal{G}}(\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta,\mathcal{V}_\gamma)=\frac{1}{2}\Big(\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}+\frac{\partial\mathcal{G}_{\alpha\gamma}}{\partial\mathbf{y}^\beta}-\frac{\partial\mathcal{G}_{\alpha\beta}}{\partial\mathbf{y}^\gamma}\Big)=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha},$$

which gives us

$$\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\mathcal{V}_\mu.\tag{3.56}$$

Since $\overset{\mathrm{C}}{D}$ is d-connection, using the above equation we obtain

$$\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\delta_\beta=\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}F\mathcal{V}_\beta=F\ \overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}F(\mathcal{V}_\mu),$$

which gives us

$$\overset{\mathrm{C}}{D}_{\mathcal{V}_\alpha}\delta_\beta=\frac{1}{2}\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\delta_\mu.\tag{3.57}$$

Similarly, using (3.55) we get

$$\begin{aligned}\overset{c}{D}_{\delta_\alpha}\mathcal{V}_\beta =&\overset{c}{D}_{\delta_\alpha}J\delta_\beta = J\ \overset{c}{D}_{\delta_\alpha}\delta_\beta\\ =&\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\alpha\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}\Big)J(\delta_\mu),\end{aligned}$$

which gives us

$$\overset{c}{D}_{\delta_\alpha}\mathcal{V}_\beta=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho^i_\alpha\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}^\lambda_\alpha\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}\Big)\mathcal{V}_\mu. \tag{3.58}$$

Relations (3.55)–(3.58) prove the existence and uniqueness of $\overset{c}{D}$.

Proposition 3.14 *Let $(E,\mathcal{F})$ be a Finsler algebroid, h be a torsion free and conservative horizontal endomorphism on $\pounds^\pi E$ and $\overset{c}{D}$ be the d-connection given by the above theorem. If h-deflection of $\overset{c}{D}$ is zero, h is the Barthel endomorphism.*

Proof It is sufficient to show that h is homogeneous. Since h-deflection of $(\overset{c}{D},h)$ is zero, using (3.25) and (3.58) we obtain

$$\begin{aligned}0=h^*(\overset{c}{D}C)(\delta_\alpha)=&\overset{c}{D}_{h\delta_\alpha}(C)=\overset{c}{D}_{\delta_\alpha}(\mathbf{y}^\beta\mathcal{V}_\beta)\\ =&\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho^i_\alpha\circ\pi)\frac{\partial^2\mathcal{F}}{\partial\mathbf{x}^i\partial\mathbf{y}^\gamma}-\mathbf{y}^\beta\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\mathbf{y}^\beta\mathcal{G}_{\lambda\beta}\Big)\mathcal{V}_\mu+\mathcal{B}^\mu_\alpha\mathcal{V}_\mu.\end{aligned}$$

Since h is conservative, we have (3.19). Using this equation in the above equation we deduce

$$0=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big(-\mathcal{B}^\beta_\alpha\frac{\partial^2\mathcal{F}}{\partial\mathbf{y}^\beta\partial\mathbf{y}^\gamma}-\mathbf{y}^\beta\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}\Big)\mathcal{V}_\mu+\mathcal{B}^\mu_\alpha\mathcal{V}_\mu=\frac{1}{2}\Big(\mathcal{B}^\mu_\alpha-\mathbf{y}^\beta\frac{\partial\mathcal{B}^\mu_\alpha}{\partial\mathbf{y}^\beta}\Big)\mathcal{V}_\mu.$$

The above equation shows that h is homogeneous.

If h is the Barthel endomorphism of Finsler algebroid $(E,\mathcal{F})$, the d-connection $\overset{c}{D}$ given by (3.55)–(3.58) is called the Cartan connection of $(E,\mathcal{F})$.

