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Rarefaction Wave Interaction and Shock-Rarefaction Composite Wave Interaction for a Two-Dimensional Nonlinear Wave System*

Geng LAI[1] Sisi XIE[1]

Abstract In order to construct global solutions to two-dimensional (2D for short) Riemann problems for nonlinear hyperbolic systems of conservation laws, it is important to study various types of wave interactions. This paper deals with two types of wave interactions for a 2D nonlinear wave system with a nonconvex equation of state: Rarefaction wave interaction and shock-rarefaction composite wave interaction. In order to construct solutions to these wave interactions, the authors consider two types of Goursat problems, including standard Goursat problem and discontinuous Goursat problem, for a 2D self-similar nonlinear wave system. Global classical solutions to these Goursat problems are obtained by the method of characteristics. The solutions constructed in the paper may be used as building blocks of solutions of 2D Riemann problems.

Keywords Nonlinear wave system, Rarefaction wave, Shock-rarefaction composite wave, Wave interaction, Characteristic decomposition
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1 Introduction

The 2D nonlinear wave system takes the form

$$\begin{cases}\rho_t+(\rho u)_x+(\rho v)_y=0,\\ (\rho u)_t+p_x=0,\\ (\rho v)_t+p_y=0,\end{cases}\tag{1.1}$$

where ρ represents the density, (u,v) represents the velocity, and $p=p(\rho)$ is the pressure. This system is derived from the compressible Euler system by neglecting the quadratic terms in the velocity, or by writing the nonlinear wave system as a first order system (see [5] for more details). This system is similar to the pressure gradient system which is also derived from the compressible Euler system (see [1, 38]).

The global existence of solution to the Cauchy problem for multi-dimensional nonlinear hyperbolic systems of conservation laws is still a complicated open problem. Thus it has been profitable to consider some special problems, such as 2D Riemann problems, which refer to Cauchy problems with special initial data that are constant along each ray from the origin. Recently, several types of 2D Riemann problems for the compressible Euler system and system (1.1) have been studied by many researchers, see [3, 6, 8–11, 18, 20, 37, 39] for shock reflection

Manuscript received October 14, 2018. Revised May 14, 2020.
[1]Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail: laigeng@shu.edu.cn SHU17xss@shu.edu.cn
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problems; [7] for shock diffraction problem; [4, 15] for supersonic flows around a convex wedge; [19] for the interaction of transonic shock and rarefaction wave; and [2, 14, 16–17, 21, 27–29, 31, 36] for the interactions of rarefaction waves.

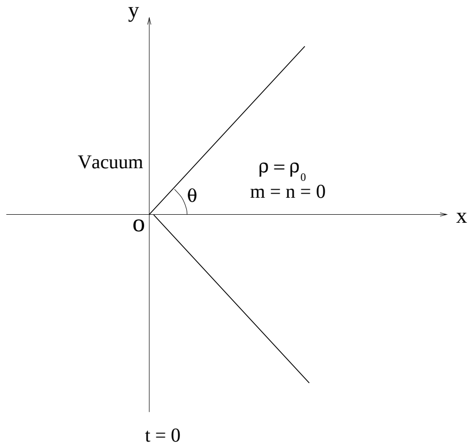

Figure 1 Initial data of the expansion of a wedge of gas to vacuum.

In this paper, we consider the system (1.1) with the following initial data:

$$(\rho, m, n)(0, x, y) = \begin{cases} (\rho_0, 0, 0), & (x, y) \in \{x > 0, -x \tan\theta < y < x \tan\theta\}; \\ \text{vacuum}, & \text{otherwise}, \end{cases} \tag{1.2}$$

where $(m, n) = (\rho u, \rho v)$ is the momentum, $\rho_0 > 0$, and $\theta \in \left(0, \frac{\pi}{2}\right)$ (see Figure 1). Here, the momentum in vacuum is not specified. This problem describes the expansion of a wedge of gas at rest into vacuum. It also plays an important role in 2D Riemann problems, since it catches several important types of wave interactions.

2D Riemann problems allow us to consider the so-called self-similar solutions, that are the solutions which depend only on the self-similar variables $\xi = \frac{x}{t}$ and $\eta = \frac{y}{t}$. Then by self-similar transformation, system (1.1) can be changed into the form

$$\begin{cases} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta = 0, \\ -\xi m_\xi - \eta m_\eta + p_\xi = 0, \\ -\xi n_\xi - \eta n_\eta + p_\eta = 0, \end{cases} \tag{1.3}$$

which is called the 2D self-similar nonlinear wave system. The greatest feature of the system (1.3) is that its type is a priori unknown, and the type is determined by the local Mach number $M = \frac{\sqrt{\xi^2+\eta^2}}{c}$, where $c = \sqrt{p'(\rho)}$ represents the speed of sound. The system (1.3) is hyperbolic if and only if $M > 1$, and elliptic-hyperbolic if and only if $M < 1$.

In this paper, we consider a nonconvex equation of state $p = p(\rho)$ which is assumed to satisfy:

$$p'(\rho) > 0 \quad \text{as } \rho > 0, \quad p''(\rho) > 0 \quad \text{as } 0 < \rho < \rho_c, \quad p''(\rho) < 0 \quad \text{as } \rho > \rho_c, \quad p'(0) = 0. \tag{1.4}$$

Nonconvex equations of state frequently appear in van der Waals gases (see [22–25]). We divide the discussions into the following two cases: $0 < \rho_0 \le \rho_c$ and $\rho_0 > \rho_c$. Let us briefly describe the results of the paper.

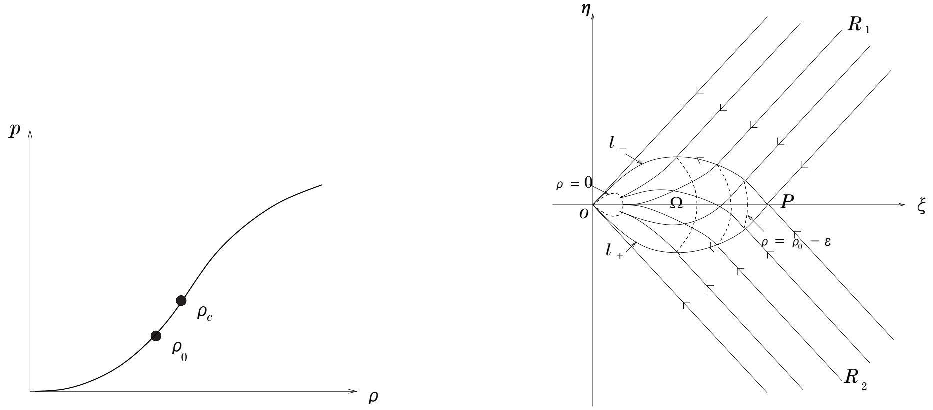

Figure 2 Interaction of rarefaction waves.

If $0 < \rho_0 \leq \rho_c$ then the gas away from the sharp corner of the wedge expands to vacuum as two symmetrical planar rarefaction waves R_1 and R_2. As illustrated in Figure 2(right), the rarefaction waves R_1 and R_2 meet at some point P, then interaction starts. Through P draw a C_- (C_+, resp.) cross characteristic curve l_- (l_+, resp.) in R_1 (R_2, resp.). Then, by solving a standard Goursat problem (SGP for short) for the 2D self-similar nonlinear wave system (1.3) with l_+ and l_- as the characteristic boundaries (see the SGP (1.3), (3.2) in Subsection 3.2), we can construct the solution in a region Ω bounded by characteristic curves l_+, l_-, and a level curve $\rho = 0$, where the two rarefaction waves interact. The main result about this interaction is stated as Theorem 3.1 where we obtain the existence of global classical solution to the SGP (1.3), (3.2).

