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Metrics and Connections on the Bundle of Affinor Frames

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Abstract In this paper the authors consider the bundle of affinor frames over a smooth manifold, define the Sasaki metric on this bundle, and investigate the Levi-Civita connection of Sasaki metric. Also the authors determine the horizontal lifts of symmetric linear connection from a manifold to the bundle of affinor frames and study the geodesic curves corresponding to the horizontal lift of the linear connection.

Keywords Bundle of affinor frames, Riemannian manifold, Sasaki metric, Horizontal lift, Geodesic curve
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1 Introduction

Fiber bundles play an important role in all major areas of modern differential geometry. Prime examples of fiber bundles are tangent, cotangent and tensor bundles over differentiable manifolds. The geometry of the tangent bundle was first investigated in the fundamental paper [16] of Sasaki. He used the metric g, given on the differentiable manifold M_n, to construct the Riemannian metric G on the tangent bundle $T(M_n)$ of M_n. This metric is called the Sasaki metric (or the diagonal lift of g to $T(M_n)$). The curvature properties of the Sasaki metric on $T(M_n)$ are studied by Kowalski in [6]. Interesting relationships between the geometric properties of the base manifold (M_n, g) and its tangent bundle $(T(M_n), G)$ with the Sasaki metric are investigated by Aso [1], Musso and Tricerri [11]. The Sasaki metric on the cotangent bundle was studied by Mok [9], Salimov and Agca [13]. In [3, 12, 14–15] the Sasaki metric was investigated on the tensor bundles. The Sasaki metrics on the linear frame and linear coframe bundles were studied by Mok [10], Kowalski and Sekizawa [7], Fattayev and Salimov [5]. In the study of fiber bundles, a special place is also occupied by lifts of linear connections and their geodesic curves (see, for example, [2, 8–9, 17–18]).

In this paper, we consider a bundle of affinor frames over a smooth manifold, and using the diagonal lift of the Riemannian metric, we study some questions of the differential geometry of this bundle, also we define the horizontal lifts of the symmetric linear connection and investigate their geodesic curves.

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2 Preliminaries

In this section, we summarize all the basic definitions and results that are needed later. In the following all manifolds, maps, tensor fields, connections and metrics in question are supposed to be differentiable of class C^{∞}.

Let M_n be an n-dimensional differentiable manifold and $L_1^1(M_n)$ the bundle of an affinor frames of M_n (see [4]). The bundle of an affinor frames $L_1^1(M_n)$ consists of all pairs (x, A_x), where x is a point of M_n and A_x is a basis (an affinor frame) for the linear space $T_1^1(x)$ of all affinors at point x. We denote by $\pi : L_1^1(M_n) \to M_n$ the projection map, defined by $\pi(x, A_x) = x$. For the coordinate system (U, x^i) in M_n, we put $L_1^1(U) = \pi^{-1}(U)$ and an affinor X_β^α of the frame A_x can be uniquely expressed in the form $X_\beta^\alpha = X_{\beta i}^{\alpha j}\left(\frac{\partial}{\partial x^j}\right)_x \otimes (\mathrm{d}x^i)_x$, so that $\{L_1^1(U), (x^i, X_{\beta i}^{\alpha j})\}$ is a coordinate system in $L_1^1(M_n)$ (see [4]). Indices $i, j, k, \cdots, \alpha, \beta, \gamma, \cdots$ have range in $\{1, 2, \cdots, n\}$, while indices $A, B, C, \cdots$ have range in $\{1, \cdots, n, n+1, \cdots, n+n^4\}$ and indices $i_{\alpha\beta}, l_{\gamma\delta}, k_{\sigma\tau}, \cdots$ have range in $\{n+1, \cdots, n+n^4\}$. For simplicity, we give the following notation: $(x^I) = (x^i, x^{i_{\alpha\beta}}) = (x^i, X_{\beta i}^{\alpha j})$. Summation over repeated indices is always implied.

We denote by $\Im_s^r(M_n)$ the set of all differentiable tensor fields of type (r, s) on M_n. Let ∇ be a linear connection, $V \in \Im_0^1(M_n)$ a vector field and $B \in \Im_1^1(M_n)$ an affinor field on M_n with local components Γ_{ij}^k, V^i and B_i^j, respectively. Then there is exactly one vector field ${}^H V$ on $L_1^1(M_n)$, called the horizontal lift of V, and exactly one vector field ${}^{V_{\alpha\beta}} B$ on $L_1^1(M_n)$ for each pair $\alpha, \beta = 1, 2, \cdots, n$, called the $\alpha\beta^{th}$ -vertical lift of B, that are known to be defined in $L_1^1(U)$ (see [4]) by

$$ {}^H V = V^i \frac{\partial}{\partial x^i} + V^k (X_{\beta m}^{\alpha j} \Gamma_{ki}^m - X_{\beta i}^{\alpha m} \Gamma_{km}^j) \frac{\partial}{\partial X_{\beta i}^{\alpha j}}, \tag{2.1} $$

$$ {}^{V_{\alpha\beta}} B = \delta_\alpha^\gamma \delta_\sigma^\beta B_i^j \frac{\partial}{\partial X_{\sigma i}^{\gamma j}} \tag{2.2} $$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_{\beta i}^{\alpha j}}\right\}$ in $L_1^1(M_n)$, where δ_α^γ is the Kronecker delta.

If f is a differentiable function on M_n, ${}^V f = f \circ \pi$ denotes its canonical vertical lift to $L_1^1(M_n)$.

Let (U, x^i) be a coordinate system in M_n. In $U \subset M_n$, we put

$$ X_{(i)} = \frac{\partial}{\partial x^i} = \delta_i^h \frac{\partial}{\partial x^h} \in \Im_0^1(M_n), $$
$$ \Lambda_i^j = \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j = \delta_i^h \delta_k^j \partial_h \otimes \mathrm{d}x^k \in \Im_1^1(M_n), \quad i, j = 1, 2, \cdots, n. $$

From (2.1)–(2.2), we have

$$ {}^H X_{(i)} = \delta_i^h \partial_h + (X_{\beta m}^{\alpha k} \Gamma_{hi}^m - X_{\beta i}^{\alpha m} \Gamma_{hm}^k) \frac{\partial}{\partial X_{\beta h}^{\alpha k}}, \tag{2.3} $$

$$ {}^{V_{\alpha\beta}} \Lambda_i^j = \delta_\gamma^\alpha \delta_\beta^\sigma \delta_i^h \delta_k^j \frac{\partial}{\partial X_{\sigma h}^{\gamma k}} \tag{2.4} $$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\}$ in $L_1^1(M_n)$. These $n+n^4$ vector fields are linearly independent and generate, respectively, the horizontal distribution of linear connection ∇ and the vertical distribution of $L_1^1(M_n)$. We call the set $\{{}^H X_{(i)}, {}^{V_{\alpha\beta}}\Lambda_i^j\}$ the frame adapted to the linear connection ∇ on $\pi^{-1}(U) \subset L_1^1(M_n)$. Putting

$$D_i = {}^H X_{(i)}, \quad D_{i_{\alpha\beta}} = {}^{V_{\alpha\beta}}\Lambda_i^j,$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_{\alpha\beta}}\}$. From (2.3)–(2.4) we see that ${}^H V$ and ${}^{V_{\alpha\beta}}B$ have respectively, components

$$ {}^H V = V^i D_i = ({}^H V^I) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \tag{2.5}$$

$$ {}^{V_{\alpha\beta}}B = B_i^j \delta_\alpha^\gamma \delta_\sigma^\beta D_{i_{\gamma\sigma}} = ({}^{V_{\alpha\beta}}B^I) = \begin{pmatrix} 0 \\ \delta_\alpha^\gamma \delta_\sigma^\beta B_i^j \end{pmatrix} \tag{2.6}$$

with respect to the adapted frame $\{D_I\}$, V^i and B_i^j are the local components of V and B on M_n, respectively.

