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Volterra Type Operators on Weighted Dirichlet Spaces*

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Abstract The Carleson measures for weighted Dirichlet spaces had been characterized by Girela and Peláez, who also characterized the boundedness of Volterra type operators between weighted Dirichlet spaces. However, their characterizations for the boundedness are not complete. In this paper, the author completely characterizes the boundedness and compactness of Volterra type operators from the weighted Dirichlet spaces D^p_α to D^q_β ($-1 < \alpha, \beta$ and $0 < p < q < \infty$), which essentially complete their works. Furthermore, the author investigates the order boundedness of Volterra type operators between weighted Dirichlet spaces.

Keywords Volterra type operator, Boundedness, Compactness, Weighted Dirichlet space, Order boundedness
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1 Introduction

Let $\mathbb{D}$ be the unit disk of a complex plane and let $H(\mathbb{D})$ be the space consisting of all the analytic functions on $\mathbb{D}$. For $0 < p < \infty, -1 < \alpha$, the weighted Bergman space A^p_α on the unit disk $\mathbb{D}$ is the space consisting of all the functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z)\Big)^{\frac{1}{p}} < \infty,$$

where $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ is the normalized Lebesgue area measure (see [8, 12, 34] for references). Furthermore, the weighted Dirichlet space D^p_α on $\mathbb{D}$ is the space consisting of all the functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{D^p_\alpha} = \Big(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z)\Big)^{\frac{1}{p}} < \infty.$$

For any fixed function $g \in H(\mathbb{D})$, the Volterra type operator T_g and its companion operator S_g are defined, respectively, by

$$(T_g f)(z) = \int_0^z f(\omega) g'(\omega) \mathrm{d}\omega, \quad (S_g f)(z) = \int_0^z f'(\omega) g(\omega) \mathrm{d}\omega$$

for any $f \in H(\mathbb{D})$.

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Let $|I|$ be the normalized Lebesgue length of I, which is an interval of $\partial\mathbb{D}$. The Carleson square $S(I)$ is defined by

$$S(I) := \{r\mathrm{e}^{\mathrm{i}\theta} :\ \mathrm{e}^{\mathrm{i}\theta} \in I\,, 1-|I| \le r < 1\}\,.$$

For any $s>0$ and any positive Borel measure μ in $\mathbb{D}$, we say that μ is an s-Carleson measure if there is a positive constant C such that

$$\mu(S(I)) \le C|I|^s \quad \text{for all interval } I \subset \partial\mathbb{D}\,.$$

For a space X of analytic functions on $\mathbb{D}$, it is often useful to know the integrability properties of the functions $f \in X$. That is to determine for which positive Borel measure μ on $\mathbb{D}$ there is a continuous inclusion $X \subset L^p(\mathrm{d}\mu)$, or equivalently, by the closed graph theorem, there exists a positive constant C such that for any $f \in X$,

$$\|f\|_{L^q(\mathrm{d}\mu)} \le C\|f\|_X\,.$$

Duren [7] proved that the Hardy space $H^p \subset L^q(\mathrm{d}\mu),\ 0<p\le q<\infty$, if and only if μ is a $\frac{q}{p}$-Carleson measure, which extends the result obtained by Carleson [4] where the case $p=q$ was proven. For the weighted Bergman spaces, Luecking [23] proved that, for $0<p\le q<\infty$ and $-1<\alpha$, $A^p_\alpha \subset L^q(\mathrm{d}\mu)$ if and only if μ is a $\frac{q(\alpha+2)}{p}$-Carleson measure.

For $0<p<q<\infty$ and $-1<\alpha$, Girela and Peláez [11] gave the characterizations of the measures μ for which $D^p_\alpha \subset L^q(\mathrm{d}\mu)$. Indeed, they proved the following theorem.

