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Congruence Pairs of Decomposable MS-Algebras*

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Abstract In this paper, the authors first introduce the concept of congruence pairs on the class of decomposable MS-algebras generalizing that for principal MS-algebras (see [13]). They show that every congruence relation θ on a decomposable MS-algebra L can be uniquely determined by a congruence pair (θ_1, θ_2), where θ_1 is a congruence on the de Morgan subalgebra $L^{\circ\circ}$ of L and θ_2 is a lattice congruence on the sublattice $D(L)$ of L. They obtain certain congruence pairs of a decomposable MS-algebra L via central elements of L. Moreover, they characterize the permutability of congruences and the strong extensions of decomposable MS-algebras in terms of congruence pairs.

Keywords MS-Algebras, Decomposable MS-algebras, Congruence pairs, Strong extension, Permutability, Congruence lattices
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1 Introduction

Blyth and Varlet [18] studied Morgan Stone algebras (briefly MS-algebras) as a generalization of the classes of de Morgan and Stone algebras. Such algebras are bounded distributive lattices with additional unary operation. Blyth and Varlet [19] described the lattice of subvarieties of the variety **MS** of all MS-algebras. Badawy, Guffova and Haviar [12] introduced and characterized the class of decomposable MS-algebras by means of decomposable MS-triples. They observed that every decomposable MS-algebra L has two auxiliary substructures, namely, the de Morgan subalgebra $L^{\circ\circ}$ of all closed elements of L and the sublattice $D(L)$ of all dense elements of L. Also, they introduced and characterized principal MS-algebras by means of principal MS-triples. They observed that the class of decomposable MS-algebras contains the class of principal MS-algebras. Badawy [1–7] investigated a relationship between congruences and special filters of a principal MS-algebra and a decomposable MS-algebra, respectively. Also, Badawy and El-Fawal [9] studied homomorphisms and subalgebras of decomposable MS-algebras in terms of decomposable MS-triples. Recently, Badawy and Atallah [8] introduced and characterized the set $B(L)$ of all central elements of an MS-algebra L and established the relationship between its MS-intervals and congruences. For recent studies of (decomposable) MS-algebras and double MS-algebras see also [2, 4–6, 10–11, 14–15, 23, 25, 31].

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In this paper, we introduce a suitable notion of congruence pairs of decomposable MS-algebras which is a generalization of the notion of congruence pairs of both Stone algebras and principal MS-algebras. We study many properties of congruence pairs of a decomposable MS-algebra. We derive that every congruence relation θ on a decomposable MS-algebra L can be represented by a pair of congruences (θ_1, θ_2), where $\theta_1 \in \mathrm{Con}(L^{\circ\circ})$ and $\theta_2 \in \mathrm{Con}(D(L))$. We establish that there is a one to one correspondence between the lattice $\mathrm{Con}(L)$ of all congruences of L and the lattice $A(L)$ of all congruence pairs of L. Also, we investigate the relationship between the central elements of a decomposable MS-algebra L and the congruence pairs of the form $(\theta[a\downarrow], \theta[a\varphi(L)])$ for $a \in L^{\circ\circ}$. Using the concept of congruence pairs, we prove that a decomposable MS-algebra L is congruence permutable if and only if both $L^{\circ\circ}$ and $D(L)$ are congruence permutable. If L is a subalgebra of a decomposable MS-algebra L_1, we show that L_1 is a strong extension of L if and only if $L_1^{\circ\circ}$ is a strong extension of $L^{\circ\circ}$ and $D(L_1)$ is a strong extension of $D(L)$.

2 Preliminaries

In this section, we give the definitions and the main results which are needed through this work. We refer the readers to [8–9, 12–13, 18–20, 29–31] for more details.

A de Morgan algebra is an algebra $(L; \vee, \wedge, ^-, 0, 1)$ of type (2,2,1,0,0), where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^-$ is the unary operation of involution satisfying:

$$\overline{\overline{x}} = x, \quad \overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \quad \overline{(x \wedge y)} = \overline{x} \vee \overline{y}.$$

A Stone algebra is a universal algebra $(L; \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$, where the unary operation * of pseudocomplementation has the properties that $x \wedge a = 0 \Leftrightarrow x \leq a^*$ and $x^{**} \vee x^* = 1$.

An MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type (2,2,1,0,0), where a unary operation $^\circ$ satisfies :

$$x \leq x^{\circ\circ}, \quad (x \wedge y)^\circ = x^\circ \vee y^\circ, \quad 1^\circ = 0.$$

The class **MS** of all MS-algebras is equational. A de Morgan algebra is an MS-algebra satisfying the identity, $x = x^{\circ\circ}$. The class **S** of Stone algebras is a subclass of **MS** and is characterized by the identity $x \wedge x^\circ = 0$.

We recall some of the basic properties of MS-algebras which were proved in [18] or [20].

Theorem 2.1 *For any two elements a, b of an* MS-*algebra L, we have*
(1) $0^\circ = 1$,
(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

We recall special subsets of an MS-algebra L which play an important role in the construction:

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ is the set of closed elements of L which is a de Morgan subalgebra of L (see [18]),

(2) $D(L) = \{x \in L : x^{\circ} = 0\}$ is the set of dense elements of L which is a filter of L (see [12]),

(3) $a\uparrow = \{x \in L : x \geq a\}$ is the principal filter of L generated by the element a of L,

(4) $a\downarrow = \{x \in L : x \leq a\}$ is the principal ideal of L generated by the element a of L.

Now, we recall from [12] the definition of a decomposable MS-algebra and some related properties.

Definition 2.1 (see [12]) *An* MS-*algebra* $(L; \vee, \wedge, ^{\circ}, 0, 1)$ *is called a decomposable* MS-*algebra if for every* $x \in L$ *there exists* $d \in D(L)$ *such that* $x = x^{\circ\circ} \wedge d$.

The class of decomposable MS-algebras contains both the class **M** of all de Morgan algebras and the class **S** of all Stone algebras.

Let L be a decomposable MS-algebra. Define a map $\varphi(L) : L^{\circ\circ} \to F(D(L))$ (the lattice of all filters of $D(L)$) by

$$a\varphi(L) = a^{\circ}\uparrow \cap D(L), \quad \text{for all } a \in L^{\circ\circ}.$$

It is known that $\varphi(L)$ is a (0,1)-lattice homomorphism (see [12]).

An equivalence relation θ on a lattice L is called a lattice congruence on L if it is compatible with the lattice operations, that is, $(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a \vee c, b \vee d) \in \theta$ and $(a \wedge c, b \wedge d) \in \theta$.

Let θ be a lattice congruence on a bounded lattice (a lattice with the smallest element 0 and the greatest element 1) L. Then the subset $\{x \in L : (x, 0) \in \theta\}$ is called the Kernel of θ and is denoted by Ker θ. Also, the subset $\{x \in L : (x, 1) \in \theta\}$ is called the Cokernel of θ and is denoted by Coker θ. It is clear that Ker θ and Coker θ are ideal and filter of L, respectively.

