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Locally Conformal Kähler and Hermitian Yang-Mills Metrics

Jieming YANG[1]

Abstract The author shows that if a locally conformal Kähler metric is Hermitian Yang-Mills with respect to itself with Einstein constant $c \leq 0$, then it is a Kähler-Einstein metric. In the case of $c > 0$, some identities on torsions and an inequality on the second Chern number are derived.

Keywords Hermitian Yang-Mills metric, Locally conformal Kähler metric, Torsion, Chern number inequality
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1 Introduction

Let (X, g) be a compact Hermitian manifold of complex dimension $n \geq 2$. Let $\omega = i\Sigma g_{i\overline{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$ be the associated positive definite (1,1)-form, which is also called a Hermitian metric.

Let R_ω be the curvature of the Chern connection of ω. A Hermitian metric ω is a Hermitian Yang-Mills (HYM for short) metric with respect to itself if

$$n \cdot iR_\omega \wedge \omega^{n-1} = c \cdot I_{T^{1,0}X} \otimes \omega^n, \tag{1.1}$$

where $c = \int_X i\mathrm{tr}R_\omega \wedge \omega^{n-1} / \int_X \omega^n$ is the Einstein constant. In this paper we will always assume that a Hermitian metric ω is Hermitian Yang-Mills with respect to itself. It is also called an Einstein-Hermitian metric in [4]. In fact, in [4] Gauduchon and Ivanov proved that when $n = 2$, ω is a HYM metric if and only if ω is a Kähler-Einstein metric or is the natural metric on the Hopf surface, i.e., is locally isometric to the product $\mathbb{R} \times S^3$ (up to homothety).

In this paper we consider how to generalize Gauduchon and Ivanov's result to the higher dimensional case. We need some definitions.

A Hermitian metric ω is called a Gauduchon metric if $i\partial\overline{\partial}\omega^{n-1} = 0$. A well-known result in [3] says that there exists a unique Gauduchon metric, up to a constant conformal factor, in the conformal class of a Hermitian metric.

A Hermitian metric ω is called a locally conformal Kähler (l.c.K for short) metric if for any point $x \in X$, there exist an open neighbourhood U of x and a smooth function $\varphi \in \mathcal{A}^0_{\mathbb{R}}(U)$ such that $\omega' = \mathrm{e}^\varphi \omega$ is a Kähler metric on U.

Denote torsions of the Chern connection of a Hermitian metric ω to be

$$T_{ki\overline{j}} = \partial_k g_{i\overline{j}} - \partial_i g_{k\overline{j}} \quad \text{and} \quad T_i = \Sigma g^{k\overline{l}} T_{ik\overline{l}}.$$

Then $\tau = \Sigma T_i \mathrm{d}z^i$ is the torsion 1-form of ω. A Hermitian metric ω is l.c.K if and only if equations

$$(n-1)\partial\omega = \tau \wedge \omega \tag{1.2}$$

Manuscript received September 28, 2020. Revised April 14, 2021.
[1]School of Mathematical Science, Fudan University, Shanghai 200433, China.
E-mail: 15110180009@fudan.edu.cn

and

$$\mathrm{d}(\tau+\overline{\tau})=0 \tag{1.3}$$

hold. Note that when $n=2$, equation (1.2) always holds for any Hermitian metric ω and when $n\geq 3$, (1.2) implies (1.3). These results can be consulted in [2].

As we will see, the natural metric ω on the Hopf manifold of complex dimension $n\geq 2$ is a Gauduchon, l.c.K and HYM metric. Our main result is as follows.

Theorem 1.1 *Let ω be a* l.c.K *and* HYM *metric on a compact complex manifold X of dimension $n\geq 2$. If $c\leq 0$, then ω is a* Kähler-Einstein *metric; If $c>0$ and ω is also a* (*non*-Kähler) Gauduchon *metric, then $|\tau|^2=(n-1)c$ and*

$$(n-2)\|D'\tau\|^2=n(c\|\tau\|^2-\|D''\tau\|^2). \tag{1.4}$$

Hence the real 1-form $\tau+\overline{\tau}$ is a non-vanishing d-closed form and so the Euler characterization of X is equal to zero. We wonder whether the case of $c>0$ implies ω is a Kähler-Einstein metric or is the natural metric (up to homothety) on the Hopf manifold.

