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On a Logarithmic Type Nonlocal Plane Curve Flow*

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Abstract In this paper the author devotes to studying a logarithmic type nonlocal plane curve flow. Along this flow, the convexity of evolving curve is preserved, the perimeter decreases, while the enclosed area expands. The flow is proved to exist globally and converge to a finite circle in the C^∞ metric as time goes to infinity.

Keywords Convex curve, Nonlocal flow, Logarithmic type, Parallel curve
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1 Introduction

The curve evolution problems have received much attention during the last few decades (see [1–4,17,24]). In particular, the popular curve shortening flow in the plane studied by Gage [8–10], Gage and Hamilton [11] and Grayson [12], etc., is a very interesting and subtle case which, except for its own significance, provides a background for understanding the mean curvature flow [26] and other geometric evolution problems in higher dimensions. The book by Chou (Tso) and Zhu [6] provides an excellent and unified account of many results related to flowing curves by curvature.

Recently, there has been some interest in the nonlocal flow of convex closed plane curves, for example, the area-preserving flows are studied by Gage [11], Ma-Cheng [18] and Mao-Pan-Wang [20], the perimeter-preserving flows are investigated by Ma-Zhu [19] and Pan-Yang [23]. In [16], Lin and Tsai summarized previous nonlocal curve flows to the following general form:

$$\begin{cases}\dfrac{\partial X}{\partial t}(u,t)=[F(k(u,t))-\lambda(t)]\mathbb{N}_{\rm in}(u,t),\\ X(u,0)=X_0(u),\end{cases}$$

where $X_0(u)\subset\mathbb{R}^2$ is a given smooth closed strictly convex curve, parametrized by $u\in S^1$, and $X(u,t):S^1\times[0,T)\to\mathbb{R}^2$ is a family of closed planar curves moving along its inward normal direction $\mathbb{N}_{\rm in}(u,t)$ with given speed function $F(k(u,t))-\lambda(t)$. $F(k)$ is a function of curvature $k(u,t)$ of the evolution curve $X(u,t)$ satisfying the parabolic condition $F'(z)>0$ for all z in its domain, and $\lambda(t)$ is a function of time which depends on certain global quantities of $X(u,t)$, say its length $L(t)$, enclosed area $A(t)$, or other possible global quantities of the integral of curvature over the entire curve in certain ways. In such a case the flow has nonlocal

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character. In the paper [25], Tsai-Wang gave a systematical study of area-preserving flows and perimeter-preserving flows if $F(k) = k^\alpha,\ \alpha > 0$.

Different from the previous models, the purpose of this paper is to investigate a logarithmic type nonlocal curve flow which takes the form:

$$\begin{cases} \dfrac{\partial X}{\partial t}(u,t) = \Big[\ln\Big(\dfrac{L(t)}{2\pi}k(u,t)\Big)\Big]\mathbb{N}_{\rm in}(u,t), \\ X(u,0) = X_0(u), \end{cases} \tag{1.1}$$

where $X_0(u)$ is a smooth closed strictly convex curve in the plane, $L(t)$ is the perimeter of the evolution curve at time t and k is its relative curvature. Most previous papers in the topic of nonlocal curve flow considered the flows with velocity of $k^\alpha + \lambda(t)$ ($\alpha \in \mathbb{R},\ \alpha \neq 0$), $\lambda(t)$ being a constraint term. The velocity in the flow (1.1) has a different form, that is, a logarithmic form. The nonlocal flow (1.1) is not suitable for non-convex simple closed curves. The circle will be stable under this flow. If the initial curve is not circle, the points on the part where curvature is large will move inwards and the points on the part where curvature is small will move outwards.

The main result of the present paper is as follows.

Main Theorem *A closed convex planar curve $X_0(u)$ which evolves according to* (1.1) *remains convex, decreases its length but enlarges its enclosed area, becomes more and more circular during the evolution process, and finally converges to a finite circle in the C^∞ metric as t goes to infinity.*

To obtain the main theorem, we follow the classical proof steps for the study of curve flows. The first step is to show that the convexity of evolving curve is preserved along the flow via maximum principle. The second step is to obtain the local existence of flow by converting the problem to an equivalent parabolic type PDE. Then by showing that the curvature of evolving curve does not blow up (surely, a time-independent upper bound estimate on curvature, but not necessary, implies this), we can obtain the long time existence of the flow. Combining with the structure of the flow, we can employ Bonnesen inequality to prove that a globally existing flow converges to a circle in Hausdorff metric as time goes to infinity. Furthermore, by establishing a time-independent upper bound on curvature and its derivatives, we can finally show the smooth convergence of the flow.

Though the proof steps are standard, we have to overcome some difficulties which are caused by the logarithmic form velocity. To this end, we establish two new type integral inequalities (see Theorem 2.1), in order to show the monotonicity of perimeter and enclosed area of the evolving curve. The upper bound of curvature is obtained via Tso's support function method (see [5]). The gradient estimate of curvature is obtained via energy method. After a delicate combination of various new and classical techniques, the main theorem is proved finally.