Theorem 2.2 (see [26]) *An equivalence relation on a lattice* L *is a lattice congruence on* L *if and only if* $(a, b) \in \theta$ *implies* $(a \vee c, b \vee c) \in \theta$ *and* $(a \wedge c, b \wedge c) \in \theta$ *for all* $c \in L$.

A lattice congruence θ on an MS-algebra $(L; ^{\circ})$ is called a congruence on L if $(a, b) \in \theta$ implies $(a^{\circ}, b^{\circ}) \in \theta$.

The symbols ∇_L and Δ_L will be used, as usual, for the universal congruence $L \times L$ and the equality congruence on L, respectively.

Let L be an MS-algebra. Then, we use Con(L) to denote the congruence lattice of L and we also use $\theta_{L^{\circ\circ}}, \theta_{D(L)}$ to denote the restrictions of a congruence $\theta \in \text{Con}(L)$ to $L^{\circ\circ}$ and $D(L)$, respectively. Evidently, $(\theta_{L^{\circ\circ}}, \theta_{D(L)}) \in \text{Con}(L^{\circ\circ}) \times \text{Con}(D(L))$.

Now, we restrict the definition of a congruence pair of quasi-modular p-algebras (see [29, Definition 7]) to Stone algebras.

Definition 2.2 *Let* L *be a* Stone *algebra. Then the pair* $(\theta_1, \theta_2) \in \text{Con}(L^{\circ\circ}) \times \text{Con}(D(L))$ *is called a congruence pair if* $a \in L^{\circ\circ}, u \in D(L), u \geq a$ *and* $a \equiv 1(\theta_1)$ *imply* $u \equiv 1(\theta_2)$.

Definition 2.3 (see [12]) *An* MS-*algebra* $(L; \vee, \wedge, ^{\circ}, 0, 1)$ *is called a principal* MS-*algebra if it satisfies the following conditions*:

(i) *The filter* $D(L)$ *is principal, i.e., there exists an element* $d_L \in L$ *such that* $D(L) = [d_L)$,

(ii) $x = x^{\circ\circ} \wedge (x \vee d_L)$ *for any* $x \in L$.

It is known that any principal MS-algebra is a decomposable MS-algebra (see [12]). From [13], we recall the definition of a congruence pair of a principal MS-algebra.

Definition 2.4 (see [13]) *Let* L *be a principal* MS-*algebra with a smallest dense element* d_L. *A pair of congruences* $(\theta_1, \theta_2) \in \mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(D(L))$ *will be called a congruence pair if*

$$(a, b) \in \theta_1 \text{ implies } (a \vee d_L, b \vee d_L) \in \theta_2.$$

3 Congruence Pairs of a Decomposable MS-Algebra

The notion of a congruence pair was studied on various classes of algebras containing the class **S** of all Stone algebras. Katriňák [27, 29] studied the congruence pairs and the lattices of congruence pairs of certain p-algebras, El-Assar [21] characterized the congruence lattices of quasi-modular p-algebras, Badawy and Shume [16] considered the congruence pairs and related properties of principal p-algebras. Also, Badawy [3] presented a characterization of the congruence lattices of principal p-algebras. Beazear [17] introduced the notion of congruence pairs on MS-algebras from the subvariety $\mathbf{K_2}$ (K_2-algebras). Recently, Badawy, Haviar and Ploščica [13] studied the concept of congruence pairs of principal MS-algebras. Also, they characterized the congruence lattices of principal MS-algebras in terms of congruence pairs.

In this section we introduce the concept of congruence pairs on decomposable MS-algebras generalizing that for principal MS-algebras. Some properties of congruence pairs of a decomposable MS-algebra L will be investigated.

Definition 3.1 *Let* L *be a decomposable* MS-*algebra. An arbitrary pair* (θ_1, θ_2) *in* $\mathrm{Con}(L^{\circ\circ})$ $\times\mathrm{Con}(D(L))$ *is called a congruence pair if* $a \equiv b(\theta_1)$ *implies* $a \vee d \equiv b \vee d(\theta_2)$ *for all* $d \in D(L)$.

It is clear that if L is a principal MS-algebra with a smallest dense element d_L, then Definition 2.6 implies Definition 3.1.

Lemma 3.1 *Let* L *be a decomposable* MS-*algebra and* (θ_1, θ_2) *be a congruence pair. Then we have the following property*:

$$a \equiv b(\theta_1) \text{ and } c \equiv d(\theta_2) \text{ imply } a \vee c \equiv b \vee d(\theta_2).$$

Proof Let $a \equiv b(\theta_1)$. Thus by Definition 3.1, we get $a \vee c \equiv b \vee c(\theta_2)$, $a \vee d \equiv b \vee d(\theta_2)$ and hence $a \vee c \vee d \equiv b \vee c \vee d(\theta_2)$ as $c, d, c \vee d \in D(L)$. Then $a \vee c \equiv b \vee c(\theta_2)$ and $c \equiv d(\theta_2)$ imply $a \vee c \equiv b \vee c \vee d$. Also $a \vee d \equiv b \vee d(\theta_2)$ and $c \equiv d(\theta_2)$ imply $a \vee c \vee d \equiv b \vee d(\theta_2)$. Consequently $a \vee c \equiv b \vee d(\theta_2)$.

For a Stone algebra, the following lemma shows that Definitions 2.4 and 3.1 are equivalent.

Lemma 3.2 *Let* L *be a* Stone *algebra. Then* $(\theta_1, \theta_2) \in \mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(D(L))$ *is a congruence pair according to Definition* 2.4 *if and only if it is a congruence pair by Definition* 3.1.

Proof Let L be a Stone algebra. Then $L^{\circ\circ}$ is a Boolean subalgebra of L. Thus $a \vee a^{\circ} = 1$ for all $a \in L^{\circ\circ}$. Let $(\theta_1, \theta_2) \in \mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(D(L))$ be a congruence pair by Definition 2.4.

Suppose that $a \equiv b(\theta_1)$. Let $\alpha = (a \vee b^\circ) \wedge (a^\circ \vee b)$. Then $\alpha \in L^{\circ\circ}$ and $\alpha \wedge a = \alpha \wedge b = a \wedge b$. Since $a \vee b^\circ \equiv b \vee b^\circ(\theta_1) = 1$ and $a^\circ \vee b \equiv a^\circ \vee a(\theta_1) = 1$, we have $\alpha \equiv 1(\theta_1)$ and by Definition 2.4, $\alpha \leq \alpha \vee d \in D(L)$ implies $\alpha \vee d \equiv 1(\theta_2)$ for all $d \in D(L)$. Since L is a distributive lattice, we have

$$a \vee d = (a \vee d) \wedge 1 \equiv (a \vee d) \wedge (\alpha \vee d)(\theta_2) = (a \wedge \alpha) \vee d = (a \wedge b) \vee d.$$

In a similar way, we get $b \vee d \equiv (a \wedge b) \vee d(\theta_2)$. Thus $a \vee d \equiv b \vee d(\theta_2)$. For the converse, let $a \in L^{\circ\circ}, a \leq u \in D(L)$ and $a \equiv 1(\theta_1)$. Then we have $a \vee d \equiv 1 \vee d(\theta_2)$ for all $d \in D(L)$ by Definition 3.1. Without loss of generality we can take $u \geq d$. Then $u = a \vee d \vee u \equiv 1 \vee u \vee d(\theta_2) = 1$. Therefore (θ_1, θ_2) is a congruence pair according to Definition 2.4.