Theorem 1.2 *Let ω be a* Gauduchon*,* l.c.K *and* HYM *metric on a compact complex manifold X of dimension $n\geq 2$. Then*

$$\int_X c_2(X,\omega)\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0. \tag{1.5}$$

The equality holds if and only if ω is either a flat Kähler *metric or the natural metric on the* Hopf *surface.*

A Kähler-Einstein metric ω satisfies the Miyaoka-Yau inequality

$$4\pi^2(2(n+1)\cdot c_2(X,\omega)-n\cdot c_1(X,\omega)^2)\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0, \tag{1.6}$$

from which we can easily get

$$c_2(X,\omega)\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0.$$

When ω is non-Kähler and HYM, it satisfies the Bogomolov-Lübke inequality

$$4\pi^2(2n\cdot c_2(X,\omega)-(n-1)\cdot c_1(X,\omega)^2)\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0, \tag{1.7}$$

where the equality holds if and only if ω is projectively flat. Under the assumption in Theorem 1.2, we will show that $\int_X c_1(X,\omega)^2\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0$, hence the inequality (1.5) follows.

This paper is arranged as follows. In Section 2, the geometry of the natural metric on the Hopf manifold of dimension n is studied. In Section 3, some identities on torsion of a Gauduchon and HYM metric are derived and in particular identity (1.4) in Theorem 1.1 is proved. In Section 4, we finish the proof of Theorem 1.1 and in Section 5 we prove Theorem 1.2.

We follow the notations in [5]. For a Hermitian metric ω, we denote R_ω to be the curvature of the Chern connection of ω. Locally, its components are

$$R^p_{ik\overline{l}}=-\Sigma g^{p\overline{j}}\partial_{\overline{l}}\partial_k g_{i\overline{j}}+\Sigma g^{p\overline{j}}g^{m\overline{q}}\partial_{\overline{l}}g_{m\overline{j}}\partial_k g_{i\overline{q}}$$

and $R_{i\overline{j}k\overline{l}}=\Sigma g_{p\overline{j}}R^p_{ik\overline{l}}$. Denote $R_{i\overline{j}}=\Sigma g^{k\overline{l}}R_{k\overline{l}i\overline{j}}$ and $K_{i\overline{j}}=\Sigma g^{k\overline{l}}R_{i\overline{j}k\overline{l}}$. Then $\rho_\omega=i\Sigma R_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$ is the Ricci curvature and $K_\omega=i\Sigma K_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$ is the mean curvature (see [5, p. 26]). Hence the equation (1.1) is equivalent to $K_{i\overline{j}}=c\cdot g_{i\overline{j}}$.

2 Hopf Manifolds

Let $H^n = S^{2n-1} \times S^1$ with $n \geq 2$ be the standard Hopf manifold (see [6, Section 6]), equipped with the natural metric

$$\omega = i\Sigma \frac{4\delta_{ij}}{|z|^2} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j.$$

It is direct to check that ω is both Gauduchon and l.c.K.

The torsions of the Chern connection of ω are

$$T_{ik\overline{j}} = -\frac{4}{|z|^4}(\overline{z}^i \delta_{kj} - \overline{z}^k \delta_{ij}) \quad \text{and} \quad T_i = -\frac{n-1}{|z|^2}\overline{z}^i,$$

and hence $|\tau|^2 = \frac{(n-1)^2}{4}$. Further calculation yields

$$\nabla_k T_i = 0 \quad \text{and} \quad \nabla_{\overline{j}} T_i = -\frac{n-1}{|z|^2}\Big(\delta_{ij} - \frac{\overline{z}^i z^j}{|z|^2}\Big), \tag{2.1}$$

which imply $D'\tau = 0$ and $|D''\tau|^2 = \frac{(n-1)^3}{16}$.

The curvature R_ω is

$$R_{i\overline{j}k\overline{l}} = \frac{4\delta_{ij}}{|z|^4}\Big(\delta_{kl} - \frac{\overline{z}^k z^l}{|z|^2}\Big), \tag{2.2}$$

and the mean curvature K_ω is

$$K_{i\overline{j}} = \frac{n-1}{|z|^2}\delta_{ij} = \frac{n-1}{4} g_{i\overline{j}}.$$

Hence ω satisfies the HYM equation (1.1) with $c = \frac{n-1}{4}$.