This paper is organized as follows. In Section 2, two integral inequalities are proved to help us understand the final shape of the evolving curve under the flow (1.1). Section 3 is devoted to prove that if the evolving curve does not develop singularities then it converges to a circle in the Hausdorff metric as t goes to infinity. In Section 4, the long time existence of the evolving curve is proved. In Section 5, it is shown that the limiting curve of the flow (1.1) is a finite circle in the C^∞ metric.

2 Two Integral Inequalities

In this section, if a convex curve evolving under (1.1) remains convex, we will prove two integral inequalities to help us obtain the monotonicity of the perimeter $L(t)$ and the area $A(t)$ bounded by the evolving curve.

Lemma 2.1 (Entropy Inequality for Curvature, see [13, Theorem 0.2]) *For a C^2 closed and strictly convex curve γ, let L and A be the perimeter of γ and the area it encloses respectively. One gets that*

$$\oint_\gamma k \ln \Big(k\sqrt{\frac{A}{\pi}}\Big) \mathrm{d}s \geq 0. \tag{2.1}$$

The equality in (2.1) *holds if and only if γ is a circle.*

Theorem 2.1 *For a C^2 closed and strictly convex curve γ, one gets*

$$\oint_\gamma k(\ln k)\mathrm{d}s + 2\pi \ln \frac{L}{2\pi} \geq 0, \tag{2.2}$$

$$\oint_\gamma \ln k \mathrm{d}s + L \ln \frac{L}{2\pi} \leq 0, \tag{2.3}$$

and moreover, the equality in (2.2) *or* (2.3) *holds if and only if the curve is a circle.*

Proof From the classical isoperimetric inequality $L^2 \geq 4\pi A$, we have

$$0 \leq \oint_\gamma k \ln \Big(k\sqrt{\frac{A}{\pi}}\Big)\mathrm{d}s \leq \oint_\gamma k \ln \Big(k\sqrt{\frac{1}{\pi}\cdot\frac{L^2}{4\pi}}\Big)\mathrm{d}s = \oint_\gamma k(\ln k)\mathrm{d}s + 2\pi \ln \frac{L}{2\pi}.$$

So (2.2) is proved and the equality holds if and only if the curve is a circle. It remains to prove (2.3).

Inspired by the method of the paper [15, Lemma 18], we use the monotonicity formula for convex parallel curves to prove (2.3). Suppose that γ_β is a convex curve outer parallel to the curve γ with distance $r > 0$. Their perimeter, enclosed area, and curvature are related by (see [7, p. 47])

$$L_\beta = L + 2\pi r, \quad A_\beta = A + rL + \pi r^2, \quad k_\beta(s) = \frac{k(s)}{1 + rk(s)}, \tag{2.4}$$

where s is an arc length parameter of γ. As a consequence, we have the infinitesimal identities

$$\frac{\mathrm{d}L_\beta}{\mathrm{d}r} = 2\pi, \quad \frac{\mathrm{d}A_\beta}{\mathrm{d}r} = L_\beta, \quad \frac{\mathrm{d}k_\beta}{\mathrm{d}r}(s) = -k_\beta^2(s). \tag{2.5}$$

Let $H(r) = \oint_{\gamma_\beta} \ln(k_\beta)\mathrm{d}s_\beta + L_\beta \ln \frac{L_\beta}{2\pi}$. Then from (2.3)–(2.5), we have

$$\begin{aligned}\frac{\mathrm{d}H(r)}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\Big[\oint_{\gamma_\beta} \ln(k_\beta)\mathrm{d}s_\beta + L_\beta \ln\Big(\frac{L_\beta}{2\pi}\Big)\Big] = \frac{\mathrm{d}}{\mathrm{d}r}\Big[\int_0^{2\pi} \frac{\ln(k_\beta)}{k_\beta}\mathrm{d}\theta + L_\beta \ln\Big(\frac{L_\beta}{2\pi}\Big)\Big] \\ &= \int_0^{2\pi} \frac{k_\beta \cdot \frac{1}{k_\beta}(-k_\beta^2) - \ln(k_\beta)\cdot(-k_\beta^2)}{k_\beta^2}\mathrm{d}\theta + (L_\beta)_r \cdot \ln\frac{L_\beta}{2\pi} + L_\beta \cdot \frac{2\pi}{L_\beta}\cdot (L_\beta)_r \\ &= \oint_{\gamma_\beta} k_\beta \cdot \ln(k_\beta)\mathrm{d}s_\beta + 2\pi \ln\Big(\frac{L_\beta}{2\pi}\Big) \geq 0.\end{aligned}$$

From (2.4), we also note that

$$\begin{aligned}\lim_{r\to+\infty} H(r) &= \lim_{r\to+\infty} \Big[\oint_{\gamma_\beta} \ln(k_\beta)\mathrm{d}s_\beta + L_\beta \ln\Big(\frac{L_\beta}{2\pi}\Big)\Big] \\ &= \lim_{r\to+\infty} \oint_{\gamma_\beta} \Big[\ln\Big(k_\beta \cdot \frac{L_\beta}{2\pi}\Big)\Big]\mathrm{d}s_\beta \\ &= \lim_{r\to+\infty} \oint_{\gamma_\beta} \ln\Big(\frac{kL+2\pi kr}{2\pi+2\pi kr}\Big)\mathrm{d}s_\beta = 0.\end{aligned}$$

So we have $H(r) \le 0$ for all $r \ge 0$. Specially, when $r = 0$,

$$H(0) = \oint_\gamma \ln k\mathrm{d}s + L\ln\Big(\frac{L}{2\pi}\Big) \le 0.$$

The equality holds if and only if γ is a circle.