If $\rho_0 > \rho_c$ then the gas away from the sharp corner of the wedge expands to vacuum as two symmetrical planar shock-rarefaction composite waves $S_1 \cup R_1$ and $S_2 \cup R_2$, where S and R represent shock and rarefaction wave, respectively. Here, the shock-rarefaction composite waves consist of a rarefaction shock from the front side state $(\rho_0, 0, 0)$ to an intermediate state with the density ρ_* which is defined so that $p'(\rho_*) = \frac{p(\rho_0)-p(\rho_*)}{\rho_0-\rho_*}$, followed by a rarefaction wave from the intermediate state to the vacuum (see Figure 3). As illustrated in Figure 3(right), these two composite waves meet at some point P, then interaction starts. Through P draw a C_- (C_+, resp.) cross characteristic curve l_- (l_+, resp.) in R_1 (R_2, resp.). Then, by solving a discontinuous Goursat problem (DGP for short) for system (1.3) with l_+ and l_- as the characteristic boundaries (see the DGP (1.3), (4.1) in Subsection 4.2), we can construct the solution in a region Ω bounded by characteristic curves l_+, l_-, and a level curve $\rho = 0$, where the two composite waves interact. Here, the discontinuous Goursat problem means that the boundary data is discontinuous at P. The main result about this interaction is stated as Theorem 4.1 where we obtain the existence of global piecewise smooth solution to the DGP (1.3), (4.1).

In [16–17, 21], the authors considered rarefaction wave interactions for the nonlinear wave system for polytropic gases $p = \rho^\gamma$. They used the idea of Dai and Zhang [14] to convert the 2D self-similar nonlinear wave system (1.3) into the following second order equation:

$$(\gamma p^{\frac{\gamma-1}{\gamma}} - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (\gamma p^{\frac{\gamma-1}{\gamma}} - \eta^2)p_{\eta\eta} + \frac{\gamma-1}{\gamma p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0, \quad (1.5)$$

and obtained global solutions of rarefaction wave interactions by solving some standard Goursat

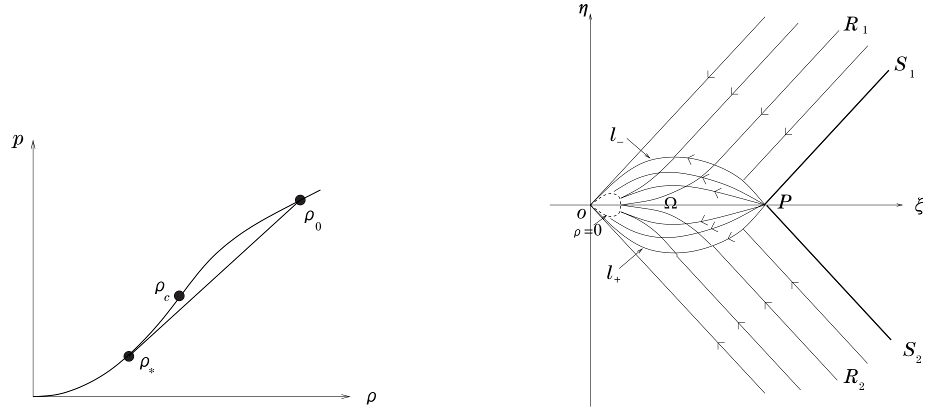

Figure 3 Interaction of shock-rarefaction composite waves.

problems for (1.5). However, for the general equation of state $p = p(\rho)$, if we still use this way to study wave interactions then the process will become complicated. Motivated by the results of Zheng et al. [12, 29–31] in investigating rarefaction wave interactions of the compressible Euler equations, we derive some characteristic equations and characteristic decompositions of the 2D self-similar nonlinear wave system (1.3). These characteristic equations and characteristic decompositions will be extensively used to establish the a priori C^1 norm estimates of solutions. Using these a priori C^1 norm estimates, we construct the global solutions of the Goursat problems. Since the main purposes of the paper is the wave interactions, we do not consider the flow after the interactions, i.e., we do not construct a global solution to the 2D Riemann problem (1.1)–(1.2). But, the wave structures constructed in this paper may be used as building blocks of solutions of 2D Riemann problems.

The rest of the paper is organized as follows. Section 2 is mainly concerned with the 2D self-similar nonlinear wave system (1.3). The concepts of $C_\pm$ characteristic directions, Mach angle, and $C_\pm$ characteristic angles α and β are presented in Subsection 2.1. A group of characteristic equations in terms of the variables α, β, and ρ are derived in Subsection 2.2. These equations will be extensively used to control the hyperbolicity of the system (1.3) and to establish the uniform a priori C^0 norm estimates of solutions. Characteristic decompositions for (1.3) are derived in Subsection 2.3. These decompositions will be used to establish the uniform a priori gradient estimates of solutions. Section 3 is devoted to study the interaction of the rarefaction waves. Section 4 is devoted to study the interaction of the shock-rarefaction composite waves.

2 2D Self-Similar Nonlinear Wave System

2.1 Characteristics

The eigenvalues of (1.3) are determined by

$$\Big(\lambda - \frac{\eta}{\xi}\Big)[(\eta - \lambda\xi)^2 - c^2(1+\lambda^2)] = 0, \tag{2.1}$$

which yields the eigenvalues

$$\lambda = \lambda_\pm(\xi, \eta, c) = \frac{\xi\eta \pm c\sqrt{q^2 - c^2}}{\xi^2 - c^2}, \quad \lambda = \lambda_0 = \frac{\eta}{\xi}, \tag{2.2}$$

where $q^2 = \xi^2 + \eta^2$. So, if and only if $q^2 > c^2$ (supersonic) system (1.3) is hyperbolic and has two families of wave characteristics $C_\pm$ defined as the integral curves of $\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_\pm$ and a family of stream lines C_0 defined as the integral curves of $\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\eta}{\xi}$.

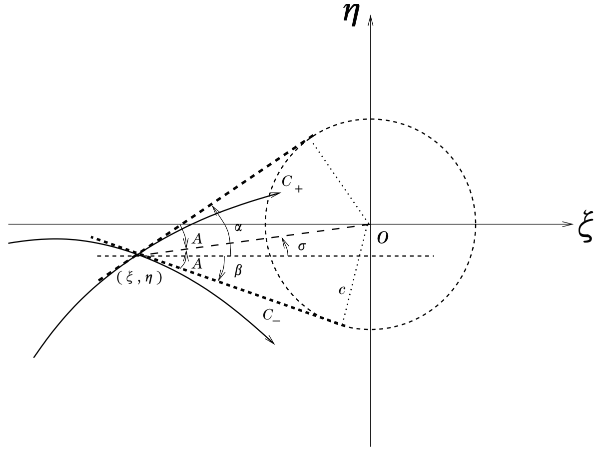

Figure 4 Characteristic directions and characteristic angles.