Let $B \in \Im_1^1(M_n)$, which is locally represented by $B = B_i^j \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^i$. The vector fields γB and $\widetilde{\gamma} B$ on $L_1^1(M_n)$ are defined by

$$\begin{cases} \gamma B = (X_{\beta i}^{\alpha m} B_m^j) \dfrac{\partial}{\partial X_{\beta i}^{\alpha j}}, \\ \widetilde{\gamma} B = (X_{\beta m}^{\alpha j} B_i^m) \dfrac{\partial}{\partial X_{\beta i}^{\alpha j}} \end{cases}$$

with respect to the natural frame $\{\partial_i, \partial_{i_{\alpha\beta}}\}$ in $L_1^1(M_n)$.

The brackets of vertical and horizontal lifts are expressed by the following formulae:

$$\begin{aligned} &[{}^{V_{\alpha\beta}}B, {}^{V_{\lambda\tau}}C] = 0, \\ &[{}^H X, {}^{V_{\alpha\beta}}B] = {}^{V_{\alpha\beta}}(\nabla_X B), \\ &[{}^H X, {}^H Y] = {}^H[X,Y] + (\widetilde{\gamma} - \gamma)(R(X,Y)) \end{aligned} \tag{2.7}$$

for all $X, Y \in \Im_0^1(M_n)$ and $B, C \in \Im_1^1(M_n)$, where $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $\widetilde{\gamma} - \gamma : B \to \Im_0^1(L_1^1(M_n))$ equals

$$(\widetilde{\gamma} - \gamma)B = \begin{pmatrix} 0 \\ X_{\beta m}^{\alpha j} B_i^m - X_{\beta i}^{\alpha m} B_m^j \end{pmatrix} \tag{2.8}$$

for any $B \in \Im_1^1(M_n)$.

Remark 2.1 Using equality (2.6), it is easy to establish that a vertical vector field $(\overline{\gamma} - \gamma)(R(X,Y)) \in \Im_1^1(L_1^1(M_n))$ can be represented as

$$(\overline{\gamma} - \gamma)(R(X,Y)) = \sum_{\alpha,\gamma=1}^{n} ({}^{V_{\alpha\gamma}} X_\gamma^\alpha \circ R(X,Y)). \tag{2.9}$$

3 Sasaki Metric on $L_1^1(M_n)$

Let (M_n, g) be a Riemannian manifold. For each $x \in M_n$, the extension of scalar product g (denoted by G) is defined on the linear space $L_1^1(x) = \pi^{-1}(x)$ by

$$G(B, C) = g_{pq} g^{ij} B_i^p C_j^q$$

for all $B, C \in \Im_1^1(M_n)$.

Definition 3.1 *The Sasaki metric* ${}^S g$ *(or diagonal lift of* g*) is defined on* $L_1^1(M_n)$ *by the following three relations*:

$$ {}^S g({}^{V_{\alpha\beta}} B, {}^{V_{\gamma\sigma}} C) = \delta^{\alpha\gamma} \delta_{\beta\sigma} {}^V (G(B, C)), \tag{3.1}$$

$$ {}^S g({}^{V_{\alpha\beta}} B, {}^H Y) = 0, \tag{3.2}$$

$$ {}^S g({}^H X, {}^H Y) = {}^V (g(X, Y)) \tag{3.3}$$

for all $X, Y \in \Im_0^1(M_n)$ *and* $B, C \in \Im_1^1(M_n)$.

We recall that any element $t \in \Im_2^0(L_1^1(M_n))$ is completely determined by its action on vector fields of type ${}^H X$ and ${}^{V_{\alpha\beta}} B$. From this it follows that ${}^S g$ is completely determined by (3.1)–(3.3).

From (3.1)–(3.3), we see that the Sasaki metric ${}^S g$ has the components

$$ ({}^S g_{IJ}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta^{\alpha\gamma} \delta_{\beta\sigma} g_{pq} g^{ij} \end{pmatrix}, \tag{3.4}$$

$$ ({}^S g^{IJ}) = \begin{pmatrix} g^{ij} & 0 \\ 0 & \delta_{\alpha\gamma} \delta^{\beta\sigma} g^{pq} g_{ij} \end{pmatrix} \tag{3.5}$$

with respect to the adapted frame $\{D_I\}$, where g_{ij} and g^{ij} are the local covariant and contravariant components of g on M_n. For cotangent bundle ${}^C T(M_n)$, linear frame bundle $F(M_n)$, linear coframe bundle $F^*(M_n)$ and $(1,1)$-tensor bundle $T_1^1(M_n)$, see [5–6, 13–14], respectively.

Now we consider local 1-form $\widetilde{\eta}^I$ in $\pi^{-1}(U)$, defined by

$$\widetilde{\eta}^I = \widetilde{A}^I{}_J \mathrm{d}x^J,$$

where

$$ A^{-1} = (\widetilde{A}^I{}_J) = \begin{pmatrix} \widetilde{A}^i{}_j & \widetilde{A}^i{}_{j_{\beta\sigma}} \\ \widetilde{A}^{i_{\alpha\gamma}}{}_j & \widetilde{A}^{i_{\alpha\gamma}}{}_{j_{\beta\sigma}} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ X_{\gamma i}^{\alpha m} \Gamma_{jm}^h - X_{\gamma m}^{\alpha h} \Gamma_{ji}^m & \delta_\beta^\alpha \delta_\gamma^\sigma \delta_k^h \delta_i^j \end{pmatrix}. \tag{3.6}$$

The matrix (3.6) is the inverse of the matrix

$$ A = (A_L{}^J) = \begin{pmatrix} A_l{}^j & A_{l_{\tau\lambda}}{}^j \\ A_l{}^{j_{\beta\sigma}} & A_{l_{\tau\lambda}}{}^{j_{\beta\sigma}} \end{pmatrix} = \begin{pmatrix} \delta_l^j & 0 \\ -X_{\sigma j}^{\beta m} \Gamma_{lm}^k + X_{\sigma m}^{\beta k} \Gamma_{lj}^m & \delta_\tau^\beta \delta_\sigma^\lambda \delta_r^k \delta_j^l \end{pmatrix} \tag{3.7}$$

of the transformation $D_L = A_L{}^J \partial_J$ (see (2.3)–(2.4)). We easily see that the set $\{\widetilde{\eta}^I\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e., $\widetilde{\eta}^I(D_K) = \widetilde{A}^I{}_J A_K{}^J = \delta_K^I$.

Since the adapted frame $\{D_I\}$ is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^K D_K$$

and then we have

$$\Omega_{IJ}{}^K = (D_I A_J{}^L - D_J A_I{}^L)\widetilde{A}^K{}_L.$$

According to (2.3)–(2.4) and (3.6)–(3.7), the components of non-holonomic object $\Omega_{IJ}{}^K$ are given by

$$\begin{cases} \Omega_{ij_{\beta\sigma}}{}^{k_{\tau\lambda}} = -\Omega_{j_{\beta\sigma}i}{}^{k_{\tau\lambda}} = \delta^\tau_\beta \delta^\sigma_\lambda (\delta^j_k \Gamma^r_{il} - \delta^r_l \Gamma^j_{ik}), \\ \Omega_{ij}{}^{k_{\tau\lambda}} = X^{\tau r}_{\lambda m} R^m_{ijk} - X^{\tau m}_{\lambda k} R^r_{ijm} \end{cases} \tag{3.8}$$

with all the others being zero, where $R_{ijk}{}^h$ are the local components of the curvature tensor R of the Riemannian metric g on M_n.

Let ${}^S\nabla$ be the Levi-Civita connection of the Sasaki metric ${}^Sg$. Putting ${}^S\nabla_{D_I} D_J = {}^S\Gamma_{IJ}{}^K D_K$, from the equation

$${}^S\nabla_{\widetilde{X}} \widetilde{Y} - {}^S\nabla_{\widetilde{Y}} \widetilde{X} = [\widetilde{X}, \widetilde{Y}], \quad \forall \widetilde{X}, \widetilde{Y} \in \Im^1_0(L^1_1(M_n)),$$

we have

$${}^S\Gamma^K_{IJ} - {}^S\Gamma^K_{JI} = \Omega_{IJ}{}^K \tag{3.9}$$

with respect to the adapted frame $\{D_I\}$, where ${}^S\Gamma^K_{IJ}$ are the components of the Levi-Civita connection ${}^S\nabla$.