3 The Final Shape of the Evolving Curve

Let $X_0(u) : S^1 \to \mathbb{R}^2$ be a convex closed curve in the plane and $X(u,t) = (x(u,t), y(u,t)) : S^1 \times [0,T) \to \mathbb{R}^2$ be a family of closed planar curves which evolve under the flow (1.1).

Let $g(u,t) = |X_u| = (x_u^2 + y_u^2)^{\frac{1}{2}}$ denote the metric along the curve. Then the arc-length element is given by $\mathrm{d}s = g(u,t)\mathrm{d}u$, or formally

$$\frac{\partial}{\partial s} = \frac{1}{g}\frac{\partial}{\partial u}, \quad \frac{\partial s}{\partial u} = g.$$

The tangent $\mathbb{T}$, normal $\mathbb{N}$, tangent angle θ, curvature k and perimeter L of the curve and area A it bounds are defined in the standard way:

$$\mathbb{T} = \frac{\partial X}{\partial s} = \frac{1}{g}\frac{\partial X}{\partial u}, \quad k = \frac{\partial \theta}{\partial s} = \frac{1}{g}\frac{\partial \theta}{\partial u}, \quad \mathbb{N} = \frac{1}{k}\frac{\partial \mathbb{T}}{\partial s} = \frac{1}{kg}\frac{\partial \mathbb{T}}{\partial u},$$

$$L(t) = \int_a^b g(u,t)\mathrm{d}u = \oint \mathrm{d}s, \quad A(t) = \frac{1}{2}\oint x\mathrm{d}y - y\mathrm{d}x = -\frac{1}{2}\oint \langle X, \mathbb{N}\rangle \mathrm{d}s.$$

Since changing the tangential components of the velocity vector X_t affects only the parametrization, not the geometric shape of the evolving curve (see [6, 11, 22]), we can choose a suitable tangential component $\alpha = -\frac{\partial}{\partial\theta}\big(\ln\big(\frac{L}{2\pi}k\big)\big)$ which makes θ independent of t, such that the geometric analysis of the evolving curve can be simplified, i.e., we may consider the following evolution problem which is equivalent to flow (1.1):

$$\begin{cases} \dfrac{\partial X}{\partial t}(u,t) = \alpha\mathbb{T} + \Big[\ln\Big(\dfrac{L(t)}{2\pi}k(u,t)\Big)\Big]\mathbb{N}_{\mathrm{in}}(u,t), \\ X(u,0) = X_0(u). \end{cases} \tag{3.1}$$

Just as Gage, Hamilton and others have done, we can derive the evolution equations of the perimeter $L(t)$, the enclosed area $A(t)$ and the curvature k of the evolving curve as follows.

Lemma 3.1 *Under the flow* (3.1) *the geometric quantities of the evolving curve evolve as*

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\oint k\Big(\ln\Big(\frac{L}{2\pi}k\Big)\Big)\mathrm{d}s = -\Big[\oint k\ln k\mathrm{d}s + 2\pi\ln\Big(\frac{L}{2\pi}\Big)\Big], \tag{3.2}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\oint\Big(\ln\Big(\frac{L}{2\pi}k\Big)\Big)\mathrm{d}s = -\Big[\oint \ln k\mathrm{d}s + L\cdot\ln\frac{L}{2\pi}\Big], \tag{3.3}$$

$$\frac{\partial k}{\partial t} = k^2\Big[\frac{\partial^2}{\partial\theta^2}\Big(\ln\Big(\frac{L}{2\pi}k\Big)\Big) + \ln\Big(\frac{L}{2\pi}k\Big)\Big]. \tag{3.4}$$

In the following, suppose that the flow (3.1) exists on time interval $[0,\infty)$. It is first proved that the evolving curve is convex, i.e., the curvature $k = \frac{1}{\rho}$ (ρ is the curvature radius of the evolution curve) is always positive. Then the Hausdorff convergence of the evolving curve is proved by showing that the isoperimetric ratio $\frac{L(t)^2}{A(t)}$ converges to 4π as $t\to\infty$.

Theorem 3.1 *The flow* (3.1) *keeps the convexity of the curve during the evolution process.*

Proof From the evolution equation (3.4), it follows that

$$\frac{1}{k^2}\frac{\partial k}{\partial t} = \frac{\partial^2}{\partial\theta^2}\Big(\ln\frac{L}{2\pi} - \ln\rho\Big) - \ln\Big(\frac{2\pi}{L}\rho\Big),$$

namely,

$$\frac{\partial\rho}{\partial t} = (\ln\rho)_{\theta\theta} + \ln\Big(\frac{2\pi}{L}\rho\Big) = \frac{\rho_{\theta\theta}}{\rho} - \Big(\frac{\rho_\theta}{\rho}\Big)^2 + \ln\Big(\frac{2\pi}{L}\rho\Big) \le \frac{\rho_{\theta\theta}}{\rho} - \Big(\frac{\rho_\theta}{\rho}\Big)^2 + \frac{2\pi}{L}\rho.$$

Let $\rho_{\max}(t) \triangleq \max\{\rho(\theta,t)\mid\theta\in[0,2\pi]\}$. The maximum principle implies that

$$\frac{\partial\rho_{\max}}{\partial t} \le \frac{2\pi}{L}\rho_{\max} \le \sqrt{\frac{\pi}{A_0}}\rho_{\max},$$

and

$$\rho_{\max}(t) \le \rho_{\max}(0)\mathrm{e}^{\sqrt{\frac{\pi}{A_0}}t}.$$

Thus the curvature

$$k(\theta,t) \ge k_{\min}(\theta,t) \ge k_{\min}(\theta,0)\mathrm{e}^{-\sqrt{\frac{\pi}{A_0}}t} > 0.$$

We get that the evolving curve is convex during the evolution process.

