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Turán Problems for Berge-(k,p)-Fan Hypergraph*

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Abstract Let F be a graph. A hypergraph $\mathcal{H}$ is Berge-F if there is a bijection $f: E(F) \to E(\mathcal{H})$ such that $e \subset f(e)$ for every $e \in E(F)$. A hypergraph is Berge-F-free if it does not contain a subhypergraph isomorphic to a Berge-F hypergraph. The authors denote the maximum number of hyperedges in an n-vertex r-uniform Berge-F-free hypergraph by $ex_r(n, \text{Berge-}F)$.

A (k,p)-fan, denoted by $F_{k,p}$, is a graph on $k(p-1)+1$ vertices consisting of k cliques with p vertices that intersect in exactly one common vertex. In this paper they determine the bounds of $ex_r(n, \text{Berge-}F)$ when F is a (k,p)-fan for $k \geq 2$, $p \geq 3$ and $r \geq 3$.

Keywords Berge-hypergraph, Turán number
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1 Introduction

Let F be a graph and $\mathcal{H}$ an r-uniform hypergraph. The hypergraph $\mathcal{H}$ is Berge-F if there is a bijection $f: E(F) \to E(\mathcal{H})$ such that $e \subset f(e)$ for every $e \in E(F)$. In general, Berge-F is a family of hypergraphs. An r-uniform hypergraph $\mathcal{H}$ is Berge-F-free if it does not contain a subhypergraph isomorphic to a Berge-F hypergraph. For an integer $r \geq 2$, write $ex_r(n, \text{Berge-}F)$ for the maximum number of hyperedges in an r-uniform Berge-F-free hypergraph on n vertices.

Let G be a graph. The chromatic number of G is denoted by $\chi(G)$. The number of clique of size s in G is denoted by $N_s(G)$. Following Alon and Shikhelman [1], let us denote the maximum number of copies of G in an n-vertex F-free graph by $ex(n, G, F)$.

The Berge-Turán problem is of interest because it is closely related to the subgraph-counting problem. If G is an F-free graph on n vertices, then we can define an r-uniform hypergraph $\mathcal{H}$ on $V(G)$, and an r-subset of $V(G)$ forms a hyperedge in $\mathcal{H}$ if and only if that the set forms a clique of size r in G. Since G is F-free, $\mathcal{H}$ is Berge-F-free. Therefore,

$$ex(n, K_r, F) \leq ex_r(n, \text{Berge-}F). \tag{1.1}$$

Alon and Shikhenlman [1] gave the following result.

Lemma 1.1 (see [1]) *For any graph H, $ex(n, K_t, H) = \Omega(n^t)$ if and only if $\chi(H) > t$. Furthermore, if indeed $\chi(H) = p > t$, then $ex(n, K_t, H) = (1+o(1))\binom{p-1}{t}(\frac{n}{p-1})^t$.*

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Given a positive integer k and a graph F, the vertex disjoint union of k copies of the graph F is denoted by kF. Let $H = kK_p$. The following results, given by Gerbner, Methuku and Vizer [6], determined the order of magnitude of $ex(n, K_{r-1}, kK_p)$ for all $r \geq 2$, p and k (as n tends to infinity).

Theorem 1.1 (see [6]) *For $r \leq p$,*

$$ex(n, K_{r-1}, kK_p) = (1 + o(1))\binom{p-1}{r-1}\left(\frac{n}{p-1}\right)^{r-1}.$$

Theorem 1.2 (see [6]) *For $p + 1 \leq r \leq p + k - 1$,*

$$ex(n, K_{r-1}, kK_p) = (1 + o(1))\binom{k-1}{r-1-p+1}\left(\frac{n}{p-1}\right)^{p-1}.$$

Theorem 1.3 (see [6]) *Let $r \geq p + 1 \geq 3$ and $k \geq 1$ be arbitrary integers and let $x = \lceil \frac{kp-r+1}{k-1} \rceil - 1$. Then*

$$ex(n, K_{r-1}, kK_p) = \Theta(n^x).$$

A (k, p)-fan, denoted by $F_{k,p}$, is a graph on $k(p - 1) + 1$ vertices consisting of k cliques with p vertices that intersect in exactly one common vertex. The extremal number for $F_{k,p+1}$ was determined by Chen et al. [3] when $p \geq 2$.