By (2.2), the Ricci curvature of ω is

$$R_{k\overline{l}} = \frac{n}{|z|^2}\Big(\delta_{kl} - \frac{\overline{z}^k z^l}{|z|^2}\Big),$$

and hence

$$R_{i\overline{j}k\overline{l}} = \frac{1}{n} R_{k\overline{l}} g_{i\overline{j}},$$

i.e., ω is projectively flat. So the equality in the Bogomolov-Lübke inequality (1.7) holds.

Now we assume $n > 2$. Since

$$\rho_\omega \wedge \rho_\omega \wedge \frac{\omega^{n-2}}{(n-2)!} = \frac{n^2(n-1)(n-2)}{16} \frac{\omega^n}{n!},$$

and ω is projectively flat, by the formula in [5, p. 42], we have

$$\begin{aligned} 8\pi^2 \cdot c_2(H^n, \omega) \wedge \frac{\omega^{n-2}}{(n-2)!} &= \frac{n-1}{n} 4\pi^2 \cdot c_1(H^n, \omega)^2 \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &= \frac{n(n-1)^2(n-2)}{16} \frac{\omega^n}{n!} > 0. \end{aligned}$$

Moreover, we calculate

$$\begin{aligned} &4\pi^2(2(n+1) \cdot c_2(H^n, \omega) - n \cdot c_1(H^n, \omega)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &= -\frac{1}{n} \rho_\omega \wedge \rho_\omega \wedge \frac{\omega^{n-2}}{(n-2)!} = -\frac{n(n-1)(n-2)}{16} \frac{\omega^n}{n!} < 0. \end{aligned}$$

Hence the natural metric ω on H^n does not satisfy the Miyaoka-Yau inequality (1.6), but satisfies the inequality (1.5).

3 Some Identities on Torsion

The start point of Theorem 1.1 is the following identities. Let ω be a HYM metric. Denote

$$|T|^2 = \Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} T_{ip\overline{n}} \overline{T_{jq\overline{m}}}.$$

We have

$$\begin{aligned} i\Lambda_\omega \partial\overline{\partial} |T|^2 &= \Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} (\nabla_k \nabla_{\overline{l}} T_{ip\overline{n}} \overline{T_{jq\overline{m}}} + T_{ip\overline{n}} \nabla_k \nabla_{\overline{l}} \overline{T_{jq\overline{m}}} \\ &\quad + \nabla_{\overline{l}} T_{ip\overline{n}} \nabla_k \overline{T_{jq\overline{m}}} + \nabla_k T_{ip\overline{n}} \nabla_{\overline{l}} \overline{T_{jq\overline{m}}}) \\ &= 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} \nabla_k \nabla_{\overline{l}} T_{ip\overline{n}} \overline{T_{jq\overline{m}}}) + |D'T|^2 + |D''T|^2 \\ &\quad + \Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} T_{ip\overline{n}} [\nabla_k, \nabla_{\overline{l}}] \overline{T_{jq\overline{m}}}, \end{aligned} \tag{3.1}$$

where

$$\Sigma g^{k\overline{l}} [\nabla_k, \nabla_{\overline{l}}] \overline{T_{jq\overline{m}}} = \Sigma g^{k\overline{l}} (\Sigma \overline{T_{sq\overline{m}} R^s_{jl\overline{k}}} + \Sigma \overline{T_{js\overline{m}} R^s_{ql\overline{k}}} - \Sigma \overline{T_{jq\overline{r}}} R^r_{mk\overline{l}}) = c \cdot \overline{T_{jq\overline{m}}}.$$

Let ω be a Gauduchon metric. Integrating (3.1) over X yields

$$-\int_X 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} \nabla_k \nabla_{\overline{l}} T_{ip\overline{n}} \overline{T_{jq\overline{m}}}) \frac{\omega^n}{n!} = \|D'T\|^2 + \|D''T\|^2 + c\|T\|^2,$$

where the left hand side, after integration by parts, is equal to

$$\int_X 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} T_k \overline{T_{jq\overline{m}}} \nabla_{\overline{l}} T_{ip\overline{n}}) \frac{\omega^n}{n!} + 2\|D''T\|^2. \tag{3.2}$$

Thus we obtain the following result.