The equation $({}^S\nabla_{\widetilde{X}} {}^Sg)(\widetilde{Y}, \widetilde{Z}) = 0$, $\forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(L^1_1(M_n))$, has the form

$$D_L {}^Sg_{IJ} - {}^S\Gamma^K_{LI} {}^Sg_{KJ} - {}^S\Gamma^K_{LJ} {}^Sg_{IK} = 0 \tag{3.10}$$

with respect to the adapted frame $\{D_K\}$. Thus, we have from (3.9)–(3.10)

$${}^S\Gamma^K_{IJ} = \frac{1}{2} {}^Sg^{KL}(D_I {}^Sg_{LJ} + D_J {}^Sg_{IL} - D_L {}^Sg_{IJ}) + \frac{1}{2}(\Omega_{IJ}{}^K + \Omega^K_{\cdot\ IJ} + \Omega^K_{\cdot\ JI}), \tag{3.11}$$

where $\Omega^K_{\cdot\ IJ} = {}^Sg^{KL} {}^Sg_{PJ} \Omega_{LI}{}^P$.

Taking account of (3.4)–(3.5), (3.8) and (3.11), we immediately get the following theorem.

Theorem 3.1 *Let (M_n, g) be a Riemannian manifold and ${}^S\nabla$ be the Levi-Civita connection of the bundle of affinor frames $L^1_1(M_n)$ equipped with the Sasaki metric ${}^Sg$. The particular values of ${}^S\Gamma^K_{IJ}$ for different indices, are then found to be*

$$\begin{aligned} {}^S\Gamma^k_{ij} &= \Gamma^k_{ij}, \\ {}^S\Gamma^{k_{\tau\lambda}}_{ij} &= \frac{1}{2}(X^{\tau r}_{\lambda m} R^m_{ijk} - X^{\tau m}_{\lambda k} R^r_{ijm}), \\ {}^S\Gamma^k_{i_{\alpha\gamma} j_{\beta\sigma}} &= {}^S\Gamma^{k_{\tau\lambda}}_{i_{\alpha\beta} j} = {}^S\Gamma^{k_{\tau\lambda}}_{i_{\alpha\gamma} j_{\beta\sigma}} = 0, \\ {}^S\Gamma^k_{i j_{\beta\sigma}} &= \frac{1}{2}(g_{la} X^{\beta a}_{\sigma m} R^{mj}_{\cdot\,\cdot\ i}{}^k - g^{jb} X^{\beta m}_{\sigma b} R^k_{lmi}), \end{aligned}$$

$$
\begin{aligned}
{}^S\Gamma^{k_{\tau\lambda}}_{ij_{\beta\sigma}} &= \delta^\tau_\beta \delta^\sigma_\lambda (\delta^j_k \Gamma^r_{il} - \delta^r_l \Gamma^j_{ik}),\\
{}^S\Gamma^{k}_{i_{\alpha\gamma}j} &= \frac{1}{2}(g_{ha} X^{\alpha a}_{\gamma m} R^{mi\ \ k}_{\cdot\ \cdot\ j} - g^{ib} X^{\alpha m}_{\gamma b} R^k_{hmj})
\end{aligned}
$$

with respect to the adapted frame $\{D_K\}$, *where* $R^{mj\ \ k}_{\cdot\ \cdot\ i} = g^{ml} g^{js} R_{lsi}^{\ \ \ k}$.

It is well-known that the Levi-Civita connection ∇ of a Riemannian metric g is given by Koszul formula

$$
\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z)\\
&\quad - g([Y,Z],X) + g([Z,X],Y)
\end{aligned}
\tag{3.12}
$$

for all vector fields $X, Y, Z \in \Im^1_0(M_n)$.

The following theorem holds.

Theorem 3.2 *Let* M_n *be a Riemannian manifold with the metric* g *and* $L^1_1(M_n)$ *be the bundle of affinor frames of* M_n *equipped with the Sasaki metric* ${}^S g$. *Then the corresponding Levi-Civita connection* ${}^S\nabla$ *satisfies the following relations*:

(i) ${}^S\nabla_{{}^H X}{}^H Y = {}^H(\nabla_X Y) + \frac{1}{2}\sum\limits_{\tau,\lambda=1}^{n} {}^{V_{\tau\lambda}}(X^\tau_\lambda \circ R(X,Y) - R(X,Y)\circ X^\tau_\lambda)$,

(ii) ${}^S\nabla_{{}^H X}{}^{V_{\beta\sigma}} B = {}^{V_{\beta\sigma}}(\nabla_X B) + \frac{1}{2}\sum\limits_{\tau,\lambda=1}^{n} \delta^{\tau\beta}\delta_{\lambda\sigma}{}^H(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\ \ ,X)^s_i))B^{iq} + g^{ij}(R(X^\tau_{\lambda i}, X))B_j)$,

(iii) ${}^S\nabla_{V_{\alpha\gamma}A}{}^H Y = \frac{1}{2}\sum\limits_{\tau,\lambda=1}^{n} \delta^{\tau\alpha}\delta_{\lambda\gamma}{}^H(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\ \ ,Y)^s_i))A^{iq} + g^{ij}(R(X^\tau_{\lambda i}, Y))A_j)$,

(iv) ${}^S\nabla_{V_{\alpha\gamma}A}{}^{V_{\beta\sigma}} B = 0$ *for all* $X, Y \in \Im^1_0(M_n)$ *and* $A, B \in \Im^1_1(M)$, *where* $B_j = (B^q_j)$, $B^{iq} = g^{ij} B^q_j$, $X^\tau_{\lambda i} = (X^{\tau s}_{\lambda i})$, $R(\ \ ,X)Y \in \Im^1_1(M_n)$ *and* $g^{-1}\circ R(\ \ ,X)Y \in \Im^2_0(M_n)$.

Proof (i) By the help of Koszul formula (3.12), (2.5)–(2.7), (2.9) and (3.1)–(3.3), we have

$$
\begin{aligned}
2{}^S g({}^S\nabla_{{}^H X}{}^H Y, {}^H Z) &= {}^H X({}^S g({}^H Y, {}^H Z)) + {}^H Y({}^S g({}^H Z, {}^H X)) - {}^H Z({}^S g({}^H X, {}^H Y))\\
&\quad - {}^S g({}^H X, [{}^H Y, {}^H Z]) + {}^S g({}^H Y, [{}^H Z, {}^H X]) + {}^S g({}^H Z, [{}^H X, {}^H Y])\\
&= 2g(\nabla_X Y, Z)
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
2{}^S g(\widetilde{\nabla}_{{}^H X}{}^H Y, {}^{V_{\beta\sigma}}C) &= {}^H X({}^S g({}^H Y, {}^{V_{\beta\sigma}}C)) + {}^H Y({}^S g({}^{V_{\beta\sigma}}C, {}^H X))\\
&\quad - {}^{V_{\beta\sigma}}C({}^S g({}^H X, {}^H Y)) - {}^S g({}^H X, [{}^H Y, {}^{V_{\beta\sigma}}C])\\
&\quad + {}^S g({}^H Y, [{}^{V_{\beta\sigma}}C, {}^H X]) + {}^S g({}^{V_{\beta\sigma}}C, [{}^H X, {}^H Y])\\
&= -{}^{V_{\beta\sigma}}C({}^S g({}^H X, {}^H Y)) - {}^S g({}^H X, {}^{V_{\beta\sigma}}(\nabla_Y C))\\
&\quad + {}^S g({}^H Y, -{}^{V_{\beta\sigma}}(\nabla_X C)) + {}^S g({}^{V_{\beta\sigma}}C, {}^H[X,Y] + (\overline{\gamma}-\gamma)(R(X,Y)))\\
&= {}^S g({}^{V_{\beta\sigma}}C, (\overline{\gamma}-\gamma)(R(X,Y)))\\
&= {}^S g\Big({}^{V_{\beta\sigma}}C, \sum_{\tau,\lambda=1}^{n} {}^{V_{\tau\lambda}}(X^\tau_\lambda \circ R(X,Y) - R(X,Y)\circ X^\tau_\lambda)\Big).
\end{aligned}
\tag{3.14}
$$