Theorem 1.4 (see [3]) *For every $k \geq 1$ and for every $n \geq 16k^3(p + 1)^8$, if a graph G on n vertices has more than*

$$ex(n, K_{p+1}) + \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \dfrac{3k}{2}, & \text{if } k \text{ is even} \end{cases}$$

edges, then G contains a copy of a $F_{k,p+1}$-fan. Further, the number of edges is the best possible.

Many results are known for $ex_r(n, \text{Berge-}F)$. Győri et al. [10] generalized the Erdős-Gallai theorem to Berge-paths. Győri and Lemons [11] proved that the maximum number of hyperedges in an n-vertex r-uniform Berge-C_{2k}-free hypergraph (for $r \geq 3$) is $O(n^{1+\frac{1}{k}})$. Gerbner and Palmer [7] gave bounds on $ex_r(n, \text{Berge-}K_{s,t})$. Gerbner et al. [5] established new bounds for a Berge-K_r and Berge-trees. For general results on the maximum size of a Berge-F-free hypergraph for an arbitrary graph F, see Gerbner and Palmer [7] and Grösz et al. [9]. For a short survey on Turán problems on Berge hypergraphs, see [8].

In this paper, we give a general lemma and establish some bounds on $ex_r(n, \text{Berge-}F_{k,p+1})$ for $r \geq 3$, $k \geq 2$ and $p \geq 2$. Our main results are the following.

Theorem 1.5 *For given integers $r \geq 3$, $k \geq 2$ and sufficiently large n,*

$$ex_r(n, \text{ Berge}-F_{k,3}) \leq \begin{cases} (1 + o(1))\dfrac{1}{4}n^2, & \text{if } r \geq 2k - 1, \\ (1 + o(1))\dfrac{1}{2r(r-1)}\dbinom{2k-2}{r-2}n^2, & \text{if } r \leq 2k - 2. \end{cases}$$

Theorem 1.6 *For $p \geq 3$ and sufficiently large n, if $r \leq p$, then*

$$(1 + o(1))\binom{p}{r}\left(\frac{n}{p}\right)^r \leq ex_r(n, \text{ Berge}-F_{k,p+1}) \leq (1 + o(1))\frac{r(r-1)}{2}\binom{p}{r}\left(\frac{n}{p}\right)^r.$$

Theorem 1.7 *For $p \geq 3$ and sufficiently large n, if $r \geq p+1$, let $x = \lceil \frac{kp-r+1}{k-1} \rceil - 1$. Then*

$$ex_r(n, \text{Berge}-F_{k,p+1}) \leq \begin{cases} (1+o(1))\dfrac{r-1}{2}\dbinom{k-1}{r-p}\left(\dfrac{1}{p}\right)^{p-1} n^p, & \text{if } p+1 \leq r \leq p+k-1, \\ (1+o(1))\dfrac{(r-1)c}{2}\left(\dfrac{p-1}{p}\right)^{x} n^{x+1}, & \text{if } p+k \leq r \leq pk-2k+2, \\ \dfrac{kp(p+1)}{2}(1+\sqrt{n})n^2, & \text{if } pk-2k+3 \leq r \leq pk-k+1, \end{cases}$$

where c is a positive constant depending on r, p and k.

The structure of the remaining part of the paper is as follows: In next section we provide the bound on $ex_r(n, \text{Berge-}F_{k,3})$. In Section 3 we give a general lemma and use it to establish the bounds on $ex_r(n, \text{Berge-}F_{k,p+1})$ for $p \geq 3$.

2 Berge-$F_{k,3}$

Let $\nu(G)$ denote the matching number of G. For an integer t, the t-closure of G is the graph obtained from G by iteratively joining non-adjacent vertices with degree sum at least t until there is no more such a pair of vertices. The following lemma was given by Bondy and Chvátal [2].

