Chin. Ann. Math. Ser. B
42(1), 2021, 85–104
DOI: 10.1007/s11401-021-0246-9

Chinese Annals of Mathematics, Series B

Boundedness of Solutions of a Quasi-periodic Sublinear Duffing Equation*

Yaqun PENG[1] Xinli ZHANG[2] Daxiong PIAO[3]

Abstract The authors study the Lagrangian stability for the sublinear Duffing equations $\ddot{x} + e(t)|x|^{\alpha-1}x = p(t)$ with $0 < \alpha < 1$, where e and p are real analytic quasi-periodic functions with frequency ω. It is proved that if the mean value of e is positive and the frequency ω satisfies Diophantine condition, then every solution of the equation is bounded.

Keywords Hamiltonian system, Sublinear Duffing equation, Boundedness, Quasi-periodic solution, Invariant curve
2000 MR Subject Classification 34C11, 34D20, 37E40, 37J40

1 Introduction

In 1976, Morris [1], by using Moser's twist theorem, proved that all solutions of the equation

$$\ddot{x} + 2x^3 = p(t)$$

are bounded when p is periodic and piecewise continuous. Since then, KAM theory has been the most powerful tool to study Littlewood's boundedness problem for Duffing type equations

$$\ddot{x} + \psi(x, t) = 0, \tag{1.1}$$

where ψ is periodic in t. And fruitful achievements have been made by many authors (see for examples [2–6] and references therein).

In 1999, Küpper-You [7] proved that all solutions of the equation

$$\ddot{x} + |x|^{\alpha-1}x = p(t)$$

are bounded, where $0 < \alpha < 1$ and $p \in C^\infty(\mathbb{T})$.

In 2001, Liu [8] investigated the sublinear equation in the more general form

$$\ddot{x} + \varphi(x) = p(t),$$

Manuscript received March 4, 2019. Revised October 28, 2019.
[1]School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai 519082, Guangdong, China.
E-mail: pengyq35@mail.sysu.edu.cn
[2]School of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao 266061, Shandong, China. E-mail: zxl@qust.edu.cn
[3]Corresponding author. School of Mathematical Sciences, Ocean University of China, Qingdao 266100, Shandong, China. E-mail: dxpiao@ouc.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11571327, 11971059).

and concluded that all solutions of the equation are bounded with $p \in C^5(\mathbb{T})$ and $\varphi \in C^6(\mathbb{R})$ satisfying the sublinear condition:

$$\operatorname{sign}(x) \cdot \varphi(x) \to +\infty, \quad \frac{\varphi(x)}{x} \to 0 \quad \text{as } |x| \to \infty.$$

In 2009, Wang [9] studied the sublinear equation

$$\ddot{x} + e(t)|x|^{\alpha-1}x = p(t), \tag{1.2}$$

where $0 < \alpha < 1$, $e, p \in C^5(\mathbb{T})$, $\int_0^1 e(s)\mathrm{d}s \neq 0$. He proved that the necessary and sufficient condition that the equation posses the Lagrangian stability is $\int_0^1 e(s)\mathrm{d}s > 0$.

In the dynamical point of view, it is natural to study Littlewood's boundedness problem for (1.1) with ψ quasi-periodic in t.

In 2000, Zharnitsky [10] proved an invariant curve theorem for a quasi-periodic planar mapping and applied it to answering a question asked by Levi-Zehnder [11], that is the boundedness of solutions of the Fermi-Ulam model.

In 2005, Liu [12] established some invariant curve theorems for some planar reversible mappings with quasi-periodic perturbations. As an application, he proved the existence of quasi-periodic solutions and the boundedness of all solutions of an asymmetric oscillation

$$\ddot{x} + \widehat{a}x^+ - \widehat{b}x^- = p(t), \tag{1.3}$$

when p is a real analytic, even and quasi-periodic function with the frequency ω satisfying the Diophantine condition.

Recently Huang-Li-Liu [13–14] proved the existence of invariant curves for quasi-periodic smooth mappings and used the theory to get the existence of quasi-periodic solutions and the boundedness of all solutions of (1.3) when p is a smooth quasi-periodic function with the frequency satisfying the Diophantine condition (see the results in Appendix).

Motivated by the above references, especially by Wang [9] and Huang-Li-Liu [13–14], we are going to investigate the boundedness problem of the special quasi-periodic subilinear Duffing equations

$$\ddot{x} + e(t)|x|^{\alpha-1}x = p(t) \tag{1.4}$$

with $0 < \alpha < 1$, where e and p are real analytic quasi-periodic functions and their frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ satisfies the Diophantine condition

$$|\langle k, \omega\rangle| \geq \frac{\widetilde{c}}{|k|^{\widetilde{\sigma}}}, \quad k \in \mathbb{Z}^n \backslash \{0\} \tag{1.5}$$

for two positive constants $\widetilde{c}$, $\widetilde{\sigma}$.

It is well known that for any quasi-periodic function f, its mean value $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T f(t)\mathrm{d}t$ always exists. Denote it by $[f]$.

Our main result is the following theorem.

Theorem 1.1 *Assume that* e, p *are real analytic quasi-periodic functions with the frequency* $\omega = (\omega_1, \cdots, \omega_n)$ *satisfying the Diophantine condition* (1.5). *If* $[e] > 0$, *then* (1.4) *has quasi-periodic solutions and all the solutions of* (1.4) *are bounded, i.e., every solution* $x(t)$ *of* (1.4) *exists for* $t \in \mathbb{R}$ *and* $\sup\limits_{t\in\mathbb{R}}(|x(t)| + |\dot{x}(t)|) < +\infty$.

Remark 1.1 The main idea of the proof of Theorem 1.1 is similar to the one in [9]. But here, due to the quasi-periodicity of e and p, we meet the so called "small divisor" problems, so we need much more regularity estimates and introducing new function class $\mathcal{F}_\omega(r_0, l_0)$ of quasi-periodic functions as a tool. To meet the requirements of the invariant curve theorem established by Huang-Li-Liu in [13], we must suppose that e and p are analytic quasi-periodic functions. It seems an interesting question to consider the smooth case.

The rest of our paper is organized as follows. In Section 2, we will give some definitions and proprieties and the integral proposition of quasi-periodic functions. In Section 3, we will introduce the action-angle variables and the new function class $\mathcal{F}_\omega(r_0, l_0)$ of quasi-periodic functions, then change action-angle variables. In Section 4, we will make further canonical transformations and obtain a new transformed Hamiltonian system. In Section 5, we will prove the existence of quasi-periodic solutions and the boundedness of all solutions for (1.4). Here we point out that though our proof appears a simple variant of [9], there is a huge difference between our quasi-periodic case and the periodic case in [9]. In fact, in our proof we use the integral proposition of quasi-periodic functions in Section 2 and the proprieties of the new function class $\mathcal{F}_\omega(r_0, l_0)$ in Section 3.

2 Preliminaries

We first recall some basic knowledge on the analytic quasi-periodic functions. For further contents, one can refer to [15, Chapter 3].

Definition 2.1 (see [15]) *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a real analytic quasi-periodic function with the frequency* ω, *if it can be represented by a Fourier series*

$$f(t) = \sum_{k\in\mathbb{Z}^n} f_k \mathrm{e}^{2\pi \mathrm{i}\langle k,\omega\rangle t},$$

where $k = (k_1, k_2, \cdots, k_n)$, $\langle k, \omega\rangle = k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n \neq 0$ *if* $k \neq 0$, *and* f_k *exponentially decays with* $|k|$, *where* $|k| = |k_1| + |k_2| + \cdots + |k_n|$.

The set of all such functions is denoted by $Q(\omega)$.

It is not difficult to see $f_0 = [f]$.

For each $f \in Q(\omega)$, there is a real analytic function $F(\theta) = F(\theta_1, \theta_2, \cdots, \theta_n) : \mathbb{R}^n \to \mathbb{R}$ which is 1-periodic in each variable θ_j $(1 \leq j \leq n)$ and bounded in a complex neighborhood $\Pi_r^n = \{(\theta_1, \theta_2, \cdots, \theta_n) \in \mathbb{C}^n : |\mathrm{Im}\,\theta_j| \leq r,\ j = 1, 2, \cdots, n\}$ of $\mathbb{R}^n$ for some $r > 0$ such that

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_n t), \quad \forall\, t \in \mathbb{R}.$$

Then F has a Fourier expansion

$$F(\theta)=\sum_{k\in\mathbb{Z}^n} f_k \mathrm{e}^{2\pi\mathrm{i}\langle k,\theta\rangle}.$$

This F is called the shell function of f.

Let $Q_r(\omega)\subseteq Q(\omega)$ be the set of real analytic quasi-periodic function f such that the corresponding shell functions F is bounded on the subset $\Pi_r^n=\{(\theta_1,\theta_2,\cdots,\theta_n)\in\mathbb{C}^n : |\mathrm{Im}\,\theta_j|\le r,\ j=1,2,\cdots,n\}$ with the supremum norm

$$|F|_r=\sup_{\theta\in\Pi_r^n}|F(\theta)|=\sup_{\theta\in\Pi_r^n}\Big|\sum_k f_k\mathrm{e}^{2\pi\mathrm{i}\langle k,\theta\rangle}\Big|<+\infty.$$

Define $|f|_r=|F|_r$.

It is well known that indefinite integral of a periodic function is still a periodic function if the mean value of the function is zero. It is easy to prove that this conclusion is not valid for a quasi-periodic function. However we have the following result for a real analytic quasi-periodic function.

Proposition 2.1 *If $f\in Q(\omega)$ with the frequency ω satisfing the Diophantine condition* (1.5), *and*

$$g(t):=\int_0^t (f(s)-[f])\mathrm{d}s,$$

then $g\in Q(\omega)$.