Theorem 3.2 *Under the flow* (3.1) *with an initial convex curve, the perimeter* $L(t)$ *of the evolving curve decreases and its enclosed area* $A(t)$ *increases. Furthermore, the isoperimetric ratio* $\frac{L^2}{A}$ *is decreasing to* 4π *which means that the evolving curve becomes more and more circular.*

Proof From (2.2)–(2.3) of Theorem 2.2, we can see that $L_t \le 0$ and $A_t \ge 0$. So the perimeter $L(t)$ decreases and the enclosed area $A(t)$ increases during the evolution process. Since $L \le L_0$ and $A \ge A_0$, together with the classical isoperimetric inequality, we can get

$$4\pi A_0 \le 4\pi A \le L^2 \le L_0^2.$$

So we can conclude that there exists a positive number R such that

$$\lim_{t\to\infty} L(t) = 2\pi R \triangleq L(\infty), \quad \lim_{t\to\infty} A(t) = \pi R^2 \triangleq A(\infty). \tag{3.5}$$

By the evolution equations of (3.2)–(3.3), and the inequality $\frac{x}{1+x} \le \ln(1+x),\ x > 0$, the isoperimetric ratio $\frac{L^2}{A}$ of the evolving curve evolves according to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{L^2}{A}\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{L^2}{A} - 4\pi\Big) = \frac{2LAL_t - L^2A_t}{A^2} \\
&\le \frac{2LAL_t}{A^2} = \frac{2L}{A}\Big[-\oint k\ln k\mathrm{d}s - 2\pi\ln\Big(\frac{L}{2\pi}\Big)\Big] \\
&\le \frac{2L}{A}\Big[\pi\ln\Big(\frac{A}{\pi}\Big) - 2\pi\Big(\frac{L}{2\pi}\Big)\Big] = -\frac{2\pi L}{A}\Big[\ln\Big(1 + \frac{L^2}{4\pi A} - 1\Big)\Big] \\
&\le -\frac{2\pi L}{A}\frac{\frac{L^2}{4\pi A} - 1}{1 + \frac{L^2}{4\pi A} - 1} = -\frac{2\pi}{L}\Big(\frac{L^2}{A} - 4\pi\Big) \\
&\le -\frac{2\pi}{L_0}\Big(\frac{L^2}{A} - 4\pi\Big).
\end{aligned}$$

So $\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{L^2}{A} - 4\pi\big) \le 0$. Moreover, we have

$$0 \le \frac{L^2}{A} - 4\pi \le \Big(\frac{L^2(0)}{A(0)} - 4\pi\Big)\mathrm{e}^{-\frac{2\pi}{L_0}t}.$$

As $t \to \infty$ we have the decay of the isoperimetric ratio $\frac{L^2}{A} - 4\pi \to 0$, i.e., $\frac{L^2}{A} \to 4\pi$.

Employing Theorem 3.2 and the Bonnesen inequality (see [21]), we can get the following Theorem 3.3. We omit the details of the proof (which are similar to that of [11, Corollary 2.5]) here.

Theorem 3.3 *If an evolving curve under the flow* (3.1) *does not develop singularities, then it converges to a circle in the Hausdorff metric.*

4 The Long Time Existence

Following the standard method of curve flows (see [11, 14, 23]), we study the long time behavior of the flow (1.1) which is equivalent to a Cauchy problem of curvature.

Lemma 4.1 *The nonlocal flow* (1.1) *is equivalent to the following Cauchy problem*:

$$\begin{cases}
\dfrac{\partial k}{\partial t} = k^2(\ln k)_{\theta\theta} + k^2\Big(\ln\dfrac{L}{2\pi}k\Big), \\
\dfrac{\mathrm{d}L}{\mathrm{d}t} = -\displaystyle\int_0^{2\pi}\ln\Big(\frac{L}{2\pi}k\Big)\mathrm{d}\theta, \\
k(\theta, 0) = k_0(\theta) > 0, \\
L(0) = \displaystyle\int_0^{2\pi}\frac{1}{k_0(\theta)}\mathrm{d}\theta > 0,
\end{cases} \tag{4.1}$$

where $(\theta, t) \in [0, 2\pi] \times [0, T)$.

The local existence of the flow (1.1) is an application of the classical theory of parabolic equations. In this section, we will use the method in the paper [25] to obtain a time-independent upper bound of the curvature. Then the flow (1.1) can be extended on time interval $[0, +\infty)$.