Lemma 2.1 (see [2]) *Let G be a graph and G' be the $(2k-1)$-closure of G. Then $\nu(G') \geq k$ implies $\nu(G) \geq k$.*

For integers k, r and n, let $h_{r-1}(n, k-1, \delta) = \binom{2k-1-\delta}{r-1} + (n-2k+1+\delta)\binom{\delta}{r-2}$. The following result determines the number of $(r-1)$-cliques in a graph G with matching number $\nu(G) \leq k-1$ and minimum degree $\delta(G)$.

Theorem 2.1 (see [4]) *If G is a graph with $n \geq 2k$ vertices, minimum degree δ, and $\nu(G) \leq k-1$, then $N_{r-1}(G) \leq \max\{h_{r-1}(n, k-1, \delta), h_{r-1}(n, k-1, k-1)\}$ for each $r \geq 3$.*

We now obtain the following lemma by combining Lemma 2.1 and Theorem 2.1.

Lemma 2.2 *If $n \leq 2k-1$, then $ex(n, K_{r-1}, kK_2) = \binom{n}{r-1}$. If $n \geq 2k$, then $ex(n, K_{r-1}, kK_2) \leq \max\{h_{r-1}(n, k-1, 0), h_{r-1}(n, k-1, k-1)\}$.*

Proof If $n \leq 2k-1$, it is easily seen that $\nu(K_n) \leq k-1$. Then for any graph G on n vertices, $\nu(G) \leq k-1$ and $N_{r-1}(G) \leq N_{r-1}(K_n) = \binom{n}{r-1}$. Thus, $ex(n, K_{r-1}, kK_2) = \binom{n}{r-1}$.

If $n \geq 2k$, for any graph G on n vertices with $\nu(G) \leq k-1$, let G' be the $(2k-1)$-closure of G. We first claim that the minimum degree $\delta(G)$ of G is at most $k-1$. Indeed, assume that $\delta(G) \geq k$, we have $d_G(u) + d_G(v) \geq 2k$ for each pair of vertices $u, v \in V(G)$. This implies $G' \cong K_n$ and $\nu(G') = \nu(K_n) \geq k$. Then $\nu(G) \geq k$ by Lemma 2.1, a contradiction.

Applying Theorem 2.1, we have $N_{r-1}(G) \leq \max\{h_{r-1}(n, k-1, \delta(G)), h_{r-1}(n, k-1, k-1)\}$. Since $0 \leq \delta(G) \leq k-1$, by the convexity of $h_{r-1}(n, k-1, \delta)$,

$$N_{r-1}(G) \leq \max\{h_{r-1}(n, k-1, 0), h_{r-1}(n, k-1, k-1)\}.$$

This completes the proof of the lemma.

Lemma 2.3 *For every n, $ex(n, K_{r-1}, kK_2) \le \frac{1}{r-1}\binom{2k-2}{r-2}n$.*

Proof If $n \le 2k-1$, by Lemma 2.2,

$$ex(n, K_{r-1}, kK_2) = \binom{n}{r-1} = \frac{1}{r-1}\binom{n-1}{r-2}n \le \frac{1}{r-1}\binom{2k-2}{r-2}n.$$

If $n \ge 2k$, we have

$$h_{r-1}(n, k-1, 0) = \binom{2k-1}{r-1} = \binom{2k-2}{r-2}\frac{2k-1}{r-1} \le \frac{1}{r-1}\binom{2k-2}{r-2}n$$

and

$$\begin{aligned} h_{r-1}(n, k-1, k-1) &= \binom{k-1}{r-2}(n-k) + \binom{k}{r-1} \\ &= \binom{k-1}{r-2}n - (r-2)\binom{k}{r-1} \\ &\le \binom{k-1}{r-2}n \\ &\le \frac{1}{r-1}\binom{2k-2}{r-2}n. \end{aligned}$$

By Lemma 2.2, $ex(n, K_{r-1}, kK_2) \le \frac{1}{r-1}\binom{2k-2}{r-2}n$. The lemma follows.

Gerbner et al. [5] gave the following general lemma that will be used in the proof of Theorem 1.5.