Proof From Definition 2.1,

$$f(t)-[f]=\sum_{k\in\mathbb{Z}^n\setminus\{0\}} f_k\mathrm{e}^{2\pi\mathrm{i}\langle k,\theta\rangle t}.$$

Suppose $|f_k|\le|f|_r\mathrm{e}^{-\varrho|k|}$ for some $r>0$ and $\varrho>0$. Then from (1.5), we have

$$g(t)=\int_0^t (f(s)-[f])\mathrm{d}s=\int_0^t\sum_{k\in\mathbb{Z}^n\setminus\{0\}} f_k\mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle s}\mathrm{d}s=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{f_k}{\mathrm{i}\langle k,\omega\rangle}(\mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle t}-1).$$

So

$$|g(t)|\le\Big|2\sum_{k\in\mathbb{Z}^n\setminus\{0\}}|f|_r\mathrm{e}^{-\varrho|k|}\Big(\frac{\widetilde{c}}{|k|^{\widetilde{\sigma}}}\Big)^{-1}\Big|\le C\sum_{k\in\mathbb{Z}^n\setminus\{0\}}|f|_r\cdot\mathrm{e}^{-\varrho|k|}\cdot|k|^{\widetilde{\sigma}}<+\infty,$$

which implies that the function g is well defined, where C is a positive constant. Since $g(t)=\sum\limits_{k\in\mathbb{Z}^n} g_k\mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle t}$ with $g_0=\sum\limits_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{f_k}{\mathrm{i}\langle k,\omega\rangle}$ and $g_k=\frac{f_k}{\mathrm{i}\langle k,\omega\rangle}$ for $k\ne0$, noting the fact that f_k decay exponentially, we see $g\in Q(\omega)$.

Lemma 2.1 (see [13]) *The set $Q(\omega)$ has the following properties*:

(1) *If f, $g\in Q(\omega)$, then $f\pm g$, $g(\cdot+f(\cdot))\in Q(\omega)$.*

(2) *If ω satisfies Diophantine condition, $f\in Q(\omega)$ and $\tau=\beta t+f(t)$ with $\beta+f'>0$, then the inverse relation is given by $t=\beta^{-1}\tau+g(\tau)$ where $g\in Q\big(\frac{\omega}{\beta}\big)$. In particular, if $\beta=1$, then $g\in Q(\omega)$.*

Lemma 2.2 *If $f,\ g \in Q(\omega)$, then $f \cdot g \in Q(\omega)$.*

Proof Since $f,\ g \in Q(\omega)$, we denote f, g respectively as

$$f(t) = \sum_{k\in\mathbb{Z}^n} f_k \mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle t}, \quad g(t) = \sum_{k\in\mathbb{Z}^n} g_k \mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle t},$$

where $f_k,\ g_k$ satisfy $|f_k| \le M_1\mathrm{e}^{-|k|\rho_1},\ |g_k| \le M_2\mathrm{e}^{-|k|\rho_2}$ for positive constants $M_1,\ M_2,\ \rho_1,\ \rho_2$.

Let $\zeta(t) = f(t)g(t)$. Then

$$\zeta(t) = \sum_{k\in\mathbb{Z}^n} \zeta_k \mathrm{e}^{2\pi\mathrm{i}\langle k,\omega\rangle t},$$

where $\zeta_k = \sum\limits_{m\in\mathbb{Z}^n} f_{k-m}g_m$ or $\zeta_k = \sum\limits_{m\in\mathbb{Z}^n} f_m g_{k-m}$.

(i) Consider the case $\rho_2 > \rho_1 > 0$. We have

$$\begin{aligned}
|\zeta_k| &= \Big|\sum_{m\in\mathbb{Z}^n} f_{k-m}g_m\Big| \le \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|k-m|\rho_1} M_2\mathrm{e}^{-|m|\rho_2} \\
&= \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|k-m|\rho_1} M_2\mathrm{e}^{-|m|\rho_1}\mathrm{e}^{-|m|(\rho_2-\rho_1)} \\
&\le \sum_{m\in\mathbb{Z}^n} M_1M_2\mathrm{e}^{-|k|\rho_1}\mathrm{e}^{-|m|(\rho_2-\rho_1)} \\
&= M_1M_2\mathrm{e}^{-|k|\rho_1} \sum_{m\in\mathbb{Z}^n} \mathrm{e}^{-|m|(\rho_2-\rho_1)} \\
&\le \widehat{M}M_1M_2\mathrm{e}^{-|k|\rho_1} \le M\mathrm{e}^{-|k|\rho_1},
\end{aligned}$$

therefore, $\zeta \in Q(\omega)$.

(ii) Consider the case $\rho_1 > \rho_2 > 0$. We have

$$\begin{aligned}
|\zeta_k| &= \Big|\sum_{m\in\mathbb{Z}^n} f_m g_{k-m}\Big| \le \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|m|\rho_1} M_2\mathrm{e}^{-|k-m|\rho_2} \\
&= \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|m|\rho_2}\mathrm{e}^{-|m|(\rho_1-\rho_2)} M_2\mathrm{e}^{-|k-m|\rho_2} \\
&\le \sum_{m\in\mathbb{Z}^n} M_1M_2\mathrm{e}^{-|k|\rho_2}\mathrm{e}^{-|m|(\rho_1-\rho_2)} \\
&= M_1M_2\mathrm{e}^{-|k|\rho_2} \sum_{m\in\mathbb{Z}^n} \mathrm{e}^{-|m|(\rho_1-\rho_2)} \\
&\le \widehat{M}M_1M_2\mathrm{e}^{-|k|\rho_2} \le M\mathrm{e}^{-|k|\rho_2},
\end{aligned}$$

therefore, $\zeta \in Q(\omega)$.

(iii) Consider the case $\rho_1 = \rho_2$ and choose a constant $0 < \rho < \rho_1 = \rho_2$. We have

$$\begin{aligned}
|\zeta_k| = \Big|\sum_{m\in\mathbb{Z}^n} f_{k-m}g_m\Big| &\le \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|k-m|\rho_1} M_2\mathrm{e}^{-|m|\rho_2} \le \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|k-m|\rho} M_2\mathrm{e}^{-|m|\rho_2} \\
&= \sum_{m\in\mathbb{Z}^n} M_1\mathrm{e}^{-|k-m|\rho} M_2\mathrm{e}^{-|m|\rho}\mathrm{e}^{-|m|(\rho_2-\rho)} \\
&\le \sum_{m\in\mathbb{Z}^n} M_1M_2\mathrm{e}^{-|k|\rho}\mathrm{e}^{-|m|(\rho_2-\rho)}
\end{aligned}$$

$$\leq M_1 M_2 \mathrm{e}^{-|k|\rho} \sum_{m\in\mathbb{Z}^n} \mathrm{e}^{-|m|(\rho_2-\rho)} \leq \widehat{M} M_1 M_2 \mathrm{e}^{-|k|\rho} \leq M \mathrm{e}^{-|k|\rho},$$

therefore, $\zeta \in Q(\omega)$. We complete the proof now.

3 Action-Angle Variables

We will first introduce the action-angle variables after two canonical transformations and then change action-angle variables in this section. Moreover, we give the definition and properties of a new function class $\mathcal{F}_\omega(r_0, l_0)$ of quasi-periodic functions in order to estimate the Hamiltonian.

3.1 A canonical transformation

(1.4) can be written as a Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y} = y + q(t), \quad \dot{y} = -\frac{\partial H}{\partial x} = -e(t)x|x|^{\alpha-1} + [p]. \tag{3.1}$$

The corresponding Hamiltonian is

$$\begin{aligned} H(x,y,t) &= \frac{1}{2}y^2 + q(t)y + \frac{e(t)}{\alpha+1}|x|^{\alpha+1} - [p]x \\ &= \frac{1}{2}y^2 + \frac{[e]}{\alpha+1}|x|^{\alpha+1} + q(t)y + \frac{e_1(t)}{\alpha+1}|x|^{\alpha+1} - [p]x, \end{aligned} \tag{3.2}$$

where e_1, q are as the following:

$$e_1(t) := e(t) - [e], \quad q(t) := \int_0^t (p(s) - [p])\mathrm{d}s.$$

Since e, $p \in Q(\omega)$, from Proposition 2.1, q is well defined and $q \in Q(\omega)$.

To make the Hamiltonian system simple, we introduce a transformation

$$\Phi_1: \ x = x, \quad y = z + \frac{\partial G_1}{\partial x}(x,t),$$

where $G_1(x,t)$ will be determined later. Under Φ_1, Hamiltonian function (3.2) is transformed to

$$\begin{aligned} H(x,z,t) &= \frac{1}{2}\Big(z + \frac{\partial G_1}{\partial x}\Big)^2 + \frac{[e]}{\alpha+1}|x|^{\alpha+1} + q(t)\cdot\Big(z + \frac{\partial G_1}{\partial x}\Big) + \frac{e_1(t)}{\alpha+1}|x|^{\alpha+1} - [p]x + \frac{\partial G_1}{\partial t} \\ &= \frac{1}{2}z^2 + \frac{[e]}{\alpha+1}|x|^{\alpha+1} + z\frac{\partial G_1}{\partial x} + \frac{1}{2}\Big(\frac{\partial G_1}{\partial x}\Big)^2 + q(t)\Big(z + \frac{\partial G_1}{\partial x}\Big) + \frac{e_1(t)}{\alpha+1}|x|^{\alpha+1} \\ &\quad + \frac{\partial G_1}{\partial t} - [p]x. \end{aligned}$$

Let

$$\frac{e_1(t)}{\alpha+1}|x|^{\alpha+1} + \frac{\partial G_1}{\partial t} = 0,$$

then

$$G_1(x,t) = -\frac{1}{\alpha+1}|x|^{\alpha+1}\int_0^t e_1(s)\mathrm{d}s.$$

Define

$$E(t) = -\int_0^t e_1(s)\mathrm{d}s = -\int_0^t (e(s) - [e])\mathrm{d}s.$$

From Proposition 2.1, we see $E \in Q(\omega)$. From Lemma 2.2, $E^2 \in Q(\omega)$. Therefore, $G_1(x,\cdot) \in Q(\omega)$ for every $x \in \mathbb{R}$. Then the Hamiltonian function (3.2) becomes

$$\begin{aligned} H(x,z,t) =& \frac{1}{2}z^2 + \frac{[e]}{\alpha+1}|x|^{\alpha+1} + z|x|^{\alpha-1}xE(t) + \frac{1}{2}|x|^{2\alpha}E(t)^2 \\ &+ q(t)(z + |x|^{\alpha-1}xE(t)) - [p]x, \end{aligned} \tag{3.3}$$

and the corresponding Hamiltonian system is

$$\begin{aligned} \dot{x} &= z + |x|^{\alpha-1}xE(t) + q(t), \\ \dot{z} &= -[e]|x|^{\alpha-1}x - \alpha|x|^{2\alpha-2}xE(t)^2 - \alpha|x|^{\alpha-2}x(q(t)+z)E(t) + [p]. \end{aligned} \tag{3.4}$$

3.2 Introducing action-angle variables

In order to introduce the action-angle variables, firstly consider the corresponding autonomous Hamiltonian system of (3.4),

$$\dot{x} = z, \quad \dot{z} = -[e]|x|^{\alpha-1}x \tag{3.5}$$

with the Hamiltonian $h_0(x,z) = \frac{1}{2}z^2 + \frac{[e]}{\alpha+1}|x|^{\alpha+1}$.