See Figure 4. The direction of the wave characteristics is defined as the tangent direction that forms an acute angle A with the vector $(-\xi, -\eta)$. By simple computation, we see that the C_+ characteristic direction forms with the direction $(-\xi, -\eta)$ the angle A from C_+ to $(-\xi, -\eta)$ in the clockwise direction, and the C_- characteristic direction forms with the direction $(-\xi, -\eta)$ the angle A from C_- to $(-\xi, -\eta)$ in the counterclockwise direction. By computation, we have

$$c^2 = q^2 \sin^2 A. \tag{2.3}$$

The angle A is called the Mach angle.

From (2.1), we have $c = \frac{|(\xi,\eta)\cdot(\lambda,-1)|}{|(\lambda,-1)|}$ which implies that the component of the vector $(-\xi, -\eta)$ normal to the direction of a characteristic C_+ (or C_-) is equal to the sound speed. Equivalently, it can be stated as that the tangent line of a C_+ (or C_-) characteristic at a point is tangent to the sonic circle of the state $\xi^2 + \eta^2 = p'(u)$ at that point. Furthermore, a C_+ (or C_-) characteristic must be straight if u is constant along it.

Following [13] and [31], we use the concept of characteristic angles. The C_+ (C_-) characteristic angle is defined as the counterclockwise angle from the positive ξ-axis to the C_+ (C_-) characteristic direction. We denote by α and β the C_+ and C_- characteristic angle, respectively. Let σ be the counterclockwise angle from the positive ξ-axis to the direction $(-\xi, -\eta)$. Then, we have

$$\alpha = \sigma + A, \quad \beta = \sigma - A, \quad \sigma = \frac{\alpha+\beta}{2}, \quad A = \frac{\alpha-\beta}{2}, \tag{2.4}$$

and

$$(\xi, \eta) = \Big(-c\frac{\cos\sigma}{\sin A}, \ -c\frac{\sin\sigma}{\sin A}\Big). \tag{2.5}$$

The first equation of (1.3) can be written as

$$(c^2 - \xi^2)m_\xi - \xi\eta n_\xi - \xi\eta m_\eta + (c^2 - \eta^2)n_\eta = 0 \tag{2.6}$$

by the last two equations of (1.3).

Let $\omega = m_\eta - n_\xi$. Then by the last two equations of (1.3) we have

$$-\xi\omega_\xi - \eta\omega_\eta = \omega. \tag{2.7}$$

The left eigenvectors corresponding to the eigenvalues $\lambda_\pm$ are $(1, \ \mp c\sqrt{\xi^2+\eta^2-c^2})$. Multiplying

$$\begin{pmatrix} c^2-\xi^2 & -\xi\eta \\ 0 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}_\xi + \begin{pmatrix} -\xi\eta & c^2-\eta^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}_\eta = \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

by $(1, \ \mp c\sqrt{\xi^2+\eta^2-c^2})$, we get

$$\begin{cases} \overline{\partial}_+ m + \lambda_- \overline{\partial}_+ n = \dfrac{\omega \sin A \cos A}{\cos\beta}, \\ \overline{\partial}_- m + \lambda_+ \overline{\partial}_- n = -\dfrac{\omega \sin A \cos A}{\cos\alpha}, \end{cases} \tag{2.8}$$

where

$$\overline{\partial}_+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \quad \overline{\partial}_- = \cos\beta\partial_\xi + \sin\beta\partial_\eta. \tag{2.9}$$

2.2 2D self-similar nonlinear wave system with $\omega \equiv 0$

If $\omega \equiv 0$, we can introduce a potential function $\varphi(\xi,\eta)$ such that $\varphi_\xi = m$, $\varphi_\eta = n$. Hence, from the last two equations of (1.3), we obtain the Bernoulli law

$$-\eta n - \xi m + p(\rho) + \varphi = 0. \tag{2.10}$$

Moreover, system (1.3) can be reduced to

$$\overline{\partial}_\pm m + \lambda_\mp \overline{\partial}_\pm n = 0, \tag{2.11}$$

supplemented by (2.10).

From (2.5) we have

$$\cos(\sigma \pm A) + \frac{\cos\sigma}{\sin A}\overline{\partial}_\pm c + \frac{c\cos\alpha\overline{\partial}_\pm\beta - c\cos\beta\overline{\partial}_\pm\alpha}{2\sin^2 A} = 0, \tag{2.12}$$

$$\sin(\sigma \pm A) + \frac{\sin\sigma}{\sin A}\overline{\partial}_\pm c + \frac{c\sin\alpha\overline{\partial}_\pm\beta - c\sin\beta\overline{\partial}_\pm\alpha}{2\sin^2 A} = 0. \tag{2.13}$$

From (2.12)–(2.13) we have

$$c\overline{\partial}_+\alpha = \tan A\overline{\partial}_+ c, \tag{2.14}$$

$$c\overline{\partial}_+\beta = -\sin 2A\tan A - \tan A\overline{\partial}_+ c, \tag{2.15}$$

$$c\overline{\partial}_-\beta = -\tan A\overline{\partial}_- c, \tag{2.16}$$

$$c\overline{\partial}_-\alpha = \sin 2A\tan A + \tan A\overline{\partial}_- c. \tag{2.17}$$

From (2.10) we have

$$-\xi\overline{\partial}_\pm m - \eta\overline{\partial}_\pm n + \overline{\partial}_\pm p = 0. \tag{2.18}$$

Combining this with (2.11) and (2.5), we have

$$\overline{\partial}_\pm n = \mp\sqrt{p'(\rho)}\cos(\sigma \mp A)\overline{\partial}_\pm\rho, \quad \overline{\partial}_\pm m = \pm\sqrt{p'(\rho)}\sin(\sigma \mp A)\overline{\partial}_\pm\rho. \tag{2.19}$$

Proposition 2.1 (Commutator Relation) *We have*

$$\overline{\partial}_-\overline{\partial}_+ - \overline{\partial}_+\overline{\partial}_- = \frac{1}{\sin 2A}[(\cos 2A\overline{\partial}_+\beta - \overline{\partial}_-\alpha)\overline{\partial}_- - (\overline{\partial}_+\beta - \cos 2A\overline{\partial}_-\alpha)\overline{\partial}_+]. \tag{2.20}$$

Proof See [30] and we omit the proof.

Proposition 2.2 *For the variable ρ, we have the following characteristic decompositions*:

$$\begin{cases} c\overline{\partial}_+\overline{\partial}_-\rho = \sin 2A\overline{\partial}_-\rho + \dfrac{p''}{4c\cos^2 A}[(\overline{\partial}_-\rho)^2 + (2\sin^2 A - 1)\overline{\partial}_-\rho\overline{\partial}_+\rho], \\ c\overline{\partial}_-\overline{\partial}_+\rho = \sin 2A\overline{\partial}_+\rho + \dfrac{p''}{4c\cos^2 A}[(\overline{\partial}_+\rho)^2 + (2\sin^2 A - 1)\overline{\partial}_-\rho\overline{\partial}_+\rho]. \end{cases} \tag{2.21}$$

Proof We apply the commutator relation (2.19) for n and use $\overline{\partial}_\pm c = \frac{p''}{2c}\overline{\partial}_\pm\rho$ to obtain

$$\begin{aligned} &\overline{\partial}_+\Big[\frac{2p'}{p''}\cos\alpha\overline{\partial}_-c\Big] + \overline{\partial}_-\Big[\frac{2p'}{p''}\cos\beta\overline{\partial}_+c\Big] \\ &= -\frac{1}{\sin 2A}\Big[(\cos 2A\overline{\partial}_+\beta - \overline{\partial}_-\alpha)\frac{2p'}{p''}\cos\alpha\overline{\partial}_-c \\ &\quad + (\overline{\partial}_+\beta - \cos 2A\overline{\partial}_-\alpha)\frac{2p'}{p''}\cos\beta\overline{\partial}_+c\Big]. \end{aligned} \tag{2.22}$$