Lemma 2.4 (see [5]) *Let F be a graph and let F' be a graph resulting from the deletion of a vertex from F. Let $c = c(n)$ be such that $ex(n, K_{r-1}, F') \le cn$ for every n. Then*

$$ex_r(n, \text{ Berge}{-}F) \le \max\Big\{\frac{2c}{r}, 1\Big\} ex(n, F).$$

Proof of Theorem 1.5 Let $\mathcal{H}$ be an r-uniform Berge-$F_{k,3}$-free hypergraph on n vertices. Let $u \in V(F_{k,3})$ be the center of $F_{k,3}$. Then $F' = F_{k,3} - u \cong kK_2$. By Lemma 2.3, we have $ex(n, K_{r-1}, F') \le \frac{1}{r-1}\binom{2k-2}{r-2}n$ for all n. Thus $c = \frac{1}{r-1}\binom{2k-2}{r-2}$.

First we consider the case $r \ge 2k-1$. Then we have

$$\max\Big\{\frac{2c}{r}, 1\Big\} = \max\Big\{\frac{2}{r(r-1)}\binom{2k-2}{r-2}, 1\Big\} = 1.$$

Hence, Lemma 2.4 and Theorem 1.4 give

$$ex_r(n, \text{Berge-}F_{k,3}) \le ex(n, F) = (1 + o(1))\frac{1}{4}n^2.$$

Let us continue with the case $r \le 2k-2$. If $4 \le r \le 2k-2$, then $\binom{2k-2}{r-2} \ge \binom{2k-2}{2} \ge \binom{r}{2}$. If $r = 3$, then $\binom{2k-2}{r-2} = 2k-2 \ge r = 3 = \binom{r}{2}$. Hence,

$$\max\Big\{\frac{2c}{r}, 1\Big\} = \frac{2}{r(r-1)}\binom{2k-2}{r-2}.$$

Then Lemma 2.4 and Theorem 1.4 give

$$ex_r(n, \text{Berge-}F_{k,3}) \le \frac{2}{r(r-1)}\binom{2k-2}{r-2} ex(n, F) = (1 + o(1))\frac{1}{2r(r-1)}\binom{2k-2}{r-2}n^2.$$

This yields the needed result.

3 Berge-$F_{k,p+1}$ for $p \geq 3$

For an r-uniform hypergraph $\mathcal{H}$, we define $\partial\mathcal{H}$ to be the graph induced by the pairs of vertices in $\mathcal{H}$ which contained in at least one hyperedge of $\mathcal{H}$, i.e., $V(\partial\mathcal{H}) = V(\mathcal{H})$ and

$$E(\partial\mathcal{H}) = \{\{u,v\} \subset V(\mathcal{H}) : \ \{u,v\} \subset e \text{ for some } e \in E(\mathcal{H})\}.$$

For $\{u,v\} \in \partial\mathcal{H}$, let $d_H(u,v) = |\{e \in E(\mathcal{H}) : \ \{u,v\} \subset e\}|$. An r-uniform hypergraph $\mathcal{H}$ is called d-full if $d_H(u,v) \geq d$ for all $\{u,v\} \in \partial\mathcal{H}$. The following lemmas were given by Palmer et al. [12], which are useful for Turán problems involving expansion.

Lemma 3.1 (see [12]) *For any positive integer d, the r-uniform hypergraph $\mathcal{H}$ has a d-full sub-hypergraph $\mathcal{H}_1$ with $e(\mathcal{H}_1) \geq e(\mathcal{H}) - (d-1)|\partial\mathcal{H}|$.*

Lemma 3.2 (see [12]) *Let $r \geq 3$ be an integer and $\mathcal{H}$ be an r-uniform hypergraph with no Berge-F. If $\partial\mathcal{H}$ contains a copy of F, then there is a pair of vertices u,v such that $d_{\mathcal{H}}(u,v) < e(F)$.*

Lemma 3.3 *Suppose F is a graph with $ex(n,F) = \beta n^\alpha$, where $1 \leq \alpha \leq 2$ and β is a positive constant, and there is a vertex $v \in V(F)$ such that for large enough m, $ex(m, K_{r-1}, F - v) \leq cm^i$ for some positive constant c and integer $i \geq 1$. If $r \geq 3$ and $e(F)$ is the number of edges of F, then for large enough n we have*

$$ex_r(n, \text{ Berge} - \text{F}) \leq \max\Big\{c(r-1)2^{i-1}\Big(1 + \frac{1}{\sqrt{n}}\Big)\frac{ex(n,F)^i}{n^{i-1}}, e(F)(\sqrt{n}+1)n^2\Big\}.$$