Theorem 1.1 *Suppose that $0<p<q<\infty, -1<\alpha$ and μ is a positive Borel measure in $\mathbb{D}$. Then*

(1) *if $p<\alpha+2$, then $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ if and only if μ is a $\frac{q(\alpha+2-p)}{p}$-Carleson measure*;

(2) *if $p=\alpha+2$, then $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ if and only if there exists a positive constant C such that for all interval $I \subset \partial\mathbb{D}$, it holds that $\mu(S(I)) \le C\big(\log\frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}$*;

(3) *if $p>\alpha+2$, then $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ if and only if μ is a finite measure.*

For the case of $p\ge q$, the corresponding characterizations were partly investigated in [9, 26, 31], where several questions were still open.

In Section 2, we completely characterize the boundedness of Volterra type operators T_g and S_g from the weighted Dirichlet spaces D^p_α to D^q_β ($-1<\alpha,\beta$ and $0<p<q<\infty$), which extends the works by Girela and Peláez in [11], where the original characterizations only covered the case $\alpha<p<\alpha+2$. In Section 3, we investigate the compactness of the Volterra type operators T_g and S_g from D^p_α to D^q_β ($-1<\alpha,\beta$ and $0<p<q<\infty$). Finally, in Section 4, we investigate the order boundedness of Volterra type operators between weighted Dirichlet spaces. Throughout the paper, C will represent a positive constant which may be different at different occurrences.

2 Boundedness of Volterra Type Operators

The Volterra type operator T_g was introduced by Pommerenke [27] to study the exponentials of BMOA functions and in the meantime, he proved that T_g acting on the Hardy-Hilbert space H^2 is bounded if and only if $g \in \mathrm{BMOA}$. After his work, Aleman, Siskakis and Cima [1–2] studied the boundedness and compactness of T_g on the Hardy space H^p, where they showed that T_g is bounded (compact) on H^p, $0 < p < \infty$, if and only if $g \in$ BMOA ($g \in$ VMOA). For the related works, see [16]. Furthermore, Aleman and Siskakis [3] studied the boundedness and compactness of T_g on the Bergman spaces while Galanopoulos et al. [10–11] investigated the boundedness of T_g and S_g on the Dirichlet type spaces, and Xiao [32] studied the Volterra type operators on Q_p spaces through the characterizations of the Carleson measures. It should be noted that Li, Liu and Lou [17] dealt with T_g and S_g operators whose range is the Morrey space and whose domain is either the Hardy space or the Morrey space.

Recently, Lin et al. [20–22] characterized the boundedness and the strict singularities of the Volterra type operators acting on the (derivative) Hardy spaces and weighted Banach spaces with general weights. Li and Stević [18–19] introduced the generalized composition operators (also called generalized Volterra type operators) acting on Zygmund spaces and Bloch type spaces and so forth, which had attracted intensive attentions. For instance, Mengestie [24] obtained a complete description of the boundedness and compactness of the product of the Volterra type operators and composition operators on the weighted Fock spaces, and recently, he studied the topological structure of the space of Volterra-type integral operators on the Fock spaces endowed with the operator norm (see [25]). Furthermore, by applying the Carleson embedding theorem and the Littlewood-Paley formula, Constantin and Peláez [5] obtained the boundedness and compactness of T_g on the weighted Fock spaces and investigated the invariant subspaces of the classical Volterra operator T_z on such spaces.

The multiplication operator M_g is defined by

$$(M_g f)(z) := g(z)f(z) \quad \text{for } f \in H(\mathbb{D}),\ z \in \mathbb{D}\,.$$

The following relation holds:

$$(M_g f)(z) = f(0)g(0) + (T_g f)(z) + (S_g f)(z)\,.$$

Then we characterize the boundedness of these operators.