Theorem 4.1 *Under the flow* (1.1), *the curvature* k *has a uniform upper bound for all* $(\theta, t) \in [0, 2\pi] \times [0, \infty)$.

Proof Given a convex curve in the plane, the Bonnesen inequality (see [21]) says

$$rL - A - \pi r^2 \geq 0$$

for all $r_{\text{in}} \leq r \leq r_{\text{out}}$ where r_{in}, r_{out} are the inradius and the outradius of the given convex curve. So

$$0 < \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq r_{\text{in}} \leq r_{\text{out}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}. \tag{4.2}$$

Let $I = \frac{L^2}{4\pi A}$. It follows from (4.2) that

$$I \leq \frac{r_{\text{out}}}{r_{\text{in}}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{L - \sqrt{L^2 - 4\pi A}} = (\sqrt{I} + \sqrt{I + 1})^2.$$

Since the isoperimetric ratio $I(t)$ of $X(\theta, t)$ is decreasing as well as the isoperimetric deficit under the flow (1.1), we have $1 \leq I(t) \leq I(0)$ for all $t \in [0, \varpi)$. Let $r_{\text{in}}(t)$ and $r_{\text{out}}(t)$ denote the inradius and the outradius of $X(\theta, t)$ respectively. Let $\sigma = (\sqrt{I(0)} + \sqrt{I(0) + 1})^2$. Therefore

$$r_{\text{out}}(t) \leq \sigma r_{\text{in}}(t).$$

Since $A(0) \leq A(t) \leq \pi r_{\text{out}}^2(t)$, we get

$$r_{\text{in}}(t) \geq \sigma^{-1} \sqrt{\frac{A(0)}{\pi}}, \quad \forall\, t \in [0, \varpi].$$

Both the inradius and the outradius have time-independent positive bounds.

Let $E(0)$ be the inscribed circle of $X(\theta, 0)$ with radius $r_{\text{in}}(0)$. Let $E(0)$ shrink according to $X_t = \big(\frac{L(0)}{2\pi} k\big)\mathbb{N}_{\text{in}}$, where $L(0)$ is the length of $X(\theta, 0)$. Then $E(t)$ is enclosed by $X(\theta, t)$ for all $t \in [0, \min\{\varpi, t_1\})$ by the maximum principle, where $t_1 = \frac{4\pi}{L(0)} \sigma^{-1} \sqrt{\frac{A(0)}{\pi}}$. The radius $r(t)$ of $E(t)$ is given by

$$r(t) = \Big(r^2(0) - \frac{L(0)}{4\pi} t\Big)^{\frac{1}{2}}.$$

Let $p(\theta, t) = -\langle X(\theta, t), \mathbb{N}_{\text{in}}(\theta, t)\rangle$ be the support function of $X(\theta, t)$. If the center of $E(0)$ is chosen to be the origin O, then $p(\theta, t)$ (with respective to O) of $X(\theta, t)$ is defined on $[0, 2\pi] \times [0, \min\{\varpi, t_1\})$ and

$$p(\theta, t) \geq \Big(r^2(0) - \frac{L(0)}{4\pi} t\Big)^{\frac{1}{2}}. \tag{4.3}$$

Set $T_0 = \frac{1}{2} \min(\varpi, t_1)$. It follows from (4.3) and $r(0) \geq \sigma^{-1} \sqrt{\frac{A(0)}{\pi}}$ that $p(\theta, t) \geq 2\beta$ holds on $[0, 2\pi] \times [0, T_0]$, where $\beta > 0$ is a constant only depending on the initial curve $X(\theta, 0)$. And β can be explicitly given by $\beta = \frac{1}{2\sqrt{2}} \sigma^{-1} \sqrt{\frac{A(0)}{\pi}}$. Since $L(t)$ is decreasing, we have an upper bound of the support function $p(\theta, t) \leq \frac{L(t)}{2} \leq \frac{L(0)}{2}$. So there is a constant $C = \frac{L(0)}{2} > 0$ depending only on $X(\theta, 0)$ such that $p(\theta, t) \leq C$ on $S^1 \times [0, T_0]$. We can get

$$0 < 2\beta \leq p(\theta, t) \leq C \quad \text{ on } S^1 \times [0, T_0). \tag{4.4}$$

Define

$$Q(\theta,t)=\frac{\ln k(\theta,t)}{p(\theta,t)-\beta},\quad (\theta,t)\in S^1\times[0,T_0).$$

The first and the second derivations of Q with respect to θ are

$$Q_\theta=\frac{(\ln k)_\theta}{p-\beta}-\frac{p_\theta\cdot\ln k}{(p-\beta)^2},$$

$$Q_{\theta\theta}=\frac{(\ln k)_{\theta\theta}}{p-\beta}-\frac{2p_\theta\cdot(\ln k)_\theta}{(p-\beta)^2}+\Big[\frac{2p_\theta^2}{(p-\beta)^3}-\frac{p_{\theta\theta}}{(p-\theta)^2}\Big]\cdot\ln k.$$