Proof Let $\mathcal{H}$ be an r-uniform Berge-F-free hypergraph on n vertices. If $e(\mathcal{H}) \leq e(F)(\sqrt{n}+1)n^2$, then we are done. Otherwise, $e(\mathcal{H}) > e(F)(\sqrt{n}+1)n^2$. Let θ be a real number such that $e(\mathcal{H}) = e(F)(\sqrt{n}+1)n^{r-\theta}$. Note that $r - \theta > 2$.

Since $\partial\mathcal{H}$ is a subgraph of K_n, $|\partial\mathcal{H}| \leq \frac{n^2}{2} < \frac{n^{r-\theta}}{2}$. Thus, by Lemma 3.1, there exists an $e(F)$-full sub-hypergraph $\mathcal{H}_1$ of $\mathcal{H}$ satisfying

$$e(\mathcal{H}_1) \geq e(\mathcal{H}) - e(F)|\partial\mathcal{H}| \geq \Big(\sqrt{n} + \frac{1}{2}\Big)e(F)n^{r-\theta}.$$

Since $\mathcal{H}_1$ is $e(F)$-full, if $\partial\mathcal{H}_1$ contains a copy of F, then there exists a Berge-F in $\mathcal{H}_1$ by Lemma 3.2, a contradiction. Thus, $\partial\mathcal{H}_1$ is F-free, which implies that $|\partial\mathcal{H}_1| \leq ex(n,F)$.

Let $d = \frac{e(F)n^{r-\theta+\frac{1}{2}}}{ex(n,F)}$. Applying Lemma 3.1, we obtain a d-full sub-hypergraph $\mathcal{H}_2$ of $\mathcal{H}_1$ with

$$e(\mathcal{H}_2) \geq e(\mathcal{H}_1) - \frac{e(F)n^{r-\theta+\frac{1}{2}}}{ex(n,F)}|\partial\mathcal{H}_1| \geq \frac{1}{2}e(F)n^{r-\theta}.$$

Let $\mathcal{H}_3$ be the hypergraph obtained from $\mathcal{H}_2$ by removing all isolated vertices. Let $G = \partial\mathcal{H}_3$ and $n' = |V(G)|$. Note that $e(G) \leq ex(n',F)$, since G is a subgraph of $\partial\mathcal{H}_1$. Thus, there exists a vertex $u \in V(G)$ such that

$$d_G(u) \leq \frac{2ex(n',F)}{n'} \leq \frac{2ex(n,F)}{n}, \tag{3.1}$$

since $ex(n,F) = \beta n^\alpha$ with $1 \leq \alpha \leq 2$.

Let $G' = G[N_G(u)]$. Since $\mathcal{H}_3$ is d-full, there are at least d edges in $\mathcal{H}_3$ that contain both u and u' for any vertex $u' \in N_G(u)$. If $e = \{u, u', w_1, \cdots, w_{r-2}\}$ is an edge of $\mathcal{H}_3$, then

$\{u', w_1, \cdots, w_{r-2}\}$ forms an $(r-1)$-clique in G'. On the other hand, although there are at least d edges e in $\mathcal{H}_3$ that contain u and u', edge e is counted $r-1$ times since there are $r-1$ vertices in e that are neighbors of u in G. Therefore,

$$N_{r-1}(G') \geq \frac{d_G(u) \cdot d}{r-1} \tag{3.2}$$

and

$$d_G(u) \geq d = \frac{e(F)n^{r-\theta+\frac{1}{2}}}{ex(n,F)}.$$

Since G is F-free and $G' = G[N_G(u)]$, G' is $(F-v)$-free, where v is any vertex in F. So $N_{r-1}(G') \leq ex(d_G(u), K_{r-1}, F-v)$. Thus, for large enough n, we have

$$N_{r-1}(G') \leq ex(d_G(u), K_{r-1}, F-v) \leq cd_G(u)^i. \tag{3.3}$$

By (3.1)–(3.3), we have

$$d \leq c(r-1)d_G(u)^{i-1} \leq c(r-1)\Big(\frac{2ex(n,F)}{n}\Big)^{i-1}.$$

Since $e(\mathcal{H}) = (\sqrt{n}+1)e(F)n^{r-\theta}$ and $d = \frac{e(F)n^{r-\theta+\frac{1}{2}}}{ex(n,F)}$,

$$e(\mathcal{H}) \leq \Big(1+\frac{1}{\sqrt{n}}\Big)c(r-1)2^{i-1}\frac{ex(n,F)^i}{n^{i-1}},$$

which completes the proof.

