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Carleson Measures on the Weighted Bergman Spaces with Békollé Weights*

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Abstract In this paper, the authors characterize Carleson measures for the weighted Bergman spaces with Békollé weights on the unit ball. They apply the Carleson embedding theorem to study the properties of Toeplitz-type operators and composition operators acting on such spaces.

Keywords Békollé weight, Bergman space, Carleson measure, Toeplitz operator, Composition operator
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1 Introduction

For a positive integer n, we let $\mathbb{B}_n$ denote the open unit ball in n-dimensional complex Euclidean space $\mathbb{C}^n$. Let $\mathrm{d}V$ be the standard Lebesgue measure on $\mathbb{B}_n$. For $b > -1$, the constant c_b is chosen so that $\int_{\mathbb{B}_n} c_b(1-|z|^2)^b \mathrm{d}V(z) = 1$. We define $\mathrm{d}v_b(z) = c_b(1-|z|^2)^b \mathrm{d}V(z)$. The Bergman space $A_b^p(\mathbb{B}_n)$ is defined to be the space of holomorphic functions on $\mathbb{B}_n$ with finite L_b^p norm. That is $f \in A_b^p$ if it is holomorphic and

$$\|f\|_{A_b^p}^p := \int_{\mathbb{B}_n} |f(z)|^p \mathrm{d}v_b(z) < \infty.$$

If $b > -1$ and $u \in L^1(\mathrm{d}v_b)$ is a weight, let $L_b^p(u)$ denote the space of measurable functions on $\mathbb{B}_n$ that are pth power integrable with respect to $u(z)\mathrm{d}v_b(z)$. That is

$$\|f\|_{L_b^p(u)} := \Big(\int_{\mathbb{B}_n} |f(z)|^p u(z) \mathrm{d}v_b(z)\Big)^{\frac{1}{p}} < \infty.$$

Recall that for $r > 0$ and $z \in \mathbb{B}_n$, the set

$$B_\beta(z,r) := \{w \in \mathbb{B}_n : \beta(z,w) < r\}$$

is a Bergman metric ball centered at z with radius r.

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The Carleson tent over a non-zero $z \in \mathbb{B}_n$ is defined to be the set:

$$T_z := \Big\{ w \in \mathbb{B}_n : \Big| 1 - \frac{z\overline{w}}{|z|} \Big| < 1 - |z| \Big\},$$

where $z\overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j$. The Carleson tent over 0 is $\mathbb{B}_n$.

We define the $D_{p,a,b}$ characteristic of two weights u, σ by

$$[u,\sigma]_{D_{p,a,b}} := \sup_{z\in\mathbb{B}_n} \Big(\frac{\int_{T_z} \sigma \mathrm{d}v_b}{\int_{T_z} \mathrm{d}v_b} \Big)^{p-1} \frac{\int_{T_z} u \mathrm{d}v_{pa+b}}{\int_{T_z} \mathrm{d}v_{pa+b}}.$$

Throughout the paper, we will use the notation defined in Section 2 to make this more compactly as:

$$[u,\sigma]_{D_{p,a,b}} = \sup_{z\in\mathbb{B}_n} (\langle\sigma\rangle_{T_z}^{\mathrm{d}v_b})^{p-1} \langle u\rangle_{T_z}^{\mathrm{d}v_{pa+b}}.$$

Let p' be the conjugate number of p. We denote by $u \in B_{p,b}$ if $[u]_{B_{p,b}} := [u,\sigma]_{D_{p,0,b}} < \infty$, where $\sigma = u^{-\frac{p'}{p}}$ is the dual weight of u. If $p > 1$, according to the Hölder inequality one can obtain that $[u]_{B_{p,b}} \geq 1$. To be more precise,

$$\begin{aligned} \int_{T_z} \mathrm{d}v_b &= \int_{T_z} u^{\frac{1}{p}} \cdot u^{-\frac{1}{p}} \mathrm{d}v_b \\ &\leq \Big(\int_{T_z} u \mathrm{d}v_b \Big)^{\frac{1}{p}} \cdot \Big(\int_{T_z} u^{-\frac{p'}{p}} \mathrm{d}v_b \Big)^{-\frac{1}{p'}}. \end{aligned} \tag{1.1}$$

Békollé and Bonami introduced these weights in [1–2], and characterized the boundedness of the Bergman projection. The sharp dependence of the operator norm on the $B_{p,b}$ characteristic was given by Pott and Reguera [10] and Rahm, Tchoundja and Wick [11]. This was proven for the upper half plane of $\mathbb{C}$ in [10] and for the ball in [11].

Constantin proved Carleson-type embedding theorems for weighted Bergman spaces with Békollé weights on the unit disk, and characterized the boundedness, compactness and Schatten class of Toeplitz type operators, integral operators and composition operators in [3]. The goal of this paper is to generalize these results to the setting of the unit ball. The key tool is the "test function" $(1 - z\overline{w})^{-s}$ in the weighted Bergman spaces with Békollé weights.

The paper is organized as follows. In Section 2, we briefly give the preliminaries and background information. We recall a covering lemma and prove the key lemma on the norm estimate of the test function $(1 - z\overline{w})^s$. In Section 3, we completely characterize the Carleson embedding theorem from $A_b^p(u)$ to $L^q(\mathrm{d}\mu)$. In Section 4, we use the Carleson measure to study the Toeplitz type operators. In Section 5, the boundedness and compactness of composition operators are characterized.

Throughout the paper, for real positive quantities Q_1 and Q_2, we write $Q_1 \lesssim Q_2$ (or $Q_2 \gtrsim Q_1$) if there is a positive constant C (independent of the "key" variables) such that $Q_1 \leq C \cdot Q_2$. And we write $Q_1 \simeq Q_2$ if $Q_1 \lesssim Q_2$ and $Q_1 \gtrsim Q_2$.

2 Preliminaries

The following notations will be used throughout the paper. For a weight u and $E \subset \mathbb{B}_n$, we set $u_b(E) = \int_E u \mathrm{d}v_b$ and $\mathrm{vol}_b(E) = \int_E \mathrm{d}v_b$ and define $\langle f \rangle_E^{\mathrm{d}\mu} := \mu(E)^{-1} \int_E f(z) \mathrm{d}\mu(z)$ for integrable f and measure μ.

Let Φ_z be the involution of $\mathbb{B}_n$. Using Φ_z, we define the so-called Bergman metric, β on $\mathbb{B}_n$, by:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\Phi_z(w)|}{1 - |\Phi_z(w)|}.$$

Let $B_\beta(z, r)$ be the ball in the Bergman metric of radius r centered at z. It is well known that for $w \in B_\beta(z, r)$ there holds:

$$\mathrm{vol}_b(B_\beta(z, r)) \simeq |1 - w\overline{z}|^{n+1+b} \simeq (1 - |z|^2)^{n+1+b} \simeq (1 - |w|^2)^{n+1+b},$$

and the characteristic functions

$$\mathbb{1}_{B_\beta(z,r)}(w) = \mathbb{1}_{B_\beta(w,r)}(z).$$

We need the following covering lemma in the proofs of our main results.

Lemma 2.1 (see [13, Theorem 2.23]) *There exists a positive N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ with the following properties*:

(1) $\mathbb{B}_n = \bigcup_k B_\beta(a_k, r)$;

(2) *the sets $B_\beta\left(a_k, \frac{r}{4}\right)$ are mutually disjoint*;

(3) *each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $B_\beta(a_k, 2r)$.*

We will also use the following class of weights which is denoted by $C_{p,b}$. A positive locally integrable weight u belongs to $C_{p,b}$, or say u satisfies $C_{p,b}$ condition if

$$[u]_{C_{p,b}} := \sup_{z \in \mathbb{B}_n} \langle u \rangle_{B_\beta(z,r)}^{\mathrm{d}v_b} (\langle u^{-\frac{p'}{p}} \rangle_{B_\beta(z,r)}^{\mathrm{d}v_b})^{p-1} \lesssim 1$$

for some $0 < r < 1$. Condition $C_{p,b}$ seems to depend on a choice of r, but it is known that the same class of weights is obtained for any $r \in (0, 1)$ and $B_{p,b} \subset C_{p,b}$ for every $b > -1$. To see this, we note that for a given r, there is a $a' \in \mathbb{B}_n$ such that $B_\beta(a, r) \subset T_{a'}$ with comparable volumes. It follows that

$$[u]_{B_{p,b}} \geq C[u]_{C_{p,b}},$$

where the constant $C > 0$ may depend on r. See the details in [6]. Interested readers can also refer in [11] for further discussions on the $D_{p,a,b}$ weights.