Using (2.30)–(2.32) and (3.55)–(3.58), we can obtain

$$\begin{aligned}\overset{c}{R}{}_{\alpha\beta\gamma}{}^{\lambda}=&-(\rho^i_\alpha\circ\pi)\frac{\partial}{\partial\mathbf{x}^i}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\beta}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma}\Big)\\ &-\mathcal{B}^\mu_\alpha\frac{\partial}{\partial\mathbf{y}^\mu}\Big(\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\gamma}+\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\beta}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}\Big)\\ &+(\rho^i_\beta\circ\pi)\frac{\partial}{\partial\mathbf{x}^i}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\alpha}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\Big)\\ &+\mathcal{B}^\mu_\beta\frac{\partial}{\partial\mathbf{y}^\mu}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\alpha}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\gamma}\Big)\\ &+\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\beta}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\mu\kappa}+\frac{\partial\mathcal{B}^\mu_\beta}{\partial\mathbf{y}^\gamma}\Big)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\iota_\alpha}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\iota}\mathcal{G}^{\lambda\sigma}+\frac{\partial\mathcal{B}^\lambda_\alpha}{\partial\mathbf{y}^\mu}\Big)\\ &-\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\alpha}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\mu\kappa}+\frac{\partial\mathcal{B}^\mu_\alpha}{\partial\mathbf{y}^\gamma}\Big)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\iota_\beta}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\iota}\mathcal{G}^{\lambda\sigma}+\frac{\partial\mathcal{B}^\lambda_\beta}{\partial\mathbf{y}^\mu}\Big)\\ &+(L^\mu_{\alpha\beta}\circ\pi)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}^\nu_\mu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}^\lambda_\mu}{\partial\mathbf{y}^\gamma}\Big)-\frac{1}{2}R^\mu_{\alpha\beta}\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\mu}\mathcal{G}^{\lambda\kappa},\end{aligned}$$

$$\overset{\mathrm{C}}{P}{}_{\alpha\beta\gamma}^{\ \ \lambda}=\frac{1}{2}(\rho_\alpha^i\circ\pi)\frac{\partial}{\partial\mathbf{x}^i}\Big(\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\Big)+\frac{1}{2}\mathcal{B}_\alpha^\mu\frac{\partial}{\partial\mathbf{y}^\mu}\Big(\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\Big)-\frac{1}{2}\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\beta}\mathcal{G}^{\mu\kappa}\Big(\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\mu}$$
$$+\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\sigma}\Big)+\frac{\partial}{\partial\mathbf{y}^\beta}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\Big)$$
$$+\frac{1}{2}\frac{\partial\mathcal{G}_{\mu\kappa}}{\partial\mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\Big(\frac{\partial\mathcal{B}_\alpha^\mu}{\partial\mathbf{y}^\gamma}+\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\iota}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\mu\iota}\Big)+\frac{1}{2}\frac{\partial\mathcal{B}_\alpha^\mu}{\partial\mathbf{y}^\beta}\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\mu}\mathcal{G}^{\lambda\kappa},$$
$$\overset{\mathrm{C}}{S}{}_{\alpha\beta\gamma}^{\ \ \lambda}=\frac{1}{2}\Big(\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\beta}\frac{\partial\mathcal{G}^{\lambda\kappa}}{\partial\mathbf{y}^\alpha}-\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{G}^{\lambda\kappa}}{\partial\mathbf{y}^\beta}\Big)+\frac{1}{4}\Big(\mathcal{G}^{\sigma\mu}\mathcal{G}^{\lambda\kappa}\frac{\partial\mathcal{G}_{\gamma\sigma}}{\partial\mathbf{y}^\beta}\frac{\partial\mathcal{G}_{\mu\kappa}}{\partial\mathbf{y}^\alpha}$$
$$-\mathcal{G}^{\mu\kappa}\mathcal{G}^{\lambda\sigma}\frac{\partial\mathcal{G}_{\gamma\kappa}}{\partial\mathbf{y}^\alpha}\frac{\partial\mathcal{G}_{\mu\sigma}}{\partial\mathbf{y}^\beta}\Big).$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be sections of $\overset{\circ}{\pounds^\pi} E$. Using (3.55)–(3.58) we can obtain the following formula for Cartan connection:

$$\overset{\mathrm{C}}{D}_{\widetilde{X}}\widetilde{Y}=\overset{\mathrm{C}}{D}_{v\widetilde{X}}v\widetilde{Y}+\overset{\mathrm{C}}{D}_{v\widetilde{X}}h\widetilde{Y}+\overset{\mathrm{C}}{D}_{h\widetilde{X}}v\widetilde{Y}+\overset{\mathrm{C}}{D}_{h\widetilde{X}}h\widetilde{Y},$$