Let $(x_0(t), z_0(t))$ be the periodic solution of (3.5) satisfying the initial value

$$(x_0(0), z_0(0)) = (1, 0)$$

and $T_0 > 0$ be its minimal period. Introduce the functions $\mathtt{C}$ and $\mathtt{S}$ by

$$(\mathtt{C}(t), \mathtt{S}(t)) = \Big(x_0\Big(\frac{t}{T_0}\Big), z_0\Big(\frac{t}{T_0}\Big)\Big).$$

The functions $\mathtt{C}$, $\mathtt{S}$ satisfy

(1) $\mathtt{C} \in C^2(\mathbb{T}),\ \mathtt{S} \in C^1(\mathbb{T}),\ \mathtt{C}(0) = 1,\ \mathtt{S}(0) = 0$;

(2) $\mathtt{C}(-t) = \mathtt{C}(t),\ \mathtt{S}(-t) = -\mathtt{S}(t),\ \mathtt{C}\big(\frac{1}{2} - t\big) = -\mathtt{C}(t),\ \mathtt{S}\big(\frac{1}{2} - t\big) = \mathtt{S}(t)$;

(3) $\mathtt{C}(t) = 0 \Leftrightarrow t\big(\mathrm{mod}\ \frac{1}{2}\big) = \frac{1}{4}$;

(4) $\dot{\mathtt{C}} = \frac{1}{T_0}\mathtt{S},\ \dot{\mathtt{S}} = -\frac{[e]}{T_0}|\mathtt{C}|^{\alpha-1}\mathtt{C}$;

(5) $\frac{1}{2}\mathtt{S}(t)^2 + \frac{[e]}{\alpha+1}|\mathtt{C}(t)|^{\alpha+1} = \frac{[e]}{\alpha+1}$.

The action and angle variables are introduced by the canonical transformation

$$\Phi_2: \ x = d^b I^b \mathtt{C}(\theta), \quad z = d^{\frac{a}{2}} I^{\frac{a}{2}} \mathtt{S}(\theta),$$

where $b = \frac{2}{\alpha+3}$, $a = 2 - 2b = \frac{2(\alpha+1)}{\alpha+3}$ and $d = b[e]T_0$. It is obvious that $\frac{1}{2} < b < \frac{2}{3} < a < 1$ if $0 < \alpha < 1$. We claim that Φ_2 is a symplectic diffeomorphism from $\mathbb{R}^+ \times \mathbb{T}$ onto $\mathbb{R}^2/\{0\}$ for the following reason. $(\mathtt{C}, \mathtt{S})$ is a solution of (3.5) with the minimal period T_0, so Φ_2 is one to one and onto. Moreover, Φ_2 is measure preserving.

Under Φ_2, the Hamiltonian (3.3) is transformed into

$$H(\theta,I,t)=d_0I^a+(d^{\frac{a}{2}}\mathtt{S}(\theta)I^{\frac{a}{2}}+q(t))(d^b\mathtt{C}(\theta))^{\alpha}E(t)I^{b\alpha}+\frac{1}{2}(d^b\mathtt{C}(\theta))^{2\alpha}E(t)^2I^{2b\alpha}$$
$$+d^{\frac{a}{2}}\mathtt{S}(\theta)q(t)I^{\frac{a}{2}}-[p]d^b\mathtt{C}(\theta)I^b, \tag{3.6}$$

where $d_0=\frac{[e]}{\alpha+1}d^b$.

We introduce the quasi-periodic function space $\mathcal{F}_\omega(r_0,l_0)$ as follows.

3.3 A function class $\mathcal{F}_\omega(r_0,l_0)$

Given $r_0\in\mathbb{R}$, $l_0\geq 0$, denote $\mathcal{F}_\omega(r_0,l_0)$ the set of functions in $(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}$: f is C^∞ in λ, C^{l_0} in θ, $f(\lambda,\theta,\cdot)\in Q(\omega)$ for all $(\lambda,\theta)\in\mathbb{R}^+\times\mathbb{T}$ and satisfies

$$\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j-r_0}|D_\lambda^jD_t^kD_\theta^lf(\lambda,\theta,t)|)<\infty,\quad l\leq l_0.$$

Lemma 3.1 *$\mathcal{F}_\omega(r_0,l_0)$ has the following properties*:

(i) *If $r_1<r_2$, then $\mathcal{F}_\omega(r_1,l_0)\subset\mathcal{F}_\omega(r_2,l_0)$.*

(ii) *If $f\in\mathcal{F}_\omega(r_0,l_0)$, then $D_\lambda^{j_0}f\in\mathcal{F}_\omega(r_0-j_0,l_0)$.*

(iii) *If $f_1\in\mathcal{F}_\omega(r_1,l_1)$ and $f_2\in\mathcal{F}_\omega(r_2,l_2)$, then $f_1\cdot f_2\in\mathcal{F}_\omega(r_1+r_2,\min\{l_1,l_2\})$.*

(iv) *If $f\in\mathcal{F}_\omega(r_0,l_0)$ satisfies $|f(\lambda,\cdot,\cdot)|\geq c\lambda^{r_0}$ for $\lambda>\lambda_0$, then $\frac{1}{f}\in\mathcal{F}_\omega(-r_0,l_0)$.*

Proof (i) $f\in\mathcal{F}_\omega(r_1,l_0)$, $r_1<r_2$, then

$$\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j-r_2}|D_\lambda^jD_t^kD_\theta^lf(\lambda,\theta,t)|)$$
$$=\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j-r_1}\lambda^{r_1-r_2}|D_\lambda^jD_t^kD_\theta^lf(\lambda,\theta,t)|)<\infty,\quad l\leq l_0.$$

(ii) $f\in\mathcal{F}_\omega(r_0,l_0)$, then

$$\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j-(r_0-j_0)}|D_\lambda^jD_t^kD_\theta^l(D_\lambda^{j_0}f(\lambda,\theta,t))|)$$
$$=\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j+j_0-r_0}|D_\lambda^{j+j_0}D_t^kD_\theta^lf(\lambda,\theta,t)|)<\infty,\quad l\leq l_0.$$

(iii) $f_1\in\mathcal{F}_\omega(r_1,l_1)$ and $f_2\in\mathcal{F}_\omega(r_2,l_2)$, from Lemma 2.2, $f_1\cdot f_2\in Q(\omega)$.

$$\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}(\lambda^{j-(r_1+r_2)}|D_\lambda^jD_t^kD_\theta^l(f_1f_2)(\lambda,\theta,t)|)$$
$$=\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}}\sum_{\substack{j_1+j_2=j,\\k_1+k_2=k,\\l_{01}+l_{02}=l}}\prod_{i=1}^{2}(\lambda^{j_i-r_i}|D_\lambda^{j_i}D_t^{k_i}D_\theta^{l_{0i}}f_i(\lambda,\theta,t)|)<\infty,\quad l\leq\min\{l_1,l_2\}.$$

(iv) $f\in\mathcal{F}_\omega(r_0,l_0)$, then

$$\sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R},\ \lambda>\lambda_0}\left(\lambda^{j-(-r_0)}\left|D_\lambda^jD_t^kD_\theta^l\frac{1}{f(\lambda,\theta,t)}\right|\right)$$

$$\leq \sup_{(\lambda,\theta,t)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R},\ \lambda>\lambda_0} (\lambda^{j+r_0}|D_\lambda^j D_t^k D_\theta^l (c^{-1}\lambda^{-r_0})|) < \infty, \quad l \leq l_0.$$

For $f \in \mathcal{F}_\omega(r_0, l_0)$, denote the mean value over t-variables by $[f]$:

$$[f](\lambda,\theta) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\lambda,\theta,t)\mathrm{d}t > 0.$$

Define $\mathtt{C}_1(\theta) = (d^b \mathtt{C}(\theta))^\alpha$. It is obvious that $\mathtt{C}_1 \in C^0$ for $0 < \alpha < 1$. Rewrite (3.6) as

$$\begin{aligned} H(\theta, I, t) = {} & d_0 I^a + (d^{\frac{a}{2}}\mathtt{S}(\theta) I^{\frac{a}{2}} + q(t))\mathtt{C}_1(\theta)E(t)I^{b\alpha} + \frac{1}{2}\mathtt{C}_1(\theta)^2 E(t)^2 I^{2b\alpha} \\ & + d^{\frac{a}{2}}\mathtt{S}(\theta)q(t)I^{\frac{a}{2}} - [p]d^b\mathtt{C}(\theta)I^b. \end{aligned} \tag{3.7}$$

Denote

$$\begin{aligned} \widehat{H}_0(I) &= d_0 I^a, \\ \widehat{H}_1(I, \mathtt{C}_1(\theta), \mathtt{S}(\theta), t) &= (d^{\frac{a}{2}}\mathtt{S}(\theta)I^{\frac{a}{2}} + q(t))\mathtt{C}_1(\theta)E(t)I^{b\alpha} + \frac{1}{2}\mathtt{C}_1(\theta)^2E(t)^2I^{2b\alpha}, \\ \widehat{H}_2(I, \mathtt{S}(\theta), t) &= d^{\frac{a}{2}}\mathtt{S}(\theta)q(t)I^{\frac{a}{2}}, \\ \widehat{H}_3(I, \mathtt{C}(\theta)) &= -[p]d^b\mathtt{C}(\theta)I^b. \end{aligned}$$

From the definition of the function space $\mathcal{F}_\omega(r_0, l_0)$ and Lemma 3.1, we have

$$\widehat{H}_0 \in \mathcal{F}_\omega(a, +\infty), \quad \widehat{H}_1 \in \mathcal{F}_\omega(2a-1, 0), \quad \widehat{H}_2 \in \mathcal{F}_\omega\Big(\frac{a}{2}, 1\Big), \quad \widehat{H}_3 \in \mathcal{F}_\omega(b, 2). \tag{3.8}$$

Define

$$H_0(I, \mathtt{C}(\theta)) = \widehat{H}_0(I) + \widehat{H}_3(I, \mathtt{C}(\theta)), \quad H_1(I, \mathtt{C}_1(\theta), \mathtt{S}(\theta), t) = \widehat{H}_1(I, \mathtt{C}_1(\theta), \mathtt{S}(\theta), t) + \widehat{H}_2(I, \mathtt{S}(\theta), t).$$

Then the Hamiltonian (3.7) becomes

$$H(\theta, I, t) = H_0(I, \mathtt{C}(\theta)) + H_1(I, \mathtt{C}_1(\theta), \mathtt{S}(\theta), t), \tag{3.9}$$

where

$$H_0 \in \mathcal{F}_\omega(a, 2), \quad H_1 \in \mathcal{F}_\omega(2a-1, 0). \tag{3.10}$$

Since the Hamiltonian (3.9) is only C^0 on θ, we cannot guarantee that the Poincaré map of (3.9) is smooth enough as required in the quasi-periodic invariant curve theorem obtained by Huang-Li-Liu in [13]. To solve this probelm, we will exchange the role of θ and t in the following part.