Theorem 2.1 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p < q < \infty$. *Define* $\mathrm{d}\mu_{g,q,\beta}(z) := (1-|z|^2)^\beta |g'(z)|^q \mathrm{d}A(z)$. *Then the following statements hold*:

(1) *If* $p < \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is bounded if and only if* $\mu_{g,q,\beta}(z)$ *is a* $\frac{q(\alpha+2-p)}{p}$*-Carleson measure*;

(2) *if* $p = \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is bounded if and only if there exists a positive constant* C *such that for all interval* $I \subset \partial\mathbb{D}$, *it holds that* $\mu_{g,q,\beta}(S(I)) \le C\big(\log \frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}$;

(3) *if* $p > \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is bounded if and only if* $\mu_{g,q,\beta}$ *is a finite measure, or equivalently,* $g \in D^q_\beta$.

Proof This follows directly from Theorem 1.1 and the closed graph theorem.

Theorem 2.2 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p < q < \infty$. *Then* $S_g : D^p_\alpha \to D^q_\beta$ *is bounded if and only if* $|g(z)| = O((1-|z|^2)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}})$, *as* $|z| \to 1^-$.

Proof First, suppose that $|g(z)| = O((1-|z|^2)^{\frac{2+\alpha}{p} - \frac{2+\beta}{q}})$. If $f \in D^p_\alpha$, then $f' \in A^p_\alpha$ by definition. It is a well-known fact (see [8, 34]) that if $h \in A^p_\alpha$, then for all $z \in \mathbb{D}$, we have

$$|h(z)| \le C \frac{\|h\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}\,.$$

Then it holds that

$$\begin{aligned}\|S_gf\|_{D^q_\beta}&=\Big(\int_{\mathbb{D}}|f'(z)g(z)|^q(1-|z|^2)^\beta\mathrm{d}A(z)\Big)^{\frac{1}{q}}\\&\le C\Big(\int_{\mathbb{D}}|f'(z)|^p|f'(z)|^{q-p}(1-|z|^2)^{\frac{q(2+\alpha)}{p}-2}\mathrm{d}A(z)\Big)^{\frac{1}{q}}\\&\le C\Big(\int_{\mathbb{D}}|f'(z)|^p\Big(\frac{\|f\|_{D^p_\alpha}}{(1-|z|^2)^{\frac{(2+\alpha)}{p}}}\Big)^{q-p}(1-|z|^2)^{\frac{q(2+\alpha)}{p}-2}\mathrm{d}A(z)\Big)^{\frac{1}{q}}\\&\le C\|f\|^{\frac{q-p}{q}}_{D^p_\alpha}\Big(\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^\alpha\mathrm{d}A(z)\Big)^{\frac{1}{q}}\\&\le C\|f\|_{D^p_\alpha}\,.\end{aligned}$$

Hence, $S_g:D^p_\alpha\to D^q_\beta$ is bounded.

Conversely, suppose that $S_g:D^p_\alpha\to D^q_\beta$ is bounded. Given $a\in\mathbb{D}$, define the function f_a by

$$f_a(z):=\frac{(1-|a|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{a}z)^{\frac{2(\alpha+2)}{p}-1}}\,.$$

It is easy to prove that $f_a\in D^p_\alpha$ and there exists a positive constant C such that for all $a\in\mathbb{D}$, $\|f_a\|_{D^p_\alpha}\le C$. Denoting $\Delta(a,r)$ as the pseudo-hyperbolic disk with center a and radius r, we have

$$\begin{aligned}(1-|a|^2)^{2+\beta-\frac{q(2+\alpha)}{p}}|g(a)|^q&\le C(1-|a|^2)^{\beta-\frac{q(2+\alpha)}{p}}\int_{\Delta(a,r)}|g(\omega)|^q\mathrm{d}A(\omega)\\&\le C|a|^{-q}\int_{\Delta(a,r)}|(S_gf_a)'(\omega)|^q(1-|\omega|^2)^\beta\mathrm{d}A(\omega)\\&\le C|a|^{-q}\|S_gf_a\|^q_{D^q_\beta}\\&\le C|a|^{-q}\|S_g\|^q\|f_a\|^q_{D^p_\alpha}\\&\le C|a|^{-q}\,.\end{aligned}$$

Thus, $|g(a)|=O((1-|a|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, as $|a|\to1^-$.