where

$$\overset{\mathrm{C}}{D}_{h\widetilde{X}}h\widetilde{Y}=hF[h\widetilde{X},J\widetilde{Y}]_\pounds+F\widetilde{\mathcal{C}}(\widetilde{X},\widetilde{Y}),\tag{3.59}$$
$$\overset{\mathrm{C}}{D}_{v\widetilde{X}}v\widetilde{Y}=J[v\widetilde{X},F\widetilde{Y}]_\pounds+\mathcal{C}(F\widetilde{X},F\widetilde{Y}),\tag{3.60}$$
$$\overset{\mathrm{C}}{D}_{v\widetilde{X}}h\widetilde{Y}=h[v\widetilde{X},\widetilde{Y}]_\pounds+F\mathcal{C}(F\widetilde{X},\widetilde{Y}),\tag{3.61}$$
$$\overset{\mathrm{C}}{D}_{h\widetilde{X}}v\widetilde{Y}=v[h\widetilde{X},v\widetilde{Y}]_\pounds+\widetilde{\mathcal{C}}(\widetilde{X},F\widetilde{Y}).\tag{3.62}$$

Theorem 3.6 *Let $(E,\mathcal{F})$ be a Finsler algebroid, h be a torsion free and conservative horizontal endomorphism on $\pounds^\pi E$ and $\widetilde{\mathcal{G}}$ be the prolongation of $\mathcal{G}$ along h. Then, there exists a unique d-connection $\overset{\mathrm{CR}}{D}$ on $(E,\mathcal{F})$ such that $\overset{\mathrm{CR}}{D}$ is h-metrical, (i.e., $\forall\,\widetilde{X}\in\Gamma(\overset{\circ}{\pounds^\pi} E)$, $\overset{\mathrm{CR}}{D}_{h\widetilde{X}}\widetilde{\mathcal{G}}=0$), $J^*\overset{\mathrm{CR}}{D}=J^*\overset{\mathrm{BF}}{D}$ and the h-horizontal torsion of $\overset{\mathrm{CR}}{D}$ is zero. Moreover, if the h-deflection of $\overset{\mathrm{CR}}{D}$ is zero, h is the Barthel endomorphism.*

Proof There exists a d-connection $\overset{\mathrm{CR}}{D}$ on $(E,\mathcal{F})$ such that $\overset{\mathrm{CR}}{D}$ is h-metrical, $J^*\overset{\mathrm{CR}}{D}=J^*\overset{\mathrm{B}}{D}$ and the h-horizontal torsions of $\overset{\mathrm{CR}}{D}$ is zero. Since $\overset{\mathrm{CR}}{D}$ is h-metrical and h-horizontal torsion of $\overset{\mathrm{CR}}{D}$ is zero, similar to the proof of Theorem 3.5 we can deduce

$$\overset{\mathrm{CR}}{D}_{\delta_\alpha}\delta_\beta=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho_\alpha^i\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}_\alpha^\lambda\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}\Big)\delta_\mu.\tag{3.63}$$

Also, since $\overset{\mathrm{CR}}{D}$ is d-connection, the above equation gives us

$$\overset{\mathrm{CR}}{D}_{\delta_\alpha}\mathcal{V}_\beta=\frac{1}{2}\mathcal{G}^{\mu\gamma}\Big((\rho_\alpha^i\circ\pi)\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{x}^i}+\mathcal{B}_\alpha^\lambda\frac{\partial\mathcal{G}_{\beta\gamma}}{\partial\mathbf{y}^\lambda}-\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\beta}\mathcal{G}_{\lambda\gamma}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\mathcal{G}_{\lambda\beta}\Big)\mathcal{V}_\mu.\tag{3.64}$$

The condition $J^*\overset{\mathrm{CR}}{D}=J^*\overset{\mathrm{BF}}{D}$ and (3.48) give us

$$\overset{\mathrm{CR}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=\overset{\mathrm{CR}}{D}_{J\delta_\alpha}J\delta_\beta=\overset{\mathrm{BF}}{D}_{J\delta_\alpha}J\delta_\beta=\overset{\mathrm{BF}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta=0,\tag{3.65}$$

and consequently

$$\overset{\mathrm{CR}}{D}_{\mathcal{V}_\alpha}\delta_\beta=0.\tag{3.66}$$

Relations (3.63)–(3.66) prove the existence and uniqueness of $\overset{\text{CR}}{D}$. The proof of the second part of the assertion is similar to Proposition 3.14.

If h is the Barthel endomorphism of Finsler algebroid $(E,\mathcal{F})$, the d-connection $\overset{\text{CR}}{D}$ given by (3.63)–(3.66) is called the Chern-Rund connection of $(E,\mathcal{F})$.