Hence, we have

$$\begin{aligned} &(\cos\alpha + \cos\beta)\frac{1}{c}\Big(\frac{2p'}{p''}\Big)'\overline{\partial}_+c\overline{\partial}_-c + \cos\alpha\overline{\partial}_+\overline{\partial}_-c + \cos\beta\overline{\partial}_-\overline{\partial}_+c \\ &= -\frac{1}{\sin 2A}[(\cos\beta\overline{\partial}_+\beta - \cos\beta\cos 2A\overline{\partial}_-\alpha - \sin\beta\sin 2A\overline{\partial}_-\beta)\overline{\partial}_+c \\ &\quad - (\cos\alpha\overline{\partial}_-\alpha - \cos\alpha\cos 2A\overline{\partial}_+\beta + \sin\alpha\sin 2A\overline{\partial}_+\alpha)\overline{\partial}_-c]. \end{aligned} \tag{2.23}$$

Applying the commutator relation (2.20) for c, we get

$$\overline{\partial}_-\overline{\partial}_+c - \overline{\partial}_+\overline{\partial}_-c = \frac{1}{\sin 2A}[(\cos 2A\overline{\partial}_+\beta - \overline{\partial}_-\alpha)\overline{\partial}_-c - (\overline{\partial}_+\beta - \cos 2A\overline{\partial}_-\alpha)\overline{\partial}_+c].$$

Inserting this into (2.23) and using (2.14)–(2.17) we can get

$$\begin{cases} c\overline{\partial}_+\overline{\partial}_-c = \Big\{\sin 2A + \dfrac{1}{2\cos^2 A}\overline{\partial}_-c + \Big(\dfrac{1}{2\cos^2 A} - \Big(\dfrac{2p'}{p''}\Big)'\Big)\overline{\partial}_+c\Big\}\overline{\partial}_-c, \\ c\overline{\partial}_-\overline{\partial}_+c = \Big\{\sin 2A + \dfrac{1}{2\cos^2 A}\overline{\partial}_+c + \Big(\dfrac{1}{2\cos^2 A} - \Big(\dfrac{2p'}{p''}\Big)'\Big)\overline{\partial}_-c\Big\}\overline{\partial}_+c. \end{cases} \tag{2.24}$$

Combining this with $\overline{\partial}_\pm c = \frac{p''}{2c}\overline{\partial}_\pm\rho$ we can get (2.21).

3 Interaction of Rarefaction Waves

3.1 Planar rarefaction waves

If $0 < \rho_0 \le \rho_c$ then the gas away from the sharp corner of the wedge expands to vacuum as two symmetrical planar rarefaction waves R_1 and R_2 (see Figure 2). In the (ξ, η) plane, R_1

and R_2 can be represented by

$$
\begin{aligned}
&R_1: \begin{cases}
\rho_1(\xi,\eta) = \widehat{\rho}(\xi\sin\theta - \eta\cos\theta),\\
m_1(\xi,\eta) = \sin\theta \displaystyle\int_{\rho_0}^{\rho_1(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho,\\
n_1(\xi,\eta) = -\cos\theta \displaystyle\int_{\rho_0}^{\rho_1(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho,\\
0 < \xi\sin\theta - \eta\cos\theta \le \zeta_0,
\end{cases}\\
&R_2: \begin{cases}
\rho_2(\xi,\eta) = \widehat{\rho}(\xi\sin\theta + \eta\cos\theta),\\
m_2(\xi,\eta) = \sin\theta \displaystyle\int_{\rho_0}^{\rho_2(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho,\\
n_2(\xi,\eta) = \cos\theta \displaystyle\int_{\rho_0}^{\rho_2(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho,\\
0 < \xi\sin\theta + \eta\cos\theta \le \zeta_0,
\end{cases}
\end{aligned}
\tag{3.1}
$$

where $\zeta_0 = \sqrt{p'(\rho_0)}$ and the function $\widehat{\rho}(\zeta)$ $(0 \le \zeta \le \zeta_0)$ is defined so that $\sqrt{p'(\widehat{\rho}(\zeta))} = \zeta$. Here, R_1 and R_2 are obtained by solving a one-dimensional Riemann problem. Since it is very classical, we omit the details.

3.2 Goursat problem

Referring to Figure 2, the rarefaction waves R_1 and R_2 start to interact from the point $P = (0, \xi_P) = \big(0, \frac{\zeta_0}{\sin\theta}\big)$. Through P draw a C_- (C_+, resp.) cross characteristic curve l_- (l_+, resp.) in R_1 (R_2, resp.). Using (2.2) and (3.1), we know that l_- and l_+ can be determined by

$$
\begin{cases}
l_-: \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_-(\xi,\eta,c_1(\xi,\eta)), \quad \eta(\xi_P) = 0, \quad 0 < \xi < \xi_P,\\
l_+: \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_+(\xi,\eta,c_2(\xi,\eta)), \quad \eta(\xi_P) = 0, \quad 0 < \xi < \xi_P,
\end{cases}
$$

where $c_i(\xi,\eta) = \sqrt{p'(\rho_i(\xi,\eta))}$ $(i = 1, 2)$. In order to construct the solution to the interaction of R_1 and R_2, we consider system (1.3) with the boundary data

$$
(\rho, m, n)(\xi,\eta) = \begin{cases}
(\rho_1, m_1, n_1)(\xi,\eta) & \text{on } l_-,\\
(\rho_2, m_2, n_2)(\xi,\eta) & \text{on } l_+.
\end{cases}
\tag{3.2}
$$

Problem (1.3), (3.2) is a standard Goursat problem (SGP for short). By the definition of characteristic angle, we can set

$$
\alpha\,|_{l_-} = \pi + \theta, \quad \beta\,|_{l_+} = \pi - \theta.
$$

3.3 Global classical solution to the SGP (1.3), (3.2)

Lemma 3.1 (Local Solution) *When $\varepsilon > 0$ is sufficiently small, the SGP (1.3), (3.2) admits a unique C^1 solution on a triangle domain Ω_ε closed by l_-, l_+, and a level curve $\rho = \rho_0 - \varepsilon$. Moreover, this solution satisfies*

$$
\omega = 0, \quad \overline{\partial}_-\rho < 0, \quad \overline{\partial}_+\rho < 0, \quad \overline{\partial}_-\beta > 0, \quad \overline{\partial}_+\alpha < 0.
\tag{3.3}
$$

Proof The local classical solution can be obtained by the classical theory for boundary value problems for quasilinear hyperbolic system (see for example Li and Yu [33]).

By computation, we have $\omega = 0$ in the rarefaction waves R_1 and R_2. Then by (2.8) we have

$$\omega = 0 \quad \text{on } l_- \cup l_+. \tag{3.4}$$

Combining this with (2.7) we have that the solution satisfies $\omega = 0$.