The following theorem gives one of the main results of this paper. We give a characterization of congruence pairs of a decomposable MS-algebra.

Theorem 3.1 *Let L be a decomposable* MS-*algebra. Then every congruence relation θ of L determines a congruence pair $(\theta_{L^{\circ\circ}}, \theta_{D(L)})$. Conversely, every congruence pair (θ_1, θ_2) uniquely determines a congruence relation θ on L satisfying $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_{D(L)} = \theta_2$ by the rule*

$$x \equiv y(\theta) \text{ if and only if } x^{\circ\circ} \equiv y^{\circ\circ}(\theta_1) \text{ and } x \vee d \equiv y \vee d(\theta_2) \quad \text{for all } d \in D(L).$$

Proof Let $\theta \in \mathrm{Con}(L)$ and $a \equiv b(\theta_{L^{\circ\circ}})$ for $a, b \in L^{\circ\circ}$. Then $a \equiv b(\theta)$. This result implies that $a \vee d \equiv b \vee d(\theta)$. Hence, $a \vee d \equiv b \vee d(\theta_{D(L)})$, where $a \vee d, b \vee d \in D(L)$. This shows that $(\theta_{L^{\circ\circ}}, \theta_{D(L)})$ is a congruence pair. Conversely, let (θ_1, θ_2) be a congruence pair and let θ be defined as above. It is clear that θ is an equivalence relation. We now proceed to show that θ is a congruence on L. Let $a \equiv b(\theta)$ and $c \equiv f(\theta)$. Then we get $a^{\circ\circ} \equiv b^{\circ\circ}(\theta_1), c^{\circ\circ} \equiv f^{\circ\circ}(\theta_1)$ and $a \vee d \equiv b \vee d(\theta_2), c \vee d \equiv f \vee d(\theta_2)$ for all $d \in D(L)$. Now, we have

$$(a \wedge c)^{\circ\circ} = a^{\circ\circ} \wedge c^{\circ\circ} \equiv b^{\circ\circ} \wedge f^{\circ\circ}(\theta_1) = (b \wedge f)^{\circ\circ},$$
$$(a \wedge c) \vee d = (a \vee d) \wedge (c \vee d) \equiv (b \vee d) \wedge (f \vee d)(\theta_2) = (b \wedge f) \vee d \quad \text{for all } d \in D(L).$$

Then $a \wedge c \equiv b \wedge f(\theta)$, and therefore θ preserves the meet operation of L. Also, θ preserves the join operation of L since the following equalities hold on L:

$$(a \vee c)^{\circ\circ} = a^{\circ\circ} \vee c^{\circ\circ} \equiv b^{\circ\circ} \vee f^{\circ\circ}(\theta_1) = (b \vee f)^{\circ\circ},$$
$$(a \vee c) \vee d = (a \vee d) \vee (c \vee d) \equiv (b \vee d) \vee (f \vee d)(\theta_2) = (b \vee f) \vee d, \quad \forall d \in D(L).$$

In order to show that θ preserves the unary operation $^\circ$, we let $a \equiv b(\theta)$, then $a^{\circ\circ} \equiv b^{\circ\circ}(\theta_1)$. Hence, $a^\circ = a^{\circ\circ\circ} \equiv b^{\circ\circ\circ}(\theta_1) = b^\circ$. Thus by Definition 3.1, we have shown that $a^\circ \vee d \equiv b^\circ \vee d(\theta_2)$ for all $d \in D(L)$. Therefore, $a^\circ \equiv b^\circ(\theta)$.

Now, we proceed to show that $\theta_{L^{\circ\circ}} = \theta_1$ and $\theta_{D(L)} = \theta_2$. If $a, b \in L^{\circ\circ}$ and $a \equiv b(\theta_1)$, then $a^{\circ\circ} \equiv b^{\circ\circ}(\theta_1)$ and $a \vee d \equiv b \vee d(\theta_2)$, the latter holds by Definition 3.1 since (θ_1, θ_2) is a congruence pair. It follows that $a \equiv b(\theta_{L^{\circ\circ}})$, thus $\theta_1 \leq \theta_{L^{\circ\circ}}$. The inequality $\theta_{L^{\circ\circ}} \leq \theta_1$ as well as the equality $\theta_{D(L)} = \theta_2$ follow straight from the definition of θ. For the uniqueness of θ, let θ and $\acute{\theta}$ be two congruences on L with $\theta_{L^{\circ\circ}} = \acute{\theta}_{L^{\circ\circ}} = \theta_1$ and $\theta_{D(L)} = \acute{\theta}_{D(L)} = \theta_2$. Let $x \equiv y(\theta)$. Then $x^{\circ\circ} \equiv y^{\circ\circ}(\theta_{L^{\circ\circ}})$ and $x \vee d \equiv y \vee d(\theta_{D(L)})$. Now, we have $x^{\circ\circ} \equiv y^{\circ\circ}(\acute{\theta}_{L^{\circ\circ}})$ and

$x \vee d \equiv y \vee d(\acute{\theta}_{D(L)})$ for all $d \in D(L)$. Thus $x \equiv y(\acute{\theta})$ and $\theta \leq \acute{\theta}$. Similarly, we can prove that $\acute{\theta} \leq \theta$. Hence $\theta = \acute{\theta}$ and our proof is completed.

Corollary 3.1 *Let L be a decomposable* MS*-algebra. Then the set $A(L)$ of congruence pairs of L is a bounded sublattice of* $\mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(D(L))$ *and $\theta \mapsto (\theta_{L^{\circ\circ}}, \theta_{D(L)})$ is an isomorphism of* $\mathrm{Con}(L)$ *and $A(L)$.*

Proof It is clear that $(\triangle_{L^{\circ\circ}}, \triangle_{D(L)}), (\bigtriangledown_{L^{\circ\circ}}, \bigtriangledown_{D(L)}) \in A(L)$. Let $(\theta_1, \theta_2), (\psi_1, \psi_2) \in A(L)$. Then, it is easy to verify that $(\theta_1 \wedge \psi_1, \theta_2 \wedge \psi_2) \in A(L)$. Now, we proceed to show that $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$. Let $a \equiv b(\theta_1 \vee \psi_1)$. Then there is a finite sequence $a = a_0, a_1, \cdots, a_n = b$ in $L^{\circ\circ}$ such that, for each i with $0 \leq i \leq n-1$, either $a_{i-1} \equiv a_i(\theta_1)$ or $a_i \equiv a_{i+1}(\psi_1)$. Then $a_{i-1} \vee d \equiv a_i \vee d(\theta_2)$ or $a_i \vee d \equiv a_{i+1} \vee d(\psi_2)$, for every $d \in D(L)$ by Definition 3.1. Thus we have the sequence

$$a \vee d = a_0 \vee d, a_1 \vee d, \cdots, a_n \vee d = b \vee d \quad \text{in } D(L).$$

The above result leads to $a \vee d \equiv b \vee d(\theta_2 \vee \psi_2)$ and hence $(\theta_1 \vee \psi_1, \theta_2 \vee \psi_2) \in A(L)$. Thus we conclude that $A(L)$ is a bounded sublattice of $\mathrm{Con}(L^{\circ\circ}) \times \mathrm{Con}(D(L))$. It is clear that the map $\theta \mapsto (\theta_{L^{\circ\circ}}, \theta_{D(L)})$ of $\mathrm{Con}(L)$ into $A(L)$ is an isomorphism.