As an immediate corollary, we obtain the known results originally proven by Zhao [33].

Corollary 2.1 *Let $-1<\alpha,\beta$, $g\in H(\mathbb{D})$ and $0<p<q<\infty$. Then $M_g:A^p_\alpha\to A^q_\beta$ is bounded if and only if $|g(z)|=O((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, as $|z|\to1^-$.*

Proof This follows immediately from the fact that $DS_g=M_gD$, where D is the differential operator.

Theorem 2.3 *Let $-1<\alpha,\beta$, $g\in H(\mathbb{D})$ and $0<p<q<\infty$. Define $\mathrm{d}\mu_{g,q,\beta}(z):=(1-|z|^2)^\beta|g'(z)|^q\mathrm{d}A(z)$. Then the following statements hold*:

(1) *If $p<\alpha+2$, then $M_g:D^p_\alpha\to D^q_\beta$ is bounded if and only if $\mu_{g,q,\beta}(z)$ is a $\frac{q(\alpha+2-p)}{p}$-Carleson measure and $|g(z)|=O((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, as $|z|\to1^-$;*

(2) *if $p=\alpha+2$, then $M_g:D^p_\alpha\to D^q_\beta$ is bounded if and only if $|g(z)|=O((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ as $|z|\to1^-$ and there exists a positive constant C such that for all interval $I\subset\partial\mathbb{D}$, it holds that $\mu_{g,q,\beta}(S(I))\le C\big(\log\frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}$;*

(3) *if* $p > \alpha+2$, *then* $M_g : D^p_\alpha \to D^q_\beta$ *is bounded if and only if* $|g(z)| = O((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ *as* $|z| \to 1^-$ *and* $g \in D^q_\beta$.

Proof Since $(M_g f)(z) = f(0)g(0)+(T_g f)(z)+(S_g f)(z)$, the sufficiency follows immediately from Theorems 2.1–2.2. It remains to prove the necessity. In this case, it is obvious that if we can prove that $|g(z)| = O((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ as $|z| \to 1^-$, then all the other statements follow immediately from Theorems 2.1–2.2 again.

Given $a \in \mathbb{D}$, define the function F_a by

$$F_a(z) := \frac{(1-|a|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{a}z)^{\frac{2(\alpha+2)}{p}-1}} - (1-|a|^2)^{\frac{p-\alpha-2}{p}}.$$

Then $F_a(a) = 0$, and the remainder of the proof is essentially similar to the converse part of the proof of Theorem 2.2.

3 Compactness of Volterra Type Operators

For any $s > 0$ and μ a positive Borel measure in $\mathbb{D}$, we say μ is a vanishing s-Carleson measure if

$$\mu(S(I)) = o(|I|^s) \quad \text{as } |I| \to 0.$$

Theorem 3.1 *Suppose that* $0 < p < q < \infty, -1 < \alpha$ *and* μ *is a positive Borel measure in* $\mathbb{D}$. *Then*

(1) *if* $p < \alpha+2$, *then* $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ *is compact if and only if* μ *is a vanishing* $\frac{q(\alpha+2-p)}{p}$-*Carleson measure*;

(2) *if* $p = \alpha+2$, *then* $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ *is compact if and only if* $\mu(S(I)) = o\big(\big(\log\frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}\big)$ *as* $|I| \to 0$;

(3) *if* $p > \alpha+2$, *then* $D^p_\alpha \subset L^q(\mathrm{d}\mu)$ *is compact if and only if* μ *is a finite measure.*

Proof (1) is known (see [15] for example).

For (2), we notice that this condition is, in deed, a vanishing $\big((1-\frac{1}{p})q, 0\big)$-logarithmic Carleson measure and the proof of it is basically similar to that of [26, Theorem 3.1(ii)].