By (2.15) and (2.17) we have

$$\begin{cases} \overline{\partial}_- \rho = -\dfrac{2c\sin 2A}{p''(\rho)} < 0 \quad \text{on } l_-, \\ \overline{\partial}_+ \rho = -\dfrac{2c\sin 2A}{p''(\rho)} < 0 \quad \text{on } l_+. \end{cases} \tag{3.5}$$

Combining this with (2.21) we have that the solution satisfies $\overline{\partial}_-\rho < 0$ and $\overline{\partial}_+\rho < 0$. Consequently, by (2.14) and (2.16) we have $\overline{\partial}_+\alpha < 0$ and $\overline{\partial}_-\beta > 0$, respectively. We then have this lemma.

Lemma 3.2 (Hyperbolicity) *Assume that the SGP* (1.3), (3.2) *admits a* C^1 *solution on* Ω_ε *where* $0 < \varepsilon < \rho_0$. *Then the solution satisfies*

$$\arcsin\Big(\frac{\sin\theta\sqrt{p'(\rho_0-\varepsilon)}}{\zeta_0}\Big) \ < \ A \ < \ \theta \quad \text{in } \Omega_\varepsilon. \tag{3.6}$$

Proof From $\overline{\partial}_-\beta > 0$ and $\overline{\partial}_+\alpha < 0$ we have

$$\alpha < \pi + \theta, \quad \beta > \pi - \theta \quad \text{in } \Omega_\varepsilon.$$

It is easy to check by $\overline{\partial}_0 q = -1$ that $q < \frac{\zeta_0}{\sin\theta}$ in Ω_ε. Thus, by $A = \arcsin\frac{c}{q}$ and $p'' > 0$ as $\rho < \rho_0$ we have $A > \arcsin\big(\frac{\sin\theta\sqrt{p'(\rho_0-\varepsilon)}}{\zeta_0}\big)$ in Ω_ε. We then have this lemma.

Lemma 3.3 (A priori C^0 Norm Estimate) *Assume that the SGP* (1.3), (3.2) *admits a* C^1 *solution on* Ω_ε *where* $0 < \varepsilon < \rho_0$. *Then there exists a positive constant* $\mathcal{H}_0$ *independent of* ε, *such that*

$$\| \, (m, n, \rho) \, \|_{C^0(\Omega_\varepsilon)} < \mathcal{H}_0. \tag{3.7}$$

Proof This lemma can be proved by integrating (2.19) along $C_\pm$ characteristic curves.

Lemma 3.4 *Assume that the SGP* (1.3), (3.2) *admits a unique* C^1 *solution on* Ω_ε *where* $0 < \varepsilon < \rho_0$. *Then the solution satisfies*

$$(\overline{\partial}_+\rho, \overline{\partial}_-\rho) \in (-\mathcal{M}(\varepsilon),\ 0) \times (-\mathcal{M}(\varepsilon),\ 0) \quad \textit{in } \Omega_\varepsilon, \tag{3.8}$$

where

$$\mathcal{M}(\varepsilon) = \frac{4\sqrt{2}\zeta_0}{\sin\theta\sqrt{p'(\rho_0-\varepsilon)}} \cdot \max_{\rho\in[\rho_0-\varepsilon,\rho_0]} \Big\{\frac{\sqrt{p'(\rho)}}{p''(\rho)}\Big\}.$$

Proof By (3.5) we have that $\overline{\partial}_-\rho \in (-\mathcal{M}(\varepsilon), 0)$ along $\widetilde{PB_\varepsilon}$ and $\overline{\partial}_+\rho \in (-\mathcal{M}(\varepsilon), 0)$ along $\widetilde{PD_\varepsilon}$, where B_ε and D_ε are the points on l_- and l_+, respectively, such that $\rho(B_\varepsilon) = \rho(D_\varepsilon) = \rho_0 - \varepsilon$.

Let E be an arbitrary point in Ω_ε. If $\overline{\partial}_-\rho(E) = -\mathcal{M}(\varepsilon)$ and $\overline{\partial}_+\rho(E) \in [-\mathcal{M}(\varepsilon), 0)$, then by the first equation of (2.21) we have

$$\begin{aligned} c\overline{\partial}_+\overline{\partial}_-\rho &= \sin 2A\overline{\partial}_-\rho + \frac{p''}{4c\cos^2 A}[(\overline{\partial}_-\rho)^2 + (2\sin^2 A - 1)\overline{\partial}_-\rho\overline{\partial}_+\rho] \\ &> -\mathcal{M}(\varepsilon)\sin 2A + \frac{p''\sin^2 A}{2c\cos^2 A}\mathcal{M}^2(\varepsilon) \\ &> 2\mathcal{M}\sin A\Big(-1 + \frac{p''(\rho)}{\sqrt{p'(\rho)}} \cdot \frac{\sqrt{2}\zeta_0 \sin A}{\sin\theta\sqrt{p'(\rho_0-\varepsilon)}} \cdot \max_{\rho\in[\rho_0-\varepsilon,\rho_0]}\Big\{\frac{\sqrt{p'(\rho)}}{p''(\rho)}\Big\}\Big) \ > \ 0 \end{aligned}$$

at the point E. Similarly, if $\overline{\partial}_+\rho(E) = -\mathcal{M}(\varepsilon)$ and $\overline{\partial}_-\rho(E) \in [-\mathcal{M}(\varepsilon), 0)$, then by (3.6) and the second equation of (2.21) we have $c\overline{\partial}_-\overline{\partial}_+\rho \ > 0$ at the point E. Therefore, by an argument of continuity we can get (3.8). We then have this lemma.

Lemma 3.5 (A priori Gradient Estimate) *Assume that the SGP* (1.3), (3.2) *admits a* C^1 *solution on* Ω_ε *where* $0 < \varepsilon < \rho_0$. *Then there exists a positive constant* $\mathcal{H}_1$ *depending on* ε, *such that*

$$\| (Dm, Dn, D\rho) \|_{C^0(\Omega_\varepsilon)} < \mathcal{H}_1. \tag{3.9}$$

Proof By computation, we get

$$\partial_\xi = -\frac{\sin\beta\overline{\partial}_+ - \sin\alpha\overline{\partial}_-}{\sin 2A}, \quad \partial_\eta = \frac{\cos\beta\overline{\partial}_+ - \cos\alpha\overline{\partial}_-}{\sin 2A}.$$

Then the lemma can be obtained by (2.19) and Lemmas 3.2 and 3.4.

Theorem 3.1 *The SGP* (1.3), (3.2) *admits a* C^1 *solution on the domain* $\Omega = \bigcup_{\varepsilon\in(0,\rho_0)} \Omega_\varepsilon$.

Proof It is easy to check by $\overline{\partial}_\pm\rho < 0$ that the level curves of ρ are non-characteristic. Using Lemmas 3.3 and 3.5, and the standard extension method of [32], we can prove that for any $\varepsilon \in (0, \rho_0)$, if the SGP (1.3), (3.2) admits a C^1 solution on Ω_ε then there exists a $e > 0$ which depends on ε, such that the solution can be extend to $\Omega_{\varepsilon+e}$. We then have this theorem.

Remark 3.1 Hu and Wang [17, Section 5] studied the level curve $\rho(\xi,\eta) = 0$. They proved that the level curve $\rho(\xi,\eta) = 0$ is not a point but a closed curve.