The next corollary follows immediately.

Corollary 3.2 *Let L be a decomposable* MS*-algebra. Then the following statements hold*:
(1) $(\forall\, \Phi \in \mathrm{Con}(D(L)))(\triangle_{L^{\circ\circ}}, \Phi) \in A(L)$,
(2) $(\forall\, \Psi \in \mathrm{Con}(L^{\circ\circ})\,(\Psi, \bigtriangledown_{D(L)}) \in A(L)$.

4 Congruence Pairs via Central Elements of a Decomposable MS-algebra

In this section, we investigate the relationship between the central elements of a decomposable MS-algebra L and the congruence pairs of L.

From [8], we recall the following.

Definition 4.1 (see [8]) *An element a of an* MS*-algebra L is called a central element of L if $a \vee a^{\circ} = 1$. The set of all central elements of L is denoted by $B(L)$.*

Theorem 4.1 (see [8]) *Let L be an* MS*-algebra. Then $B(L)$ is a* Boolean *subalgebra of $L^{\circ\circ}$.*

For each central element a of an MS-algebra L, we define a relation $\theta[a\downarrow]$ on $L^{\circ\circ}$ as follows:

$$(x, y) \in \theta[a\downarrow] \Leftrightarrow x \wedge a^{\circ} = y \wedge a^{\circ}.$$

For each central element a of a decomposable MS-algebra L, we define a relation $\theta[a\varphi(L)]$ on $D(L)$ as follows:

$$(x, y) \in \theta[a\varphi(L)] \Leftrightarrow x \wedge d = y \wedge d \quad \text{for some } d \in a\varphi(L).$$

The properties of the above two relations are given in the following two lemmas, respectively.

Lemma 4.1 *Let L be an* MS-*algebra. Then for every a, b of $B(L)$, we have*
(1) *$\theta[a\downarrow]$ is a congruence on $L^{\circ\circ}$ with* $\mathrm{Ker}(\theta[a\downarrow]) = a\downarrow$,
(2) *$a \leq b$ if and only if $\theta[a\downarrow] \subseteq \theta[b\downarrow]$,*
(3) *$a = b$ if and only if $\theta[a\downarrow] = \theta[b\downarrow]$,*
(4) *$\theta[0\downarrow] = \triangle_{L^{\circ\circ}}$ and $\theta[1\downarrow] = \nabla_{L^{\circ\circ}}$,*
(5) *$\theta[a\downarrow] \vee \theta[b\downarrow] = \theta[(a \vee b)\downarrow]$,*
(6) *$\theta[a\downarrow] \cap \theta[b\downarrow] = \theta[(a \wedge b)\downarrow]$.*

Proof (1) It is clear that $\theta[a\downarrow]$ is an equivalence relation on $L^{\circ\circ}$ for every $a \in B(L)$. Now let $(x, y) \in \theta[a\downarrow]$ and $c \in L^{\circ\circ}$. Then $x \wedge a^\circ = y \wedge a^\circ$ and hence

$$\begin{aligned}(x \vee c) \wedge a^\circ &= (x \wedge a^\circ) \vee (c \wedge a^\circ)\\ &= (y \wedge a^\circ) \vee (c \wedge a^\circ)\\ &= (y \vee c) \wedge a^\circ.\end{aligned}$$

Therefore $(x \vee c, y \vee c) \in \theta[a\downarrow]$ for all $c \in L^{\circ\circ}$. Also, we can deduce that $(x \wedge c, y \wedge c) \in \theta[a\downarrow]$. Then by Theorem 2.6, $\theta[a\downarrow]$ is a lattice congruence on $L^{\circ\circ}$. To show that $\theta[a\downarrow]$ is preserved by a unary operation $^\circ$ on $L^{\circ\circ}$, let $(x, y) \in \theta[a\downarrow]$. Then we have:

$$\begin{aligned}(x, y) \in \theta[a\downarrow] &\Rightarrow x \wedge a^\circ = y \wedge a^\circ\\ &\Rightarrow (x \wedge a^\circ) \vee a = (y \wedge a^\circ) \vee a\\ &\Rightarrow (x \vee a) \wedge (a^\circ \vee a) = (y \vee a) \wedge (a^\circ \vee a)\\ &\Rightarrow x \vee a = y \vee a \quad \text{as } a^\circ \vee a = 1\\ &\Rightarrow (x \vee a)^\circ = (y \vee a)^\circ\\ &\Rightarrow x^\circ \wedge a^\circ = y^\circ \wedge a^\circ\\ &\Rightarrow (x^\circ, y^\circ) \in \theta[a\downarrow].\end{aligned}$$

Further,

$$\begin{aligned}\mathrm{Ker}(\theta[a\downarrow]) &= \{x \in L^{\circ\circ} : (x, 0) \in \theta[a\downarrow]\}\\ &= \{x \in L^{\circ\circ} : x \wedge a^\circ = 0\}\\ &= \{x \in L^{\circ\circ} : x \leq a\} = a\downarrow,\end{aligned}$$

as $a = a \vee 0 = a \vee (x \wedge a^\circ) = a \vee x$ implies $x \leq a$.

(2) Let $a \leq b$ and $(x, y) \in \theta[a\downarrow]$. Then $x \wedge a^\circ = y \wedge a^\circ$. Thus $x \wedge a^\circ \wedge b^\circ = y \wedge a^\circ \wedge b^\circ$ and $b^\circ \leq a^\circ$ imply $x \wedge b^\circ = y \wedge b^\circ$. So $(x, y) \in \theta[b\downarrow]$ and hence $\theta[a\downarrow] \subseteq \theta[b\downarrow]$. Conversely, let $\theta[a\downarrow] \subseteq \theta[b\downarrow]$. As a is a central element of L, then $(a \wedge b) \wedge a^\circ = 0 = a \wedge a^\circ$. Hence $(a \wedge b, a) \in \theta[a\downarrow]$. By hypotheses, $(a \wedge b, a) \in \theta[b\downarrow]$. Since b is a central element of L, then $(a \wedge b) \wedge b^\circ = a \wedge b^\circ$ implies $a \wedge b^\circ = 0$. Now, since $a \wedge b^\circ = 0$ and a, b belong to the Boolean algebra $B(L)$ then $a \leq b^{\circ\circ} = b$.