Under the nonlocal flow (1.1), the evolution of the support function is

$$\frac{\partial p}{\partial t}=-\ln\Big(\frac{L}{2\pi}k\Big).$$

We compute the evolution of Q on $S^1\times[0,T_0)$ to get

$$\begin{aligned}Q_t&=\frac{(\ln k)_t(p-\beta)-\ln k\cdot p_t}{(p-\beta)^2}\\&=\frac{k\Big[(\ln k)_{\theta\theta}+\ln\Big(\frac{L}{2\pi}k\Big)\Big]}{p-\beta}-\frac{\ln k\Big(-\ln\Big(\frac{L}{2\pi}k\Big)\Big)}{(p-\beta)^2}\\&=kQ_{\theta\theta}+\frac{2p_\theta k}{p-\beta}Q_\theta-\frac{\beta}{p-\beta}Q\mathrm{e}^{(p-\beta)Q}+\frac{\ln\Big(\frac{L}{2\pi}\Big)}{p-\beta}\mathrm{e}^{(p-\beta)Q}\\&\quad+Q^2+\Big(\frac{1}{p-\beta}+\ln\Big(\frac{L}{2\pi}\Big)\Big)Q.\end{aligned}$$

Apparently, there is a large enough constant Q^* independent of θ and t, such that when $Q>Q^*$, we have

$$-\frac{\beta}{p-\beta}Q\mathrm{e}^{(p-\beta)Q}+\frac{\ln\frac{L}{2\pi}}{p-\beta}\mathrm{e}^{(p-\beta)Q}+Q^2+\Big(\frac{1}{p-\beta}+\ln\frac{L}{2\pi}\Big)Q<0.$$

Let $Q_{\max}(t)=\max\{Q(\theta,t)\mid\theta\in[0,2\pi]\}$. Then $Q_{\max}(t)$ is decreasing if $Q>Q^*$. By the maximum principle, there exists a positive constant $\Phi=\max\{Q_{\max}(0),Q^*\}$ such that $Q(\theta,t)\le\Phi$ for $(\theta,t)\in[0,2\pi]\times[0,T_0)$. Combining with (4.4), we can get that the curvature k has a uniform upper bound $k(\theta,t)\le\mathrm{e}^{\Phi(C-\beta)}\triangleq\overline{C}$ for $t\in[0,T_0)$. Using the above method, we can prove that $k(\theta,t)\le\overline{C}$ holds on time intervals $[T_0,2T_0]$, $[3T_0,4T_0],\cdots$, where $\overline{C}$ is a constant depending on the initial curve X_0. So the proof is done.

Corollary 4.1 *The solution to the flow* (1.1) *exists for* $t\in[0,\infty)$.

Proof By Theorem 3.1, the evolving curve is convex under the flow (1.1). (4.1) is uniformly parabolic in any finite time interval. Theorem 4.1 tells us that the curvature has a uniform upper bound for all $t\ge0$. The regularity theory of parabolic equations ensures that all derivatives of k are bounded. So the flow can be extended to the time interval $[0,\infty)$.

5 C^∞ Convergence

It is shown that the evolving curve converges to a circle in the C^0 sense (Hausdorff convergence). And C^∞ convergence means all the derivatives of curvature k (or curvature radius ρ)

converge to 0 as $t \to \infty$. In this section, we investigate convergence of the flow (1.1) and finish the proof of Main Theorem by showing that the evolving curve converges to a circle in the C^∞ metric.

Lemma 5.1 *Under the flow* (1.1), *we have*

$$\int_0^{2\pi} \Big(\frac{\partial k}{\partial \theta}\Big)^2 \mathrm{d}\theta \le M_1, \tag{5.1}$$

where M_1 is a positive constant independent of θ and t.

Proof Set $\Phi = \big(\frac{\partial k}{\partial \theta}\big)^2 + \alpha k$, where α is a constant to be chosen later. It follows from (4.1) that

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{\partial k}{\partial \theta}\Big)^2 &= 2k\frac{\partial k}{\partial \theta}\frac{\partial^3 k}{\partial \theta^3} - 2\Big(\frac{\partial k}{\partial \theta}\Big)^2\frac{\partial^2 k}{\partial \theta^2} + 2k\Big(\frac{\partial k}{\partial \theta}\Big)^2\Big(2\ln\Big(\frac{L}{2\pi}k\Big)+1\Big)\\ &= k\frac{\partial^2}{\partial \theta^2}\Big(\frac{\partial k}{\partial \theta}\Big)^2 - 2k\Big(\frac{\partial^2 k}{\partial \theta^2}\Big)^2 - \frac{\partial k}{\partial \theta}\frac{\partial}{\partial \theta}\Big(\frac{\partial k}{\partial \theta}\Big)^2 + 2k\Big(\frac{\partial k}{\partial \theta}\Big)^2\Big(2\ln\Big(\frac{L}{2\pi}k\Big)+1\Big),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} k\mathrm{d}\theta &= -2\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\mathrm{d}\theta + \int_0^{2\pi} k^2\ln\Big(\frac{L}{2\pi}k\Big)\mathrm{d}\theta,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\Phi\mathrm{d}\theta &= \int_0^{2\pi}\Big[k\frac{\partial^2}{\partial \theta^2}\Big(\frac{\partial k}{\partial \theta}\Big)^2 - 2k\Big(\frac{\partial^2 k}{\partial \theta^2}\Big)^2 - \frac{\partial k}{\partial \theta}\frac{\partial}{\partial \theta}\Big(\frac{\partial k}{\partial \theta}\Big)^2\Big]\mathrm{d}\theta\\ &\quad + \int_0^{2\pi} 2k\Big(\frac{\partial k}{\partial \theta}\Big)^2\Big(2\ln\Big(\frac{L}{2\pi}k\Big)+1\Big)\mathrm{d}\theta\\ &\quad - 2\alpha\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\mathrm{d}\theta + \alpha\int_0^{2\pi} k^2\ln\Big(\frac{L}{2\pi}k+1\Big)\mathrm{d}\theta\\ &\le 2\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\frac{\partial^2 k}{\partial \theta^2}\mathrm{d}\theta + \int_0^{2\pi} 2k\Big(\frac{\partial k}{\partial \theta}\Big)^2\Big(2\ln\Big(\frac{L}{2\pi}k\Big)+1\Big)\mathrm{d}\theta\\ &\quad - 2\alpha\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\mathrm{d}\theta + \alpha\int_0^{2\pi} k^2\ln\Big(\frac{L}{2\pi}k\Big)\mathrm{d}\theta.\end{aligned}$$