Proposition 3.1 *If a Gauduchon metric ω satisfies the* HYM *equation* (1.1), *then*

$$\int_X 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{p\overline{q}} g^{m\overline{n}} g^{k\overline{l}} T_k \overline{T_{jq\overline{m}}} \nabla_{\overline{l}} T_{ip\overline{n}}) \frac{\omega^n}{n!} = \|D'T\|^2 - \|D''T\|^2 + c\|T\|^2.$$

For any Hermitian metric ω, we obtain from the calculation (3.1) that

$$i\Lambda_\omega \partial\overline{\partial} |\tau|^2 = 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{m\overline{n}} g^{k\overline{l}} \nabla_k \nabla_{\overline{l}} T_i \overline{T_j}) + |D'\tau|^2 + |D''\tau|^2 + \Sigma g^{i\overline{j}} g^{k\overline{l}} T_i [\nabla_k, \nabla_{\overline{l}}] \overline{T_j}.$$

From the HYM equation (1.1) and the calculation (3.2), we obtain the following result.

Proposition 3.2 *If a Gauduchon metric ω satisfies the* HYM *equation* (1.1), *then*

$$\int_X 2\mathrm{Re}(\Sigma g^{i\overline{j}} g^{k\overline{l}} T_k \nabla_{\overline{l}} T_i \overline{T_j}) \frac{\omega^n}{n!} = \|D'\tau\|^2 - \|D''\tau\|^2 + c\|\tau\|^2. \tag{3.3}$$

The curvature R_ω of the Chern connection of a Hermitian metric ω satisfies the following Bianchi identity

$$R_{i\overline{j}k\overline{l}} - R_{k\overline{j}i\overline{l}} = \nabla_{\overline{l}} T_{ik\overline{j}}, \tag{3.4}$$

which imples

$$\Sigma g^{k\overline{l}} \nabla_k \nabla_{\overline{l}} T_{ip\overline{j}} = \Sigma g^{k\overline{l}} \nabla_k (R_{i\overline{j}p\overline{l}} - R_{p\overline{j}i\overline{l}}). \tag{3.5}$$

Combining the Bianchi identity

$$\nabla_p R_{i\bar{j}k\bar{l}} - \nabla_k R_{i\bar{j}p\bar{l}} = \Sigma R_{i\bar{j}m\bar{l}} T^m_{kp}$$

with the HYM equation (1.1), we obtain

$$\Sigma g^{k\bar{l}} \nabla_k R_{i\bar{j}p\bar{l}} = \Sigma g^{k\bar{l}} \nabla_k R_{i\bar{j}p\bar{l}} - c \cdot \nabla_p g_{i\bar{j}} = \Sigma g^{k\bar{l}} R_{i\bar{j}m\bar{l}} T^m_{pk}.$$

Inserting it into (3.5) yields

$$\Sigma g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} T_{ip\bar{j}} = \Sigma g^{k\bar{l}} (\Sigma R_{i\bar{j}m\bar{l}} T^m_{pk} + \Sigma R_{p\bar{j}m\bar{l}} T^m_{ki}).$$

Moreover, we have

$$\Sigma g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} T_i = \Sigma g^{k\bar{l}} (\Sigma R^p_{i\bar{j}m\bar{l}} T^m_{pk} + \Sigma R_{m\bar{l}} T^m_{ki}). \tag{3.6}$$

Let ω be a l.c.K metric. By (1.2), we have

$$T_{ki\bar{j}} = \frac{1}{n-1}(T_k g_{i\bar{j}} - T_i g_{k\bar{j}}). \tag{3.7}$$

Notice that inserting (3.7) into Proposition 3.1 recovers (3.3). Inserting (3.7) and the HYM equation (1.1) into (3.6), we obtain

$$\begin{aligned}(n-1)\Sigma g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} T_i &= -(n-1)c \cdot T_i - \Sigma g^{k\bar{l}} R^p_{ip\bar{l}} T_k + \Sigma g^{k\bar{l}} R_{i\bar{l}} T_k \\ &= -(n-1)c \cdot T_i - \Sigma g^{k\bar{l}} T_k \nabla_{\bar{l}} T_i. \quad \text{(by (3.4))}\end{aligned}$$