Lemma 2.2 *Suppose that $u \in C_{p,b}$ for some $p > 1$, and let $t, s \in (0, 1)$, and $z, w \in \mathbb{B}_n$ with $\beta(z, w) < r$ for some $r > 0$. Then we have*

$$u_b(B_\beta(z, t)) \simeq u_b(B_\beta(w, s)),$$

where the constant is independent of z and w.

Proof Notice that if $B_\beta(z, t) \subset B_\beta(w, s)$, then $u \in C_{p,b}$ and $\beta(z, w) < r$ imply that

$$u_b(B_\beta(z, t))^{\frac{1}{p}} \leq u_b(B_\beta(w, s))^{\frac{1}{p}} \lesssim \mathrm{vol}_b(B_\beta(w, s)) \sigma_b(B_\beta(w, s))^{-\frac{1}{p'}}$$

$$\leq \mathrm{vol}_b(B_\beta(w,s))\sigma_b(B_\beta(z,t))^{-\frac{1}{p'}} \lesssim \frac{\mathrm{vol}_b(B_\beta(w,s))}{\mathrm{vol}_b(B_\beta(z,t))} u_b(B_\beta(z,t))^{\frac{1}{p}}$$
$$\simeq u_b(B_\beta(z,t))^{\frac{1}{p}}.$$

For general case, we have $B_\beta(z,t), B_\beta(w,s) \subset B_\beta(w,t+s+r)$, and hence we have

$$u_b(B_\beta(z,t)) \simeq u_b(B_\beta(w,t+s+r)) \simeq u_b(B_\beta(w,s)).$$

The proof is completed.

The point evaluations on $A_b^p(u)$ are bounded linear functionals for $p > 0$. To be more precisely, we have the following estimate.

Lemma 2.3 (see [6, Lemma 3.1]) *If $p_0 > 1, 0 < r < 1$ and a weight $u \in C_{p_0,b}$, $\sigma = u^{-\frac{p_0'}{p_0}}$, we have the following estimate*

$$|f(z)|^p \lesssim u_b(B_\beta(z,r))^{-1} \int_{B_\beta(z,r)} |f(w)|^p u(w) \mathrm{d}v_b(w) \lesssim \frac{\|f\|^p_{L_b^p(u)}}{u_b(B_\beta(z,r))},$$

where the constant involved is independent of $z \in \mathbb{B}_n$.

If $s > 0$, we denote $G_w^s(z) = (1 - z\overline{w})^{-s}$ for $z, w \in \mathbb{B}_n$. We will make heavy use of this function in our discussion. It is necessary to estimate its $L_b^p(u)$ norm.

Lemma 2.4 *Let $p > 0$, $p_0 > 1$, $b > -1$ and the weight $u \in B_{p_0,b}$. We have*

$$\frac{u_b(T_w)^{\frac{1}{p}}}{(1-|w|)^s} \lesssim \|G_w^s\|_{L_b^p(u)} \lesssim \frac{u_b(T_w)^{\frac{1}{p}}}{(1-|w|)^{\max\left\{\frac{(n+1+b)p_0}{p}, s\right\}}}, \tag{2.1}$$

where the constant involved is independent of $w \in \mathbb{B}_n$.

Proof If $z \in T_w$, then

$$1 - |w| \geq \left|1 - \frac{z\overline{w}}{|w|}\right| \geq |1 - z\overline{w}| - \left|z\overline{w} - \frac{z\overline{w}}{|w|}\right|$$
$$\geq |1 - z\overline{w}| - (1 - |w|).$$

That is $1 - |w| \geq \frac{|1-z\overline{w}|}{2}$, and it is obvious that

$$\frac{u_b(T_w)}{(1-|w|)^{ps}} \lesssim \int_{T_w} \frac{1}{|1-z\overline{w}|^{ps}} u(z) \mathrm{d}v_b(z) \leq \|G_w^s\|^p_{L_b^p(u)}.$$

On the other hand, we firstly consider the case when $s > (n+1+b)\frac{p_0}{p}$. Denote by

$$E_0 = T_w, \quad E_k = \left\{z \in \mathbb{B}_n : \left|1 - \frac{z\overline{w}}{|w|}\right| < 2^k(1-|w|)\right\}, \quad k = 1, 2, \cdots,$$

and $\widetilde{E}_0 = E_0, \widetilde{E}_k = E_k \setminus E_{k-1}$, $k = 1, 2, \cdots$. It is easy to see that

$$\mathrm{vol}_b(E_k) \simeq (2^k(1-|w|))^{n+1+b}.$$

Then we can obtain the following estimate under this decomposition of $\mathbb{B}_n$.

- If $z \in \widetilde{E}_0$, $|1 - z\overline{w}| \geq 1 - |w|$, and
- if $z \in \widetilde{E}_k$ for $k \geq 1$,

$$|1 - z\overline{w}| \geq \left|1 - \frac{z\overline{w}}{|w|}\right| - (1 - |w|) \gtrsim 2^k(1 - |w|).$$

Denote by

$$w_k = \begin{cases} (1 - 2^k(1 - |w|))\frac{w}{|w|}, & \text{if } 2^k(1 - |w|) < 1; \\ 0, & \text{if } 2^k(1 - |w|) \geq 1. \end{cases}$$

One can easily find that

$$E_k = T_{w_k} = \left\{z \in \mathbb{B}_n : \left|1 - \frac{z\overline{w}_k}{|w_k|}\right| < (1 - |w_k|)\right\} \quad \text{if } 2^k(1 - |w|) < 1,$$

and $E_k \subset T_{w_k} = T_0 = \mathbb{B}_n$ if $2^k(1 - |w|) \geq 1$. Since $u \in B_{p_0,b}$, for every positive integer k, we have

$$\begin{aligned} \int_{E_k} u(z)\mathrm{d}v(z) &\leq \int_{T_{w_k}} u(z)\mathrm{d}v_b(z) \lesssim \frac{\mathrm{vol}_b(T_{w_k})^{p_0}}{\left(u^{-\frac{p_0'}{p_0}}\right)_b(T_{w_k})^{p_0-1}} \leq \frac{\mathrm{vol}_b(T_{w_k})^{p_0}}{\left(u^{-\frac{p_0'}{p_0}}\right)_b(T_w)^{p_0-1}} \\ &\leq \left(\frac{\mathrm{vol}_b(T_{w_k})}{\mathrm{vol}_b(T_w)}\right)^{p_0} u_b(T_w) \lesssim \left(\frac{(2^k(1 - |w|))^{n+1+b}}{(1 - |w|)^{n+1+b}}\right)^{p_0} u_b(T_w) \\ &= 2^{kp_0(n+1+b)} u_b(T_w). \end{aligned}$$