3.4 Changing action-angle variables

From (3.8), $\widehat{H}_0 \in \mathcal{F}_\omega(a, +\infty)$, $\widehat{H}_3 \in \mathcal{F}_\omega(b, 2)$, it is obvious that $\frac{\partial H_0}{\partial I} = \frac{\partial(\widehat{H}_0 + \widehat{H}_3)}{\partial I} \neq 0$ for large enough $I > 0$, then there exists a function $\mathcal{I}_0(\sigma, \mathtt{C}(\theta))$ which is C^∞ on $\mathtt{C}$ such that

$$\sigma = H_0(\mathcal{I}_0(\sigma, \mathtt{C}(\theta)), \mathtt{C}(\theta)). \tag{3.11}$$

Similarly, from (3.10), $H_0 \in \mathcal{F}_\omega(a,2)$, $H_1 \in \mathcal{F}_\omega(2a-1,0)$, $\frac{\partial H}{\partial I} = \frac{\partial(H_0+H_1)}{\partial I} \neq 0$ for large enough $I>0$, so there exists a function $\mathcal{I}(H,t,\theta)$ such that

$$H = H_0(\mathcal{I}(H,t,\theta), \mathtt{C}(\theta)) + H_1(\mathcal{I}(H,t,\theta), \mathtt{C}_1(\theta), \mathtt{S}(\theta), t), \tag{3.12}$$

which can be rewritten as

$$H - H_1(\mathcal{I}(H,t,\theta), \mathtt{C}_1(\theta), \mathtt{S}(\theta), t) = H_0(\mathcal{I}(H,t,\theta), \mathtt{C}(\theta)). \tag{3.13}$$

From (3.11) and (3.13), it is obvious that

$$\mathcal{I}(H,t,\theta) = \mathcal{I}_0(H - H_1(\mathcal{I}(H,t,\theta), \mathtt{C}_1(\theta), \mathtt{S}(\theta), t), \mathtt{C}(\theta)). \tag{3.14}$$

Consider the function

$$H(I, \mathtt{C}_1, \mathtt{C}, \mathtt{S}, t) = H_0(I, \mathtt{C}) + H_1(I, \mathtt{C}_1, \mathtt{S}, t).$$

From (3.10) and $\frac{\partial H}{\partial I} = \frac{\partial(H_0+H_1)}{\partial I} \neq 0$ for large enough $I>0$, there exists a function $\widetilde{\mathcal{I}}(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S})$ with $\widetilde{\mathcal{I}}$ being C^∞ on $\mathtt{C}$, $\mathtt{C}_1$, $\mathtt{S}$ such that

$$H = H_0(\widetilde{\mathcal{I}}(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S}), \mathtt{C}) + H_1(\widetilde{\mathcal{I}}(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S}), \mathtt{C}_1, \mathtt{S}, t). \tag{3.15}$$

From these definitions, $\mathcal{I}(H,t,\theta) = \widetilde{\mathcal{I}}(H,t,\mathtt{C}(\theta),\mathtt{C}_1(\theta),\mathtt{S}(\theta))$.

Let

$$\begin{aligned}\mathcal{I}_1(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S}) = -\int_0^1 &\frac{\partial \mathcal{I}_0}{\partial \sigma}(H - \mu H_1(\widetilde{\mathcal{I}}(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S}),t,\mathtt{C},\mathtt{C}_1,\mathtt{S}),\ \mathtt{C}) \\ &\cdot H_1(\widetilde{\mathcal{I}}(H,t,\mathtt{C},\mathtt{C}_1,\mathtt{S}),\mathtt{C}_1,\mathtt{S},t)\mathrm{d}\mu.\end{aligned} \tag{3.16}$$

It is easy to deduce that $\mathcal{I}_1$ is C^∞ on $\mathtt{C}$, $\mathtt{C}_1$ and $\mathtt{S}$ respectively. From the definition of H_1, $\mathcal{I}_0$ and Lemmas 2.1–2.2, $\mathcal{I}_1 \in Q(\omega)$.

The Hamiltonian (3.9) becomes

$$\mathcal{I}(H,t,\theta) = \mathcal{I}_0(H, \mathtt{C}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_1(\theta),\mathtt{S}(\theta)) \tag{3.17}$$

with θ, t, H being the new time variables, new angle variables and new action variables respectively. From the proprieties of the function space $\mathcal{F}_\omega(r_0,l_0)$ (see Lemma 3.1), we have

$$\mathcal{I}_0 \in \mathcal{F}_\omega\Big(\frac{1}{a}, 2\Big), \quad \mathcal{I}_1 \in \mathcal{F}_\omega(1,0). \tag{3.18}$$

Furthermore, for a positive constant C_0,

$$\partial_H^i \mathcal{I}_0 \geq C_0 H^{\frac{1}{a}-i}. \tag{3.19}$$

4 More Transformations

We will make more transformations since the Poincaré mapping of the Hamiltonian system (3.17) is not a small perturbation of a stand quasi-periodic twist mapping. Notice that all these transformations are quasi-periodic in the time variable. We will discuss the quasi-periodicity after every transformation.

It should be noticed that $\mathtt{C}_1 \in C^0$. In order to make more transformations, we will improve the smoothness of $\mathtt{C}_1$ by constructing a smooth approximation function $\mathtt{C}_2$ of $\mathtt{C}_1$. Denote $\mathtt{S}_1(\theta) = \mathtt{S}'(\theta) = -\frac{[e]}{T_0}\mathtt{C}(\theta) \cdot |\mathtt{C}(\theta)|^{\alpha-1}$. For the same reason, we also find a smooth approximation function $\mathtt{S}_2$ of $\mathtt{S}_1$. The method can be found in [9], so we state the two conclusion in the following Lemma 4.1 without detail proof for simplicity. Here we point out that though our proof appears a simple variant of [9], there is a big difference between our quasi-periodic case and the periodic case in [9]. In fact, in our proof we use the integral proposition of quasi-periodic functions in Section 2 and the proprieties of the new function class $\mathcal{F}_\omega(r_0, l_0)$ in Section 3.

Lemma 4.1 (see [9]) *For any $\varepsilon > 0$, there exist C^1 periodic functions $\mathtt{C}_2$, $\mathtt{S}_2$ such that*

$$|\mathtt{C}_2(\theta) - \mathtt{C}_1(\theta)| \le D_1 \cdot \varepsilon^\alpha, \quad |\mathtt{C}_2'(\theta)| \le D_1 \cdot \varepsilon^{\alpha-1}, \tag{4.1}$$

$$\mathtt{C}_2(\theta) = \mathtt{C}_1(\theta) \quad \text{if } \left|\theta \left(\operatorname{mod} \frac{1}{2}\right) - \frac{1}{4}\right| \ge D \cdot \varepsilon, \tag{4.2}$$

and

$$|\mathtt{S}_2(\theta) - \mathtt{S}_1(\theta)| \le D_2 \cdot \varepsilon^\alpha, \quad |\mathtt{S}_2'(\theta)| \le D_2 \cdot \varepsilon^{\alpha-1}, \tag{4.3}$$

where constants D_1, $D_2 > 0$ are independent of ε.

From (3.17),

$$\begin{aligned}
\mathcal{I}(H,t,\theta) &= \mathcal{I}_0(H,\mathtt{C}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_1(\theta),\mathtt{S}(\theta)) \\
&= \mathcal{I}_0(H,\mathtt{C}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_2(\theta),\mathtt{S}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_1(\theta),\mathtt{S}(\theta)) \\
&\quad - \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_2(\theta),\mathtt{S}(\theta)) \\
&= \mathcal{I}_0(H,\mathtt{C}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_2(\theta),\mathtt{S}(\theta)) \\
&\quad + \int_0^1 \frac{\partial \mathcal{I}_1}{\partial \mathtt{C}_1}(H,t,\mathtt{C}(\theta),\mu(\mathtt{C}_1(\theta)-\mathtt{C}_2(\theta)),\mathtt{S}(\theta)) \cdot (\mathtt{C}_1(\theta)-\mathtt{C}_2(\theta))\mathrm{d}\mu.
\end{aligned}$$

Let

$$\mathcal{I}_2(H,t,\theta) = \int_0^1 \frac{\partial \mathcal{I}_1}{\partial \mathtt{C}_1}(H,t,\mathtt{C}(\theta),\mu(\mathtt{C}_1(\theta)-\mathtt{C}_2(\theta)),\mathtt{S}(\theta)) \cdot (\mathtt{C}_1(\theta)-\mathtt{C}_2(\theta))\mathrm{d}\mu. \tag{4.4}$$

Then (3.17) is rewritten as

$$\mathcal{I} = \mathcal{I}_0(H,\mathtt{C}(\theta)) + \mathcal{I}_1(H,t,\mathtt{C}(\theta),\mathtt{C}_2(\theta),\mathtt{S}(\theta)) + \mathcal{I}_2(H,t,\theta). \tag{4.5}$$