By combining of (3.13) and (3.14), we obtain

$$ {}^{S}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}\sum_{\tau,\lambda=1}^{n} {}^{V_{\tau\lambda}}(X^{\tau}_{\lambda}\circ R(X,Y) - R(X,Y)\circ X^{\tau}_{\lambda}). $$

(ii) The statement is obtained as follows:

$$ \begin{aligned} 2^{S}g({}^{S}\nabla_{{}^{H}X}{}^{V_{\beta\sigma}}B, {}^{H}Y) &= {}^{H}X({}^{S}g({}^{V_{\beta\sigma}}B, {}^{H}Y)) + {}^{V_{\beta\sigma}}B({}^{S}g({}^{H}Y, {}^{H}X)) \\ &\quad - {}^{H}Y({}^{S}g({}^{H}X, {}^{V_{\beta\sigma}}B)) - {}^{S}g({}^{H}X, [{}^{V_{\beta\sigma}}B, {}^{H}Y]) \\ &\quad + {}^{S}g({}^{V_{\beta\sigma}}B, [{}^{H}Y, {}^{H}X]) + {}^{S}g({}^{H}Y, [{}^{H}X, {}^{V_{\beta\sigma}}B]) \\ &= {}^{S}g({}^{V_{\beta\sigma}}B, (\overline{\gamma} - \gamma)(R(Y,X))) \\ &= {}^{S}g\Big({}^{V_{\beta\sigma}}B, \sum_{\tau,\lambda=1}^{n} {}^{V_{\tau\lambda}}(X^{\tau}_{\lambda}\circ R(Y,X) - R(Y,X)\circ X^{\tau}_{\lambda})\Big) \\ &= {}^{S}g\Big(\sum_{\tau,\lambda=1}^{n} {}^{V_{\tau\lambda}}(X^{\tau}_{\lambda}\circ R(Y,X) - R(Y,X)\circ X^{\tau}_{\lambda}\Big), {}^{V_{\beta\sigma}}B\Big) \\ &= \sum_{\tau,\lambda=1}^{n} {}^{S}g({}^{V_{\tau\lambda}}(X^{\tau}_{\lambda}\circ R(Y,X) - R(Y,X)\circ X^{\tau}_{\lambda}), {}^{V_{\beta\sigma}}B) \\ &= \sum_{\tau,\lambda=1}^{n} \delta^{\tau\beta}\delta_{\lambda\sigma} G(X^{\tau}_{\lambda}\circ R(Y,X) - R(Y,X)\circ X^{\tau}_{\lambda}, B). \end{aligned} $$

Using

$$ \begin{aligned} &G(X^{\tau}_{\lambda}\circ R(Y,X) - R(Y,X)\circ X^{\tau}_{\lambda}, B) \\ &= g_{pq}g^{ij}(X^{\tau}_{\lambda}\circ R(Y,X))^{p}_{i}B^{q}_{j} - g_{pq}g^{ij}(R(Y,X)\circ X^{\tau}_{\lambda})^{p}_{i}B^{q}_{j} \\ &= g_{pq}g^{ij}X^{\tau p}_{\lambda s}R^{s}_{kli}Y^{k}X^{l}B^{q}_{j} - g_{pq}g^{ij}R^{p}_{kls}X^{\tau s}_{\lambda i}Y^{k}X^{l}B^{q}_{j} \\ &= g_{km}g_{pq}X^{\tau p}_{\lambda s}R^{m}_{\cdot\, li}{}^{s}Y^{k}X^{l}B^{iq}_{\cdot} - g_{km}g^{ij}R^{m}_{lsq}X^{\tau s}_{\lambda i}Y^{k}X^{l}B^{q}_{j} \\ &= g_{km}g_{pq}X^{\tau p}_{\lambda s}g^{nr}R_{\ rli}{}^{s}X^{l}B^{iq}_{\cdot}Y^{k} + g_{km}g^{ij}R^{m}_{lsq}X^{\tau s}_{\lambda i}Y^{k}X^{l}B^{q}_{j} \\ &= g(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, X)^{s}_{i}))B^{iq}, Y) + g(g^{ij}(R(X^{\tau}_{\lambda i}, X))B_{j}, Y) \\ &= {}^{S}g({}^{H}(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, X)^{s}_{i}))B^{iq}), {}^{H}Y) \\ &\quad + {}^{S}g({}^{H}(g^{ij}(R(X^{\tau}_{\lambda i}, X))B_{j}), {}^{H}Y), \end{aligned} $$

we have

$$ \begin{aligned} 2^{S}g({}^{S}\nabla_{{}^{H}X}{}^{V_{\beta\sigma}}B, {}^{H}Y) &= \sum_{\tau,\lambda=1}^{n} \delta^{\tau\beta}\delta_{\lambda\sigma}{}^{S}g({}^{H}(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, X)^{s}_{i}))B^{iq}), {}^{H}Y) \\ &\quad + \sum_{\tau,\lambda=1}^{n} \delta^{\tau\beta}\delta_{\lambda\sigma}{}^{S}g({}^{H}(g^{ij}(R(X^{\tau}_{\lambda i}, X))B_{j}), {}^{H}Y). \end{aligned} $$

On the other hand,

$$ 2^{S}g({}^{S}\nabla_{{}^{H}X}{}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}C) = {}^{H}X({}^{S}g({}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}C)) + {}^{V_{\beta\sigma}}B({}^{S}g({}^{V_{\tau\lambda}}C, {}^{H}X)) $$

$$
\begin{aligned}
&- {}^{V_{\tau\lambda}}C({}^Sg({}^HX, {}^{V_{\beta\sigma}}B)) - {}^Sg({}^HX, [{}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}C]) \\
&+ {}^Sg({}^{V_{\beta\sigma}}B, [{}^{V_{\tau\lambda}}C, {}^HX]) + {}^Sg({}^{V_{\tau\lambda}}C, [{}^HX, {}^{V_{\beta\sigma}}B]) \\
= {}&{}^Sg({}^{V_{\beta\sigma}}(\nabla_X B), {}^{V_{\tau\lambda}}C) + {}^Sg({}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}(\nabla_X C)) \\
&- {}^Sg({}^{V_{\beta\sigma}}B, {}^{\tau\lambda}(\nabla_X C)) + {}^Sg({}^{V_{\tau\lambda}}C, {}^{V_{\beta\sigma}}(\nabla_X B)) \\
= {}&2{}^Sg({}^{V_{\beta\sigma}}(\nabla_X B), {}^{V_{\tau\lambda}}C).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&{}^S\nabla_{{}^HX}{}^{V_{\beta\sigma}}B \\
&= {}^{V_{\beta\sigma}}(\nabla_X B) + \frac{1}{2}\sum_{\tau,\lambda=1}^{n} \delta^{\tau\beta}\delta_{\lambda\sigma}{}^H(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, X)^s_i))B^{iq} + g^{ij}(R(X^{\tau}_{\lambda i}, X))B_j).
\end{aligned}
$$

(iii) By calculations analogy to those in (ii), we obtain

$$
\begin{aligned}
&2{}^Sg({}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^HY, {}^{V_{\beta\sigma}}B) \\
&= {}^HY({}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}B)) - {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Y B)) \\
&\quad - ({}^Sg({}^{V_{\beta\sigma}}B, {}^{V_{\alpha\gamma}}(\nabla_Y A)) \\
&= {}^Sg({}^{V_{\alpha\gamma}}(\nabla_Y A), {}^{V_{\beta\sigma}}B) + {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Y B)) \\
&\quad - {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Y B)) - ({}^Sg({}^{V_{\beta\sigma}}B, {}^{V_{\alpha\gamma}}(\nabla_Y A)) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&2{}^Sg({}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^HY, {}^HZ) \\
&= \sum_{\tau,\lambda=1}^{n} \delta^{\tau\alpha}\delta_{\lambda\gamma}{}^Sg({}^H(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, Y)^s_i))A^{iq}), {}^HZ) \\
&\quad + \sum_{\tau,\lambda=1}^{n} \delta^{\tau\alpha}\delta_{\lambda\gamma}{}^Sg({}^H(g^{ij}(R(X^{\tau}_{\lambda i}, Y))A_j), {}^HZ).
\end{aligned}
$$