By Theorem 4.1, k has a uniform upper bound, and the length $L(t)$ has an upper bound L_0 which is independent of time. So there exists a constant $\widetilde{C}$ independent of time such that $2k\big(2\ln\big(\frac{L}{2\pi}k\big)+1\big) \le \widetilde{C}$. Now we choose $\alpha = \widetilde{C}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\Phi\mathrm{d}\theta &\le \widetilde{C}\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\mathrm{d}\theta - 2\alpha\int_0^{2\pi}\Big(\frac{\partial k}{\partial \theta}\Big)^2\mathrm{d}\theta + \widetilde{M}\\ &\le -\alpha\int_0^{2\pi}\Phi\mathrm{d}\theta + \alpha^2\int_0^{2\pi} k\mathrm{d}\theta + \widetilde{M}\\ &\le -\alpha\int_0^{2\pi}\Phi\mathrm{d}\theta + C(\alpha),\end{aligned}$$

where $\widetilde{M}$ and $C(\alpha)$ are independent of t and θ. Hence $\int_0^{2\pi}\Phi\mathrm{d}\theta$ is bounded above by a constant independent of t and θ. $\int_0^{2\pi}\big(\frac{\partial k}{\partial \theta}\big)^2\mathrm{d}\theta$ is also bounded above by a constant independent of t and θ. We have completed the proof.

Since $k(\theta,t)$ is uniformly bounded and $\int_0^{2\pi}(\frac{\partial k}{\partial\theta})^2\mathrm{d}\theta$ is also uniformly bounded, $k(\theta,t)$ is equicontinuous. So for any sequence $k(\theta,t_i)$, we can choose a subsequence $k(\theta,t_{i_n})$ converging uniformly to a function $k_\infty(\theta)$. We know that the curve converges to a circle in the Hausdorff sense, so $k_\infty(\theta)=\frac{1}{R}$, where R is a positive constant mentioned in (3.5). Since every subsequence converges to $\frac{1}{R}$, we get that $k(\theta,t)$ converges to $\frac{1}{R}$. We can obtain the following C^∞ convergence.

Theorem 5.1 *Under the flow* (1.1), *we have*

$$\lim_{t\to\infty}\Big\|k(\theta,t)-\frac{1}{R}\Big\|_{C^n(S^1)}=0,\quad n=0,1,2,\cdots. \tag{5.2}$$

Proof Evidently, the curvature equation is uniformly parabolic and the regularity theory ensures that all the space and time derivatives of k are bounded by constants depending only on the order of differentiations. By induction and the Ascoli-Arzelá theorem, we obtain (5.2).

Next, we can imitate the method in [8, Subsections 5.7.6–5.7.14] to estimate the convergence of derivatives of the radius curvature.

By (4.1), the evolution equation of the curvature radius $\rho(\theta,t)$ is

$$\frac{\partial\rho}{\partial t}=\frac{\rho_{\theta\theta}}{\rho}-\Big(\frac{\rho_\theta}{\rho}\Big)^2+\ln\Big(\frac{L}{2\pi}\rho\Big).$$

Lemma 5.2 *Let $\rho^{(i)}$ denote the ith derivative of $\rho(\theta,t)$ with respect to θ. We have*

$$\int_0^{2\pi}(\rho^{(i+1)})^2\mathrm{d}\theta\ge 4\int_0^{2\pi}(\rho^{(i)})^2\mathrm{d}\theta,\quad i=1,2,3,\cdots. \tag{5.3}$$

Proof In fact, this is just a special version of the Wirtinger inequality.

Lemma 5.3 *There exists a positive constant C_1 depending on the initial curve such that*

$$\|\rho^{(1)}\|_2\le\sqrt{C_1}\mathrm{e}^{-\frac{t}{2R}} \tag{5.4}$$

holds for sufficiently large t.