Using (2.30)–(2.32) and (3.63)–(3.66), we can get

$$\begin{aligned}
\overset{\text{CR}}{R}{}_{\alpha\beta\gamma}^{\ \ \lambda}= &-(\rho_\alpha^i\circ\pi)\frac{\partial}{\partial\mathbf{x}^i}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\beta^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\beta^\lambda}{\partial\mathbf{y}^\gamma}\Big)\\
&-\mathcal{B}_\alpha^\mu\frac{\partial}{\partial\mathbf{y}^\mu}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\beta^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\beta^\lambda}{\partial\mathbf{y}^\gamma}\Big)\\
&+(\rho_\beta^i\circ\pi)\frac{\partial}{\partial\mathbf{x}^i}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\Big)\\
&+\mathcal{B}_\beta^\mu\frac{\partial}{\partial\mathbf{y}^\mu}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\Big)\\
&+\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\beta^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\mu\kappa}+\frac{\partial\mathcal{B}_\beta^\mu}{\partial\mathbf{y}^\gamma}\Big)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\iota}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\iota}\mathcal{G}^{\lambda\sigma}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\mu}\Big)\\
&-\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\mu\kappa}+\frac{\partial\mathcal{B}_\alpha^\mu}{\partial\mathbf{y}^\gamma}\Big)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\beta^\iota}{\partial\mathbf{y}^\mu\partial\mathbf{y}^\sigma}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\iota}\mathcal{G}^{\lambda\sigma}+\frac{\partial\mathcal{B}_\beta^\lambda}{\partial\mathbf{y}^\mu}\Big)\\
&+(L_{\alpha\beta}^\mu\circ\pi)\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\mu^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\mu^\lambda}{\partial\mathbf{y}^\gamma}\Big),\\
\overset{\text{CR}}{P}{}_{\alpha\beta\gamma}^{\ \ \lambda}= &\frac{\partial}{\partial\mathbf{y}^\beta}\Big(\frac{1}{2}\frac{\partial^2\mathcal{B}_\alpha^\nu}{\partial\mathbf{y}^\gamma\partial\mathbf{y}^\kappa}\frac{\partial\mathcal{F}}{\partial\mathbf{y}^\nu}\mathcal{G}^{\lambda\kappa}+\frac{\partial\mathcal{B}_\alpha^\lambda}{\partial\mathbf{y}^\gamma}\Big),\\
\overset{\text{CR}}{S}{}_{\alpha\beta\gamma}^{\ \ \lambda}= &0.
\end{aligned}$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be sections of $\pounds^{\pi}\overset{\circ}{E}$. Then, using (3.63)–(3.66) we can obtain the following formula for Chern-Rund connection:

$$\overset{\text{CR}}{D}_{\widetilde{X}}\widetilde{Y}=\overset{\text{CR}}{D}_{v\widetilde{X}}v\widetilde{Y}+\overset{\text{CR}}{D}_{v\widetilde{X}}h\widetilde{Y}+\overset{\text{CR}}{D}_{h\widetilde{X}}v\widetilde{Y}+\overset{\text{CR}}{D}_{h\widetilde{X}}h\widetilde{Y},$$

where

$$\overset{\text{CR}}{D}_{h\widetilde{X}}h\widetilde{Y}=hF[h\widetilde{X},J\widetilde{Y}]_\pounds+F\widetilde{\mathcal{C}}(\widetilde{X},\widetilde{Y}),\tag{3.67}$$

$$\overset{\text{CR}}{D}_{v\widetilde{X}}v\widetilde{Y}=J[v\widetilde{X},F\widetilde{Y}]_\pounds,\tag{3.68}$$

$$\overset{\text{CR}}{D}_{v\widetilde{X}}h\widetilde{Y}=h[v\widetilde{X},\widetilde{Y}]_\pounds,\tag{3.69}$$

$$\overset{\text{CR}}{D}_{h\widetilde{X}}v\widetilde{Y}=v[h\widetilde{X},v\widetilde{Y}]_\pounds+\widetilde{\mathcal{C}}(\widetilde{X},F\widetilde{Y}).\tag{3.70}$$