Thus,

$$
{}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^HY = \frac{1}{2}\sum_{\tau,\lambda=1}^{n} \delta^{\tau\alpha}\delta_{\lambda\gamma}{}^H(g_{pq}(X^{\tau p}_{\lambda s}(g^{-1}\circ R(\quad, Y)^s_i))A^{iq} + g^{ij}(R(X^{\tau}_{\lambda i}, Y))A_j).
$$

(iv) By using the Koszul formula (3.12), (2.7) and (2.9), we yield

$$
\begin{aligned}
&2{}^Sg({}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^{V_{\beta\sigma}}B, {}^HZ) \\
&= -{}^HZ({}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}B)) + {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Z B)) + {}^Sg({}^{V_{\beta\sigma}}B, {}^{V_{\alpha\gamma}}(\nabla_Z A)) \\
&= -{}^Sg({}^{V_{\alpha\gamma}}(\nabla_Z A), {}^{V_{\beta\sigma}}B) - {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Z B)) \\
&\quad + {}^Sg({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}(\nabla_Z B)) + {}^Sg({}^{V_{\beta\sigma}}B, {}^{V_{\alpha\gamma}}(\nabla_Z A)) \\
&= 0
\end{aligned}
$$

and

$$\begin{aligned}
&2\,{}^S g({}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}C) \\
&= {}^{V_{\alpha\gamma}}A({}^S g({}^{\beta\sigma}B, {}^{V_{\tau\lambda}}C)) + {}^{\beta\sigma}B({}^S g({}^{V_{\tau\lambda}}C, {}^{V_{\alpha\gamma}}A)) \\
&\quad - {}^{V_{\tau\lambda}}C({}^S g({}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}B)) - {}^S g({}^{V_{\alpha\gamma}}A, [{}^{V_{\beta\sigma}}B, {}^{V_{\tau\lambda}}C]) \\
&\quad + {}^S g({}^{V_{\beta\sigma}}B, [{}^{V_{\tau\lambda}}C, {}^{V_{\alpha\gamma}}A]) + {}^S g({}^{V_{\tau\lambda}}C, [{}^{V_{\alpha\gamma}}A, {}^{V_{\beta\sigma}}B]) \\
&= 0.
\end{aligned}$$

Therefore, we have

$${}^S\nabla_{{}^{V_{\alpha\gamma}}A}{}^{V_{\beta\sigma}}B = 0$$

and Theorem 3.2 is proved.

4 Horizontal Lifts of Linear Connections

Let ∇ be the symmetric linear connection on M_n and Γ_{ij}^k its components.

Definition 4.1 *A horizontal lift of the symmetric linear connection ∇ on M_n to the bundle of affinor frames $L_1^1(M_n)$ is the linear connection ${}^H\nabla$ defined by*

$$\begin{aligned}
&{}^H\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y), \quad {}^H\nabla_{{}^HX}{}^{V_{\beta\sigma}}B = {}^{V_{\beta\sigma}}(\nabla_X B), \\
&{}^H\nabla_{{}^{V_{\alpha\gamma}}A}{}^HY = 0, \quad {}^H\nabla_{{}^{V_{\alpha\gamma}}A}{}^{V_{\beta\sigma}}B = 0
\end{aligned} \tag{4.1}$$

for all $X, Y \in \Im_0^1(M_n)$ and $A, B \in \Im_1^1(M)$.

We note that the horizontal lifts of linear connections to tangent, cotangent, and linear frame bundles were defined in [2, 8–9, 17, 19].

The components of the horizontal lift ${}^H\nabla$ of ∇ on M_n with components Γ_{ij}^k in the natural frame $\{\frac{\partial}{\partial x^i}\}$, are defined in the adapted frame $\{D_I\}$ by decomposition

$${}^H\nabla_{D_I}D_J = {}^H\Gamma_{IJ}^K D_K. \tag{4.2}$$

From (4.1)–(4.2), by using (2.5)–(2.6), we obtain

(1) ${}^H\nabla_{{}^{V_{\alpha\gamma}}A}{}^{V_{\beta\sigma}}B = 0,$

$${}^H\nabla_{A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma D_{i_{\tau\lambda}}}(B_k^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}}) = 0,$$
$$A_i^j B_k^l\delta_\alpha^\tau\delta_\lambda^\gamma\delta_\beta^\omega\delta_\mu^\sigma\,{}^H\nabla_{D_{i_{\tau\lambda}}}D_{k_{\omega\mu}} + A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma\delta_\beta^\omega\delta_\mu^\sigma D_{i_{\tau\lambda}}(B_k^l)D_{k_{\omega\mu}} = 0,$$
$$A_i^j B_k^l\delta_\alpha^\tau\delta_\lambda^\gamma\delta_\beta^\omega\delta_\mu^\sigma\,{}^H\Gamma_{i_{\tau\lambda}k_{\omega\mu}}^P D_P = 0,$$
$$A_i^j B_k^l\delta_\alpha^\tau\delta_\lambda^\gamma\delta_\beta^\omega\delta_\mu^\sigma({}^H\Gamma_{i_{\tau\lambda}k_{\omega\mu}}^p D_p + {}^H\Gamma_{i_{\tau\lambda}k_{\omega\mu}}^{p_{\eta\varepsilon}} D_{p_{\eta\varepsilon}}) = 0,$$
$${}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^p D_p + {}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^{p_{\eta\varepsilon}} D_{p_{\eta\varepsilon}} = 0,$$

consequently,

$${}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^p = 0, \quad {}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^{p_{\eta\varepsilon}} = 0.$$

(2) ${}^H\nabla_{{}^{V_{\alpha\gamma}}A}{}^HY = 0,$

$${}^H\nabla_{A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma D_{i_{\tau\lambda}}}(Y^k D_k) = 0,$$

$$A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma Y^{k\,H}\nabla_{D_{i_{\tau\lambda}}}D_k + A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma D_{i_{\tau\lambda}}(Y^k)D_k = 0,$$
$$A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma Y^{k\,H}\Gamma_{i_{\tau\lambda}k}^P D_P = A_i^j\delta_\alpha^\tau\delta_\lambda^\gamma Y^k({}^H\Gamma_{i_{\tau\lambda}k}^p D_p + {}^H\Gamma_{i_{\tau\lambda}k}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}}) = 0,$$
$${}^H\Gamma_{i_{\alpha\gamma}k}^p D_p + {}^H\Gamma_{i_{\tau\gamma}k}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}} = 0,$$

from which we find

$${}^H\Gamma_{i_{\alpha\gamma}k}^p = 0, \quad {}^H\Gamma_{i_{\tau\gamma}k}^{p_{\eta\varepsilon}} = 0.$$

(3) ${}^H\nabla_{{}^HX}{}^HY = {}^H(\nabla_XY),\ {}^H\nabla_{X^iD_i}(Y^kD_k) = (\nabla_XY)^pD_p,$

$$X^iY^{k\,H}\Gamma_{ik}^P D_P + X^i\partial_iY^pD_p + X^i(\Gamma_{hi}^sX_{\varepsilon s}^{\eta j} - \Gamma_{hs}^jX_{\varepsilon i}^{\eta s})\frac{\partial}{\partial X_{\eta h}^{\varepsilon j}}(Y^p)D_p$$
$$= X^m\partial_mY^pD_p + X^m\Gamma_{mr}^pY^rD_p$$
$$X^iY^{k\,H}\Gamma_{ik}^pD_p + X^iY^{k\,H}\Gamma_{ik}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}} = X^m\Gamma_{mr}^pY^rD_p,$$

consequently,

$${}^H\Gamma_{ik}^p = \Gamma_{ik}^p, \quad {}^H\Gamma_{ik}^{p_{\eta\varepsilon}} = 0.$$