Now for (3), since when $p > \alpha+2$, it holds that $D^p_\alpha \subset H^\infty$, where H^∞ is the space of all the bounded analytic functions on $\mathbb{D}$, the compactness follows easily by the standard arguments.

Then we characterize the compactness of these operators.

Theorem 3.2 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p < q < \infty$. *Define* $\mathrm{d}\mu_{g,q,\beta}(z) := (1-|z|^2)^\beta |g'(z)|^q \mathrm{d}A(z)$. *Then the following statements hold*:

(1) *If* $p < \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is compact if and only if* $\mu_{g,q,\beta}(z)$ *is a vanishing* $\frac{q(\alpha+2-p)}{p}$-*Carleson measure*;

(2) *if* $p = \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is compact if and only if* $\mu_{g,q,\beta}(S(I)) = o\big(\big(\log\frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}\big)$ *as* $|I| \to 0$;

(3) *if* $p > \alpha+2$, *then* $T_g : D^p_\alpha \to D^q_\beta$ *is compact if and only if* $\mu_{g,q,\beta}$ *is a finite measure, or equivalently,* $g \in D^q_\beta$.

Proof This follows directly from Theorem 3.1.

Theorem 3.3 *Let* $-1<\alpha,\beta$, $g\in H(\mathbb{D})$ *and* $0<p<q<\infty$. *Then* $S_g: D^p_\alpha\to D^q_\beta$ *is compact if and only if* $|g(z)|=o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, *as* $|z|\to 1^-$.

Proof First suppose that $|g(z)|=o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$. Then, for any $\epsilon>0$, there exists r with $0<r<1$ such that $\frac{|g(z)|}{((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})}<\epsilon$, whenever $|z|>r$. Now, for any bounded sequence $\{f_n\}_{n=0}^\infty\subset D^p_\alpha$ such that f_n converges to 0 locally uniformly, it holds that

$$\begin{aligned}
&\limsup_{n\to\infty}\|S_gf_n\|_{D^q_\beta}\\
&=\limsup_{n\to\infty}\Big(\int_{\mathbb{D}}|f_n'(z)g(z)|^q(1-|z|^2)^\beta \mathrm{d}A(z)\Big)^{\frac1q}\\
&\le\limsup_{n\to\infty}\Big(\int_{\mathbb{D}\setminus r\overline{\mathbb{D}}}|f_n'(z)g(z)|^q(1-|z|^2)^\beta \mathrm{d}A(z)\Big)^{\frac1q}\\
&\le\limsup_{n\to\infty}C\epsilon^{\frac1q}\Big(\int_{\mathbb{D}}|f_n'(z)|^p|f'(z)|^{q-p}(1-|z|^2)^{\frac{q(2+\alpha)}{p}-2}\mathrm{d}A(z)\Big)^{\frac1q}\\
&\le\limsup_{n\to\infty}C\epsilon^{\frac1q}\Big(|f_n'(z)|^p\Big(\frac{\|f_n\|_{D^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}\Big)^{q-p}(1-|z|^2)^{\frac{q(2+\alpha)}{p}-2}\mathrm{d}A(z)\Big)^{\frac1q}\\
&\le\limsup_{n\to\infty}C\epsilon^{\frac1q}\|f_n\|_{D^p_\alpha}^{\frac{q-p}{q}}\Big(\int_{\mathbb{D}}|f_n'(z)|^p(1-|z|^2)^\alpha\mathrm{d}A(z)\Big)^{\frac1q}\\
&\le\limsup_{n\to\infty}C\epsilon^{\frac1q}\|f_n\|_{D^p_\alpha}\\
&\le C\epsilon^{\frac1q}.
\end{aligned}$$

Since ϵ is arbitrary, it follows that $S_g: D^p_\alpha\to D^q_\beta$ is compact.