(3) It is obvious.

(4) Let $(x, y) \in \theta[0\downarrow]$. Then $x = x \wedge 0^\circ = y \wedge 0^\circ = y$. Therefore $\theta[0\downarrow] = \triangle_{L^{\circ\circ}}$. For all $x, y \in L$, we have $x \wedge 1^\circ = 0 = y \wedge 1^\circ$ and hence $(x, y) \in \theta[1\downarrow]$. Then $\theta[1\downarrow] = \nabla_{L^{\circ\circ}}$.

(5) Since $a, b \leq a \vee b$, then by (2), $\theta[a\downarrow], \theta[b\downarrow] \subseteq \theta[(a \vee b)\downarrow]$. Therefore $\theta[(a \vee b)\downarrow]$ is an upper bound of both $\theta[a\downarrow]$ and $\theta[b\downarrow]$. Suppose that $\theta[c\downarrow]$ is an upper bound of $\theta[a\downarrow]$ and $\theta[b\downarrow]$. Then $\theta[a\downarrow], \theta[b\downarrow] \subseteq \theta[c\downarrow]$. Thus by (2) we get $a, b \leq c$. Then $a \vee b \leq c$. Again by (2), $\theta[(a \vee b)\downarrow] \subseteq \theta[(c)\downarrow]$. Therefore $\theta[(a \vee b)\downarrow]$ is the least upper bound of both $\theta[a\downarrow]$ and $\theta[b\downarrow]$. This deduces that $\theta[a\downarrow] \vee \theta[b\downarrow] = \theta[(a \vee b)\downarrow]$.

(6) Since $a \wedge b \leq a, b$, then by (2), $\theta[(a \wedge b)\downarrow] \subseteq \theta[a\downarrow], \theta[a\downarrow]$. Thus $\theta[(a \wedge b)\downarrow] \subseteq \theta[a\downarrow] \cap \theta[a\downarrow]$. Conversely, let $(x, y) \in \theta[a\downarrow] \cap \theta[b\downarrow]$. Then

$$\begin{aligned}(x, y) \in \theta[a\downarrow] \cap \theta[b\downarrow] &\Rightarrow (x, y) \in \theta[a\downarrow] \text{ and } (x, y) \in \theta[b\downarrow] \\ &\Rightarrow x \wedge a^\circ = y \wedge a^\circ \text{ and } x \wedge b^\circ = y \wedge b^\circ \\ &\Rightarrow (x \wedge a^\circ) \vee (x \wedge b^\circ) = (y \wedge a^\circ) \vee (y \wedge b^\circ) \\ &\Rightarrow x \wedge (a^\circ \vee b^\circ) = y \wedge (a^\circ \vee b^\circ) \quad \text{by distributivity of } L \\ &\Rightarrow x \wedge (a \wedge b)^\circ = y \wedge (a \wedge b)^\circ \\ &\Rightarrow (x, y) \in \theta[(a \wedge b)\downarrow].\end{aligned}$$

Therefore $\theta[(a\downarrow] \cap \theta[b\downarrow] \subseteq \theta[(a \wedge b)\downarrow]$ and hence $\theta[(a \wedge b)\downarrow] = \theta[a\downarrow] \cap \theta[b\downarrow]$.

Lemma 4.2 *Let L be a decomposable* MS*-algebra. Then for every a, b of $B(L)$, we have*
(1) *$\theta[a\varphi(L)]$ is a congruence on $D(L)$ with* $\mathrm{Coker}(\theta[a\varphi(L)]) = a\varphi(L)$,
(2) *$a \leq b$ implies $\theta[a\varphi(L)] \subseteq \theta[b\varphi(L)]$,*
(3) *$\theta[(0\varphi(L)] = \triangle_{D(L)}$ and $\theta[1\varphi(L)] = \bigtriangledown_{D(L)}$,*
(4) *$\theta[a\varphi(L)] \vee \theta[b\varphi(L)] = \theta[(a \vee b)\varphi(L)]$,*
(5) *$\theta[a\varphi(L)] \wedge \theta[b\varphi(L)] = \theta[(a \wedge b)\varphi(L)]$.*

Proof (1) We know that $a\varphi(L) = a^\circ \uparrow \cap D(L)$ is a filter of $D(L)$. Obviously, $\theta[a\varphi(L)]$ is an equivalence relation on $D(L)$. Let $(x, y), (x', y') \in \theta[a\varphi(L)]$. Thus $x \wedge d = y \wedge d$ and $x' \wedge e = y' \wedge e$ for some $d, e \in a\varphi(L)$. Then

$$\begin{aligned}(x \vee x') \wedge (d \wedge e) &= (x \wedge d \wedge e) \vee (x' \wedge d \wedge e) \\ &= (y \wedge d \wedge e) \vee (y' \wedge d \wedge e) \\ &= (y \vee y') \wedge (d \wedge e) \quad \text{where } d \wedge e \in a\varphi(L).\end{aligned}$$

Hence $(x \vee x', y \vee y') \in \theta[a\varphi(L)]$. Using a similar way, we get $(x \wedge x', y \wedge y') \in \theta[a\varphi(L)]$, so $\theta[a\varphi(L)]$ is lattice congruence on $D(L)$. Also, we have

$$\begin{aligned}\mathrm{Coker}(\theta[a\varphi(L)]) &= \{x \in D(L) : (x, 1) \in \theta[a\varphi(L)]\} \\ &= \{x \in D(L) : x \wedge d = 1 \wedge d = d \quad \text{for some } d \in a\varphi(L)\} \\ &= \{x \in D(L) : x \geq d \in a\varphi(L)\} \\ &= a\varphi(L).\end{aligned}$$

(2) Let $a \leq b$. Then $a\varphi(L) \subseteq b\varphi(L)$. Let $(x, y) \in \theta[a\varphi(L)]$. Then $x \wedge d = y \wedge d$ for some $d \in a\varphi(L)$. Since $d \in a\varphi(L)$ and $a\varphi(L) \subseteq b\varphi(L)$, then $d \in b\varphi(L)$. So, $(x, y) \in \theta[b\varphi(L)]$. Therefore $\theta[a\varphi(L)] \subseteq \theta[ab\varphi(L)]$.

(3) Let $(x, y) \in \theta[0\varphi(L)]$. Since $0\varphi(L) = (1]$, then $x = y$ and hence $\theta[0\varphi(L)] = \triangle_{D(L)}$. Since $1\varphi(L) = D(L)$, then $\theta[1\varphi(L)] = \theta[D(L)] = D(L) \times D(L) = \triangledown_{D(L)}$.