(4) ${}^H\nabla_{{}^HX}{}^{V_{\beta\sigma}}B = {}^{V_{\beta\sigma}}(\nabla_XB),$

$${}^H\nabla_{X^iD_i}(B_k^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}}) = (\nabla_XB)_k^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}},$$
$$X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\nabla_{D_i}D_{k_{\omega\mu}} + X^i\delta_\beta^\omega\delta_\mu^\sigma D_i(B_k^l)D_{k_{\omega\mu}}$$
$$(\partial_mB_k^l + \Gamma_{mr}^lB_k^r - \Gamma_{mk}^rB_r^l)X^m\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}},$$
$$X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^PD_P + X^i\delta_\beta^\omega\delta_\mu^\sigma(\partial_iB_k^l + (\Gamma_{hi}^sX_{\varepsilon s}^{\eta j} - \Gamma_{hs}^jX_{\varepsilon i}^{\eta s})D_{\eta\varepsilon}(B_k^l))D_{k_{\omega\mu}}$$
$$= X^m\partial_mB_k^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}} + X^m\Gamma_{mr}^lB_k^r\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}} - X^m\Gamma_{mk}^rB_r^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}},$$
$$X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^pD_p + X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}} + X^i\delta_\beta^\omega\delta_\mu^\sigma\partial_iB_k^lD_{k_{\omega\mu}}$$
$$+X^i\delta_\beta^\omega\delta_\mu^\sigma\Gamma_{hi}^sX_{\varepsilon s}^{\eta j}D_{\eta\varepsilon}(B_k^l)D_{k_{\omega\mu}} - X^i\delta_\beta^\omega\delta_\mu^\sigma\Gamma_{hs}^jX_{\varepsilon i}^{\eta s}D_{\eta\varepsilon}(B_k^l)D_{k_{\omega\mu}}$$
$$= X^m\partial_mB_k^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}} + X^m\Gamma_{mr}^lB_k^r\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}} - X^m\Gamma_{mk}^rB_r^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}},$$
$$X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^pD_p + X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}}$$
$$= X^m\Gamma_{mr}^lB_k^r\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}} - X^m\Gamma_{mk}^rB_r^l\delta_\beta^\omega\delta_\mu^\sigma D_{k_{\omega\mu}},$$
$$X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^pD_p + X^iB_k^l\delta_\beta^\omega\delta_\mu^{\sigma\,H}\Gamma_{ik_{\omega\mu}}^{p_{\eta\varepsilon}}D_{p_{\eta\varepsilon}}$$
$$= X^iB_k^l\delta_\beta^\omega\delta_\mu^\sigma(\delta_\omega^\eta\delta_\varepsilon^\mu\delta_p^k\Gamma_{il}^q - \delta_\omega^\eta\delta_\varepsilon^\mu\delta_l^q\Gamma_{ip}^k)D_{p_{\eta\varepsilon}}$$

from which we obtain that

$${}^H\Gamma_{ik_{\beta\sigma}}^p = 0, \quad {}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} = \delta_\beta^\eta\delta_\varepsilon^\sigma\delta_p^k\Gamma_{il}^q - \delta_\beta^\eta\delta_\varepsilon^\sigma\delta_l^q\Gamma_{ip}^k.$$

Thus, the following theorem holds.

Theorem 4.1 *The horizontal lift ${}^H\nabla$ of the symmetric linear connection ∇ given on M_n, to the bundle of affinor frames $L_1^1(M_n)$ has the components*

$$\begin{aligned}
{}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^p &= 0,\\
{}^H\Gamma_{i_{\alpha\gamma}k_{\beta\sigma}}^{p_{\eta\varepsilon}} &= {}^H\Gamma_{i_{\alpha\gamma}k}^p = {}^H\Gamma_{i_{\tau\gamma}k}^{p_{\eta\varepsilon}} = 0,\\
{}^H\Gamma_{ik}^p &= \Gamma_{ik}^p,\\
{}^H\Gamma_{ik}^{p_{\eta\varepsilon}} &= {}^H\Gamma_{ik_{\beta\sigma}}^p = 0,\\
{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} &= \delta_\beta^\eta\delta_\varepsilon^\sigma\delta_p^k\Gamma_{il}^q - \delta_\beta^\eta\delta_\varepsilon^\sigma\delta_l^q\Gamma_{ip}^k
\end{aligned} \qquad (4.3)$$

with respect to the adapted frame $\{D_I\}$.

Now consider the following transformation of frames on $L_1^1(M_n)$:

$$\{D_i, D_{i_{\alpha\gamma}}\} = \left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_{\sigma j}^{\beta k}}\right\} \begin{pmatrix} \delta_i^j & 0 \\ -X_{\sigma j}^{\beta m}\Gamma_{im}^k + X_{\sigma m}^{\beta k}\Gamma_{ij}^m & \delta_\alpha^\beta \delta_\sigma^\gamma \delta_h^k \delta_j^i \end{pmatrix}, \tag{4.4}$$

i.e.,

$$D_I = A_I^{\ \ J}\partial_J.$$

Note that matrix $(A_I^{\ \ J})$ and its inverse matrix $(\widetilde{A}_J^{\ \ K})$ are defined of the forms (3.7) and (3.6), respectively.

We denote the components of the linear connection ${}^H\nabla$ with respect to the natural frame $\{\partial_I\}$ by ${}^H\overline{\Gamma}_{IK}^P$, i.e.,

$${}^H\nabla_{\partial_I}\partial_K = {}^H\overline{\Gamma}_{IK}^P \partial_P.$$

Then

$${}^H\overline{\Gamma}_{JL}^S = A_P^{\ \ S}\,{}^H\Gamma_{IK}^P \widetilde{A}^I_{\ \ J}\widetilde{A}^K_{\ \ L} - (D_I A_K^{\ \ S})\widetilde{A}^I_{\ \ J}\widetilde{A}^K_{\ \ L}. \tag{4.5}$$