Moreover, we have

$$(n-1) \cdot 2\mathrm{Re}(\Sigma g^{i\bar{j}} g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} T_i \overline{T_j}) = -2(n-1)c \mid \tau \mid^2 - 2\mathrm{Re}(\Sigma g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{l}} T_i \overline{T_j}).$$

Integrating it over X and using integration by parts as in (3.2) to the left hand side yields

$$\begin{aligned}&-2(n-1)\|D''\tau\|^2 - (n-1)\int_X 2\mathrm{Re}(\Sigma g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{l}} T_i \overline{T_j}) \frac{\omega^n}{n!} \\ &= -2(n-1)c\|\tau\|^2 - \int_X 2\mathrm{Re}(\Sigma g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{l}} T_i \overline{T_j}) \frac{\omega^n}{n!},\end{aligned}$$

which implies

$$(n-2)\int_X 2\mathrm{Re}(\Sigma g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{l}} T_i \overline{T_j}) \frac{\omega^n}{n!} = 2(n-1)(c\|\tau\|^2 - \|D''\tau\|^2). \tag{3.8}$$

Comparing it with (3.3), we obtain the following result.

Proposition 3.3 *Let $n > 2$ and ω be a* Gauduchon *and* l.c.K *metric. If ω satisfies the* HYM *equation* (1.1)*, then identity* (1.4) *in Theorem* 1.1 *holds.*

For $n = 2$, by [4] the Hopf surface (H^2, ω) is the only non-Kähler HYM metric with respect to itself. By (2.1), we have $\mid D'\tau \mid = 0$ and $c \mid \tau \mid^2 = \mid D''\tau \mid^2$. Hence, the identity (1.4) also holds for $n = 2$.

4 Proof of Theorem 1.1

Let us recall a well-known result in [3].

Lemma 4.1 *Let* (X,ω) *be a compact* Hermitian *manifold of complex dimension* $n \geq 2$. *Then* $\dim_{\mathbb{R}} \ker((i\Lambda_\omega \overline{\partial}\partial)^*) = 1$ *and any function* $f \in \ker((i\Lambda_\omega \overline{\partial}\partial)^*)$ *has constant sign. Moreover, if* ω *is a* Gauduchon *metric, then* $\ker((i\Lambda_\omega \overline{\partial}\partial)^*) = \mathbb{R}$.

Let ω be a l.c.K and HYM metric on a compact complex manifold X of dimension $n \geq 2$. We follow the idea in [1] to prove Theorem 1.1.

Proof There are two scalar curvatures of any Hermitian metric ω:

$$s = \Sigma g^{i\overline{j}} g^{k\overline{l}} R_{i\overline{j}k\overline{l}}, \quad \widehat{s} = \Sigma g^{i\overline{j}} g^{k\overline{l}} R_{i\overline{l}k\overline{j}}.$$

Since ω satisfies the HYM equation (1.1), and so $s = nc$. By (3.4), we have

$$\widehat{s} - s = \Sigma g^{i\overline{j}} \nabla_{\overline{j}} T_i.$$

By (1.3), we have $\partial\overline{\tau} + \overline{\partial}\tau = 0$, which implies

$$\overline{\partial}\, \overline{\partial}^* \omega = i\overline{\partial}\tau = -i\partial\overline{\tau} = \partial\partial^*\omega.$$

Inserting these and the HYM equation (1.1) into Proposition 3.2 in [1] yields

$$(\widehat{s} - c)\omega = (n-1)\rho_\omega - n\overline{\partial}\, \overline{\partial}^* \omega, \tag{4.1}$$

which implies $\mathrm{d}((\widehat{s} - c)\omega) = 0$. Then

$$(i\Lambda_\omega \overline{\partial}\partial)^*(\widehat{s} - c)^{n-1} = \frac{i}{(n-1)!} * \overline{\partial}\partial((\widehat{s} - c)\omega)^{n-1} = 0. \tag{4.2}$$

By Lemma 4.1, we have $\widehat{s} - c \equiv 0$ or $\pm(\widehat{s} - c) > 0$.