4 Interaction of Shock-Rarefaction Composite Waves

4.1 Planar shock-rarefaction composite waves

If $\rho_0 > \rho_c$ then the gas away from the sharp corner of the wedge expands to vacuum as two symmetrical planar shock-rarefaction composite waves $S_1 \cup R_1$ and $S_2 \cup R_2$ (see Figure 3(right)). Since $\rho_0 > \rho_c$ and $p'(0) = 0$, there exists a $0 < \rho_* < \rho_c$ such that

$$p'(\rho_*) = \frac{p(\rho_0) - p(\rho_*)}{\rho_0 - \rho_*}.$$

Then by Rankine-Hugoniot conditions for nonclassical shocks (see [26]) we know that S_1 and S_2 are located at $\xi\sin\theta - \eta\cos\theta = \sqrt{p'(\rho_*)}$ and $\xi\sin\theta + \eta\cos\theta = \sqrt{p'(\rho_*)}$, respectively. Define

$$\chi := \sqrt{(p(\rho_0) - p(\rho_*))(\rho_0 - \rho_*)}.$$

Then, the (ρ, m, n) at the backsides of S_1 and S_2 are $(\rho_*, -\chi\sin\theta, \chi\cos\theta)$ and $(\rho_*, -\chi\sin\theta, -\chi\cos\theta)$, respectively.

The rarefaction waves R_1 and R_2 can be represented by

$$R_1: \begin{cases} \rho_1(\xi,\eta) = \widehat{\rho}(\xi\sin\theta - \eta\cos\theta), \\ m_1(\xi,\eta) = \sin\theta\Big(\displaystyle\int_{\rho_*}^{\rho_1(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho - \chi\Big), \\ n_1(\xi,\eta) = \cos\theta\Big(\chi - \displaystyle\int_{\rho_*}^{\rho_1(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho\Big), \\ 0 < \xi\sin\theta - \eta\cos\theta \le \zeta_*, \end{cases}$$

and

$$R_2: \begin{cases} \rho_2(\xi,\eta) = \widehat{\rho}(\xi\sin\theta + \eta\cos\theta), \\ m_2(\xi,\eta) = \sin\theta\Big(\displaystyle\int_{\rho_*}^{\rho_2(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho - \chi\Big), \\ n_2(\xi,\eta) = \cos\theta\Big(\displaystyle\int_{\rho_*}^{\rho_2(\xi,\eta)} \sqrt{p'(\rho)}\,\mathrm{d}\rho - \chi\Big), \\ 0 < \xi\sin\theta + \eta\cos\theta \le \zeta_*, \end{cases}$$

respectively, where $\zeta_* = \sqrt{p'(\rho_*)}$, and the function $\widehat{\rho}(\zeta)$ $(0 \le \zeta \le \zeta_*)$ is defined so that $\sqrt{p'(\widehat{\rho}(\zeta))} = \zeta$.

4.2 Discontinuous Goursat problem

Referring to Figure 3, the rarefaction waves S_1 and S_2 start to interact from the point $P = (0, \xi_P) := \big(0, \frac{\zeta_*}{\sin\theta}\big)$. Through P draw a C_- (C_+, resp.) cross characteristic curve l_- (l_+, resp.) in R_1 (R_2, resp.). Similarly, l_- and l_+ can be represented by

$$\begin{cases} l_-: \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_-(\xi,\eta,c_1(\xi,\eta)), \quad \eta(\xi_P) = 0, \quad 0 < \xi < \xi_P, \\ l_+: \dfrac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda_+(\xi,\eta,c_2(\xi,\eta)), \quad \eta(\xi_P) = 0, \quad 0 < \xi < \xi_P. \end{cases}$$

In order to construct the solution to the interaction of $S_1 \cup R_1$ and $S_2 \cup R_2$, we consider system (1.3) with the boundary data

$$(\rho,m,n)(\xi,\eta) = \begin{cases} (\rho_1,m_1,n_1)(\xi,\eta) & \text{on } l_-, \\ (\rho_2,m_2,n_2)(\xi,\eta) & \text{on } l_+. \end{cases} \tag{4.1}$$

Problem (1.3), (4.1) is a discontinuous Goursat problem (DGP for short), since the data at P is discontinuous.

4.3 Centered waves for the system (1.3)

In order to solve the DGP (1.3), (4.1), we fist give the definition of centered waves for the system (1.3).

Definition 4.1 (see Figure 5) *Let $\Psi(t)$ be an angular domain with curved boundaries*:

$$\Psi(t) := \{(\xi,\eta) \mid \xi_P - t \le \xi \le \xi_P,\ \eta_2(\xi) \le \eta \le \eta_1(\xi)\}, \tag{4.2}$$

where $\eta_1(\xi_P)=\eta_2(\xi_P)=\xi_P$ *and* $\eta_1'(\xi_P)<\eta_2'(\xi_P)$. *A function* $(\rho,m,n)(\xi,\eta)$ *is called a* C_- $\big(C_+$, *resp.*$\big)$ *centered wave for the system* (1.3) *with* P *as the center point if the following properties are satisfied* (*see* [33, *pp.* 188–190]):

(1) (ρ,m,n) *can be implicitly determined by the functions* $\eta=g(\xi,\nu)$ *and* $(\rho,m,n)(\xi,\eta)=(\check\rho,\check m,\check n)(\xi,\nu)$ $((\rho,m,n)(\xi,\eta)=(\overline{\rho},\overline{m},\overline{n})(\xi,\nu)$, *resp.*$)$ *defined on a rectangular domain* $T(t):=\{(\xi,\nu)\mid \xi_P-t\le\xi\le\xi_P,\ \eta_1'(\xi_P)\le\nu\le\eta_2'(\xi_P)\}$. *Moreover,* g *and* $(\check\rho,\check m,\check n)$ $((\overline{\rho},\overline{m},\overline{n})$, *resp.*$)$ *belong to* $C^1(T(t))$, *and for any* $(\xi,\nu)\in T(t)\setminus\{\xi=\xi_P\}$ *there holds* $g_\nu(\xi,\nu)<0$.

(2) *The function* $(\rho,m,n)(\xi,\eta)$ *defined above satisfies* (1.3) *on* $\Psi(t)\setminus\{(\xi_P,0)\}$.

(3) *For any fixed* $\nu\in[\eta_1'(\xi_P),\eta_2'(\xi_P)]$, $\eta=g(\xi,\nu)$ *gives the* C_- $(C_+$, *resp.*$)$ *characteristic line passing through* P *with the slope* ν *at* P, *i.e.*,

$$g_\xi=\lambda_-\ (\lambda_+,resp.),\quad g(\xi_P,\nu)=0,\quad g_\xi(\xi_P,\nu)=\nu. \tag{4.3}$$

(4) $\nu=\eta_1'(\xi_P)$ *and* $\nu=\eta_2'(\xi_P)$ *correspond to* $\eta=\eta_1(\xi)$ *and* $\eta=\eta_2(\xi)$, *respectively.*

Let $(\widetilde\rho_-,\widetilde m_-,\widetilde n_-)(\nu)=(\check u,\check m,\check n)(\xi_P,\nu)$ $((\widetilde\rho_+,\widetilde m_+,\widetilde n_+)(\nu)=(\overline{\rho},\overline{m},\overline{n})(\xi_P,\nu)$, *resp.*$)$, $\eta_1'(\xi_P)\le\nu\le\eta_2'(\xi_P)$. *Then* $(\widetilde\rho_-,\widetilde m_-,\widetilde n_-)(\nu)$ $((\widetilde\rho_+,\widetilde m_+,\widetilde n_+)(\nu)$, *resp.*$)$ *is called the principal part of this* C_- $(C_+$, *resp.*$)$ *centered wave, and* $\eta_2'(\xi_P)-\eta_1'(\xi_P)$ *the amplitude of the centered wave.*