Now we can estimate the norm $\|G_w^s\|_{L_b^p(u)}$ as follows:

$$\begin{aligned} \|G_w^s\|_{L_b^p(u)}^p &= \int_{\mathbb{B}_n} \frac{1}{|1 - z\overline{w}|^{ps}} u(z)\mathrm{d}v_b(z) \\ &= \sum_{k=0}^{\infty} \int_{\widetilde{E}_k} \frac{1}{|1 - z\overline{w}|^{ps}} u(z)\mathrm{d}v_b(z) \\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{2^{kps}(1 - |w|)^{ps}} \int_{E_k} u(z)\mathrm{d}v_b(z) \\ &\lesssim \frac{u_b(T_w)}{(1 - |w|)^{ps}} \sum_{k=0}^{\infty} \frac{1}{2^{k(ps-(n+1+b)p_0)}} \lesssim \frac{u_b(T_w)}{(1 - |w|)^{ps}}, \end{aligned}$$

where the final inequality follows by $s > (n + 1 + b)\frac{p_0}{p}$. Now we have proved that (2.1) holds for $s > (n + 1 + b)\frac{p_0}{p}$. When $s = (n + 1 + b)\frac{p_0}{p}$, we have

$$\left\|G_w^{(n+1+b)\frac{p_0}{p}}\right\|_{A_b^p(u)} \leq 2^{\epsilon}\left\|G_w^{(n+1+b)\frac{p_0}{p}+\epsilon}\right\|_{A_b^p(u)} \lesssim \frac{u_b(T_w)^{\frac{1}{p}}}{(1 - |w|)^{(n+1+b)\frac{p_0}{p}+\epsilon}},$$

whenever $\epsilon > 0$. By letting $\epsilon \to 0$ on the right hand side, we find (2.1) holds true for $s = (n + 1 + b)\frac{p_0}{p}$. Similarly, we can obtain

$$\|G_w^s\|_{A_b^p(u)} \leq 2^{(n+1+b)\frac{p_0}{p}-s}\left\|G_w^{(n+1+b)\frac{p_0}{p}}\right\|_{A_b^p(u)} \lesssim \frac{u_b(T_w)^{\frac{1}{p}}}{(1 - |w|)^{(n+1+b)\frac{p_0}{p}}},$$

when $0 < s < (n + 1 + b)\frac{p_0}{p}$. That completes the proof.

3 Embedding Theorems

In this section, we will study the boundedness and the compactness of the embedding $I : A^p_b(u) \to L^q(\mathrm{d}\mu)$. We firstly consider the case $0 < p \le q < \infty$.

Theorem 3.1 *Suppose that* $q \ge p > 0$, $p_0 > 1$, $u \in B_{p_0,b}$ *is a weight and* μ *is a positive* Borel *measure on* $\mathbb{B}_n$. *Then the following conditions are equivalent.*

(a) *The embedding* $I : A^p_b(u) \to L^q(\mathrm{d}\mu)$ *is bounded, that is*

$$\Big(\int_{\mathbb{B}_n} |f(z)|^q \mathrm{d}\mu(z)\Big)^{\frac{1}{q}} \lesssim \Big(\int_{\mathbb{B}_n} |f(z)|^p u(z)\mathrm{d}v_b(z)\Big)^{\frac{1}{p}}$$

for all holomorphic f *in* $\mathbb{B}_n$;

(b) $\mu(T_a) \lesssim u_b(T_a)^{\frac{q}{p}}$ *for all* $a \in \mathbb{B}_n$;

(c) *there is an* $r > 0$ *such that* $\mu(B_\beta(a,r)) \lesssim u_b(B_\beta(a,r))^{\frac{q}{p}}$ *for all* $a \in \mathbb{B}_n$;

(d) *there is an* $r > 0$ *such that* $\mu(B_\beta(a_k,r)) \lesssim u_b(B_\beta(a_k,r))^{\frac{q}{p}}$ *for the sequence* $\{a_k\}$ *described in Lemma* 2.1;

(e) *whenever* $s \ge (n+1+b)\frac{p_0}{p}$,

$$\sup_{w\in\mathbb{B}_n} \int_{\mathbb{B}_n} \Big|\frac{1-|w|^2}{1-z\overline{w}}\Big|^{qs} u_b(B_\beta(w,r))^{-\frac{q}{p}}\mathrm{d}\mu(z) \lesssim 1.$$

Proof Firstly we prove (a) $\Rightarrow$ (b). By choosing an $s > n+1+b$ we get

$$\frac{\mu(T_a)}{(1-|a|)^{qs}} \lesssim \int_{T_a} \frac{1}{|1-z\overline{a}|^{qs}}\mathrm{d}\mu(z) \lesssim \|G^s_a\|^q_{L^q_b(u)} \lesssim \frac{u_b(T_a)^{\frac{q}{p}}}{(1-|a|)^{qs}},$$

where we use the condition (a) in the second inequality and Lemma 2.4 in the third inequality.

To prove (b) $\Rightarrow$ (c), we let r be sufficiently small and fixed. It will be done to prove

$$\mu(B_\beta(a,r)) \lesssim u_b(B_\beta(a,r))^{\frac{q}{p}}$$

for all $|a| \ge \tanh(2r)$. According to [13, Lemma 5.23], there is a constant $\sigma \in (0,1)$ which depends only on r, such that $B_\beta(a,r) \subset T_{a'}$ where $a' = (1 - \frac{1}{\sigma}(1-|a|))\frac{a}{|a|}$. By [11, Lemma 2.3], we find

$$\mathrm{vol}_b(T_{a'}) \simeq (1-|a'|)^{n+1+b} \simeq \sigma^{-n-1-b}(1-|a|^2)^{n+1+b} \simeq \mathrm{vol}_b(B_\beta(a,r)).$$

Then we use the condition of $B_{p_0,b}$ to get $u_b(B_\beta(a,r)) \simeq u_b(T_{a'})$ as follows, which is analogues to the proof of Lemma 2.2. That is

$$\begin{aligned} u_b(B_\beta(a,r))^{\frac{1}{p_0}} &\le u_b(T_{a'})^{\frac{1}{p_0}} \lesssim \mathrm{vol}_b(T_{a'})\sigma_b(T_{a'})^{-\frac{1}{p_0'}} \le \mathrm{vol}_b(T_{a'})\sigma_b(B_\beta(a,r))^{-\frac{1}{p_0'}} \\ &\lesssim \frac{\mathrm{vol}_b(T_{a'})}{\mathrm{vol}_b(B_\beta(a,r))} u_b(B_\beta(a,r))^{\frac{1}{p_0}} \simeq u_b(B_\beta(a,r))^{\frac{1}{p_0}}. \end{aligned}$$

Hence we have

$$\mu(B_\beta(a,r)) \le \mu(T_{a'}) \lesssim u_b(T_{a'}) \simeq u_b(B_\beta(a,r)).$$

To prove (a) $\Rightarrow$ (e), we denote by

$$g^s_w(z) = \Big(\frac{1-|w|}{1-z\overline{w}}\Big)^s u_b(B_\beta(w,r))^{-\frac{1}{p}}.$$

By following Lemma 2.4, we have

$$\|g_w^s\|_{L^q(\mathrm{d}\mu)}^q = \int_{\mathbb{B}^n} \Big|\frac{1-|w|^2}{1-z\overline{w}}\Big|^{qs} u_b(B_\beta(w,r))^{-\frac{q}{p}}\mathrm{d}\mu(z) \lesssim \|g_w^s\|_{A_b^p(u)}^q \simeq 1.$$

We can conclude (e) $\Rightarrow$ (c) by noting that $(1-|w|^2)|1-z\overline{w}|^{-1} \simeq 1$ whenever $\beta(z,w) < r$, and it implies that

$$\begin{aligned}\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))^{\frac{q}{p}}} &\simeq \int_{B_\beta(w,r)} \Big(\frac{1-|w|^2}{|1-z\overline{w}|}\Big)^{qs} u_b(B_\beta(w,r))^{-\frac{q}{p}}\mathrm{d}\mu(z)\\ &\le \int_{\mathbb{B}_n} \Big(\frac{1-|w|^2}{|1-z\overline{w}|}\Big)^{qs} u_b(B_\beta(w,r))^{-\frac{q}{p}}\mathrm{d}\mu(z) \lesssim 1.\end{aligned}$$

The proof of (c) $\Rightarrow$ (d) is obvious.