Conversely, suppose that $S_g: D^p_\alpha\to D^q_\beta$ is compact. Choose the functions f_a defined in the proof of Theorem 2.2. Then the direct computation shows that $\|f_a\|_{D^p_\alpha}$ is uniformly bounded for all $a\in\mathbb{D}$ and f_a converges to 0 locally uniformly in $\mathbb{D}$. Thus, we have

$$\begin{aligned}
(1-|a|^2)^{2+\beta-\frac{q(2+\alpha)}{p}}|g(a)|^q&\le C(1-|a|^2)^{\beta-\frac{q(2+\alpha)}{p}}\int_{\Delta(a,r)}|g(\omega)|^q\mathrm{d}A(\omega)\\
&\le C|a|^{-q}\int_{\Delta(a,r)}|(S_gf_a)'(\omega)|^q(1-|\omega|^2)^\beta\mathrm{d}A(\omega)\\
&\le C|a|^{-q}\|S_gf_a\|^q_{D^q_\beta}\to 0\quad \text{as } |a|\to 1^-.
\end{aligned}$$

Thus, $|g(a)|=o((1-|a|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, as $|a|\to 1^-$.

As an immediate corollary, we obtain the known results originally proven by Čučković and Zhao [6].

Corollary 3.1 *Let* $-1<\alpha,\beta$, $g\in H(\mathbb{D})$ *and* $0<p<q<\infty$. *Then* $M_g: A^p_\alpha\to A^q_\beta$ *is compact if and only if* $|g(z)|=o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, *as* $|z|\to 1^-$.

Theorem 3.4 *Let* $-1<\alpha,\beta$, $g\in H(\mathbb{D})$ *and* $0<p<q<\infty$. *Define* $\mathrm{d}\mu_{g,q,\beta}(z):=(1-|z|^2)^\beta|g'(z)|^q\mathrm{d}A(z)$. *Then the following statements hold*:

(1) *If* $p<\alpha+2$, *then* $M_g: D^p_\alpha\to D^q_\beta$ *is compact if and only if* $\mu_{g,q,\beta}(z)$ *is a vanishing* $\frac{q(\alpha+2-p)}{p}$*-Carleson measure and* $|g(z)|=o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$, *as* $|z|\to 1^-$;

(2) *if* $p=\alpha+2$, *then* $M_g: D^p_\alpha \to D^q_\beta$ *is compact if and only if* $|g(z)| = o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ *as* $|z|\to 1^-$ *and* $\mu_{g,q,\beta}(S(I)) = o\big(\big(\log\frac{1}{|I|}\big)^{(\frac{1}{p}-1)q}\big)$ *as* $|I|\to 0$;

(3) *if* $p>\alpha+2$, *then* $M_g: D^p_\alpha \to D^q_\beta$ *is compact if and only if* $|g(z)| = o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ *as* $|z|\to 1^-$ *and* $g\in D^q_\beta$.

Proof Since $(M_g f)(z) = f(0)g(0)+(T_g f)(z)+(S_g f)(z)$, the sufficiency follows immediately from Theorems 3.2–3.3. It remains to prove the necessary conditions and in this case, it is obvious that if we can prove that $|g(z)| = o((1-|z|^2)^{\frac{2+\alpha}{p}-\frac{2+\beta}{q}})$ as $|z|\to 1^-$, then all the other statements follow immediately from Theorems 3.2–3.3 again.

Given $a\in\mathbb{D}$, define the function F_a by

$$F_a(z) := \frac{(1-|a|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{a}z)^{\frac{2(\alpha+2)}{p}-1}} - (1-|a|^2)^{\frac{p-\alpha-2}{p}}.$$

Then $F_a(a)=0$, and the remainder of the proof is similar to that of Theorem 3.3.