Proof of Theorem 1.6 Let H be an r-uniform Berge-$F_{k,p+1}$-free hypergraph on n vertices. Let $u \in V(F_{k,p+1})$ be the center of $F_{k,p+1}$ and $F' = F_{k,p+1} - u$. Observe that $F' \cong kK_p$ and $\chi(F') = p$. Therefore, by Theorem 1.1, we have

$$ex(n, K_{r-1}, F') = (1+o(1))\binom{p-1}{r-1}\Big(\frac{n}{p-1}\Big)^{r-1}.$$

Using Theorem 1.4 and Lemma 3.3 with $i = r-1$ and $c = (1+o(1))\binom{p-1}{r-1}\big(\frac{1}{p-1}\big)^{r-1}$, we have

$$\begin{aligned}
&ex_r(n, \text{Berge-}F_{k,p+1})\\
\leq{}& \max\Big\{c(r-1)2^{i-1}\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{ex(n,F)^i}{n^{i-1}},\ e(F)(\sqrt{n}+1)n^2\Big\}\\
={}& \max\Big\{\frac{r-1}{2}(1+o(1))\binom{p-1}{r-1}\Big(\frac{1}{p}\Big)^{r-1}\Big(1+\frac{1}{\sqrt{n}}\Big)n^r,\ \frac{k}{2}p(p+1)(1+\sqrt{n})n^2\Big\}\\
={}& \frac{r(r-1)}{2}(1+o(1))\binom{p}{r}\Big(\frac{n}{p}\Big)^r.
\end{aligned}$$

On the other hand, since $\chi(F_{k,p+1}) = p+1$, by Lemma 1.1,

$$ex(n, K_r, F_{k,p+1}) = (1+o(1))\binom{p}{r}\Big(\frac{n}{p}\Big)^r.$$

Combining this with (1.1), we have

$$(1+o(1))\binom{p}{r}\Big(\frac{n}{p}\Big)^r \le ex_r(n, \text{Berge-}F_{k,p+1}).$$

The Proof of Theorem 1.6 is completed.

Proof of Theorem 1.7 Let H be an r-uniform Berge-$F_{k,p+1}$-free hypergraph on n vertices. Let $u \in V(F_{k,p+1})$ be the center of $F_{k,p+1}$ and $F' = F_{k,p+1} - u$. Observe that $F' \cong kK_p$ and $\chi(F') = p$.

If $p+1 \le r \le p+k-1$, by Theorem 1.2,

$$ex(n, K_{r-1}, F') = (1+o(1))\binom{k-1}{r-p}\Big(\frac{n}{p-1}\Big)^{p-1}.$$

Using Theorem 1.4 and Lemma 3.3 with $i = p-1$ and $c = (1+o(1))\binom{k-1}{r-p}\big(\frac{1}{p-1}\big)^{p-1}$, we have

$$\begin{aligned}
&ex_r(n, \text{Berge-}F_{k,p+1})\\
&\le \max\Big\{c(r-1)2^{i-1}\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{ex(n,F)^i}{n^{i-1}},\ e(F)(\sqrt{n}+1)n^2\Big\}\\
&= \max\Big\{(1+o(1))\frac{r-1}{2}\binom{k-1}{r-p}\Big(\frac{1}{p}\Big)^{p-1}\Big(1+\frac{1}{\sqrt{n}}\Big)n^p,\ \frac{k}{2}p(p+1)(1+\sqrt{n})n^2\Big\}\\
&= (1+o(1))\frac{r-1}{2}\binom{k-1}{r-p}\Big(\frac{1}{p}\Big)^{p-1}n^p.
\end{aligned}$$