If $\widehat{s} - c \equiv 0$, then

$$0 \leq \|\tau\|^2 = -\int_X \Sigma g^{i\overline{j}} \partial_{\overline{j}} T_i \frac{\omega^n}{n!} = \int_X (s - \widehat{s}) \frac{\omega^n}{n!} = (n-1)c \int_X \frac{\omega^n}{n!},$$

which implies $c = 0$ and $\tau = 0$. Hence ω is a Kähler metric due to (3.7).

If $\widehat{s} - c$ is not identically 0, then $\pm(\widehat{s} - c)\omega$ is a Kähler metric, i.e., ω is a globally conformal Kähler metric. In this case, ω is actually Kähler-Einstein.

Indeed, let $\omega' = \mathrm{e}^f \omega$ be a Kähler metric for some function $f \in \mathcal{A}^0_{\mathbb{R}}(X)$. By (1.2),

$$\tau = -(n-1)\partial f.$$

Since

$$\widehat{s} = s + \Sigma g^{i\overline{j}} \partial_{\overline{j}} T_i = nc - (n-1) i\Lambda_\omega \partial\overline{\partial} f,$$

we obtain from (4.1) that

$$\rho_{\omega'} = \rho_\omega - n \cdot i\partial\overline{\partial} f = (c - i\Lambda_\omega \partial\overline{\partial} f)\omega,$$

which implies $\mathrm{d}((c - i\Lambda_\omega \partial\overline{\partial} f)\omega) = 0$. By Lemma 4.1, the function $c - i\Lambda_\omega \partial\overline{\partial} f$ has constant sign.

If $c - i\Lambda_\omega \partial\overline{\partial} f > 0$, then f is a constant by the maximum principle and c is non-positive. Hence we obtain $c > 0$, a contradiction.

If $c - i\Lambda_\omega\partial\overline{\partial}f = 0$, the same reason as above yields $c = 0$ and f is a constant. Hence, ω is a Kähler metric.

If $c - i\Lambda_\omega\partial\overline{\partial}f < 0$, by the uniqueness of the Gauduchon metric in the conformal class of a Hermitian metric, the constant $\gamma = c\frac{\int_X e^{-f}\omega'^n}{\int_X \omega'^n}$ satisfies

$$\gamma e^f = c - i\Lambda_\omega\partial\overline{\partial}f < 0.$$

In this case, $c < 0$. Notice that

$$0 = n\int_X i\partial\overline{\partial}e^f \wedge \omega'^{n-1} \geq n\int_X e^f \cdot i\partial\overline{\partial}f \wedge \omega'^{n-1} = \int_X (c - \gamma e^f)\omega'^n.$$

Inserting γ into the right hand side above, we have

$$c\Big(\int_X \omega'^n\Big)^2 \leq c\Big(\int_X e^{-f}\omega'^n\Big)\Big(\int_X e^f\omega'^n\Big).$$

By the Cauchy-Schwarz inequality, we obtain

$$\Big(\int_X \omega'^n\Big)^2 \leq \Big(\int_X e^{-f}\omega'^n\Big)\Big(\int_X e^f\omega'^n\Big) \leq \Big(\int_X \omega'^n\Big)^2.$$

Hence, the above inequalities hold if and only if f is a constant. Combining the above arguments, we obtain the first part of Theorem 1.1.

As to the second part, we obtain from Lemma 4.1 and (4.2) that $\widehat{s} - c$ is a constant. If $\widehat{s} - c$ is not identically zero, then ω is Kähler. Hence $\widehat{s} - c \equiv 0$, and

$$|\tau|^2 = -\Sigma g^{i\overline{j}}\partial_{\overline{j}}T_i = s - \widehat{s} = (n-1)c > 0, \tag{4.3}$$

where the first identity holds for any Gauduchon metric.