4 Order Boundedness of Volterra Type Operators

Let X be a Banach space of holomorphic functions defined on $\mathbb{D}$, $q>0$, $(\Omega,\mathcal{A},\mu)$ be a measure space and

$$L^p(\Omega,\mathcal{A},\mu) := \Big\{f \mid f:\Omega\to\mathbb{C} \text{ is measurable and } \int_\Omega |f|^p \mathrm{d}\mu < \infty\Big\}.$$

An operator $T: X\to L^p(\Omega,\mathcal{A},\mu)$ is said to be order bounded if there exists a nonnegative function $g\in L^p(\Omega,\mathcal{A},\mu)$ such that for all $f\in X$ with $\|f\|_X\le 1$, it holds that

$$|T(f)(x)| \le g(x) \quad \text{a.e. } [\mu].$$

Order boundedness plays an important role in studying the properties of many concrete operators acting between Banach spaces like Hardy spaces, weighted Bergman spaces and so forth (see [13–14, 29–30]). Recently, order boundedness of weighted composition operators between weighted Dirichlet spaces were studied in [10, 28]. In this section, we investigate the order boundedness of Volterra type operators between weighted Dirichlet spaces. Recall that in this case, if we define the measure A_β by $\mathrm{d}A_\beta(z) = (1-|z|^2)^\beta \mathrm{d}A(z)$, then an operator $T: D^p_\alpha \to D^q_\beta$ is order bounded if and only if there exists a nonnegative function $g\in L^q(A_\beta)$ such that for all $f\in D^p_\alpha$ with $\|f\|_{D^p_\alpha}\le 1$, it holds that

$$|T(f)'(z)| \le g(z) \quad \text{a.e. } [A_\beta].$$

Before proving the results, we first give some auxiliary lemmas.

Lemma 4.1 *Let* $\alpha>-1$ *and* $0<p<\infty$. *Denote* δ_z *as the point evaluation functional on* D^p_α. *Then*

(1) *for* $p<\alpha+2$, $\|\delta_z\| \approx \frac{1}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}}$;

(2) *for* $p=\alpha+2$, $\|\delta_z\| \approx \frac{1}{\big(\log(\frac{2}{1-|z|^2})\big)^{\frac{1-p}{p}}}$;

(3) *for* $p>\alpha+2$, $\|\delta_z\| \approx 1$.

Proof (1) and (2) follow from [10, Lemmas 2.2–2.3] while (3) follows directly from the fact that $D^p_\alpha \subset H^\infty$ for $p > \alpha + 2$.

Lemma 4.2 *Let $\alpha > -1$ and $0 < p < \infty$. Denote δ'_z as the derivative point evaluation functional on D^p_α, then* $\|\delta'_z\| \approx \frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}}}$.

Proof By definition, $f \in D^p_\alpha$ if and only if $f' \in A^p_\alpha$, thus the lemma follows from [12, Lemma 3.2].

Now we are ready to prove our results.

Theorem 4.1 *Let $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ and $0 < p, q < \infty$. Then the following statements hold*:

(1) *If $p < \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|g'(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2-p)}{p}}} \mathrm{d}A_\beta < \infty ;$$

(2) *if $p = \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|g'(z)|^q}{\left(\log(\frac{2}{1-|z|^2})\right)^{\frac{q(1-p)}{p}}} \mathrm{d}A_\beta < \infty ;$$

(3) *if $p > \alpha + 2$, then $T_g : D^p_\alpha \to D^q_\beta$ is order bounded if and only if $g \in D^q_\beta$.*

Proof (1) Assume first that $T_g : D^p_\alpha \to D^q_\beta$ is order bounded. Then there exists $h \in L^q(A_\beta)$ such that for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \le 1$, it holds that

$$|f(z)g'(z)| \le h(z) \quad \text{a.e. } [A_\beta] .$$

Hence, by Lemma 4.1, the inequality holds

$$h(z) \ge |g'(z)| \|\delta_z\| \gtrsim \frac{|g'(z)|}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} \quad \text{a.e. } [A_\beta] .$$

Therefore, it holds that $\int_{\mathbb{D}} \frac{|g'(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2-p)}{p}}} \mathrm{d}A_\beta < \infty$.