Lemma 4.2 *For the initial action variable $H_0 > 0$ large enough, there exists a canonical transformation such that the Hamiltonian (4.5) is transformed into*

$$\mathcal{I} = \mathcal{J}_0(\lambda,\theta,\mathtt{C}_2(\theta)) + \mathcal{J}_1(\lambda,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta)) + \mathcal{J}_2(\lambda,\tau,\theta,\mathtt{C}_2(\theta)) \cdot \mathtt{C}_2'(\theta) + \mathcal{J}_3(\lambda,\tau,\theta), \tag{4.6}$$

where

$$\mathcal{J}_0 \in \mathcal{F}_\omega\Big(\frac{1}{a}, 2\Big), \quad \mathcal{J}_1, \, \mathcal{J}_2 \in \mathcal{F}_\omega\Big(2 - \frac{1}{a}, 0\Big), \quad H_0^{\frac{1}{a}-1+c_0}\mathcal{J}_3 \in \mathcal{F}_\omega(1, 0). \tag{4.7}$$

Moreover, for a positive constant C_0,

$$\partial_H^i \mathcal{J}_0 \geq C_0 \lambda^{\frac{1}{a}-i}. \tag{4.8}$$

Proof Let $\varepsilon = H_0^{-(\frac{1}{a}-1+c_0)\frac{1}{\alpha}}$ be the parameter in Lemma 4.1 with the constant $0 < c_0 < 1$. From (3.18), (4.4) and Lemma 4.1,

$$|\mathcal{I}_2| \leq D_1 H_0^{-(\frac{1}{a}-1+c_0)}, \tag{4.9}$$

where D_1 is a constant independent of H_0 denoted in Lemma 4.1, so

$$H_0^{-(\frac{1}{a}-1+c_0)}\mathcal{I}_2 \in \mathcal{F}_\omega(1, 0). \tag{4.10}$$

Introduce a canonical transformation

$$\Phi_3: \; H = \lambda + \frac{\partial G_2}{\partial t}(\lambda, t, \theta), \quad \tau = t + \frac{\partial G_2}{\partial \lambda}(\lambda, t, \theta),$$

where the function $G_2(\lambda, t, \theta)$ will be determined later. Under Φ_3, the Hamiltonian (4.5) is transformed into

$$\begin{aligned}
\mathcal{I} &= \mathcal{I}_0\Big(\lambda + \frac{\partial G_2}{\partial t}, \mathtt{C}(\theta)\Big) + \mathcal{I}_1\Big(\lambda + \frac{\partial G_2}{\partial t}, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)\Big) + \mathcal{I}_2\Big(\lambda + \frac{\partial G_2}{\partial t}, t, \theta\Big) + \frac{\partial G_2}{\partial \theta} \\
&= \mathcal{I}_0(\lambda, \mathtt{C}(\theta)) + [\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) + \frac{\partial \mathcal{I}_0}{\partial H}(\lambda, \mathtt{C}(\theta)) \cdot \frac{\partial G_2}{\partial t} + \mathcal{I}_1(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) \\
&\quad - [\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) + \int_0^1 \frac{\partial^2 \mathcal{I}_0}{\partial H^2}\Big(\lambda + \mu \frac{\partial G_2}{\partial t}, \mathtt{C}(\theta)\Big) \cdot \Big(\frac{\partial G_2}{\partial t}\Big)^2 \mathrm{d}\mu \\
&\quad + \int_0^1 \frac{\partial \mathcal{I}_1}{\partial H}\Big(\lambda + \mu \frac{\partial G_2}{\partial t}, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)\Big) \cdot \frac{\partial G_2}{\partial t} \mathrm{d}\mu + \mathcal{I}_2\Big(\lambda + \frac{\partial G_2}{\partial t}, t, \theta\Big) + \frac{\partial G_2}{\partial \theta},
\end{aligned} \tag{4.11}$$

where $[\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) = \lim\limits_{T \to \infty} \frac{1}{T}\int_0^T \mathcal{I}_1(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta))\mathrm{d}t$.

Let

$$\frac{\partial \mathcal{I}_0}{\partial H}(\lambda, \mathtt{C}(\theta)) \cdot \frac{\partial G_2}{\partial t} + \mathcal{I}_1(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) - [\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) = 0,$$

then

$$\begin{aligned}
&G_2(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) \\
&= -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1}(\lambda, \mathtt{C}(\theta)) \int_0^t (\mathcal{I}_1(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) - [\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)))\mathrm{d}t.
\end{aligned}$$

$\mathcal{I}_1$ is C^∞ in $\mathtt{C}$, $\mathtt{C}_2$, $\mathtt{S}$ respectively and $\mathcal{I}_1 \in Q(\omega)$ with the frequency ω satisfying the Diophantine condition (1.5). According to Proposition 2.1, $\int_0^t (\mathcal{I}_1 - [\mathcal{I}_1])\mathrm{d}t \in Q(\omega)$ and C^∞ in $\mathtt{C}$, $\mathtt{C}_2$, $\mathtt{S}$ respectively. Thus $G_2(\lambda, \cdot, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) \in Q(\omega)$ and C^∞ in $\mathtt{C}$, $\mathtt{C}_2$, $\mathtt{S}$ respectively. It is obvious that

$$\frac{\partial G_2}{\partial \theta}(\lambda, t, \theta) = -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{C}} \mathrm{d}t \cdot \mathtt{S}(\theta) - \Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{S}} \mathrm{d}t \cdot \mathtt{S}_1(\theta)$$

$$
\begin{aligned}
&- \Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{C}_2} \mathrm{d}t \cdot \mathtt{C}_2'(\theta) \\
&- \Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-2} \cdot \frac{\partial^2 \mathcal{I}_0}{\partial H \partial \mathtt{C}} \int_0^t (\mathcal{I}_1 - [\mathcal{I}_1]) \mathrm{d}t \cdot \mathtt{S}(\theta).
\end{aligned} \tag{4.12}
$$

Denote

$$
\begin{aligned}
\mathcal{J}_{11}(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) &= -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{C}} \mathrm{d}t \cdot \mathtt{S}(\theta), \\
\mathcal{J}_{12}(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta), \mathtt{S}_1(\theta)) &= -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{S}} \mathrm{d}t \cdot \mathtt{S}_1(\theta), \\
\mathcal{J}_{13}(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) &= -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-2} \cdot \frac{\partial^2 \mathcal{I}_0}{\partial H \partial \mathtt{C}} \int_0^t (\mathcal{I}_1 - [\mathcal{I}_1]) \mathrm{d}t \cdot \mathtt{S}(\theta), \\
\mathcal{J}_{2}(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) &= -\Big(\frac{\partial \mathcal{I}_0}{\partial H}\Big)^{-1} \int_0^t \frac{\partial(\mathcal{I}_1 - [\mathcal{I}_1])}{\partial \mathtt{C}_2} \mathrm{d}t.
\end{aligned}
$$

Then (4.12) can be rewritten as

$$
\frac{\partial G_2}{\partial \theta} = \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13} + \mathcal{J}_2 \cdot \mathtt{C}_2'.
$$

Define

$$
\begin{aligned}
\mathcal{J}_0(\lambda, \theta, \mathtt{C}_2(\theta)) &= \mathcal{I}_0(\lambda, \mathtt{C}(\theta)) + [\mathcal{I}_1](\lambda, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)), \\
\mathcal{J}_1(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta), \mathtt{S}_1(\theta)) &= \mathcal{J}_{11}(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) + \mathcal{J}_{12}(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta), \mathtt{S}_1(\theta)) \\
&\quad + \mathcal{J}_{13}(\lambda, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)) \\
&\quad + \int_0^1 \frac{\partial^2 \mathcal{I}_0}{\partial H^2} \Big(\lambda + \mu \frac{\partial G_2}{\partial t}, \mathtt{C}(\theta)\Big) \cdot \Big(\frac{\partial G_2}{\partial t}\Big)^2 \mathrm{d}\mu \\
&\quad + \int_0^1 \frac{\partial \mathcal{I}_1}{\partial H} \Big(\lambda + \mu \frac{\partial G_2}{\partial t}, t, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta)\Big) \cdot \frac{\partial G_2}{\partial t} \mathrm{d}\mu, \\
\widehat{\mathcal{J}_1}(\lambda, \tau, \theta) &= \mathcal{J}_2\Big(\lambda + \frac{\partial G_2}{\partial t}, t, \theta\Big).
\end{aligned}
$$

In the above denotation, $t = t(\lambda, \tau, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta))$.

It is obvious that $\mathcal{J}_0(\lambda, \theta, \mathtt{C}_2(\theta))$ is C^∞ on $\mathtt{C}_2$ and C^1 on θ. From Lemma 2.2, $\mathcal{J}_1$ is C^∞ in $\mathtt{C}$, $\mathtt{C}_2$, $\mathtt{S}$ and C^1 on θ respectively and $\mathcal{J}_1(\lambda, \cdot, \mathtt{C}(\theta), \mathtt{C}_2(\theta), \mathtt{S}(\theta), \mathtt{S}_1(\theta)) \in Q(\omega)$. From Lemma 2.1, $\widehat{\mathcal{J}_1}$ is C^1 on θ and $\widehat{\mathcal{J}_1}(\lambda, \cdot, \theta) \in Q(\omega)$.