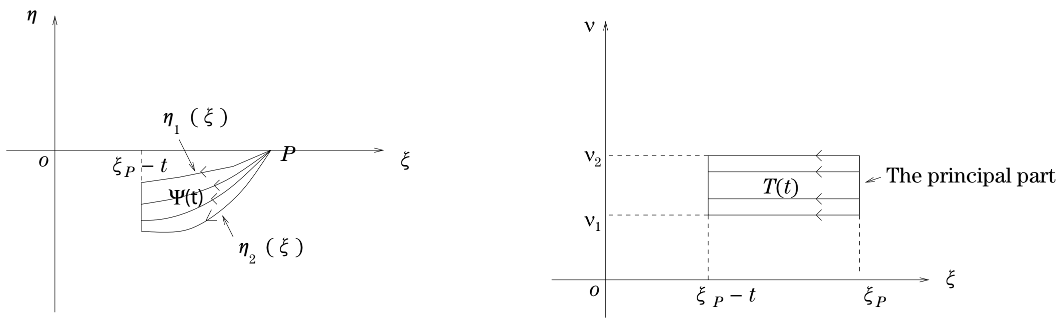

Figure 5 A C_+ centered wave, where $\nu_i=\eta_i'(\xi_P)$ $(i=1,2)$.

4.4 Principal parts of the $C_\pm$ centered waves

We first consider the principal part of the C_- centered wave. From the transformation $\xi=\xi$ and $\eta=g(\xi,\nu)$, we have the relations

$$\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\xi}-\frac{\partial g}{\partial\xi}\Big(\frac{\partial g}{\partial\nu}\Big)^{-1}\frac{\partial}{\partial\nu},\quad \frac{\partial}{\partial\eta}=\Big(\frac{\partial g}{\partial\nu}\Big)^{-1}\frac{\partial}{\partial\nu}. \tag{4.4}$$

Thus, in the (ξ,ν)-plane (1.3) can be written in the form

$$\begin{cases}-\xi\check\rho_\xi+\xi g_\xi g_\nu^{-1}\check\rho_\nu-gg_\nu^{-1}\check\rho_\nu+\check m_\xi-g_\xi g_\nu^{-1}\check m_\nu+g_\nu^{-1}\check n_\nu=0,\\ -\xi\check m_\xi+\xi g_\xi g_\nu^{-1}\check m_\nu-gg_\nu^{-1}\check m_\nu+\check p_\xi-g_\xi g_\nu^{-1}\check p_\nu=0,\\ -\xi\check n_\xi+\xi g_\xi g_\nu^{-1}\check n_\nu-gg_\nu^{-1}\check n_\nu+g_\nu^{-1}\check p_\nu=0.\end{cases} \tag{4.5}$$

From (4.3) we have that for the C_- centered wave,

$$\frac{\partial g(\xi,\nu)}{\partial\nu}=\int_{\xi_P}^{\xi}\sec^2\check\beta(\xi,\nu)\frac{\partial\check\beta(\xi,\nu)}{\partial\nu}\,\mathrm{d}\xi. \tag{4.6}$$

Therefore, using (4.5) and letting $\xi \to \xi_P$, we have that the principal part of the C_- centered waves satisfies

$$\widetilde{m}'_-(\nu) = \frac{\xi_P \nu^2 \widetilde{\rho}'_-(\nu)}{1+\nu^2}, \quad \widetilde{n}'_-(\nu) = -\frac{\xi_P \nu \widetilde{\rho}'_-(\nu)}{1+\nu^2}, \quad \widetilde{\rho}_-(\nu) = \widehat{\rho}\Big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\Big), \tag{4.7}$$

since $\nu = \lambda_- = -\frac{\sqrt{p'(\widetilde{\rho}'_-(\nu))}}{\sqrt{\xi_P^2 - \widetilde{\rho}'_-(\nu)}}$.

Lemma 4.1 *Consider the initial value problem*

$$\widetilde{n}'_-(\nu) = -\frac{\xi_P \nu \widetilde{\rho}'_-(\nu)}{1+\nu^2}, \quad \widetilde{n}_-(-\tan\theta) = \chi\cos\theta. \tag{4.8}$$

There exists a $\rho_m > \rho_c$ where ρ_m depends on θ, such that if $\rho_c < \rho_0 < \rho_m$ then there exists a $-\tan\theta < \nu_* < 0$ such that the solution of (4.8) *satisfies $\widetilde{n}_-(\nu_*) = 0$.*

Proof By integration, we have

$$\widetilde{n}_-(\nu) = \chi\cos\theta + \int_{-\tan\theta}^{\nu} \frac{\xi_P^2 \nu \widehat{\rho}'\Big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\Big)}{(1+\nu^2)^{\frac{5}{2}}}\,\mathrm{d}\nu.$$

So, when χ is not large there exists $-\tan\theta < \nu_* < 0$ such that the solution of (4.8) satisfies $\widetilde{n}_-(\nu_*) = 0$. We then have this lemma.

In what follows we shall confine ourselves to the case of $\rho_c < \rho_0 < \rho_m$.

4.5 Global piecewise smooth solution to the SGP (1.3), (4.1)

Centered wave problems for general first order quasilinear hyperbolic systems were first proposed and studied by Li and Yu [33–35]. They obtained local centered wave solutions with small amplitude (see [33, Theorem 7.1, p. 210]). Zhou [40–41] obtained local centered wave solutions with large amplitude for general first order quasilinear hyperbolic systems. In what follows, we shall use the result of Zhou [40].

Lemma 4.2 (Local Solution) *There exists a sufficiently small $\varepsilon > 0$, such that the SGP* (1.3), (4.1) *admits a solution on a triangle domain Δ closed by l_+, l_-, and the straight line $\xi = \xi_P - \varepsilon$. Moreover, the solution satisfies*

$$\omega = 0, \quad \overline{\partial}_-\rho < 0, \quad \overline{\partial}_+\rho < 0, \quad \overline{\partial}_-\beta > 0, \quad \overline{\partial}_+\alpha < 0, \quad \beta > \pi - \theta, \quad \alpha < \pi + \theta \quad \textit{in } \Delta. \tag{4.9}$$

Proof According to Lemma 4.1, the local existence of solution to the DGP (1.3), (4.1) can be obtained by Zhou [40, Theorem 2.1]. The solution contains a C_+ centered wave Δ_+ closed by l_+, $\xi = \xi_P - \varepsilon$, and a C_+ characteristic curve passing through P with the slope ν_* at P, and a C_- centered wave Δ_- closed by l_-, $\xi = \xi_P - \varepsilon$, and a C_+ characteristic curve passing through P with the slope $-\nu_*$ at P. The principal part of the C_- centered wave is

$$(\widetilde{m}_-, \widetilde{n}_-, \widetilde{\rho}_-)(\nu) = \Big(-\chi\sin\theta - \int_{-\tan\theta}^{\nu} \frac{\xi_P^2 \nu^2 \widehat{\rho}'\Big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\Big)}{(1+\nu^2)^{\frac{5}{2}}}\,\mathrm{d}\nu,\ \chi\cos\theta$$
$$+ \int_{-\tan\theta}^{\nu} \frac{\xi_P^2 \nu \widehat{\rho}'\Big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\Big)}{(1+\nu^2)^{\frac{5}{2}}}\,\mathrm{d}\nu,\ \widehat{\rho}\Big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\Big)\Big),$$