Proof By direct computation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta&=2\int_0^{2\pi}\rho^{(1)}\Big(\frac{\rho^{(2)}}{\rho}-\Big(\frac{\rho^{(1)}}{\rho}\Big)^2+\ln\rho+\ln\Big(\frac{L}{2\pi}\Big)\Big)^{(1)}\mathrm{d}\theta\\&=-2\int_0^{2\pi}\frac{(\rho^{(2)})^2}{\rho}\mathrm{d}\theta+2\int_0^{2\pi}\frac{(\rho^{(1)})^2\rho^{(2)}}{\rho^2}\mathrm{d}\theta+2\int_0^{2\pi}\frac{(\rho^{(1)})^2}{\rho}\mathrm{d}\theta\\&\le-2\int_0^{2\pi}\frac{(\rho^{(2)})^2}{\rho}\mathrm{d}\theta+\int_0^{2\pi}\Big(\frac{(\rho^{(1)})^4}{\rho^3}+\frac{(\rho^{(2)})^2}{\rho}\Big)\mathrm{d}\theta+2\int_0^{2\pi}\frac{(\rho^{(1)})^2}{\rho}\mathrm{d}\theta.\end{aligned}$$

Since $\|\frac{\partial k}{\partial\theta}\|\to 0$ as $t\to\infty$, $\frac{1}{R}-\varepsilon\le k\le\frac{1}{R}+\varepsilon$ if t is sufficiently large. Set $0<\varepsilon<\frac{1}{8R}$. By Lemma 5.2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta&\le-4\Big(\frac{1}{R}-\varepsilon\Big)\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta+2\varepsilon\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta\\&\quad+2\Big(\frac{1}{R}+\varepsilon\Big)\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta\le-\frac{1}{R}\int_0^{2\pi}(\rho^{(1)})^2\mathrm{d}\theta.\end{aligned}$$

So integrating with respect to t yields

$$\int_0^{2\pi} (\rho^{(1)})^2 \mathrm{d}\theta \le C_1 \mathrm{e}^{-\frac{t}{R}}, \quad \text{where } C_1 = \int_0^{2\pi} (\rho^{(1)}(\theta,0))^2 \mathrm{d}\theta.$$

Lemma 5.4 *For any $0<\mu<1$, we can find a constant C_2, such that*

$$\|\rho^{(2)}\|_2 \le C_2 \mathrm{e}^{-\frac{\mu t}{2R}}.$$

Proof Compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} (\rho^{(2)})^2 \mathrm{d}\theta &= 2\int_0^{2\pi} \rho^{(2)}\Big(\frac{\rho^{(2)}}{\rho} - \Big(\frac{\rho^{(1)}}{\rho}\Big)^2 + \ln\rho + \ln\Big(\frac{L}{2\pi}\Big)\Big)^{(2)} \mathrm{d}\theta \\ &= -2\int_0^{2\pi} \frac{(\rho^{(3)})^2}{\rho}\mathrm{d}\theta + 2\int_0^{2\pi} \rho^{(3)}\frac{(3\rho^{(1)})\rho^{(2)}}{\rho^2}\mathrm{d}\theta - 2\int_0^{2\pi} \frac{(\rho^{(1)})^3}{\rho^3}\mathrm{d}\theta \\ &\quad + 2\int_0^{2\pi} \frac{(\rho^{(2)})^2}{\rho}\mathrm{d}\theta - 2\int_0^{2\pi} \frac{(\rho^{(1)})^2\rho^{(2)}}{\rho^2}\mathrm{d}\theta.\end{aligned}$$

By the above estimate of $\rho^{(1)}$ and since $\|\frac{\partial k}{\partial\theta}\|_\infty \to 0$ as $t\to\infty$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} (\rho^{(2)})^2 \mathrm{d}\theta \le -\frac{1}{R}\int_0^{2\pi} (\rho^{(2)})^2 \mathrm{d}\theta + C_2 \mathrm{e}^{-\frac{1}{R}t}.$$

Denote $\int_0^{2\pi} (\rho^{(2)})^2 \mathrm{d}\theta$ by f. Then

$$\frac{\partial f}{\partial t} \le -\frac{1}{R} f + C_2 \mathrm{e}^{-\frac{1}{R}t}.$$

By [11, Lemma 5.7.6], we complete the proof.

Corollary 5.1 *Under the flow* (1.1)*, for any $0<\mu<1$, $\|\rho^{(1)}\|_\infty \le C_3 \mathrm{e}^{-\frac{\mu t}{R}}$.*

Proof In one-dimensional case

$$\max|f|^2 \le \widehat{C}\int \max|f'|^2 + f^2,$$

and we apply this to $\rho^{(1)}$.

Next, we can follow the routine of [11] and use a similar method as in the proof of Lemma 5.4 and Corollary 5.1 to obtain the estimates of higher order derivatives of ρ.

Corollary 5.2 *Under the flow* (1.1)*, for any $0<\mu<1$, $\|\rho^{(k)}\|_2 \le C_k \mathrm{e}^{-\frac{\mu t}{2R}}$ and $\|\rho^{(k-1)}\|_\infty \le C_{k-1}\mathrm{e}^{-\frac{\mu t}{2R}}$, $k = 2, 3, \cdots$.*

Since ρ is exponentially convergent to a constant, both $L(t)$ and k also exponentially converge. Integrating (1.1) shows that the evolving curve $X(\cdot, t)$ converges to a fixed limit X_∞. Therefore combining with Theorems 3.1–3.3, Corollaries 4.1 and 5.2 and Theorem 5.1, we can get the proof of Main Theorem.

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