Moreover, from (4.9),

$$
\mathcal{J}_1,\ \mathcal{J}_2 \in \mathcal{F}_\omega\Big(2 - \frac{1}{a}, 0\Big), \quad H_0^{\frac{1}{a} - 1 + c_0} \widehat{\mathcal{J}_1} \in \mathcal{F}_\omega(1, 0).
$$

From the definition of G_2, rewrite (4.11) as

$$
\begin{aligned}
\mathcal{I} &= \mathcal{J}_0(\lambda, \theta, \mathtt{C}_2(\theta)) + \mathcal{J}_1(\lambda, \tau, \theta, \mathtt{C}_2(\theta), \mathtt{S}_1(\theta)) + \mathcal{J}_2(\lambda, \tau, \theta, \mathtt{C}_2(\theta)) \cdot \mathtt{C}_2'(\theta) + \widehat{\mathcal{J}_1}(\lambda, \tau, \theta) \\
&= \mathcal{J}_0(\lambda, \theta, \mathtt{C}_2(\theta)) + \mathcal{J}_1(\lambda, \tau, \theta, \mathtt{C}_2(\theta), \mathtt{S}_2(\theta)) + \mathcal{J}_2(\lambda, \tau, \theta, \mathtt{C}_2(\theta)) \cdot \mathtt{C}_2'(\theta) \\
&\quad + \mathcal{J}_1(\lambda, \tau, \theta, \mathtt{C}_2(\theta), \mathtt{S}_1(\theta)) - \mathcal{J}_1(\lambda, \tau, \theta, \mathtt{C}_2(\theta), \mathtt{S}_2(\theta)) + \widehat{\mathcal{J}_1}(\lambda, \tau, \theta).
\end{aligned} \tag{4.13}
$$

Define

$$\widehat{\mathcal{J}_2}(\lambda,\tau,\theta)=\mathcal{J}_1(\lambda,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_1(\theta))-\mathcal{J}_1(\lambda,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta)).$$

From (4.7),

$$H_0^{\frac{1}{a}-1+c_0}\widehat{\mathcal{J}_2}\in\mathcal{F}_\omega\Big(2-\frac{1}{a},0\Big). \tag{4.14}$$

Define

$$\mathcal{J}_3(\lambda,\tau,\theta)=\widehat{\mathcal{J}_1}+\widehat{\mathcal{J}_2}.$$

Rewrite (4.13) as

$$\mathcal{I}=\mathcal{J}_0(\lambda,\theta,\mathtt{C}_2(\theta))+\mathcal{J}_1(\lambda,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))+\mathcal{J}_2(\lambda,\tau,\theta,\mathtt{C}_2(\theta))\cdot\mathtt{C}_2'(\theta)+\mathcal{J}_3(\lambda,\tau,\theta).$$

However, the Poincaré map of the Hamiltonian system corresponding to (4.5) does not have the form of the Poincaré mapping in [14], therefore we should make another transformation. In the above proof, $\mathtt{C}_2'$ is only C^0 on θ. To satisfy the smoothness requirements, we establish a C^1 function $\mathtt{C}_3$ which is an approximation of $\mathtt{C}_2'$ similar as in Lemma 4.1. Then we can use the quasi-periodic twist theorem in [14] to the Poincaré mapping. From [9], we have the following lemma.

Lemma 4.3 (see [9]) *For any $H_0>0$ and $0<\varepsilon_0<c_0$, there exists a C^1 function $\mathtt{C}_3(\theta)$ such that*

$$\int_0^1|\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta)|\mathrm{d}\theta\le D\cdot H_0^{-\varepsilon_0}, \tag{4.15}$$

$$\int_0^1|\mathtt{C}_3'(\theta)|\mathrm{d}\theta\le D\cdot H_0^{\varepsilon_0(1-\alpha)}, \tag{4.16}$$

$$\max|\mathtt{C}_3(\theta)|\le D\cdot H_0^{\varepsilon_0(1-\alpha)}, \tag{4.17}$$

where D is a constant independent of H_0.

From the above results, the Hamiltonian (4.6) is

$$\begin{aligned}\mathcal{I}=&\,\mathcal{J}_0(\lambda,\theta,\mathtt{C}_2(\theta))+\mathcal{J}_1(\lambda,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))+\mathcal{J}_2(\lambda,\tau,\theta,\mathtt{C}_2(\theta))\cdot\mathtt{C}_3(\theta)\\&+\mathcal{J}_3(\lambda,\tau,\theta)+\mathcal{J}_2(\lambda,\tau,\theta,\mathtt{C}_2(\theta))\cdot(\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta)),\end{aligned} \tag{4.18}$$

where $\mathcal{J}_0$, $\mathcal{J}_1$, $\mathcal{J}_2$ are C^1 on θ, C^∞ on $\mathtt{C}_2$ and $\mathtt{S}_2$ respectively and $\mathcal{J}_1(\lambda,\cdot,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))\in Q(\omega)$, $\mathcal{J}_2(\lambda,\cdot,\theta,\mathtt{C}_2(\theta))\in Q(\omega)$, $\mathcal{J}_3(\lambda,\cdot,\theta)\in Q(\omega)$.

Lemma 4.4 *For the initial action variable $H_0>0$ large enough (which implies that new initial action variable λ_0 large enough), there exists a canonical transformation which transforms the Hamiltonian (4.18) into*

$$\mathcal{I}=\mathcal{L}_0(\rho,\theta)+\mathcal{L}_1(\rho,\varsigma,\theta)+\mathcal{L}_2(\rho,\varsigma,\theta)+\mathcal{L}_3(\rho,\varsigma,\theta)+\mathcal{L}_4(\rho,\varsigma,\theta)\cdot(\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta))$$

$$+\mathcal{L}_5(\rho,\varsigma,\theta)\cdot \mathtt{C}_2'(\theta)+\mathcal{L}_6(\rho,\varsigma,\theta)\cdot \mathtt{S}_2'(\theta). \tag{4.19}$$

Moreover, we have

$$\mathcal{L}_0\in\mathcal{F}_\omega\Big(\frac{1}{a},2\Big),\quad \mathcal{L}_1\in\mathcal{F}_\omega\Big(3-\frac{2}{a}+c_1\varepsilon_0,0\Big),\quad \mathcal{L}_2\in\mathcal{F}_\omega\Big(3-\frac{2}{a},0\Big),\quad \mathcal{L}_3\in H_0^{1-\frac{1}{a}-c_0}\mathcal{F}_\omega(1,0),$$
$$\mathcal{L}_4\in\mathcal{F}_\omega\Big(2-\frac{1}{a},0\Big),\quad \mathcal{L}_5\in\mathcal{F}_\omega\Big(3-\frac{2}{a},0\Big),\quad \mathcal{L}_6\in\mathcal{F}_\omega\Big(3-\frac{2}{a},0\Big), \tag{4.20}$$

where $c_0>0$ *and* $c_1=\frac{2-\alpha}{\alpha}$ *are constants independent of* ε_0*.*

Proof Introduce the canonical transformation

$$\Phi_4:\ \lambda=\rho+\frac{\partial G_3}{\partial\tau}(\rho,\tau,\theta),\quad \varsigma=\tau+\frac{\partial G_3}{\partial\rho}(\rho,\tau,\theta),$$

where the function $G_3(\rho,\tau,\theta)$ will be determined later. Under Φ_4, the Hamiltonian (4.18) is transformed into

$$\begin{aligned}
\mathcal{I}=&\mathcal{J}_0\Big(\rho+\frac{\partial G_3}{\partial\tau},\theta,\mathtt{C}_2(\theta)\Big)+\mathcal{J}_1\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta)\Big)+\mathcal{J}_2\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2(\theta)\Big)\cdot\mathtt{C}_3(\theta)\\
&+\mathcal{J}_3\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta\Big)+\frac{\partial G_3}{\partial\theta}+\mathcal{J}_2\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2(\theta)\Big)\cdot(\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta))\\
=&\mathcal{J}_0(\rho,\theta,\mathtt{C}_2(\theta))+[\mathcal{J}_1]+[\mathcal{J}_2]\cdot\mathtt{C}_3(\theta)+\frac{\partial\mathcal{J}_0}{\partial\rho}(\rho,\mathtt{C}(\theta))\cdot\frac{\partial G_3}{\partial\tau}+\mathcal{J}_1(\rho,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))-[\mathcal{J}_1]\\
&+\mathcal{J}_2(\rho,\tau,\theta,\mathtt{C}_2(\theta))\cdot\mathtt{C}_3(\theta)-[\mathcal{J}_2]\cdot\mathtt{C}_3(\theta)+\mathcal{J}_3(\rho,\tau,\theta)+\frac{\partial G_3}{\partial\theta}\\
&+\mathcal{J}_2(\rho,\tau,\theta,\mathtt{C}_2(\theta))\cdot(\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta))+\int_0^1\frac{\partial^2\mathcal{J}_0}{\partial\lambda^2}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\mathtt{C}(\theta)\Big)\cdot\Big(\frac{\partial G_3}{\partial\tau}\Big)^2\mathrm{d}\mu\\
&+\int_0^1\frac{\partial[\mathcal{J}_1]}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\theta,\mathtt{C}_2(\theta),\mathtt{S}(\theta)\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\\
&+\int_0^1\frac{\partial\mathcal{J}_1}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2(\theta)\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\\
&+\int_0^1\frac{\partial\mathcal{J}_2}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2(\theta)\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\cdot\mathtt{C}_3(\theta),
\end{aligned}$$

where

$$[\mathcal{J}_1]=[\mathcal{J}_1](\rho,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathcal{J}_1(\rho,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))\mathrm{d}\tau,$$
$$[\mathcal{J}_2]=[\mathcal{J}_2](\rho,\theta,\mathtt{C}_2(\theta))=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathcal{J}_2(\rho,\tau,\theta,\mathtt{C}_2(\theta))\mathrm{d}\tau.$$

Let

$$\frac{\partial\mathcal{J}_0}{\partial\lambda}(\rho,\mathtt{C}(\theta))\cdot\frac{\partial G_3}{\partial\tau}+\mathcal{J}_1(\rho,\tau,\theta,\mathtt{C}_2(\theta),\mathtt{S}_2(\theta))-[\mathcal{J}_1]+\mathcal{J}_2(\rho,\tau,\theta,\mathtt{C}_2(\theta))\cdot\mathtt{C}_3(\theta)-[\mathcal{J}_2]\cdot\mathtt{C}_3(\theta)=0,$$

then

$$G_3=-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau(\mathcal{J}_1-[\mathcal{J}_1]+(\mathcal{J}_2-[\mathcal{J}_2])\cdot\mathtt{C}_3(\theta))\mathrm{d}\tau.$$

From the definition of $\mathcal{J}_1$, $\mathcal{J}_2$ and Proposition 2.1, $G_3(\rho,\cdot,\theta)\in Q(\omega)$. We can calculate that

$$\begin{aligned}\frac{\partial G_3}{\partial\theta}=&-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\Big(\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\mathtt{C}_2}\cdot\mathtt{C}_2'(\theta)+\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\mathtt{S}_2}\cdot\mathtt{S}_2'(\theta)+\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\theta}\Big)\mathrm{d}\tau\\&-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\Big(\frac{\partial(\mathcal{J}_2-[\mathcal{J}_2])}{\partial\mathtt{C}_2}\cdot\mathtt{C}_2'(\theta)\mathtt{C}_3(\theta)+\frac{\partial(\mathcal{J}_2-[\mathcal{J}_2])}{\partial\theta}\mathtt{C}_3(\theta)\\&+(\mathcal{J}_2-[\mathcal{J}_2])\cdot\mathtt{C}_3'(\theta)\Big)\mathrm{d}\tau+\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-2}\cdot\frac{\partial^2\mathcal{J}_0}{\partial\lambda\partial\theta}\int_0^\tau(\mathcal{J}_1-[\mathcal{J}_1]+(\mathcal{J}_2-[\mathcal{J}_2]))\mathrm{d}\tau\cdot\mathtt{C}_3(\theta).\end{aligned}$$