(4) Since $a, b \leq a \vee b$, then $a\varphi(L), b\varphi(L) \subseteq (a \vee b)\varphi(L)$. Hence by (2), we have

$$\theta[a\varphi(L)], \theta[b\varphi(L)] \subseteq \theta[(a \vee b)\varphi(L)] .$$

Then $\theta[(a\vee b)\varphi(L)]$ is an upper bound of $\theta[a\varphi(L)]$ and $\theta[b\varphi(L)]$. Let $\theta[c\varphi(L)]$ be an upper bound of $\theta[a\varphi(L)]$ and $\theta[b\varphi(L)]$. Then $\theta[a\varphi(L)], \theta[b\varphi(L)] \subseteq \theta[c\varphi(L)]$ implies $a\varphi(L), b\varphi(L) \subseteq c\varphi(L)$. Thus $(a \vee b)\varphi(L) = a\varphi(L) \vee b\varphi(L) \subseteq c\varphi(L)$ and hence $\theta[(a \vee b)\varphi(L)] \subseteq \theta[c\varphi(L)]$. Therefore $\theta[(a \vee b)\varphi(L)]$ is the least upper bound of both $\theta[a\varphi(L)]$ and $\theta[b\varphi(L)]$.

(5) Since $a \wedge b \leq a, b$, then by (2), $\theta[(a \wedge b)\varphi(L)] \subseteq \theta[a\varphi(L)], \theta[b\varphi(L)]$ and hence $\theta[(a \wedge b)\varphi(L)] \subseteq \theta[a\varphi(L)] \cap \theta[b\varphi(L)]$. Conversely, let $(x, y) \in \theta[a\varphi(L)] \cap \theta[b\varphi(L)]$. Then $(x, y) \in \theta[a\varphi(L)]$ and $(x, y) \in \theta[b\varphi(L)]$. Thus $x \wedge d = y \wedge d$ for some $d \in a\varphi(L)$ and $x \wedge e = y \wedge e$ for some $e \in b\varphi(L)$. Since $d \vee e \geq d, e$ and $d \in a\varphi(L), b \in b\varphi(L)$, then $d \vee e \in a\varphi(L) \cap b\varphi(L) = (a \wedge b)\varphi(L)$. Now

$$\begin{aligned} x \wedge (d \vee e) &= (x \wedge d) \vee (x \wedge e) \quad \text{by distributivity of } L \\ &= (y \wedge d) \vee (y \wedge e) \\ &= y \wedge (d \vee e) \quad \text{where } d \vee e \in (a \wedge b)\varphi(L). \end{aligned}$$

Therefore $(x, y) \in \theta[(a \wedge b)\varphi(L)]$ and hence $\theta[a\varphi(L)] \cap \theta[b\varphi(L)] \subseteq \theta[(a \wedge b)\varphi(L)]$.

Let L be a decomposable MS-algebra. Consider the subsets $\mathbf{B}$ and $\mathbf{D}$ of $\mathrm{Con}(L^{\circ\circ})$ and $\mathrm{Con}(D(L))$, respectively as follows:

$$\mathbf{B} = \{\theta[a \downarrow] : a \in B(L)\}, \quad \mathbf{D} = \{\theta[a\varphi(L)] : a \in B(L)\}.$$

The proof of the following theorem is a consequence of Lemmas 4.3–4.4.

Theorem 4.2 *Let L be a decomposable* MS-*algebra. Then*
(1) $(\mathbf{B}, \vee, \wedge, ', \triangle_{L^{\circ\circ}}, \triangledown_{L^{\circ\circ}})$ *is a* Boolean *algebra, where* $(\theta[a \downarrow])' = \theta[a^{\circ} \downarrow]$,
(2) $(\mathbf{D}, \vee, \wedge, ', \triangle_{D(L)}, \triangledown_{D(L)})$ *is a* Boolean *algebra, where* $(\theta[a\varphi(L)])' = \theta[a^{\circ}\varphi(L)]$.

Now, we observe that every central element a of a decomposable MS-algebra L associated with the congruence pair $(\theta[a \downarrow], \theta[a\varphi(L)])$.

Theorem 4.3 *Let L be a decomposable* MS-*algebra and $a \in L^{\circ\circ}$. Then a is a central element of L if and only if $(\theta[a \downarrow], \theta[a\varphi(L)])$ is a congruence pair of L.*

Proof Let a be a central element of L. By Lemmas 4.3(1) and 4.4(1), $\theta[a \downarrow]$ and $\theta[a\varphi(L)]$ are congruences on $L^{\circ\circ}$ and $D(L)$, respectively. To show that $(\theta[a \downarrow], \theta[a\varphi(L)])$ is a congruence pair, let $(b, c) \in \theta[a \downarrow]$. Then

$$\begin{aligned} (b, c) \in \theta[a \downarrow] &\Rightarrow b \wedge a^{\circ} = c \wedge a^{\circ} \\ &\Rightarrow (b \wedge a^{\circ}) \vee d = (c \wedge a^{\circ}) \vee d \quad \text{for all } d \in D(L) \\ &\Rightarrow (b \vee d) \wedge (a^{\circ} \vee d) = (c \vee d) \wedge (a^{\circ} \vee d) \quad \text{where } a^{\circ} \vee d \in [a^{\circ}) \cap D(L) = a\varphi(L) \\ &\Rightarrow (b \vee d, c \vee d) \in \theta[a\varphi(L)]. \end{aligned}$$

Thus $(\theta[a\downarrow],\theta[a\varphi(L)])\in A(L)$. Conversely, let $(\theta[a\downarrow],\theta[a\varphi(L)])\in A(L)$. Since $(a,0)\in\theta[a\downarrow]$, then $a\wedge a^\circ=0\wedge a^\circ=0$. Now, $a\vee a^\circ=(a^\circ\wedge a)^\circ=0^\circ=1$. Therfore $a\in B(L)$.

Let L be a decomposable MS-algebra. Consider the set

$$A'(L)=\{(\theta[a\downarrow],\theta[a\varphi(L)]):a\in B(L)\}.$$

From Theorems 4.5–4.6, we observe the following important results.

Theorem 4.4 *Let L be a decomposable* MS-*algebra. Then $(A'(L);\vee,\wedge,',0_{A'(L)},1_{A'(L)})$ is a Boolean algebra, where*

$$\begin{aligned}(\theta[a\downarrow],\theta[a\varphi(L)])\vee(\theta[b\downarrow],\theta[b\varphi(L)])&=(\theta[(a\vee b)\downarrow],\theta[(a\vee b)\varphi(L)]),\\(\theta[a\downarrow],\theta[a\varphi(L)])\wedge(\theta[b\downarrow],\theta[b\varphi(L)])&=(\theta[(a\wedge b)\downarrow],\theta[(a\wedge b)\varphi(L)]),\\(\theta[a\downarrow],\theta[a\varphi(L)])'&=(\theta[a^\circ\downarrow],\theta[a^\circ\varphi(L)]),\\1_{A'(L)}&=(\nabla_{L^{\circ\circ}},\nabla_{D(L)}),\\0_{A'(L)}&=(\triangle_{L^{\circ\circ}},\triangle_{D(L)}).\end{aligned}$$

Theorem 4.5 *Let L be a decomposable* MS-*algebra. Then $B(L)$ is isomorphic to $A'(L)$ under the isomorphism $a\mapsto(\theta[a\downarrow],\theta[a\varphi(L)])$.*

5 Congruence Permutable of Decomposable MS-Algebras

El-Assar [21] studied the notion of n-permutability of congruences of p-algebras satisfying certain condition. Also, El-Assar and Abd El-Hakim [24] characterized the permutability of congruences of modular p-algebras. Badawy and Shume [16] characterized the permutability of congruences of the class of principal p-algebras.