It remains to prove (d) $\Rightarrow$ (a). If f is holomorphic in $\mathbb{B}^n$, then by Lemma 2.3 we have

$$\begin{aligned}\int_{\mathbb{B}^n} |f(z)|^q\mathrm{d}\mu(z) &\lesssim \sum_k \int_{B_\beta(a_k,r)} \frac{1}{u_b(B_\beta(z,r))}\Big(\int_{B_\beta(z,r)} |f(w)|^q u(w)\mathrm{d}v_b(w)\Big)\mathrm{d}\mu(z)\\ &\lesssim \sum_k \int_{B_\beta(a_k,r)} \frac{1}{u_b(B_\beta(a_k,r))}\Big(\int_{B_\beta(a_k,2r)} |f(w)|^q u(w)\mathrm{d}v_b(w)\Big)\mathrm{d}\mu(z)\\ &= \sum_k \frac{\mu(B_\beta(a_k,r))}{u_b(B_\beta(a_k,r))} \int_{B_\beta(a_k,2r)} |f(w)|^q u(w)\mathrm{d}v_b(w)\\ &\lesssim \sum_k \int_{B_\beta(a_k,2r)} u_b(B_\beta(a_k,2r))^{\frac{q-p}{p}} |f(w)|^{q-p}|f(w)|^p u(w)\mathrm{d}v_b(w)\\ &\lesssim \|f\|_{A_b^p(u)}^{q-p} \sum_k \int_{B_\beta(a_k,2r)} |f(w)|^p u(w)\mathrm{d}v_b(w)\\ &\lesssim \|f\|_{A_b^p(u)}^q,\end{aligned}$$

where the last inequality is deduced by Lemma 2.1. The proof is completed.

Let us turn to the case $0 < q < p < \infty$. We will apply Luecking's approach in [8] which is based on Khinchine's inequality. Define the Rademacher functions R_n by

$$R_0(t) = \begin{cases} 1, & 0 \le t-[t] < \frac{1}{2},\\ -1, & \frac{1}{2} \le t-[t] < 1;\end{cases}$$
$$R_n(t) = R_0(2^n t), \quad n \ge 1.$$

Then Khinchine's inequality is the following.

Khinchine's Inequality For $0 < p < \infty$, there are constants

$$0 < c(p) \le C(p) < \infty$$

depending only on p such that, for all $m \in \mathbb{N}$ and $\{c_j\}_{j=1}^m \subset \mathbb{C}$, we have

$$c(p)\Big(\sum_{j=1}^m |c_j|^2\Big)^{\frac{p}{2}} \le \int_0^1 \Big|\sum_{j=1}^m c_j R_j(t)\Big|^p \mathrm{d}t \le C(p)\Big(\sum_{j=1}^m |c_j|^2\Big)^{\frac{p}{2}}.$$

Theorem 3.2 *Let $u \in B_{p_0,b}$ for some $p_0 > 1$, and μ be a positive finite Borel measure on $\mathbb{B}_n$. Assume that $p > q > 0$, then the embedding from $A_b^p(u)$ into $L^q(\mathrm{d}\mu)$ is bounded, to be more precisely,*

$$\int_{\mathbb{B}_n} |f(z)|^q \mathrm{d}\mu(z) \lesssim \|f\|^q_{A_b^p(u)}$$

if and only if the function

$$\mathbb{B}_n \ni z \mapsto \frac{\mu(B_\beta(z,r))}{u_b(B_\beta(z,r))}$$

belongs to $L_b^{\frac{p}{p-q}}(u)$ for some $r \in (0,1)$.

Proof Since $u \in B_{p_0,b}$ for some $p_0 > 1$, by Lemma 2.3 we have

$$|f(z)|^q \lesssim \frac{1}{u_b(B_\beta(z,r))} \int_{B_\beta(z,r)} |f(w)|^q u(w) \mathrm{d}v_b(w).$$

The sufficiency will be clarified by the following computation:

$$\begin{aligned}
&\int_{\mathbb{B}_n} |f(z)|^q \mathrm{d}\mu(z) \\
\lesssim& \int_{\mathbb{B}_n} \frac{\int_{B_\beta(z,r)} |f(w)|^q u(w) \mathrm{d}v_b(w)}{u_b(B_\beta(z,r))} \mathrm{d}\mu(z) \\
\lesssim& \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w)|^q}{u_b(B_\beta(w,r))} \mathbb{1}_{B_\beta(z,r)}(w) u(w) \mathrm{d}v_b(w) \mathrm{d}\mu(z) \\
=& \int_{\mathbb{B}_n} \mathbb{1}_{B_\beta(w,r)}(z) \int_{\mathbb{B}_n} \frac{|f(w)|^q}{u_b(B_\beta(w,r))} u(w) \mathrm{d}v_b(w) \mathrm{d}\mu(z) \\
=& \int_{\mathbb{B}_n} \frac{|f(w)|^q}{u_b(B_\beta(w,r))} u(w) \int_{\mathbb{B}_n} \mathbb{1}_{B_\beta(w,r)}(z) \mathrm{d}\mu(z) \mathrm{d}v_b(w) \\
=& \int_{\mathbb{B}_n} \frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))} |f(w)|^q u(w) \mathrm{d}v_b(w) \\
\leq& \Big\| \frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))} \Big\|_{L_b^{\frac{p}{p-q}}(u)} \|f\|^q_{A_b^p(u)}.
\end{aligned}$$

It remains to prove the necessity. For $\{c_j\} \in \ell^p$, we define

$$f(z) = \sum_{j=1}^{\infty} c_j \frac{(1-|a_j|)^{n+1+b}}{(1-z\overline{a}_j)^{n+1+b}} \cdot u_b(B(a_j, 2r_0))^{-\frac{1}{p}},$$

where $\{a_j\} \subset \mathbb{B}_n$ and $r_0 > 0$ satisfy the conditions in Lemma 2.1. It is followed by [6, Theorem 4.1] that

$$\|f\|^p_{A_b^p(u)} = \|\{c_j\}\|^p_{\ell^p} \simeq \sum_{j=1}^{\infty} |c_j|^p. \tag{3.1}$$

According to the embedding condition, we now get

$$\|f\|^q_{L^q(\mathrm{d}\mu)} = \int_{\mathbb{B}_n} \Big| \sum_{j=1}^{\infty} c_j \frac{(1-|a_j|)^{n+1+b}}{(1-z\overline{a}_j)^{n+1+b}} \cdot u_b(B(a_j, 2r_0))^{-\frac{1}{p}} \Big|^q \mathrm{d}\mu(z).$$

Applying Fubini's theorem and Khinchine's inequality, we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}_n}\Big(\sum_{j=1}^{\infty}|c_j|^2\frac{(1-|a_j|)^{2(n+1+b)}}{|1-z\overline{a}_j|^{2(n+1+b)}}\frac{1}{u_b(B_\beta(a_j,2r_0))^{\frac{2}{p}}}\Big)^{\frac{q}{2}}\mathrm{d}\mu(z)\\
&\lesssim\int_{\mathbb{B}_n}\int_0^1\Big|R_j(t)c_j\frac{(1-|a_j|)^{n+1+b}}{(1-z\overline{a}_j)^{n+1+b}}\cdot u_b(B(a_j,2r_0))^{-\frac{1}{p}}\Big|^q\mathrm{d}t\mathrm{d}\mu(z)\\
&=\int_0^1\int_{\mathbb{B}_n}\Big|R_j(t)c_j\frac{(1-|a_j|)^{n+1+b}}{(1-z\overline{a}_j)^{n+1+b}}\cdot u_b(B(a_j,2r_0))^{-\frac{1}{p}}\Big|^q\mathrm{d}\mu(z)\mathrm{d}t\\
&\lesssim\int_0^1\Big(\sum_{j=1}^{\infty}|c_jR_j(t)|^p\Big)^{\frac{q}{p}}\mathrm{d}t=\Big(\sum_{j=1}^{\infty}|c_j|^p\Big)^{\frac{q}{p}},
\end{aligned}$$

where the last inequality follows the condition (a) and (3.1).