Define

$$\begin{aligned}\mathcal{L}_0(\rho,\theta)=&\ \mathcal{J}_0(\rho,\theta,\mathtt{C}_2(\theta))+[\mathcal{J}_1]+[\mathcal{J}_2]\cdot\mathtt{C}_3(\theta),\\ \mathcal{L}_1(\rho,\tau,\theta)=&\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-2}\cdot\frac{\partial^2\mathcal{J}_0}{\partial\lambda\partial\theta}\int_0^\tau(\mathcal{J}_2-[\mathcal{J}_2])\mathrm{d}\tau\cdot\mathtt{C}_3(\theta)-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\frac{\partial(\mathcal{J}_2-[\mathcal{J}_2])}{\partial\theta}\mathrm{d}\tau\cdot\mathtt{C}_3(\theta)\\&-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\frac{\partial(\mathcal{J}_2-[\mathcal{J}_2])}{\partial\mathtt{C}_2}\mathrm{d}\tau\cdot\mathtt{C}_3(\theta)\cdot\mathtt{C}_2'(\theta)-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau(\mathcal{J}_2-[\mathcal{J}_2])\mathrm{d}\tau\cdot\mathtt{C}_3'(\theta),\\ \mathcal{L}_2(\rho,\tau,\theta)=&\int_0^1\frac{\partial^2\mathcal{J}_0}{\partial\lambda^2}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\mathtt{C}\Big)\cdot\Big(\frac{\partial G_3}{\partial\tau}\Big)^2\mathrm{d}\mu+\int_0^1\frac{\partial[\mathcal{J}_1]}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\theta,\mathtt{C}_2,\mathtt{S}\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\\&+\int_0^1\frac{\partial\mathcal{J}_1}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\\&+\int_0^1\frac{\partial\mathcal{J}_2}{\partial\lambda}\Big(\rho+\mu\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2\Big)\cdot\frac{\partial G_3}{\partial\tau}\mathrm{d}\mu\cdot\mathtt{C}_3(\theta)\\&+\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-2}\cdot\frac{\partial^2\mathcal{J}_0}{\partial\lambda\partial\theta}\int_0^\tau(\mathcal{J}_1-[\mathcal{J}_1])\mathrm{d}\tau-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\theta}\mathrm{d}\tau,\\ \mathcal{L}_3(\rho,\tau,\theta)=&\ \mathcal{J}_2\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta,\mathtt{C}_2\Big),\\ \mathcal{L}_4(\rho,\tau,\theta)=&\ \mathcal{J}_3\Big(\rho+\frac{\partial G_3}{\partial\tau},\tau,\theta\Big),\\ \mathcal{L}_5(\rho,\tau,\theta)=&-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\mathtt{C}_2}\mathrm{d}\tau,\\ \mathcal{L}_6(\rho,\tau,\theta)=&-\Big(\frac{\partial\mathcal{J}_0}{\partial\lambda}\Big)^{-1}\int_0^\tau\frac{\partial(\mathcal{J}_1-[\mathcal{J}_1])}{\partial\mathtt{S}_2}\mathrm{d}\tau,\end{aligned}$$

where $\tau=\tau(\rho,\varsigma,\theta,\mathtt{C}_3(\theta))$. Then, the Hamiltonian (4.18) is rewritten as

$$\begin{aligned}\mathcal{I}=&\ \mathcal{L}_0(\rho,\theta)+L_1(\rho,\varsigma,\theta)+\mathcal{L}_2(\rho,\varsigma,\theta)+\mathcal{L}_3(\varrho,\varsigma,\theta)+\mathcal{L}_4(\rho,\varsigma,\theta)\cdot(\mathtt{C}_2'(\theta)-\mathtt{C}_3(\theta))\\&+\mathcal{L}_5(\rho,\varsigma,\theta)\cdot\mathtt{C}_2'(\theta)+\mathcal{L}_6(\rho,\varsigma,\theta)\cdot\mathtt{S}_2'(\theta).\end{aligned}$$

From Proportion 2.1, $\mathcal{L}_1(\rho,\cdot,\theta)$, $\mathcal{L}_2(\rho,\cdot,\theta)$, $\mathcal{L}_3(\varrho,\cdot,\theta)$, $\mathcal{L}_4(\rho,\cdot,\theta)$, $\mathcal{L}_5(\rho,\cdot,\theta)$, $\mathcal{L}_6(\rho,\cdot,\theta)\in Q(\omega)$. Let θ^* be the number such that $\int_{\frac14-\theta^*}^{\frac14+\theta^*}\mathrm{d}\theta=2|\mathtt{C}_2(\theta)|=H_0^{-\varepsilon_0}$. Note that $\mathtt{C}_3(\theta)=0$ for $\big|\theta\,(\mathrm{mod}\,\frac12)-\frac14\big|\le\theta^*$ and there are similar results for $\mathtt{C}_3'$ and $\mathtt{C}_2'\cdot\mathtt{C}_3$. From the estimate on $\mathcal{J}_2$ in (4.7) and $\mathtt{C}_3$ in (4.3),

$$\mathcal{L}_1\in\mathcal{F}_\omega\Big(3-\frac{2}{a}+c_1\varepsilon_0,0\Big).$$

From (4.7) and for the reason that $\int_0^1|\mathtt{C}_3(\theta)|\mathrm{d}\theta$ is bounded, we can also prove other parts of (4.20).

5 Proof of Main Result

It is obvious that the solution $(H(\theta), t(\theta))$ of (3.17) with the initial condition $H(0) = H_0$, $t(0) = t_0$ satisfies

$$c \cdot H_0 \leq |H(\theta)| \leq C \cdot H_0, \quad \forall\, \theta \in [0, 1],$$

where c, $C > 0$ are two positive constants. As a consequence, for the solution $(\rho(\theta), \varsigma(\theta))$ of (4.19) with the initial condition $(\rho(0), \varsigma(0)) = (\rho(H_0, t_0, 0), \varsigma(H_0, t_0, 0))$, we have

$$c \cdot H_0 \leq |\rho(\theta)| \leq C \cdot H_0, \quad \forall\, \theta \in [0, 1].$$

Consider the Hamiltonian (4.19),

$$\begin{aligned}\mathcal{I} = {} & \mathcal{L}_0(\rho, \theta) + \mathcal{L}_1(\rho, \varsigma, \theta) + \mathcal{L}_2(\rho, \varsigma, \theta) + \mathcal{L}_3(\rho, \varsigma, \theta) + \mathcal{L}_4(\rho, \varsigma, \theta) \cdot (\mathtt{C}_2'(\theta) - \mathtt{C}_3(\theta)) \\ & + \mathcal{L}_5(\rho, \varsigma, \theta) \cdot \mathtt{C}_2'(\theta) + \mathcal{L}_6(\rho, \varsigma, \theta) \cdot \mathtt{S}_2'(\theta).\end{aligned}$$

The corresponding Hamiltonian system is

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}\theta} = -\dfrac{\partial \mathcal{I}}{\partial \varsigma} = -\dfrac{\partial \mathcal{L}_1}{\partial \varsigma} - \dfrac{\partial \mathcal{L}_2}{\partial \varsigma} - \dfrac{\partial \mathcal{L}_3}{\partial \varsigma} - \dfrac{\partial \mathcal{L}_4}{\partial \varsigma} \cdot (\mathtt{C}_2' - \mathtt{C}_3) - \dfrac{\partial \mathcal{L}_5}{\partial \varsigma} \cdot \mathtt{C}_2' - \dfrac{\partial \mathcal{L}_6}{\partial \varsigma} \cdot \mathtt{S}_2', \\ \dfrac{\mathrm{d}\varsigma}{\mathrm{d}\theta} = \dfrac{\partial \mathcal{I}}{\partial \rho} = \dfrac{\partial \mathcal{L}_0}{\partial \rho} + \dfrac{\partial \mathcal{L}_1}{\partial \rho} + \dfrac{\partial \mathcal{L}_2}{\partial \rho} + \dfrac{\partial \mathcal{L}_3}{\partial \rho} + \dfrac{\partial \mathcal{L}_4}{\partial \rho} \cdot (\mathtt{C}_2' - \mathtt{C}_3) + \dfrac{\partial \mathcal{L}_5}{\partial \rho} \cdot \mathtt{C}_2' + \dfrac{\partial \mathcal{L}_6}{\partial \rho} \cdot \mathtt{S}_2'. \end{cases} \tag{5.1}$$

The Poincaré map $\mathcal{P}$ of (5.1) is of the form

$$\mathcal{P}: \begin{cases} \rho_1 = \rho_0 + f_1(\rho_0, \varsigma_0), \\ \varsigma_1 = \varsigma_0 + \displaystyle\int_0^1 \frac{\partial \mathcal{L}_0}{\partial \rho} \mathrm{d}\theta + f_2(\rho_0, \varsigma_0), \end{cases} \tag{5.2}$$

where

$$f_1(\rho_0, \varsigma_0) = -\int_0^1 \Big(\frac{\partial \mathcal{L}_1}{\partial \varsigma} + \frac{\partial \mathcal{L}_2}{\partial \varsigma} + \frac{\partial \mathcal{L}_3}{\partial \varsigma} + \frac{\partial \mathcal{L}_4}{\partial \varsigma} \cdot (\mathtt{C}_2' - \mathtt{C}_3) + \frac{\partial \mathcal{L}_5}{\partial \varsigma} \cdot \mathtt{C}_2' + \frac{\partial \mathcal{L}_6}{\partial \varsigma} \cdot \mathtt{S}_2' \Big) \mathrm{d}\theta, \tag{5.3}$$

$$f_2(\rho_0, \varsigma_0) = \int_0^1 \Big(\frac{\partial \mathcal{L}_1}{\partial \rho} + \frac{\partial \mathcal{L}_2}{\partial \rho} + \frac{\partial \mathcal{L}_3}{\partial \rho} + \frac{\partial \mathcal{L}_4}{\partial \rho} \cdot (\mathtt{C}_2' - \mathtt{C}_3) + \frac{\partial \mathcal{L}_5}{\partial \rho} \cdot \mathtt{C}_2' + \frac{\partial \mathcal{L}_6}{\partial \rho} \cdot \mathtt{S}_2' \Big) \mathrm{d}\theta. \tag{5.4}$$

Define

$$r(\rho) = \int_0^1 \frac{\partial \mathcal{L}_0}{\partial \rho}(\rho, \theta) \mathrm{d}\theta.$$

From (4.20),

$$r \in \mathcal{F}_\omega\Big(\frac{1}{a} - 1, 2\Big). \tag{5.5}$$

Moreover, for ρ large enough, $\frac{\partial r}{\partial \rho} \neq 0$.