where $-\tan\theta \le \nu \le \nu_*$. The principal part of the C_+ centered wave is

$$(\widetilde{m}_+, \widetilde{n}_+, \widetilde{\rho}_+)(\nu)$$
$$=\Big(-\chi\sin\theta+\int_{\tan\theta}^{\nu}\frac{\xi_P^2\nu^2\widehat{\rho}'\Big(\dfrac{\nu\xi_P}{\sqrt{1+\nu^2}}\Big)}{(1+\nu^2)^{\frac{5}{2}}}\,\mathrm{d}\nu, -\chi\cos\theta+\int_{\tan\theta}^{\nu}\frac{\xi_P^2\nu\widehat{\rho}'\Big(\dfrac{\nu\xi_P}{\sqrt{1+\nu^2}}\Big)}{(1+\nu^2)^{\frac{5}{2}}}\,\mathrm{d}\nu,\ \widehat{\rho}\Big(\frac{\nu\xi_P}{\sqrt{1+\nu^2}}\Big)\Big),$$

where $-\nu_* \le \nu \le \tan\theta$. (Lemma 4.1 implies that $(\widetilde{m}_-, \widetilde{n}_-, \widetilde{\rho}_-)(\nu_*) = (\widetilde{m}_+, \widetilde{n}_+, \widetilde{\rho}_+)(-\nu_*)$.)

By a method which is similar to that of Lemma 3.3, we have $\omega \equiv 0$ in Δ.

By (2.15) and (2.17) we have

$$\begin{cases} \overline{\partial}_-\rho = -\dfrac{2c\sin 2A}{p''(\rho)} < 0 \quad \text{along } l_-, \\ \overline{\partial}_+\rho = -\dfrac{2c\sin 2A}{p''(\rho)} < 0 \quad \text{along } l_+. \end{cases} \tag{4.10}$$

By computation, we obtain

$$\overline{\partial}_+\rho = \cos\alpha\rho_\xi + \sin\alpha\rho_\eta = \Big(\frac{\partial g}{\partial\nu}\Big)^{-1}\Big\{\Big(\frac{\partial g}{\partial\nu}\Big)\cos\alpha\check{\rho}_\xi + \frac{\sin 2A}{\cos\beta}\check{\rho}_\nu\Big\} \tag{4.11}$$

in Δ_-. Since $\check{\rho}(\xi_P, \nu) = \widetilde{\rho}_-(\nu) = \widehat{\rho}\big(\frac{-\nu\xi_P}{\sqrt{1+\nu^2}}\big)$, we have $\check{\rho}_\nu(\xi_P, \nu) < 0$. Thus, by (4.11) we have that if ε is sufficiently small then $\overline{\partial}_+\rho < 0$ in Δ_-. From (2.21) and (4.10) we have that the solution satisfies $\overline{\partial}_-\rho < 0$ in Δ_-. Using (2.14) and (2.16), we also have $\overline{\partial}_+\alpha < 0$ and $\overline{\partial}_-\beta > 0$ in Δ_-. Using $\alpha\,|_{l_-} = \pi + \theta$, $\beta\,|_{l_+} = \pi - \theta$, and $\widetilde{\beta}_-(\nu) = \pi + \arctan\nu$ $(-\tan\theta < \nu < \nu_* < 0)$, we further have $\alpha < \pi + \theta$ and $\beta > \pi - \theta$ in Δ_-. By symmetry we have $\overline{\partial}_+\rho < 0$, $\overline{\partial}_-\rho < 0$, $\overline{\partial}_+\alpha < 0$, $\overline{\partial}_-\beta > 0$, $\alpha < \pi + \theta$, and $\beta > \pi - \theta$ in Δ_+.

The C_+ characteristic curve passing through P with the slope $-\nu_*$ at P intersects with the straight line $\xi = \xi_P - \varepsilon$ at a point G; the C_- characteristic curve passing through P with the slope ν_* at P intersects with the straight line $\xi = \xi_P - \varepsilon$ at a point F. Using $\overline{\partial}_-\rho\,|_{\widetilde{PG}} < 0$, $\overline{\partial}_+\rho\,|_{\widetilde{PF}} < 0$, $\alpha\,|_{\widetilde{PF}} < \pi + \theta$, and $\beta\,|_{\widetilde{PG}} > \pi - \theta$, we can get $\overline{\partial}_+\rho < 0$, $\overline{\partial}_-\rho < 0$, $\overline{\partial}_+\alpha < 0$, $\overline{\partial}_-\beta > 0$, $\alpha < \pi + \theta$, and $\beta > \pi - \theta$ in $\Delta_0 := \Delta \setminus (\Delta_- \cup \Delta_+)$.

We then have this lemma.

We are now ready to construct a global solution to the DGP (1.3), (4.1). See Figure 6. The C_+ characteristic curve through F intersects with l_- at a point E; the C_- characteristic curve through G intersects with l_+ at a point H. By solving a SGP for the system (1.3) with $\widetilde{EF}$ and l_- as the characteristic boundaries, we can find a solution in a curved quadrilateral domain closed by l_-, $\widetilde{EF}$, $\widetilde{FI}$, and $\widetilde{IO}$, where $\widetilde{FI}$ is a C_- characteristic curve passing through F and $\widetilde{IO}$ is a level curve $\rho(\xi, \eta) = 0$. Similarly, by solving a SGP for the system (1.3) with $\widetilde{HG}$ and l_+ as the characteristic boundaries we can find a solution in a curved quadrilateral domain closed by l_+, $\widetilde{HG}$, $\widetilde{GJ}$, and $\widetilde{JO}$, where $\widetilde{GJ}$ is a C_+ characteristic curve passing through G and $\widetilde{JO}$ is a level curve $\rho(\xi, \eta) = 0$. In the end, by solving a SGP for the system (1.3) with $\widetilde{PI}$ and $\widetilde{PJ}$ as the characteristic boundaries, we can find a solution in a triangle domain closed by $\widetilde{PI}$, $\widetilde{PJ}$ and $\widetilde{IJ}$, where $\widetilde{IJ}$ is a level curve $\rho(\xi, \eta) = 0$. The existence of global classical solutions to these SGPs can be obtained by the same method as in Section 3, since $\overline{\partial}_\pm\rho < 0$ are satisfied on the $C_\pm$ characteristic boundaries. We omit the details. Therefore, we have the following theorem.

Theorem 4.1 *The DGP* (1.3), (4.1) *admits a piecewise smooth solution on a region* Ω *closed by* l_+, l_-, *and a level curve* $\rho(\xi, \eta) = 0$.

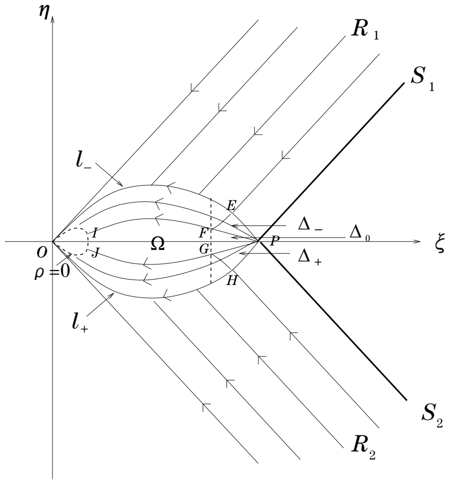

Figure 6 Global piecewise smooth solution to the SGP (1.3), (4.1).

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