It is easy to see that

$$\mathbb{1}_{B_\beta(a_j,2r_0)}(z)\lesssim\Big(\frac{1-|a_j|}{|1-z\overline{a}_j|}\Big)^{2(n+1+b)}$$

where the constant involved depends on r only. Then we can obtain

$$\begin{aligned}
\sum_{j=1}^{\infty}|c_j|^q\cdot\frac{\mu(B_\beta(a_j,2r_0))}{u_b(B_\beta(a_j,2r_0))^{\frac{q}{p}}}&=\int_{\mathbb{B}_n}\sum_{j=1}^{\infty}\frac{|c_j|^q\mathbb{1}_{B_\beta(a_j,2r_0)}(z)}{u_b(B_\beta(a_j,2r_0))^{\frac{q}{p}}}\mathrm{d}\mu(z)\\
&\lesssim\int_{\mathbb{B}_n}\Big(\sum_{j=1}^{\infty}\frac{|c_j|^2\mathbb{1}_{B_\beta(a_j,2r_0)}(z)}{u_b(B_\beta(a_j,2r_0))^{\frac{2}{p}}}\Big)^{\frac{q}{2}}\mathrm{d}\mu(z)\\
&\lesssim\int_{\mathbb{B}_n}\Big(\sum_{j=1}^{\infty}\frac{|c_j|^2}{u_b(B_\beta(a_j,2r_0))^{\frac{2}{p}}}\cdot\Big(\frac{1-|a_j|}{|1-z\overline{a}_j|}\Big)^{2(n+1+b)}\Big)^{\frac{q}{2}}\mathrm{d}\mu(z)\\
&\lesssim\Big(\sum_{j=1}^{\infty}|c_j|^p\Big)^{\frac{q}{p}}.
\end{aligned}$$

Since the sequence $\{c_j\}$ is chosen arbitrarily from ℓ^p, the sequence

$$\Big\{\frac{\mu(B_\beta(a_j,2r_0))}{u_b(B_\beta(a_j,2r_0))^{\frac{q}{p}}}\Big\}\in\ell^{\frac{p}{p-q}}=(\ell^{\frac{p}{q}})^*.$$

That is

$$\sum_{j=1}^{\infty}\Big(\frac{\mu(B_\beta(a_j,2r_0))}{u_b(B_\beta(a_j,2r_0))}\Big)^{\frac{p}{p-q}}\cdot u_b(B_\beta(a_j,2r_0))<\infty.$$

Now we consider the $L^{\frac{p}{p-q}}$ norm of the function

$$z\mapsto\frac{\mu(B_\beta(z,r))}{u_b(B_\beta(z,r))},$$

where $0<r<\frac{r_0}{r_0+1}$. It is easy to see that $B_\beta(z,r)\subset B_\beta(w,2r_0)$ for those $z\in B_\beta(w,r_0)$. Hence we obtain that

$$\begin{aligned}
&\int_{\mathbb{B}_n}\Big(\frac{\mu(B_\beta(z,r))}{u_b(B_\beta(z,r))}\Big)^{\frac{p}{p-q}}u(z)\mathrm{d}v_b(z)\\
&\lesssim\sum_{j=1}^{\infty}\int_{B_\beta(a_j,r_0)}\Big(\frac{\mu(B_\beta(z,r))}{u_b(B_\beta(z,r))}\Big)^{\frac{p}{p-q}}u(z)\mathrm{d}v_b(z)
\end{aligned}$$

$$\lesssim \sum_{j=1}^{\infty}\Big(\frac{\mu(B_\beta(a_j,2r_0))}{u_b(B_\beta(a_j,2r_0))}\Big)^{\frac{p}{p-q}}\int_{B_\beta(a_j,2r_0)}u(z)\mathrm{d}v_b(z)<\infty.$$

That completes the proof.

The measure μ characterized in Theorems 3.1–3.2 is called a $(A_b^p(u),q)$-Carleson measure. Using very similar methods to those above, one can also characterize the compactness of the embedding map from $A_b^p(u)$ to $L^q(\mathrm{d}\mu)$, where the measure μ is also called the $(A_b^p(u),q)$-vanishing Carleson measure. We just include the statements without the proofs.

Theorem 3.3 *Suppose that* $q\ge p>0, p_0>1$, $r>0$, $u\in B_{p_0,b}$ *is a weight and* μ *is a positive* Borel *measure on* $\mathbb{B}_n$*. Then the following conditions are equivalent.*

(a) *The embedding* $I: A_b^p(u)\to L^q(\mathrm{d}\mu)$ *is compact, that is*

$$\lim_{k\to\infty}\int_{\mathbb{B}_n}|f_k(z)|^q\mathrm{d}\mu(z)=0,$$

whenever $\{f_k\}$ *is bounded in* $A_b^p(u)$ *that converges to* 0 *uniformly on compact subsets of* $\mathbb{B}_n$;

(b)

$$\frac{\mu(T_a)}{u_b(T_a)^{\frac{q}{p}}}\to 0\quad \textit{as } |a|\to 1;$$

(c)

$$\frac{\mu(B_\beta(a,r))}{u_b(B_\beta(a,r))^{\frac{q}{p}}}\to 0\quad \textit{as } |a|\to 1;$$

(d)

$$\frac{\mu(B_\beta(a_k,r))}{u_b(B_\beta(a_k,r))^{\frac{q}{p}}}\to 0\quad \textit{as } k\to\infty,$$

where $\{a_k\}$ *is the sequence described in Lemma* 2.1.

Theorem 3.4 *Suppose that* $p>q>0$, $r>0$, $u\in B_{p_0,b}$ *is a weight and* μ *is a positive* Borel *measure on* $\mathbb{B}_n$*. Then the embedding* $I: A_b^p(u)\to L^q(\mathrm{d}\mu)$ *is compact if and only if* I *is bounded.*

4 Toeplitz-Type Operators

In this section, we will characterize the boundedness, compactness and Schatten class of Toeplitz type operators on $A_b^2(u)$ for the Békollé weight u.

If $f,g\in A_b^2(u)$, the inner product is given by

$$\langle f,g\rangle_{A_b^2(u)}=\int_{\mathbb{B}_n}f(w)\overline{g(w)}u(w)\mathrm{d}v_b(w).$$

According to Lemma 2.3, the reproducing kernel of $A_b^2(u)$ will be denoted by $K(z,w)$. Given a positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator T_μ associated with μ on $A_b^2(u)$ is the linear transformation defined by

$$T_\mu f(z):=\int_{\mathbb{B}_n}f(w)K(z,w)\mathrm{d}\mu(w),\quad z\in\mathbb{B}_n.$$

For every $f, g \in A_b^2(u)$, we have

$$\begin{aligned}\langle T_\mu f, g\rangle_{A_b^2(u)} &= \int_{\mathbb{B}_n} T_\mu f(z)\overline{g(z)}u(z)\mathrm{d}v_b(z)\\ &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} f(w)K(z,w)\mathrm{d}\mu(w)\overline{g(z)}u(z)\mathrm{d}v_b(z)\\ &= \int_{\mathbb{B}_n} f(w)\overline{\int_{\mathbb{B}_n} g(z)K(w,z)u(z)\mathrm{d}v_b(z)}\mathrm{d}\mu(w)\\ &= \int_{\mathbb{B}_n} f(w)\overline{g(w)}\mathrm{d}\mu(w).\end{aligned}$$

By the straightforward computation above, one can get

$$\langle T_\mu f, g\rangle_{A_b^2(u)} = \langle f, g\rangle_{L^2(\mathrm{d}\mu)}. \tag{4.1}$$

According to that observation and applying Theorems 3.1 and 3.3, one can get the following characterization of Toeplitz operators.

Theorem 4.1 *If* $p_0 > 1$ *and* $u \in B_{p_0,b}$. *Let* μ *be a positive* Borel *measure on* $\mathbb{B}_n$. *Then the following are equivalent*:

(a) *The* Toeplitz *operator* T_μ *is bounded on* $A_b^2(u)$;

(b) $\mu(T_z) \lesssim u_b(T_z)$ *for every* $z \in \mathbb{B}_n$;

(c) $\mu(B_\beta(z,r)) \lesssim u_b(B_\beta(z,r))$ *for every* $z \in \mathbb{B}_n$ *and* $r > 0$;

(d) $\mu(B_\beta(a_k,r) \lesssim u_b(B_\beta(a_k,r))$ *for the sequence* $\{a_k\}$ *described in Lemma* 2.1.