Then the Poincaré map is expressed as $\mathcal{P}_1 : \{r \mid r > r^*, r^* \gg 1\} \times \mathbb{R} \to \mathbb{R}^2$ of the following form

$$\mathcal{P}_1: \begin{cases} r_1 = r_0 + g_1(r_0, \varsigma_0), \\ \varsigma_1 = \varsigma_0 + r_0 + g_2(r_0, \varsigma_0), \end{cases} \tag{5.6}$$

where

$$r_1 = r(\rho_1) = r(\rho_0) + \int_0^1 \frac{\partial r}{\partial \rho}(\rho_0 + sf_1(\rho_0, \varsigma_0)) \cdot f_1(\rho_0, \varsigma_0)\mathrm{d}s,$$
$$r_0 = r(\rho_0).$$

From (5.5) and (5.6), $\int_0^1 |\mathtt{C}_2'(\theta)|\mathrm{d}\theta$ and $\int_0^1 |\mathtt{S}_2'(\theta)|\mathrm{d}\theta$ are bounded by a constant, therefore,

$$|\partial_{r_0}^i \partial_{\varsigma_0}^j g_k(r_0, \varsigma_0)| \le D \cdot r_0^{-\varepsilon_0}, \quad k = 1, 2, \tag{5.7}$$

when ε_0 in Lemma 4.1 is chosen small enough to make it smaller than the positive number which is dependent on c_0, c_1 and a and make the Poincaré map $\mathcal{P}_1$ satisfy (6.1) in Theorem 6.1.

Define the quasi-periodic mapping $\mathcal{M}$ as

$$\mathcal{M}: \begin{cases} \theta_1 = \theta + r + f(\theta, r), \\ r_1 = r + g(\theta, r), \end{cases} \qquad (\theta, r) \in \mathbb{R} \times [a_0, b_0], \tag{5.8}$$

where the function $f(\theta, \cdot)$, $g(\theta, \cdot)$ are quasi-periodic in θ with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$.

Definition 5.1 (see [13]) *Let $\mathcal{M}$ be a mapping defined as* (5.8). *If $\mathcal{M} : \mathbb{R} \times [a_0, b_0] \to \mathbb{R}^2$ is symplectic with respect to the usual symplectic structure* $\mathrm{d}r \wedge \mathrm{d}\theta$ *and for every curve $\Gamma : \theta = \xi + \varphi(\xi)$, $r = \psi(\xi)$, where the continuous functions φ and ψ are quasi-periodic in ξ with the frequency ω, there is*

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} r\mathrm{d}\theta = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} r_1 \mathrm{d}\theta_1,$$

we say that $\mathcal{M}$ is an exact symplectic map.

Lemma 5.1 (see [13]) *If the mapping* (5.8) *is an exact symplectic map, then it has intersection property.*

Proof of Theorem 1.1 Since all the transformations are canonical, the Poincaré map $\mathcal{P}_1$ is an exact symplectic. For all the detail above, the Poincaré map $\mathcal{P}_1$ satisfies all the requirements in Theorem 6.1. For any rotation number ϖ satisfying (6.3), we can obtain a quasi-periodic invariant curve of $\mathcal{P}_1$ with the form (6.4). Let $(x, \dot{x})$ be the solution of (1.4) staying in the interior of some quasi-periodic invariant curves with appropriate rotation number ϖ satisfying (6.3). Since all the transformations are canonical, every solution starting from $(x, \dot{x})$ is confined in the interior of the time quasi-periodic cylinder whose boundary is one of the quasi-periodic invariant curves and thus this solution is bounded. Notice that the initial value can be chosen large enough, thus, all the solutions are bounded.

6 Appendix

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^m smooth function. Define

$$|x| = \max\{|\theta|, |r|\} \quad \text{for } x = (\theta, r) \in \mathbb{R}^2,$$

$$|f|_{\mathbb{R}^2} = \sup_{\mathbb{R}^2} |f(x)|,$$
$$\|f\|_m = \sum_{|k|\le m} \sup_{x\in\mathbb{R}^2} |D^k h(x)| \quad \text{for } m \in \mathbb{N}^+.$$

In 2017, Huang-Li-Liu [13] established the twist theorem of the following smooth quasi-periodic mapping.

Theorem 6.1 (see [13]) *The quasi-periodic mapping $\mathcal{M}$ given as* (5.8),

$$\mathcal{M}: \begin{cases} \theta_1 = \theta + r + f(\theta, r), \\ r_1 = r + g(\theta, r), \end{cases} \qquad (\theta, r) \in \mathbb{R} \times [a_0, b_0],$$

is of class $\mathcal{C}^m$ $(m > 2\tau + 1 > 2n + 1)$ and satisfies the intersection property. The functions $f(\theta, r)$, $g(\theta, r)$ are quasi-periodic in θ with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ satisfying the following conditions:

$$\begin{cases} |f|_{\mathbb{R}^2} + \ |g|_{\mathbb{R}^2} \le o_1(m, \Gamma, \gamma, \tau), \\ \|f\|_m + \|g\|_m \le o_2(m, \Gamma, \gamma, \tau), \end{cases} \tag{6.1}$$

where Γ is the Gamma function, γ, τ are constants satisfying

$$0 < \gamma < \frac{1}{2} \min\{1, 12^3(b_0 - a_0)\}, \quad \tau > n \tag{6.2}$$

and $o_1(m, \Gamma, \gamma, \tau)$, $o_2(m, \Gamma, \gamma, \tau)$ are sufficiently small functions of m, Γ, γ, τ and a_0, $b_0 > 0$ are two constants.

Then for any rotation number ϖ satisfying the inequalities

$$\begin{cases} a_0 + 12^{-3}\gamma \le \varpi \le b_0 - 12^{-3}\gamma, \\ \left|\langle k, w\rangle \dfrac{\varpi}{2\pi} - j\right| \ge \dfrac{\gamma}{|\ k\ |^\tau} \quad \text{for all } k \in \mathbb{Z}^n \backslash \{0\},\ j \in \mathbb{Z}, \end{cases} \tag{6.3}$$

the quasi-periodic mapping $\mathcal{M}$ has an invariant curve Γ_0 with the form

$$\theta = \theta' + \phi(\theta'), \quad r = \psi(\theta'), \tag{6.4}$$

where ϕ, ψ are quasi-periodic with the frequency $\omega = (\omega_1, \cdots, \omega_n)$ and the invariant curve Γ_0 is continuous and quasi-periodic with the frequency ω. Moreover, the restriction of $\mathcal{M}$ onto Γ_0 is

$$\mathcal{M}\ |_{\Gamma_0}: \theta_1' = \theta' + \varpi.$$

Remark 6.1 (see [13]) If all conditions of Theorem 6.1 hold, then the mapping $\mathcal{M}$ has many invariant curves Γ_0, which can be labeled by the form

$$\mathcal{M}\ |_{\Gamma_0}: \theta_1' = \theta' + \varpi$$

of the restriction of $\mathcal{M}$ onto Γ_0. In fact, given any ϖ satisfying the inequalities (6.3), there exists an invariant curve Γ_0 of $\mathcal{M}$ which is quasi-periodic with the frequency ω, and the restriction of $\mathcal{M}$ onto Γ_0 has the form

$$\mathcal{M}\ |_{\Gamma_0}: \theta_1' = \theta' + \varpi.$$

References

[1] Morris, G. R., A case of boundedness in Littlewood's problem on oscillatory differential equations, *Bull. Aust. Math. Soc.*, **14**(1), 1976, 71–93.

[2] Alonso, J. M. and Ortega, R., Unbounded solutions of semilinear equations at resonance, *Nonlinearity*, **9**, 1996, 1099–1111.

[3] Dieckerhoff, R. and Zehnder, E., Boundedness of solutions via the twist-theorem, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **14**(1), 1987, 79–95.

[4] Liu, B., Boundedness for solutions of nonlinear periodic differential equations via Moser's twist theorem, *Acta Mathematica Sinica*, **8**(1), 1992, 91–98.

[5] Ortega, R., Boundedness in a piecewise linear oscillator and a variant of the small twist theorem, *Proc. Lond. Math. Soc.*, **79**, 1999, 381–413.

[6] You, J., Boundedness for solutions of superlinear Duffing equations via the twist theorem, *Sci. China Scientia Sinica.*, **35**(4), 1992, 399–412.

[7] Küpper, T. and You, J., Existence of quasi-periodic solutions and Littlewood's boundedness problem of Duffing equations with subquadratic potentials, *Nonlinear Anal. TMA*, **35**(5), 1999, 549–559.

[8] Liu, B., On Littlewood's boundedness problem for sublinear Duffing equations, *Trans. Amer. Math. Soc.*, **353**(4), 2001, 1567–1585.

[9] Wang, Y., Boundedness for sublinear Duffing equations with time-dependent potentials, *J. Differential Equations*, **247**(1), 2009, 104–118.

[10] Zharnitsky, V., Invariant curve theorem for quasi-periodic twist mappings and stability of motion in the Fermi-Ulam problem, *Nonlinearity*, **13**, 2000, 1123–1136.

[11] Levi, M. and Zehnder, E., Boundedness of solutions for quasi-periodic potentials, *SIAM J. Math. Anal.*, **26**, 1996, 1233–1256.

[12] Liu, B., Invariant curves of quasi-periodic reversible mappings, *Nonlinearity*, **18**(2), 2005, 685–701.

[13] Huang, P., Li, X. and Liu, B., Quasi-periodic solutions for an asymmetric oscillation, *Nonlinearity*, **29**(10), 2017, 3006–3030.

[14] Huang, P., Li, X. and Liu, B., Invariant curves of smooth quasi-periodic mappings, *Discrete Contin. Dyn. Syst.*, **38**(1), 2018, 131–154.

[15] Siegel, C. and Moser, J., Lectures on Celestial Mechanics, Springer-Verlag, Berlin, 1997.