Theorem 3.7 *Let $(E,\mathcal{F})$ be a Finsler algebroid, h be a conservative horizontal endomorphism on $\pounds^{\pi}E$ and $\widetilde{\mathcal{G}}$ be the prolongation of $\mathcal{G}$ along h. There exists a unique d-connection $\overset{\text{H}}{D}$ on $(E,\mathcal{F})$ such that $\overset{\text{H}}{D}$ is v-metrical, (i.e., $\forall\,\widetilde{X}\in\Gamma(\pounds^{\pi}\overset{\circ}{E})$, $\overset{\text{H}}{D}_{v\widetilde{X}}\widetilde{\mathcal{G}}=0$) and the v-vertical and v-mixed torsions of $\overset{\text{H}}{D}$ are zero.*

Proof There exists a d-connection $\overset{\text{H}}{D}$ on $(E,\mathcal{F})$ such that $\overset{\text{H}}{D}$ is v-metrical and the v-vertical and v-mixed torsions of $\overset{\text{H}}{D}$ are zero. Since $\overset{\text{H}}{D}$ is v-metrical and the v-vertical torsion of $\overset{\text{H}}{D}$ is zero, similar to the proof of Theorem 3.5 we can deduce

$$\overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta = \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\mathcal{V}_\mu. \tag{3.71}$$

Also, since $\overset{\mathrm{H}}{D}$ is d-connection, using the above equation we obtain

$$\overset{\mathrm{H}}{D}_{\mathcal{V}_\alpha}\delta_\beta = \frac{1}{2}\frac{\partial \mathcal{G}_{\beta\gamma}}{\partial \mathbf{y}^\alpha}\mathcal{G}^{\gamma\mu}\delta_\mu. \tag{3.72}$$

Moreover, since the v-mixed torsion of $\overset{\mathrm{H}}{D}$ is zero, we can obtain

$$\overset{\mathrm{H}}{D}_{\delta_\alpha}\ \mathcal{V}_\beta = v[\delta_\alpha, \mathcal{V}_\beta]_{\pounds} = -\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\beta}\mathcal{V}_\mu \tag{3.73}$$

and consequently

$$\overset{\mathrm{H}}{D}_{\delta_\alpha}\ \delta_\beta = -\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\beta}\delta_\mu, \tag{3.74}$$

because $\overset{\mathrm{H}}{D}$ is d-connection. Relations (3.71)–(3.74) imply the existence and uniqueness of $\overset{\mathrm{H}}{D}$.

Proposition 3.15 *Let $(E, \mathcal{F})$ be a Finsler algebroid, h be a conservative horizontal endomorphism on $\pounds^\pi E$ and $\overset{\mathrm{H}}{D}$ be the d-connection given by the above theorem. If h-horizontal torsion and h-deflection of $\overset{\mathrm{H}}{D}$ are zero, h is the Barthel endomorphism.*

Proof The proof is similar to the proof of Proposition 3.13.

If h is the Barthel endomorphism of Finsler algebroid $(E, \mathcal{F})$, the d-connection $\overset{\mathrm{H}}{D}$ given by (3.71)–(3.74) is called the Hashiguchi connection of $(E, \mathcal{F})$.