Let L be an algebra. We say that $\theta,\psi\in\mathrm{Con}(L)$ permute if for any $a,b,c\in L$ with $(a,b)\in\theta$ and $(b,c)\in\psi$, there exists $h\in L$ such that $(a,h)\in\psi$ and $(h,c)\in\theta$, that is $\theta\circ\psi=\psi\circ\theta$, where $\theta\circ\psi$ is the relational product of θ and ψ.

An algebra L is said to be congruence permutable (briefly, permutable) if every pair of congruences on it is permutable.

We characterize the congruence permutable of a decomposable MS-algebra in the following theorem.

Theorem 5.1 *Let L be a decomposable* MS-*algebra. Then the following conditions are equivalent*:

(1) *L has congruence permutable,*

(2) *$L^{\circ\circ}$ and $D(L)$ both are congruence permutable.*

Proof To show the equivalence of the conditions (1) and (2), we have to show that two congruences $\theta,\psi\in\mathrm{Con}(L)$ are permutable if and only if their restrictions $\theta_{L^{\circ\circ}},\psi_{L^{\circ\circ}}$ and $\theta_{D(L)},\psi_{D(L)}$ both are congruence permutable on $L^{\circ\circ}$ and $D(L)$, respectively. Let θ,ψ be permutable on L. Firstly, we will prove that $\theta_{L^{\circ\circ}},\psi_{L^{\circ\circ}}$ are permutable on $L^{\circ\circ}$. Let $a,b,c\in L^{\circ\circ}$ be such that $(a,b)\in\theta_{L^{\circ\circ}}$ and $(b,c)\in\psi_{L^{\circ\circ}}$. Then $(a,b)\in\theta$ and $(b,c)\in\psi$. Since θ,ψ are permutable, then there exists $x\in L$ such that $(a,x)\in\psi$ and $(x,c)\in\theta$. Thus $(a,x^{\circ\circ})\in\psi$ and

$(x^{\circ\circ}, c) \in \theta$. Then $(a, x^{\circ\circ}) \in \psi_{L^{\circ\circ}}$ and $(x^{\circ\circ}, c) \in \theta_{L^{\circ\circ}}$ as $x^{\circ\circ} \in L^{\circ\circ}$. Therefore $\theta_{L^{\circ\circ}}, \psi_{L^{\circ\circ}}$ are permutable on $L^{\circ\circ}$. Now we prove that permutability of θ and ψ implies permutability of $\theta_{D(L)}$ and $\psi_{D(L)}$. Let $x, y, z \in D(L)$ be such that $(x, y) \in \theta_{D(L)}$ and $(y, z) \in \psi_{D(L)}$. Then $(x, y) \in \theta$ and $(y, z) \in \psi$. Since θ, ψ are permutable, then there exists $a \in L$ such that $(x, a) \in \psi$ and $(a, z) \in \theta$. Then for every $d \in D(L)$, we have $(x \vee d, a \vee d) \in \psi$ and $(a \vee d, z \vee d) \in \theta$. We can choose $d \leq x, z$. Then $(x, a \vee d) \in \psi_{D(L)}$ and $(a \vee d, z) \in \theta_{D(L)}$ with $a \vee d \in D(L)$. Therefore $\theta_{D(L)}$ and $\theta_{D(L)}$ both are congruence permutable on $D(L)$.

Conversely, let $\theta, \psi \in \mathrm{Con}(L)$ such that $\theta_{L^{\circ\circ}}, \psi_{L^{\circ\circ}}$ and $\theta_{D(L)}, \psi_{D(L)}$ are congruence permutable on $L^{\circ\circ}$ and $D(L)$ respectively. Consider the elements $x, y, z \in L$ with $(x, y) \in \theta$ and $(y, z) \in \psi$. By Theorem 3.4, we get $(x^{\circ\circ}, y^{\circ\circ}) \in \theta_{L^{\circ\circ}}, (y^{\circ\circ}, z^{\circ\circ}) \in \psi_{L^{\circ\circ}}$ and $(x \vee d, y \vee d) \in \theta_{D(L)}, (y \vee d, z \vee d) \in \psi_{D(L)}$ for all $d \in D(L)$. Since $\theta_{L^{\circ\circ}}, \psi_{L^{\circ\circ}}$ are permutable, then there exists $a \in L^{\circ\circ}$ with $(x^{\circ\circ}, a) \in \psi_{L^{\circ\circ}}$ and $(a, z^{\circ\circ}) \in \theta_{L^{\circ\circ}}$. Since $\theta_{D(L)}, \psi_{D(L)}$ are permutable congruences on $D(L)$, then there exists $e \in D(L)$ such that $(x \vee d, e) \in \psi_{D(L)}$ and $(e, z \vee d) \in \theta_{D(L)}$. It follows that

$$(x^{\circ\circ}, a) \in \psi, \quad (a, z^{\circ\circ}) \in \theta, \quad \text{and} \quad (x \vee d, e) \in \psi, \quad (e, z \vee d) \in \theta.$$

Since L is a decomposable MS-algebra, then there exist $d_1, d_2 \in D(L)$ such that $x = x^{\circ\circ} \wedge d_1$ and $z = z^{\circ\circ} \wedge d_2$. Hence $x \leq d_1$ and $z \leq d_2$. Since θ and ψ are compatible with the $\wedge$ operation, then we have

$$(x^{\circ\circ}, a) \in \psi \quad \text{and} \quad (x \vee d_1, e) \in \psi \quad \text{imply} \quad (x, a \wedge e) = (x^{\circ\circ} \wedge (x \vee d_1), a \wedge e) \in \psi,$$

and

$$(a, z^{\circ\circ}) \in \theta \quad \text{and} \quad (e, z \vee d_2) \in \theta \quad \text{imply} \quad (a \wedge e, z) = (a \wedge e, z^{\circ\circ} \wedge (z \vee d_2)) \in \theta.$$

Consequently, we deduce that $(x, a \wedge e) \in \psi$ and $(a \wedge e, z) \in \theta$. Therefore θ, ψ are permutable.

Let L be an MS-algebra. Define the relation Φ on L as follows:

$$(x, y) \in \Phi \Leftrightarrow x^{\circ\circ} = y^{\circ\circ}.$$

It is known that Φ is a congruence relation on L (see [18]). Then Φ satisfies the following property.