Furthermore, the following are equivalent:

(a) *The* Toeplitz *operator* T_μ *is compact on* $A_b^2(u)$;

(b)

$$\frac{\mu(T_a)}{u_b(T_z)} \to 0 \quad as\ |z| \to 1;$$

(c)

$$\frac{\mu(B_\beta(z,r))}{u_b(B_\beta(z,r))} \to 0 \quad as\ |z| \to 1;$$

(d)

$$\frac{\mu(B_\beta(a_k,r))}{u_b(B_\beta(a_k,r))} \to 0 \quad as\ k \to \infty,$$

where $\{a_k\}$ *is the sequence described in Lemma* 2.1.

Proof We note that $T_\mu = I^*I$ from (4.1), where I^* is the adjoint operator of the embedding $I : A_b^2(u) \to L^2(\mathrm{d}\mu)$. The proof is completed.

It is well known that the Berezin trasform plays a role in the theory of Toeplitz operator. The Berezin transform of the Toeplitz operator T_μ is given by

$$\widetilde{\mu}(z) := \langle T_\mu k_z, k_z\rangle_{A_b^2(u)}, \quad z \in \mathbb{B}_n,$$

where $k_z(w) := \frac{K(w,z)}{\|K(\cdot,z)\|_{A_b^2(u)}}$ is the normalized reproducing kernel of $A_b^2(u)$. It follows (4.1) that the Berezin transform $\widetilde{\mu}$ is obtained by

$$\widetilde{\mu}(z) = \int_{\mathbb{B}_n} |k_z(w)|^2\mathrm{d}\mu(w), \quad z \in \mathbb{B}_n.$$

Proposition 4.1 *If $p_0 > 1$ and $u \in B_{p_0,b}$. Let μ be a positive* Borel *measure on $\mathbb{B}_n$. If T_μ is bounded on $A_b^2(u)$, then the* Berezin *transform $\widetilde{\mu}$ is bounded on $\mathbb{B}_n$.*

Proof Since $u \in B_{p_0,b}$, by Lemma 2.3 one gets that:

$$|k_z(w)|^2 \lesssim \frac{1}{u_b(B_\beta(w,r))} \int_{B_\beta(w,r)} |k_z(\zeta)|^2 u(\zeta) \mathrm{d}v_b(\zeta),$$

where $z \in \mathbb{B}_n$ and $r > 0$. Let $\{a_j\}$ and $r > 0$ be chosen as in Lemma 2.1. We have

$$\begin{aligned} \widetilde{\mu}(z) &\le \sum_{j=1}^{\infty} \int_{B_\beta(a_j,r)} |k_z(w)|^2 \mathrm{d}\mu(w) \\ &\lesssim \sum_{j=1}^{\infty} \int_{B_\beta(a_j,r)} \frac{1}{u_b(B_\beta(w,r))} \int_{B_\beta(w,r)} |k_z(\zeta)|^2 u(\zeta) \mathrm{d}v_b(\zeta) \mathrm{d}\mu(w) \\ &\lesssim \sup_j \frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))} \sum_{j=1}^{\infty} \int_{B_\beta(a_j,2r)} |k_z(\zeta)|^2 u(\zeta) \mathrm{d}v_b(\zeta) \\ &\lesssim \sup_j \frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}. \end{aligned}$$

The proof is completed.

Recall that if $\{e_k\}$ is an orthonormal basis of $A_b^2(u)$, then the Bergman kernel in $A_b^2(u)$ is given by

$$K(z,w) = \sum_k e_k(z)\overline{e_k(w)},$$

and

$$K(z,z) = \sum_k |e_k(z)|^2.$$

If $p \ge 1$, the Toeplitz operator $T_\mu \in \mathcal{S}^p$ if and only if

$$\sum_k |\langle T_\mu e_k, e_k \rangle_{A_b^2(u)}|^p < \infty. \tag{4.2}$$

Lemma 4.1 *Suppose that $p_0 > 1$ and $u \in B_{p_0,b}$. Then there is an $r \in (0,1)$ such that*

$$K(z,z) \simeq u_b(B_\beta(z,r))^{-1}, \quad z \in \mathbb{B}_n. \tag{4.3}$$

Proof By Lemma 2.3, we have

$$|K(w,z)| \lesssim \frac{\|K(\cdot,z)\|_{A_b^2(u)}}{u_b(B_\beta(w,r))^{\frac{1}{2}}} = \frac{\sqrt{K(z,z)}}{u_b(B_\beta(w,r))^{\frac{1}{2}}}.$$

Then we will get one of the inequalities in (4.3).

To prove the reverse inequality, by choosing $s \ge (n+1+b)\frac{p_0}{2}$, the function

$$F_w(z) := \frac{(1-|w|^2)^s}{u_b(T_w)^{\frac{1}{2}}(1-z\overline{w})^s}$$

belongs to $A_b^2(u)$ according to Lemma 2.4. We can find a dyadic tent covering T_w with comparable volume by [11, Lemma 2.5], and we denote it by $\widehat{K_w}$. Then it is followed by

[11, Lemma 2.3(i)] that there are constants r' and r depending only on λ and θ such that $B_\beta(w,r') \subset \widehat{K_w} \subset B_\beta(w,r)$. Let

$$f_w(z) := \frac{(1-|w|^2)^s}{u_b(B_\beta(w,r))^{\frac{1}{2}}(1-z\overline{w})^s},$$

then $f_w \in A^2_b(u)$ with $\|f_w\|_{A^2_b(u)} \simeq \|F_w\|_{A^2_b(u)}$. We have

$$\begin{aligned}\frac{1}{u_b(B_\beta(w,r))^{\frac{1}{2}}} &= f_w(w) = \langle f_w, K(\cdot,w)\rangle_{A^2_b(u)}\\ &\le \|f_w\|_{A^2_b(u)}\|K(\cdot,w)\|_{A^2_b(u)} \lesssim \sqrt{K(w,w)}.\end{aligned}$$

The proof is completed.

Now we turn to characterize the measure μ so that T_μ belongs to the Schatten class $\mathcal{S}^p$, $p \ge 1$. We aim to extend the results in the setting of standard Bergman spaces (see [7, 9]) to the Bergman spaces with Békollé weights.

Theorem 4.2 *If $p_0 > 1$ and $u \in B_{p_0,b}$. Let μ be a positive* Borel *measure on $\mathbb{B}_n$. Then* Toeplitz *operator T_μ belongs to the Schatten class $\mathcal{S}^p$ for some $p \ge 1$, if and only if*

$$\sum_j \Big(\frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^p < \infty, \tag{4.4}$$

where the sequence $\{a_j\}$ and $r > 0$ satisfy the conditions in Lemma 2.1.

Proof We firstly prove the sufficiency. Since $u \in B_{p_0,b}$, one can employ Lemma 2.3 to get that

$$|e_k(z)|^2 \lesssim \frac{1}{u_b(B_\beta(z,r))}\int_{B_\beta(z,r)} |e_k(w)|^2 u(w)\mathrm{d}v_b(w)$$

for every positive integer k. Since (4.4) implies that T_μ is compact on $A^2_b(u)$, we have

$$\begin{aligned}&\Big(\int_{\mathbb{B}_n} |e_k(z)|^2\mathrm{d}\mu(z)\Big)^p\\ \lesssim&\Big(\int_{\mathbb{B}_n} \frac{1}{u_b(B_\beta(z,r))}\int_{B_\beta(z,r)} |e_k(w)|^2u(w)\mathrm{d}v_b(w)\mathrm{d}\mu(z)\Big)^p\\ \lesssim&\Big(\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\mathbb{1}_{B_\beta(z,r)}(w)}{u_b(B_\beta(w,r))}|e_k(w)|^2u(w)\mathrm{d}v_b(w)\mathrm{d}\mu(z)\Big)^p\\ =&\Big(\int_{\mathbb{B}_n} \frac{1}{u_b(B_\beta(w,r))}|e_k(w)|^2u(w)\int_{\mathbb{B}_n}\mathbb{1}_{B_\beta(w,r)}(z)\mathrm{d}\mu(z)\mathrm{d}v_b(w)\Big)^p\\ =&\Big(\int_{\mathbb{B}_n} \frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}|e_k(w)|^2u(w)\mathrm{d}v_b(w)\Big)^p\\ \lesssim&\int_{\mathbb{B}_n}\Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p|e_k(w)|^2u(w)\mathrm{d}v_b(w)\cdot\Big(\int_{\mathbb{B}_n}|e_k(w)|^2u(w)\mathrm{d}v_b(w)\Big)^{p'}\\ =&\int_{\mathbb{B}_n}\Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p|e_k(w)|^2u(w)\mathrm{d}v_b(w).\end{aligned}$$