Using (2.30)–(2.32) and (3.71)–(3.74), we can obtain

$$\begin{aligned}
\overset{\mathrm{H}}{R}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= -(\rho^i_\alpha\circ\pi)\frac{\partial^2\mathcal{B}^\lambda_\beta}{\partial \mathbf{x}^i\partial \mathbf{y}^\gamma} - \mathcal{B}^\mu_\alpha\frac{\partial^2\mathcal{B}^\lambda_\beta}{\partial \mathbf{y}^\mu\partial \mathbf{y}^\gamma} + (\rho^i_\beta\circ\pi)\frac{\partial^2\mathcal{B}^\lambda_\alpha}{\partial \mathbf{x}^i\partial \mathbf{y}^\gamma} + \mathcal{B}^\mu_\beta\frac{\partial^2\mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu\partial \mathbf{y}^\gamma}\\
&\quad + \frac{\partial \mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu}\frac{\partial \mathcal{B}^\mu_\beta}{\partial \mathbf{y}^\gamma} - \frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\gamma}\frac{\partial \mathcal{B}^\lambda_\beta}{\partial \mathbf{y}^\mu} + (L^\mu_{\alpha\beta}\circ\pi)\frac{\partial \mathcal{B}^\lambda_\mu}{\partial \mathbf{y}^\gamma} - \frac{1}{2}R^\mu_{\alpha\beta}\frac{\partial g_{\gamma\kappa}}{\partial \mathbf{y}^\mu}g^{\lambda\kappa},\\
\overset{\mathrm{H}}{P}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= (\rho^i_\alpha\circ\pi)\frac{\partial}{\partial \mathbf{x}^i}\Big(\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\Big) + \frac{1}{2}\mathcal{B}^\mu_\alpha\frac{\partial}{\partial \mathbf{y}^\mu}\Big(\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\Big) - \frac{1}{2}\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\frac{\partial \mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\mu}\\
&\quad + \frac{\partial^2\mathcal{B}^\lambda_\alpha}{\partial \mathbf{y}^\beta\partial \mathbf{y}^\gamma} + \frac{1}{2}\frac{\partial \mathcal{G}_{\mu\kappa}}{\partial \mathbf{y}^\beta}\mathcal{G}^{\lambda\kappa}\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\gamma} + \frac{1}{2}\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\mu}\mathcal{G}^{\lambda\kappa}\frac{\partial \mathcal{B}^\mu_\alpha}{\partial \mathbf{y}^\beta},\\
\overset{\mathrm{H}}{S}{}_{\alpha\beta\gamma}^{\ \ \ \lambda} &= \frac{1}{2}\Big(\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\beta}\frac{\partial \mathcal{G}^{\lambda\kappa}}{\partial \mathbf{y}^\alpha} - \frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\alpha}\frac{\partial \mathcal{G}^{\lambda\kappa}}{\partial \mathbf{y}^\beta}\Big) + \frac{1}{4}\Big(\mathcal{G}^{\sigma\mu}\mathcal{G}^{\lambda\kappa}\frac{\partial \mathcal{G}_{\gamma\sigma}}{\partial \mathbf{y}^\beta}\frac{\partial \mathcal{G}_{\mu\kappa}}{\partial \mathbf{y}^\alpha}\\
&\quad - \mathcal{G}^{\mu\kappa}\mathcal{G}^{\lambda\sigma}\frac{\partial \mathcal{G}_{\gamma\kappa}}{\partial \mathbf{y}^\alpha}\frac{\partial \mathcal{G}_{\mu\sigma}}{\partial \mathbf{y}^\beta}\Big).
\end{aligned}$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be sections of $\overset{\circ}{\pounds}{}^\pi E$. Then, using (3.71)–(3.74) we can obtain the following formula for Hashiguchi connection:

$$\overset{\mathrm{H}}{D}_{\widetilde{X}}\ \widetilde{Y} = \overset{\mathrm{H}}{D}_{v\widetilde{X}}\ v\widetilde{Y} + \overset{\mathrm{H}}{D}_{v\widetilde{X}}\ h\widetilde{Y} + \overset{\mathrm{H}}{D}_{h\widetilde{X}}\ v\widetilde{Y} + \overset{\mathrm{H}}{D}_{h\widetilde{X}}\ h\widetilde{Y},$$

where

$$\overset{\mathrm{H}}{D}_{h\widetilde{X}}\, h\widetilde{Y} = hF[h\widetilde{X}, J\widetilde{Y}]_{\pounds}, \tag{3.75}$$

$$\overset{\mathrm{H}}{D}_{v\widetilde{X}}\, v\widetilde{Y} = J[v\widetilde{X}, F\widetilde{Y}]_{\pounds} + \mathcal{C}(F\widetilde{X}, F\widetilde{Y}), \tag{3.76}$$

$$\overset{\mathrm{H}}{D}_{v\widetilde{X}}\, h\widetilde{Y} = h[v\widetilde{X}, \widetilde{Y}]_{\pounds} + F\mathcal{C}(F\widetilde{X}, \widetilde{Y}), \tag{3.77}$$

$$\overset{\mathrm{H}}{D}_{h\widetilde{X}}\, v\widetilde{Y} = v[h\widetilde{X}, v\widetilde{Y}]_{\pounds}. \tag{3.78}$$

Theorem 3.8 *Let h be the Barthel endomorphism on Finsler algebroid $(E, \mathcal{F})$. Then, the Cartan connection $\overset{\mathrm{C}}{D}$*

(i) *is Chern-Rund connection if $J^*\overset{\mathrm{C}}{D} = J^*\overset{\mathrm{BF}}{D}$,*

(ii) *is Hashiguchi connection if $h^*\overset{\mathrm{C}}{D} = \overset{\mathrm{BF}}{D}$,*

(iii) *is Berwald connection if it is the Chern-Rund connection and the Hashiguchi connection at the same time.*

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