Using (4.3) and (3.6)–(3.7), from (4.5) we obtain

$$\begin{aligned}
{}^H\overline{\Gamma}_{jl}^s &= A_p^{\ \ s}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j}\widetilde{A}^k_{\ \ l} + A_{p_{\eta\varepsilon}}^{\ \ s}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} = \delta_p^s \Gamma_{ik}^p \delta_j^i \delta_l^k = \Gamma_{jl}^s, \\
{}^H\overline{\Gamma}_{j_{\tau\nu}l}^s &= A_p^{\ \ s}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l} + A_{p_{\eta\varepsilon}}^{\ \ s}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} = 0, \\
{}^H\overline{\Gamma}_{jl_{\omega\upsilon}}^s &= A_p^{\ \ s}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j}\widetilde{A}^k_{\ \ l_{\omega\upsilon}} + A_{p_{\eta\varepsilon}}^{\ \ s}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l_{\omega\upsilon}} = 0, \\
{}^H\overline{\Gamma}_{j_{\tau\nu}l_{\omega\upsilon}}^s &= A_p^{\ \ s}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l_{\omega\upsilon}} + A_{p_{\eta\varepsilon}}^{\ \ s}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l_{\omega\upsilon}} = 0, \\
{}^H\overline{\Gamma}_{j_{\tau\nu}l}^{s_{\varphi\psi}} &= A_p^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l} + A_{p_{\eta\varepsilon}}^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} \\
&\quad - (D_i A_k^{\ \ s_{\varphi\psi}})\widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l} - (D_{i_{\alpha\gamma}} A_k^{\ \ s_{\varphi\psi}})\widetilde{A}^{i_{\alpha\gamma}}_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l} \\
&\quad - (D_{i_{\alpha\gamma}} A_{k_{\beta\sigma}}^{\ \ s_{\varphi\psi}})\widetilde{A}^{i_{\alpha\gamma}}_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} \\
&= \delta_\tau^\varphi \delta_\psi^\nu \delta_s^j \Gamma_{lq}^r - \delta_\tau^\varphi \delta_\psi^\nu \delta_q^r \Gamma_{ls}^j, \\
{}^H\overline{\Gamma}_{jl_{\omega\upsilon}}^{s_{\varphi\psi}} &= A_p^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j}\widetilde{A}^k_{\ \ l_{\omega\upsilon}} + A_{p_{\eta\varepsilon}}^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l_{\omega\upsilon}} \\
&\quad - (D_{i_{\alpha\gamma}} A_{k_{\beta\sigma}}^{\ \ s_{\varphi\psi}})\widetilde{A}^{i_{\alpha\gamma}}_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} \\
&= \delta_\omega^\varphi \delta_\psi^\upsilon \delta_s^l \Gamma_{jm}^r - \delta_\omega^\varphi \delta_\psi^\upsilon \delta_m^r \Gamma_{js}^l, \\
{}^H\overline{\Gamma}_{j_{\tau\nu}l_{\omega\upsilon}}^{s_{\varphi\psi}} &= A_p^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^k_{\ \ l_{\omega\upsilon}} + A_{p_{\eta\varepsilon}}^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l_{\omega\upsilon}} \\
&\quad - (D_{i_{\alpha\gamma}} A_{k_{\beta\sigma}}^{\ \ s_{\varphi\psi}})\widetilde{A}^{i_{\alpha\gamma}}_{\ \ j_{\tau\nu}}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l_{\omega\upsilon}} \\
&= 0, \\
{}^H\overline{\Gamma}_{jl}^{s_{\varphi\psi}} &= A_p^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik}^p \widetilde{A}^i_{\ \ j}\widetilde{A}^k_{\ \ l} + A_{p_{\eta\varepsilon}}^{\ \ s_{\varphi\psi}}\,{}^H\Gamma_{ik_{\beta\sigma}}^{p_{\eta\varepsilon}} \widetilde{A}^i_{\ \ j}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} \\
&\quad - (D_I A_K^{\ \ s_{\varphi\psi}})\widetilde{A}^I_{\ \ j}\widetilde{A}^K_{\ \ l} \\
&= -\Gamma_{pm}^r X_{\psi s}^{\varphi m}\Gamma_{ik}^p \delta_j^i \delta_l^k + \Gamma_{ps}^m X_{\psi m}^{\varphi r}\Gamma_{ik}^p \delta_j^i \delta_l^k \\
&\quad + \delta_\psi^\varphi \delta_\eta^\varepsilon \delta_s^p \delta_q^r (\delta_\beta^\eta \delta_\varepsilon^\sigma \delta_p^k \Gamma_{it}^q - \delta_\beta^\eta \delta_\varepsilon^\sigma \delta_l^q \Gamma_{ip}^k)\delta_j^i (X_{\sigma k}^{\beta m}\Gamma_{lm}^t - X_{\sigma m}^{\beta t}\Gamma_{lk}^m) \\
&\quad - (D_i A_k^{\ \ s_{\varphi\psi}})\widetilde{A}^i_{\ \ j}\widetilde{A}^k_{\ \ l} - (D_i A_{k_{\beta\sigma}}^{\ \ s_{\varphi\psi}})\widetilde{A}^i_{\ \ j}\widetilde{A}^{k_{\beta\sigma}}_{\ \ l} - (D_{i_{\alpha\gamma}} A_k^{\ \ s_{\varphi\psi}})\widetilde{A}^{i_{\alpha\gamma}}_{\ \ j}\widetilde{A}^k_{\ \ l} \\
&= -\Gamma_{pm}^r X_{\psi s}^{\varphi m}\Gamma_{jl}^p + \Gamma_{ps}^m X_{\psi m}^{\varphi r}\Gamma_{jl}^p + \Gamma_{jt}^r X_{\psi s}^{\varphi m}\Gamma_{lm}^t - \Gamma_{jt}^r X_{\psi m}^{\varphi t}\Gamma_{ls}^m
\end{aligned}$$

$$
\begin{aligned}
&- \Gamma^k_{js} X^{\varphi m}_{\psi k} \Gamma^r_{lm} + \Gamma^k_{js} X^{\varphi r}_{\psi m} \Gamma^m_{lk} - (\partial_i + (\Gamma^m_{hi} X^{\rho a}_{\xi m} - \Gamma^a_{mi} X^{\rho m}_{\xi h})) \tfrac{\partial}{\partial X^{\rho a}_{\xi h}} (-X^{\varphi b}_{\psi s} \Gamma^r_{kb} \\
&+ X^{\varphi r}_{\psi b} \Gamma^b_{ks}) \delta^i_j \delta^k_l - \tfrac{\partial}{\partial X^{\alpha h}_{\gamma i}} (-X^{\varphi m}_{\psi s} \Gamma^r_{km} + X^{\varphi r}_{\psi m} \Gamma^m_{ks})(X^{\alpha p}_{\gamma i} \Gamma^h_{jp} - X^{\alpha h}_{\gamma p} \Gamma^p_{ji}) \delta^k_l \\
= &- \Gamma^r_{pm} X^{\varphi m}_{\psi s} \Gamma^p_{jl} + \Gamma^m_{ps} X^{\varphi r}_{\psi m} \Gamma^p_{jl} + \Gamma^r_{jt} X^{\varphi m}_{\psi s} \Gamma^t_{lm} - \Gamma^r_{jt} X^{\varphi t}_{\psi m} \Gamma^m_{ls} \\
&- \Gamma^k_{js} X^{\varphi m}_{\psi k} \Gamma^r_{lm} + \Gamma^k_{js} X^{\varphi r}_{\psi m} \Gamma^m_{lk} + X^{\varphi b}_{\psi s} \partial_j \Gamma^r_{lb} - X^{\varphi r}_{\psi b} \partial_j \Gamma^b_{ls} + \Gamma^m_{sj} X^{\varphi b}_{\psi m} \Gamma^r_{lb} \\
&- \Gamma^m_{bj} X^{\varphi r}_{\psi m} \Gamma^b_{ls} - \Gamma^b_{mj} X^{\varphi m}_{\psi s} \Gamma^r_{lb} + \Gamma^r_{mj} X^{\varphi m}_{\psi b} \Gamma^b_{ls} + X^{\varphi m}_{\psi s} \Gamma^r_{lh} \Gamma^h_{jm} - X^{\varphi h}_{\psi m} \Gamma^r_{lh} \Gamma^m_{js} \\
&- X^{\varphi m}_{\psi i} \Gamma^i_{ls} \Gamma^r_{jm} + X^{\varphi r}_{\psi m} \Gamma^m_{ji} \Gamma^i_{ls} \\
= & X^{\varphi b}_{\psi s} (\partial_j \Gamma^r_{lb} - \Gamma^r_{pb} \Gamma^p_{jl} + \Gamma^r_{jp} \Gamma^p_{lb}) \\
&+ X^{\varphi r}_{\psi a} (-\partial_j \Gamma^a_{ls} + \Gamma^p_{js} \Gamma^a_{lp} + \Gamma^a_{ps} \Gamma^p_{jl}) - X^{\varphi b}_{\psi a} (\Gamma^r_{lb} \Gamma^a_{sj} + \Gamma^a_{ls} \Gamma^r_{bj}).
\end{aligned}
$$

Thus, we have the following theorem.