In the case $c > 0$, we obtain from (4.1) that

$$\rho_\omega = \frac{n}{n-1}\overline{\partial}\,\overline{\partial}^*\omega, \tag{4.4}$$

which implies

$$\begin{aligned}\|D'\tau\|^2 &= \frac{n}{n-2}(c\|\tau\|^2 - \|D''\tau\|^2) \quad \text{(by (1.4))}\\ &= \frac{n}{n-1}\int_X \mathrm{Re}(\Sigma g^{i\overline{j}}g^{k\overline{l}}\nabla_{\overline{l}}T_i\overline{T_j}T_k)\frac{\omega^n}{n!} \quad \text{(by (3.8))}\\ &= -\int_X \Sigma g^{i\overline{j}}g^{k\overline{l}}R_{i\overline{l}}\overline{T_j}T_k\frac{\omega^n}{n!}.\end{aligned}$$

By these facts, it seems that the Hopf manifold (H^n, ω) is the only (non-Kähler) l.c.K metric satisfying the HYM equation (1.1) with positive Einstein constant.

5 Proof of Theorem 1.2

Let ω be a Gauduchon, l.c.K and HYM metric on a compact complex manifold X of dimension $n \geq 2$. We are ready to prove Theorem 1.2.

Proof By the Bogomolov-Lübke inequality (1.7), the inequality (1.5) holds if

$$\int_X c_1(X,\omega)^2 \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0. \tag{5.1}$$

If $c \leq 0$, then ω is Kähler-Einstein and (5.1) is obvious. For the equality, by (1.6) we have $c = 0$, and then $\rho_\omega = 0$. Hence, we obtain

$$\begin{aligned} 0 = 8\pi^2 \int_X c_2(X,\omega) \wedge \frac{\omega^{n-2}}{(n-2)!} &= \int_X \mathrm{tr}(R_\omega \wedge R_\omega) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ = \int_X (|R_\omega|^2 - |K_\omega|^2) \frac{\omega^n}{n!} &= \|R_\omega\|^2, \end{aligned}$$

where the second equality follows from the formula [5, (4.1)] and the last one follows from $K_\omega = \rho_\omega$.

If $c > 0$, then we use again the formula [5, (4.1)] to calculate

$$4\pi^2 \int_X c_1(X,\omega)^2 \wedge \frac{\omega^{n-2}}{(n-2)!} = \int_X (s^2 - |\rho_\omega|^2) \frac{\omega^n}{n!}.$$

From (4.4), (4.3) and (1.4), we obtain

$$\begin{aligned} \int_X (s^2 - |\rho_\omega|^2) \frac{\omega^n}{n!} &= \Big(\frac{n}{n-1}\Big)^2 \int_X ((n-1)^2 c^2 - |D''\tau|^2) \frac{\omega^n}{n!} \\ &= \Big(\frac{n}{n-1}\Big)^2 ((n-1)c\|\tau\|^2 - \|D''\tau\|^2) \\ &= \frac{n(n-2)}{(n-1)^2} (nc\|\tau\|^2 + \|D'\tau\|^2) \geq 0, \end{aligned} \tag{5.2}$$

which implies the inequality (5.1), and hence the inequality (1.5). For the equality, we obtain from the Bogomolov-Lübke inequality (1.7) that

$$0 \geq 4\pi^2 \int_X c_1(X,\omega)^2 \wedge \frac{\omega^{n-2}}{(n-2)!},$$

which contradicts (5.2) unless $n = 2$. By the result in [4], ω is the natural metric on the Hopf surface.

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References

[1] Chen, H., Chen, L. and Nie, X., Chern-Ricci curvatures, holomorphic sectional curvature and Hermitian metric, *Sci. China Math.*, **64**, 2021, 763–780.

[2] Dragomir, S. and Ornea, L., Locally conformal Kähler geometry, *Progress in Math.*, **155**, Birkhäuser, 1998.

[3] Gauduchon, P., La 1-forme de torsion d'une variété hermitienne compacte, *Math. Ann.*, **267**, 1984, 495–518.

[4] Gauduchon, P. and Ivanov, S., Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension 4, *Math. Z.*, **226**, 1997, 317–326.

[5] Kobayashi, S., Differential Geometry of Complex Vector Bundles, Iwanami Shoten Publishers and Princeton University Press, Princeton, 1987.

[6] Liu, K. and Yang, X., Geometry of Hermitian manifolds, *Internat. J. Math.*, **23**, 2012, 1250055, 40 pp.