If $p+k \le r \le pk-2k+2$, then $x = \lceil \frac{kp-r+1}{k-1} \rceil - 1 \ge 2$. By Theorem 1.3,

$$ex(n, K_{r-1}, F') = \Theta(n^x).$$

Using Theorem 1.4 and Lemma 3.3 with $i = x$ and $c = c_1(r,p,k)$, we have

$$\begin{aligned}
&ex_r(n, \text{Berge-}F_{k,p+1})\\
&\le \max\Big\{c(r-1)2^{i-1}\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{ex(n,F)^i}{n^{i-1}},\ e(F)(\sqrt{n}+1)n^2\Big\}\\
&= \max\Big\{(1+o(1))\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{(r-1)c_1}{2}\Big(\frac{p-1}{p}\Big)^x n^{x+1},\ \frac{k}{2}p(p+1)(1+\sqrt{n})n^2\Big\}\\
&= (1+o(1))\frac{(r-1)c_1}{2}\Big(\frac{p-1}{p}\Big)^x n^{x+1}.
\end{aligned}$$

If $pk-2k+3 \le r \le pk-k+1$, then $x = \lceil \frac{kp-r+1}{k-1} \rceil - 1 = 1$. By Theorem 1.3,

$$ex(n, K_{r-1}, F') = \Theta(n).$$

Using Theorem 1.4 and Lemma 3.3 with $i = 1$ and $c = c_1(r, p, k)$, we have

$$\begin{aligned} & ex_r(n, \text{Berge-}F_{k,p+1}) \\ \leq & \max\Big\{c(r-1)2^{i-1}\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{ex(n,F)^i}{n^{i-1}},\ e(F)(\sqrt{n}+1)n^2\Big\} \\ = & \max\Big\{(1+o(1))\Big(1+\frac{1}{\sqrt{n}}\Big)\frac{(r-1)c_1}{2}\Big(\frac{p-1}{p}\Big)n^2,\ \frac{k}{2}p(p+1)(1+\sqrt{n})n^2\Big\} \\ = & \frac{k}{2}p(p+1)(1+\sqrt{n})n^2. \end{aligned}$$

This completes the proof of Theorem 1.7.

References

[1] Alon, N. and Shikhelman, C., Many T copies in H-free graphs, *J. Combin. Theory Ser. B*, **121**, 2016, 146–172.

[2] Bondy, J. A. and Chvátal, V., A method in graph theory, *Discrete Math.*, **15**(2), 1976, 111–135.

[3] Chen, G., Gould, R. J., Pfender, F. and Wei, B., Extremal graphs for intersecting cliques, *J. Combin. Theory Ser. B*, **89**, 2003, 159–171.

[4] Duan, X., Ning, B., Peng, X., et al., Maximizing the number of cliques in graphs with given matching number, *Discrete Appl. Math.*, **287**, 2020, 110–117

[5] Gerbner, D., Methuku, A. and Palmer, C., General lemmas for Berge-Turán hypergraph problems, *European J. Combin.*, **86**, 2020, 103082.

[6] Gerbner, D., Methuku, A. and Vizer, M., Generalized Turán problems for disjoint copies of graphs, *Discrete Math.*, **342**, 2019, 3130–3141.

[7] Gerbner, D. and Palmer, C., Extremal results for Berge hypergraphs, *SIAM J. Discrete Math.*, **31**(4), 2017, 2314–2327.

[8] Gerbner, D. and Patkós, B., Extremal Finite Set Theory, CRC Press, Boca Raton, 2018.

[9] Grősz, D., Methuku, A. and Tompkins, C., Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs, *European J. Combin.*, **88**, 2020, 103109.

[10] Győri, E., Katona, G. Y. and Lemons, N., Hypergraph extensions of the Erdős-Gallai Theorem, *Electron. J. Combin.*, **58**, 2016, 238–246.

[11] Győri, E. and Lemons, N., 3-uniform hypergraphs avoiding a given odd cycle, *Combinatorica*, **32**, 2012, 187.

[12] Palmer, C., Tait, M., Timmons, C. and Wagner, A. Z., Turán number for Berge-hypergraphs and related extremal problem, *Discrete Math.*, **342**, 2019, 1553–1563.