Corollary 5.1 *Let L be a decomposable* MS-*algebra. Then the congruence relation Φ permutes with any element of* $\mathrm{Con}(L)$*, as $\Phi_{L^{\circ\circ}} = \triangle_{L^{\circ\circ}}$ and $\Phi_{D(L)} = \nabla_{D(L)}$.*

6 Strong Extensions of Decomposable MS-Algebras

It is known that the class of distributive lattices satisfies the Congruence Extension Property (CEP for short) briefly. Luo [30] proved that the class **MS** of all MS-algebras satisfies the CEP. The notion of a strong extension of algebras was first introduced by Varlet [32]. EL-Assar and Abd El-Hakim [24] studied the strong extension for modular p-algebras. Also EL-Assar [22] introduced the strong extension for quasi-modular p-algebras. Now we recall the following two definitions.

Definition 6.1 (see [28]) *An algebra A satisfies the* CEP *if for every subalgebra B of A and every θ of B, θ extends to a congruence of A.*

Definition 6.2 (see [28]) *An algebra L is said to be a strong extension of the algebra M, if M is a subalgebra of L and every congruence of M has at most one extension to L.*

In the following theorem, we study strong extensions of decomposable MS-algebras using the congruence pairs technique.

Theorem 6.1 *Let L be a subalgebra of a decomposable* MS-*algebra L_1. Then L_1 is a strong extension of L if and only if the following conditions hold*:
(1) *$D(L_1)$ is a strong extension of $D(L)$,*
(2) *$L_1^{\circ\circ}$ is a strong extension of $L^{\circ\circ}$.*

Proof Let L_1 be a strong extension of L. Let $\theta_2 \in \mathrm{Con}(D(L))$. Then θ_2 has an extension to $D(L_1)$. Since CEP holds for the class of distributive lattices, we have to verify that θ_2 has a unique extension to $D(L_1)$. Let $\overline{\theta}_2, \acute{\theta}_2 \in \mathrm{Con}(D(L_1))$ such that $\overline{\theta}_2 \mid D(L) = \acute{\theta}_2 \mid D(L) = \theta_2$. By Corollary 3.6 (1), we have $(\triangle_{L_1^{\circ\circ}}, \overline{\theta}_2), (\triangle_{L_1^{\circ\circ}}, \acute{\theta}_2) \in A(L_1)$ and $(\triangle_{L^{\circ\circ}}, \theta_2) \in A(L)$. By Theorem 3.4, there exist $\overline{\theta}$ and $\acute{\theta} \in \mathrm{Con}(L_1)$ and $\theta \in \mathrm{Con}(L)$ determined by the congruence pairs $(\triangle_{L_1^{\circ\circ}}, \overline{\theta}_2), (\triangle_{L_1^{\circ\circ}}, \acute{\theta}_2)$ and $(\triangle_{L^{\circ\circ}}, \theta_2)$, respectively. Now, we deduce that $\overline{\theta} \mid L = \acute{\theta} \mid L = \theta$, but θ has at most one extension to L_1. Thus $\overline{\theta} = \acute{\theta}$, and this result leads to $\overline{\theta}_2 = \acute{\theta}_2$, proving (1). Now we prove that $L_1^{\circ\circ}$ is a strong extension of $L^{\circ\circ}$. Let $\theta_1 \in \mathrm{Con}(L^{\circ\circ})$. Then θ_1 has an extension to $L_1^{\circ\circ}$, because the class of de Morgan algebras satisfies the CEP. We will show that this extension is unique. Let $\overline{\theta}_1, \acute{\theta}_1 \in \mathrm{Con}(L_1^{\circ\circ})$ with $\overline{\theta}_1 \mid L^{\circ\circ} = \acute{\theta}_1 \mid L^{\circ\circ} = \theta_1$. Then by Corollary 3.6 (2), it is clear that $(\overline{\theta}_1, \nabla_{D(L_1)})$ and $(\acute{\theta}_1, \nabla_{D(L_1)})$ are congruence pairs of L_1 and $(\theta_1, \nabla_{D(L)})$ is a congruence pair of L. Now, by Theorem 3.4, there exist $\overline{\theta}$ and $\acute{\theta}$ of $\mathrm{Con}(L_1)$ corresponding to $(\overline{\theta}_1, \nabla_{D(L_1)})$ and $(\acute{\theta}_1, \nabla_{D(L_1)})$ respectively and θ of $\mathrm{Con}(L)$ corresponding to $(\theta_1, \nabla_{D(L_1)})$. Then $\overline{\theta} \mid L = \acute{\theta} \mid L = \theta$, which gives $\overline{\theta}_1 = \acute{\theta}_1$. Therefore $L_1^{\circ\circ}$ is a strong extension of $L^{\circ\circ}$. Conversely, suppose that the conditions (1) and (2) hold and let $\theta \in \mathrm{Con}(L)$. Then θ has an extension to L_1, because the class of MS-algebras satisfies the CEP. We will show that this extension is unique. Assume that $\overline{\theta}$ and $\acute{\theta}$ of $\mathrm{Con}(L_1)$ such that $\overline{\theta} \mid L = \acute{\theta} \mid L = \theta$. By Theorem 3.4, these can be represented by congruence pairs as $\overline{\theta} = (\overline{\theta}_1, \overline{\theta}_2), \acute{\theta} = (\acute{\theta}_1, \acute{\theta}_2)$ and $\theta = (\theta_1, \theta_2)$, where $\overline{\theta}_1 \mid L^{\circ\circ} = \acute{\theta}_1 \mid L^{\circ\circ} = \theta_1$ and $\overline{\theta}_2 \mid D(L) = \acute{\theta}_2 \mid D(L) = \theta_2$. By the conditions (1) and (2), we get $\overline{\theta}_1 = \acute{\theta}_1$ and $\overline{\theta}_2 = \acute{\theta}_2$. Therefore $\overline{\theta} = \acute{\theta}$.

Corollary 6.1 *Let L_1 and L be decomposable* MS-*algebras. If L_1 is a strong extension of L, then* $\mathrm{Con}(L_1) \cong \mathrm{Con}(L)$.

Proof Since the class of MS-algebras satisfies the CEP, then every congruence of L has an extension. By hypotheses this extension is unique. Then $\mathrm{Con}(L_1) \cong \mathrm{Con}(L)$.

7 Conclusion

In this paper, we introduced the notion of congruence pairs of decomposable MS-algebras. It is proved that every congruence relation θ on a decomposable MS-algebra L can be represented by a unique congruence pair (θ_1, θ_2), where θ_1 is a congruence relation on the de Morgan algebra

$L^{\circ\circ}$ and θ_2 is a lattice congruence relation on the lattice $D(L)$. Also, it is observed that $\mathrm{Con}(L)$, the lattice of all congruences of a decomposable MS-algebra L, is isomorphic to $A(L)$, the lattice of all congruence pairs of L. It is observed that there is a one to one correspondence between the set $B(L)$ of central elements of a decomposable MS-algebra L and the set of congruence pairs of the form $(\theta[a\downarrow], \theta[a\varphi(L)])$, where $a \in B(L)$. Permutability of congruences and strong extensions of decomposable MS-algebras are considered in terms of congruence pairs. In a future work, we will describe the congruence lattices of decomposable MS-algebras by means of congruence pairs.

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