Plugging this into (4.2), we have

$$\sum_k |\langle T_\mu e_k, e_k\rangle_{A^2_b(u)}|^p = \sum_k\Big(\int_{\mathbb{B}_n}|e_k(z)|^2\mathrm{d}\mu(z)\Big)^p$$

$$\begin{aligned}
&\lesssim \sum_k \int_{\mathbb{B}_n} \Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p |e_k(w)|^2 u(w)\mathrm{d}v_b(w)\\
&= \int_{\mathbb{B}_n} \Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p K(w,w)u(w)\mathrm{d}v_b(w)\\
&\lesssim \sum_{j=1}^{\infty} \Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p \int_{B_\beta(a_j,r)} u_b(B_\beta(w,r))^{-1}u(w)\mathrm{d}v_b(w)\\
&\lesssim \sum_{j=1}^{\infty} \Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p u_b(B_\beta(a_j,r))^{-1} \int_{B_\beta(a_j,r)} u(w)\mathrm{d}v_b(w)\\
&= \sum_{j=1}^{\infty} \Big(\frac{\mu(B_\beta(w,r))}{u_b(B_\beta(w,r))}\Big)^p,
\end{aligned}$$

which proves the sufficiency.

To prove the necessity, assume $T_\mu \in \mathcal{S}^p$. Setting $s \geq (n+1+b)\frac{p_0}{2}$, we firstly define an operator A on the othonormal basis of $A_b^2(u)$ then extend linearly to the whole space as follows:

$$Ae_j(z) = \frac{(1-|a_j|)^s}{(1-z\overline{a}_j)^s} \cdot \frac{1}{u_b(B_\beta(a_j,r))^{\frac{1}{2}}}, \quad z \in \mathbb{B}_n,\ j = 1,2,\cdots.$$

By [6, Theorem 4.1], A is surjective, hence $A^*T_\mu A \in \mathcal{S}^p$. That is

$$\sum_j |\langle A^*T_\mu Ae_j, e_j\rangle_{A_b^2(u)}|^p < \infty.$$

On the other hand, we have

$$\begin{aligned}
\sum_j |\langle A^*T_\mu Ae_j, e_j\rangle_{A_b^2(u)}|^p &= \sum_j |\langle T_\mu Ae_j, Ae_j\rangle_{A_b^2(u)}|^p\\
&= \sum_j \Big(\int_{\mathbb{B}_n} \frac{(1-|a_j|)^{2s}}{(1-z\overline{a}_j)^{2s}} \cdot \frac{\mathrm{d}\mu(z)}{u_b(B_\beta(a_j,r))}\Big)^p\\
&\geq \sum_j \Big(\int_{B_\beta(a_j,r)} \frac{(1-|a_j|)^{2s}}{(1-z\overline{a}_j)^{2s}} \cdot \frac{\mathrm{d}\mu(z)}{u_b(B_\beta(a_j,r))}\Big)^p\\
&\simeq \sum_j \Big(\frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^p,
\end{aligned}$$

which completes the proof.

Proposition 4.2 *If $p_0 > 1$ and $u \in B_{p_0,b}$. Let μ be a positive* Borel *measure on $\mathbb{B}_n$. If $T_\mu \in \mathcal{S}^p(A_b^2(u))$, then the* Berezin *transform $\widetilde{\mu} \in L_b^p(u)$ for $0 < p < \infty$.*

Proof Firstly, we fix r as stated in Lemma 4.1. Suppose $T_\mu \in \mathcal{S}^p$, that is

$$\sum_j \Big(\frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^p < \infty,$$

where $\{a_j\}$ and $r > 0$ satisfy the conditions in Lemma 2.1. By Lemma 4.1, we have

$$\int_{\mathbb{B}_n} \widetilde{\mu}(z)^p u(z)\mathrm{d}v_b(z)$$

$$\lesssim \int_{\mathbb{B}_n} \Big(\sum_j \int_{B_\beta(a_j,r)} |k_z(w)|^2 \mathrm{d}\mu(w)\Big)^p u(z)\mathrm{d}v_b(z)$$
$$\leq \int_{\mathbb{B}_n} \Big(\sum_j \int_{B_\beta(a_j,r0)} \|K(\cdot,w)\|^2 \mathrm{d}\mu(w)\Big)^p u(z)\mathrm{d}v_b(z)$$
$$\simeq \int_{\mathbb{B}_n} \Big(\sum_j \int_{B_\beta(a_j,r)} u_b(B_\beta(w,r))^{-1} \mathrm{d}\mu(w)\Big)^p u(z)\mathrm{d}v_b(z)$$
$$\lesssim \int_{\mathbb{B}_n} \Big(\sum_j \frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^p u(z)\mathrm{d}v_b(z)$$
$$\lesssim \sum_j \Big(\frac{\mu(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^p \int_{\mathbb{B}_n} u(z)\mathrm{d}v_b(z) < \infty.$$

That completes the proof.

5 Composition Operators

Every holomorphic $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ induces a composition operator

$$C_\varphi : H(\mathbb{B}_n) \to H(\mathbb{B}_n),$$

namely, $C_\varphi f = f \circ \varphi$. When $n = 1$, it is well known that C_φ is always bounded on $A_b^p(\mathbb{D})$, and C_φ is compact on $A_b^p(\mathbb{D})$ if and only if

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

When $n > 1$, there are lots of unbounded composition operators on classical Bergman spaces with standard weights $A^p(\mathbb{B}_n, \mathrm{d}v_b)$. Interested readers can see more details in [4, 13].

For a positive weight function u on $\mathbb{B}_n$, we consider the pullback measure of $\mathrm{d}u_b = u\mathrm{d}v_b$ under the map $\varphi : \mathbb{B}_n \to \mathbb{B}_n$, given by

$$\mu_{\varphi,u,b}(E) = u_b(\varphi^{-1}(E))$$

for any Borel subset E of $\mathbb{B}_n$.

By the embedding theorems in Section 3, one can get the characterization of bounded and compact composition operators between different weighted Bergman spaces.

Theorem 5.1 *Let $0 < p \leq q < \infty$, $p_0 > 1$ and u be a $B_{p_0,b}$ weight. If $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ is a holomorphic map, and let $\mu_{\varphi,u,b}$ be the pullback measure defined above. Then the following are equivalent*:

(a) *Composition operator $C_\varphi : A_b^p(u) \to A_b^q(u)$ is bounded*;

(b) *the pullback measure $\mu_{\varphi,u,b}$ is a* Carleson *measure*:

$$\frac{\mu_{\varphi,u,b}(T_w)}{u_b(T_w)^{\frac{q}{p}}} \lesssim 1;$$

(c) *when $s \geq (n+1+b)\frac{p_0}{p}$, the equality*

$$\frac{(1-|w|^2)^{qs}}{(u_b(T_w))^{\frac{q}{p}}} \int_{\mathbb{B}_n} \frac{u(z)\mathrm{d}v_b(z)}{|1-\overline{w}\varphi(z)|^{qs}} \lesssim 1$$

holds for all $w \in \mathbb{B}_n$.

Proof The proof of (b)⇒(a) follows directly from the definition of the pullback measures and the characterization of the Carleson measures in Theorem 3.1.