Theorem 4.2 *The horizontal lift* ${}^H\nabla$ *of the symmetric linear connection* ∇ *given on* M_n*, to the bundle of affinor frames* $L^1_1(M_n)$ *has the components*

$$
\begin{aligned}
{}^H\overline{\Gamma}^s_{jl} &= \Gamma^s_{jl}, \\
{}^H\overline{\Gamma}^{s_{\varphi\psi}}_{jl} &= X^{\varphi b}_{\psi s} (\partial_j \Gamma^r_{lb} - \Gamma^r_{pb} \Gamma^p_{jl} + \Gamma^r_{jp} \Gamma^p_{lb}) \\
&\quad + X^{\varphi r}_{\psi a} (-\partial_j \Gamma^a_{ls} + \Gamma^p_{js} \Gamma^a_{lp} + \Gamma^a_{ps} \Gamma^p_{jl}) - X^{\varphi b}_{\psi a} (\Gamma^r_{lb} \Gamma^a_{sj} + \Gamma^a_{ls} \Gamma^r_{bj}), \\
{}^H\overline{\Gamma}^s_{j_{\tau\nu} l} &= {}^H\overline{\Gamma}^s_{j l_{\omega\upsilon}} = {}^H\overline{\Gamma}^s_{j_{\tau\nu} l_{\omega\upsilon}} = {}^H\overline{\Gamma}^{s_{\varphi\psi}}_{j_{\tau\nu} l_{\omega\upsilon}} = 0, \\
{}^H\overline{\Gamma}^{s_{\varphi\psi}}_{j_{\tau\nu} l} &= \delta^\varphi_\tau \delta^\nu_\psi \delta^j_s \Gamma^r_{lq} - \delta^\varphi_\tau \delta^\nu_\psi \delta^r_q \Gamma^j_{ls}, \\
{}^H\overline{\Gamma}^{s_{\varphi\psi}}_{j l_{\omega\upsilon}} &= \delta^\varphi_\omega \delta^\upsilon_\psi \delta^l_s \Gamma^r_{jm} - \delta^\varphi_\omega \delta^\upsilon_\psi \delta^r_m \Gamma^l_{js}
\end{aligned}
\tag{4.6}
$$

with respect to the natural frame $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial X^{\alpha h}_{\gamma i}} \right\}$.

5 Geodesics of the Horizontal Lift of Linear Connections

Geodesics of horizontal lifts of connections in tangent, cotangent and tensor bundles were investigated in [2] and [18, pp. 114–117, pp. 297–299]. In the present section we study geodesics of the horizontal lifts of connections in the bundle of affinor frames.

Let $\widetilde{C}$ be a geodesic curve on the bundle of affinor frames $L^1_1(M_n)$ with respect to the horizontal lift ${}^H\nabla$ of the symmetric linear connection ∇ on the M_n. In induced coordinates $(\pi^{-1}(U), x^i, X^{\alpha h}_{\gamma i})$ the equation of the geodesic curve

$$\widetilde{C} : I \to L^1_1(M_n), \quad \widetilde{C} : t \to \widetilde{C}(t) = (x^i(t), X^{\alpha h}_{\gamma i}(t)) = (x^I(t))$$

is of the form

$$\frac{\mathrm{d}^2 x^K}{\mathrm{d}t^2} + {}^H\overline{\Gamma}^K_{IJ} \frac{\mathrm{d}x^I}{\mathrm{d}t} \frac{\mathrm{d}x^J}{\mathrm{d}t} = 0, \quad I, J, K = 1, 2, \cdots, n + n^4. \tag{5.1}$$

By using formulas (4.6) for ${}^H\Gamma^K_{IJ}$, from (5.1) we obtain

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} + \Gamma^k_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} = 0,$$

and

$$\frac{\mathrm{d}^2 X^{\beta l}_{\sigma k}}{\mathrm{d}t^2} + (X^{\beta b}_{\sigma k}(\partial_i \Gamma^l_{jb} - \Gamma^l_{pb}\Gamma^p_{ij} + \Gamma^l_{ip}\Gamma^p_{jb}) + X^{\beta l}_{\sigma a}(-\partial_i \Gamma^a_{jk} + \Gamma^p_{ik}\Gamma^a_{jp} + \Gamma^a_{pk}\Gamma^p_{ij})$$
$$- X^{\beta b}_{\sigma a}(\Gamma^l_{jb}\Gamma^a_{ki} + \Gamma^a_{jk}\Gamma^r_{bi}))\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2(\Gamma^l_{jb}\delta^i_k - \Gamma^i_{jk}\delta^l_b)\frac{\mathrm{d}X^{\beta h}_{\sigma i}}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0. \tag{5.2}$$

Let us consider the covariant differentiation of $X^{\beta l}_{\sigma k}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^{\beta l}_{\sigma k}(t)) = \frac{\mathrm{d}X^{\beta l}_{\sigma k}}{\mathrm{d}t} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{\mathrm{d}x^p}{\mathrm{d}t} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{\mathrm{d}x^m}{\mathrm{d}t}. \tag{5.3}$$

Now applying

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0,$$

and taking into account the symmetry of the linear connection ∇ given on M_n, from (5.3) we get

$$\begin{aligned}
\frac{\delta^2 X^{\beta l}_{\sigma k}}{\mathrm{d}t^2} &= \frac{\delta}{\mathrm{d}t}\Big(\frac{\mathrm{d}X^{\beta l}_{\sigma k}}{\mathrm{d}t} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{\mathrm{d}x^p}{\mathrm{d}t} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{\mathrm{d}x^p}{\mathrm{d}t}\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}X^{\beta l}_{\sigma k}}{\mathrm{d}t} + \Gamma^l_{pb}X^{\beta b}_{\sigma k}\frac{\mathrm{d}x^p}{\mathrm{d}t} - \Gamma^a_{pk}X^{\beta l}_{\sigma a}\frac{\mathrm{d}x^p}{\mathrm{d}t}\Big)\\
&\quad + \Gamma^l_{ib}\Big(\frac{\mathrm{d}X^{\beta b}_{\sigma k}}{\mathrm{d}t} + \Gamma^b_{pa}X^{\beta a}_{\sigma k}\frac{\mathrm{d}x^p}{\mathrm{d}t} - \Gamma^b_{pk}X^{\beta b}_{\sigma a}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big)\frac{\mathrm{d}x^i}{\mathrm{d}t}\\
&\quad - \Gamma^b_{ik}\Big(\frac{\mathrm{d}X^{\beta l}_{\sigma b}}{\mathrm{d}t} + \Gamma^j_{pa}X^{\beta a}_{\sigma b}\frac{\mathrm{d}x^p}{\mathrm{d}t} - \Gamma^a_{pb}X^{\beta l}_{\sigma a}\frac{\mathrm{d}x^p}{\mathrm{d}t}\Big)\frac{\mathrm{d}x^i}{\mathrm{d}t}\\
&= \frac{\mathrm{d}^2 X^{\beta l}_{\sigma k}}{\mathrm{d}t^2} + (X^{\beta b}_{\sigma k}(\partial_i \Gamma^l_{jb} - \Gamma^l_{pb}\Gamma^p_{ij} + \Gamma^l_{ip}\Gamma^p_{jb}) + X^{\beta l}_{\sigma a}(-\partial_i \Gamma^a_{jk} + \Gamma^p_{ik}\Gamma^a_{jp} + \Gamma^a_{pk}\Gamma^p_{ij})\\
&\quad - X^{\beta b}_{\sigma a}(\Gamma^l_{jb}\Gamma^a_{ki} + \Gamma^a_{jk}\Gamma^r_{bi}))\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2(\Gamma^l_{jb}\delta^i_k - \Gamma^i_{jk}\delta^l_b)\frac{\mathrm{d}X^{\beta h}_{\sigma i}}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}.
\end{aligned} \tag{5.4}$$

Comparing expression (5.4) and the second equality of (5.2), we obtain the following theorem.

Theorem 5.1 *A geodesic curve on the bundle of affinor frames $L^1_1(M_n)$ with respect to the horizontal lift ${}^H\nabla$ of the symmetric linear connection ∇ on M_n has in induced coordinates $(\pi^{-1}(U), x^i, X^{\beta h}_{\gamma i})$ in $L^1_1(M_n)$ the equations of the form:*

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}^2 X^{\beta l}_{\sigma k}}{\mathrm{d}t^2} = 0.$$

From Theorem 5.1 we get the following result.

Theorem 5.2 *A curve $\widetilde{C}$ on the bundle of affinor frames $L^1_1(M_n)$ is a geodesic of horizontal lift ${}^H\nabla$ of the symmetric linear connection ∇ if the projection $C = \pi(\widetilde{C})$ on M_n is a geodesic of ∇ on M_n and the second covariant differentiation of each affinor $X^{\beta l}_{\sigma k}(t) = X^{\beta l}_{\sigma k}(t)\frac{\partial}{\partial x^l}\otimes \mathrm{d}x^k\big|_{C(t)}$, $\beta, \sigma = 1, 2, \cdots, n$ of the affinor frame A along C vanishes, where $\pi : L^1_1(M_n) \to M_n$ is the natural projection.*

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