To prove (a)⇒(c), we denote by

$$f_w(z) = \frac{(1-|w|^2)^s}{u_b(T_w)^{\frac{1}{p}}(1-\overline{w}z)^s}$$

for some $s \geq (n+1+b)\frac{p_0}{p}$ and $w \in \mathbb{B}_n$. It follows from Lemma 2.4 that $f_w \in A_b^p(u)$ for all $w \in \mathbb{B}_n$. Let C_φ act on f_w. Then we have

$$\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{qs}}{u_b(T_w)^{\frac{q}{p}}|1-\overline{w}\varphi(z)|^{qs}} u(z)\mathrm{d}v_b(z) = \|C_\varphi(f_w)(z)\|^q_{A_b^q(u)} \leq \|C_\varphi\|^q \|f_w\|^q_{A_b^p(u)}.$$

It remains to prove (c)⇒(b). Just note that

$$\begin{aligned} 1 &\gtrsim \frac{(1-|w|^2)^{qs}}{u_b(T_w)^{\frac{q}{p}}} \int_{\mathbb{B}_n} \frac{u(z)\mathrm{d}v_b(z)}{|1-\overline{w}\varphi(z)|^{qs}} \\ &= \frac{(1-|w|^2)^{qs}}{u_b(T_w)^{\frac{q}{p}}} \int_{\mathbb{B}_n} \frac{\mathrm{d}\mu_{\varphi,u,b}(z)}{|1-\overline{w}z|^{qs}} \geq \frac{(1-|w|^2)^{qs}}{u_b(T_w)^{\frac{q}{p}}} \int_{T_w} \frac{\mathrm{d}\mu_{\varphi,u,b}(z)}{|1-\overline{w}z|^{qs}} \\ &\simeq \frac{\mu_{\varphi,u,b}(T_w)}{u_b(T_w)^{\frac{q}{p}}}, \end{aligned}$$

which completes the proof.

Remark 5.1 We can deduce from the proof above that

$$\|C_\varphi\|^p_{A_b^p(u)} \gtrsim \sup_{z\in\mathbb{B}_n} \frac{\mu_{\varphi,u,b}(T_w)}{u_b(T_w)}.$$

A similar argument gives the following characterization of the compactness of C_φ on $A_b^p(u)$.

Theorem 5.2 *Let $0 < p \leq q < \infty$, $p_0 > 1$ and u be a $B_{p_0,b}$ weight. If $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ is a holomorphic map, and let $\mu_{\varphi,u,b}$ be the pullback measure. Then $C_\varphi : A_b^p(u) \to A_b^q(u)$ is compact if and only if $\mu_{\varphi,u,b}$ is a vanishing* Carleson *measure if and only if*

$$\lim_{|w|\to 1} \frac{(1-|w|^2)^{qs}}{u_b(T_w)^{\frac{q}{p}}} \int_{\mathbb{B}_n} \frac{u(z)\mathrm{d}v_b(z)}{|1-\overline{w}\varphi(z)|^{qs}} = 0 \tag{5.1}$$

for $s \geq (n+1+b)\frac{p_0}{p}$.

Now we turn to the case $0 < q < p < \infty$. The following result is a direct consequence of Theorems 3.2 and 3.4.

Theorem 5.3 *Let $0 < q < p < \infty$, $p_0 > 1$ and u be a $B_{p_0,b}$ weight. If $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ is a holomorphic map, and let $\mu_{\varphi,u,b}$ be the pullback measure defined above. Then the following are equivalent*:

(a) *The operator $C_\varphi : A_b^p(u) \to A_b^q(u)$ is bounded*;

(b) *the operator $C_\varphi : A_b^p(u) \to A_b^q(u)$ is compact*;

(c) *the function on $\mathbb{B}_n$,*

$$z \mapsto \frac{\mu_{\varphi,u,b}(B_\beta(z,r))}{u_b(B_\beta(z,r))}$$

belongs to $L^{\frac{p}{p-q}}(u)$, for some $r \in (0,1)$.

At the end of this section, we consider the Schatten class membership of $C_\varphi : A_b^2(u) \to A_b^2(u)$.

Theorem 5.4 *If $p \geq 2$, $p_0 > 1$ and $u \in B_{p_0,b}$. Suppose φ is a holomorphic self-map of $\mathbb{B}_n$, and $\mu_{\varphi,u,b}$ is the pullback measure. Then $C_\varphi : A_b^2(u) \to A_b^2(u)$ belongs to $\mathcal{S}^p$ if and only if*

$$\sum_j \Big(\frac{\mu_{\varphi,u,b}(B_\beta(a_j,r))}{u_b(B_\beta(a_j,r))}\Big)^{\frac{p}{2}} < \infty,$$

where the sequence $\{a_j\}$ and $r > 0$ satisfy the conditions in Lemma 2.1.

Proof Note that $K(z,w)$ is the reproducing kernel of $A_b^2(u)$. The adjoint operator C_φ^* may be computed as

$$\begin{aligned} C_\varphi^* f(z) &= \langle C_\varphi^* f, K(\cdot,z)\rangle_{A_b^2(u)} = \langle f, C_\varphi K(\cdot,z)\rangle_{A_b^2(u)} = \langle f, K(\varphi(\cdot),z)\rangle_{A_b^2(u)} \\ &= \int_{\mathbb{B}_n} f(w)K(z,\varphi(w))u(w)\mathrm{d}v_b(w). \end{aligned}$$

So we have

$$C_\varphi^* C_\varphi f(z) = \int_{\mathbb{B}_n} f(w)K(z,w)\mathrm{d}\mu_{\varphi,u,b}(w) = T_{\mu_{\varphi,u,b}} f(z).$$

Then it is clear that $C_\varphi \in \mathcal{S}^p$ if and only if $T_{\mu_{\varphi,u,b}} \in \mathcal{S}^{\frac{p}{2}}$. The proof is completed.

6 Final Remarks

In [14], Zhu characterizes the Schatten p class of Toeplitz operator on the standard weighted Bergman spaces A_b^2 when $0 < p < 1$. The situation on the weighted Bergman spaces $A_b^2(u)$ with Békollé weights u seems rather different. One of the obstacles is that we can not compute $u(\Phi_\zeta(w))$ directly when we change the variable by the möbius automorphisms $z = \Phi_\zeta(w)$. Meanwhile, in Zhu's paper [14], making the change of variables by the automorphisms of the unit ball is of importance to control the ℓ^p norm of the sequence $\{\widehat{\mu}_r(a_k)\}$ by the Schatten p norm of the Toeplitz operator. To make matters worse, there is no evidence that we have enough ingredients to estimate the integral after making the change of variables. According to the definition of the Békollé weights, the reason seems to be clear in the spirit of the following two aspects.

(1) We have the local conditions on the weight u. So the weight u behaves "stable" if we only do the analysis on the small local pieces of the unit ball.

(2) We lack the global property of the weight u. The möbius automorphism can map 0 to any other point in the unit ball. So u could be hard to control when we change the variables by möbius automorphisms. The original piece can be transferred to any other new piece of the ball.

It is well known that the Berezin transform is a powerful tool to study the Toeplitz operators on the standard weighted Bergman spaces. Our results Propositions 4.1–4.2 give the necessary conditions of the boundedness and Schatten class of the Toeplitz operators in terms of the Berezin transforms. The sufficient parts seem to be rather different to the case of the standard weighted Bergman spaces, especially in several complex variables. In the proof of the standard weighted Bergman spaces, the reproducing kernel of A_b^2 is $\frac{1}{(1-z\overline{w})^{2+b}}$. It is relatively easy to estimate the kernel from below. In the setting of Békollé weights, we also need to estimate the reproducing kernel of $A_b^2(u)$ from below. Because it is uncertain that the kernel coincides with any explicit function. We have to estimate the kernel without any explicit computation. By generalizing the results in [5], on the unit disk, we have settled the estimate of the kernel from below and completely characterized the Toeplitz operator in terms of the Berezin transform (see [12]). Unfortunately, the method seems to be invalid on the unit ball, because the zeros distribution of the holomorphic functions in several complex variables varies the case in the one complex variable.

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