Conversely, suppose that $\int_{\mathbb{D}} \frac{|g'(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2-p)}{p}}} \mathrm{d}A_\beta < \infty$. Let

$$h(z) = \frac{|g'(z)|}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} .$$

Then by Lemma 4.1, for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \le 1$,

$$|f(z)g'(z)| \le |g'(z)| \|\delta_z\| \lesssim h(z) \quad \text{a.e. } [A_\beta] .$$

Therefore, $T_g : D^p_\alpha \to D^q_\beta$ is order bounded.

The proofs of (2) and (3) are almost similar to that of (1), thus we omit the details.

By Theorems 2.1, 3.2 and 4.1, we obtain the following corollary.

Corollary 4.1 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $\alpha + 2 < p < q < \infty$. *Then the following statements are equivalent*:

(1) $T_g : D^p_\alpha \to D^q_\beta$ *is bounded*;

(2) $T_g : D^p_\alpha \to D^q_\beta$ *is compact*;

(3) $T_g : D^p_\alpha \to D^q_\beta$ *is order bounded*;

(4) $g \in D^q_\beta$.

Theorem 4.2 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p, q < \infty$. *Then* $S_g : D^p_\alpha \to D^q_\beta$ *is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty .$$

Proof The proof is similar to that of Theorem 4.1 except that in this case, we resort to Lemma 4.2 instead of Lemma 4.1.

Theorem 4.3 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p, q < \infty$. *Then the following statements hold*:

(1) *If* $p < \alpha + 2$, *then* $M_g : D^p_\alpha \to D^q_\beta$ *is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty ;$$

(2) *If* $p = \alpha + 2$, *then* $M_g : D^p_\alpha \to D^q_\beta$ *is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^q} \mathrm{d}A_\beta + \int_{\mathbb{D}} \frac{|g'(z)|^q}{\left(\log\left(\frac{2}{1-|z|^2}\right)\right)^{\frac{q(1-p)}{p}}} \mathrm{d}A_\beta < \infty ;$$

(3) *If* $p > \alpha + 2$, *then* $M_g : D^p_\alpha \to D^q_\beta$ *is order bounded if and only if* $g \in D^q_\beta$ *and*

$$\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty .$$

Proof (1) Suppose that $\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty$. Let $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \le 1$. Then by Lemmas 4.1–4.2, we have

$$|(f(z)g(z))'| \le |f'(z)g(z)| + |f(z)g'(z)| \lesssim \frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} + \frac{|g'(z)|}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} .$$

By taking

$$h(z) = \frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} + \frac{|g'(z)|}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} ,$$

then $h \in L^q(A_\beta)$ since

$$\int_{\mathbb{D}} \frac{|g'(z)|}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} \mathrm{d}A_\beta \lesssim \int_{\mathbb{D}} \frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \mathrm{d}A_\beta < \infty .$$

Accordingly, $M_g : D^p_\alpha \to D^q_\beta$ is order bounded.

Conversely, assume that $M_g : D^p_\alpha \to D^q_\beta$ is order bounded. Then there exists $h \in L^q(A_\beta)$ such that for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \le 1$, it holds that

$$|(fg)'(z)| \le h(z) \quad \text{a.e. } [A_\beta]\,.$$

For any $z \in \mathbb{D}$, we consider the function

$$f_z(\omega) = \frac{(1-|z|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{z}\omega)^{\frac{2(\alpha+2)}{p}-1}} - \frac{(1-|z|^2)^{\frac{\alpha+2}{p}+1}}{(1-\overline{z}\omega)^{\frac{2(\alpha+2)}{p}}}\,, \quad \omega \in \mathbb{D}\,.$$

An easy calculation shows that $\|f_z\|_{D^p_\alpha} \lesssim 1$ and

$$f_z'(\omega) = \overline{z}\Big(\frac{2(\alpha+2)-p}{p}\frac{(1-|z|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{z}\omega)^{\frac{2(\alpha+2)}{p}}} - \frac{2(\alpha+2)}{p}\frac{(1-|z|^2)^{\frac{\alpha+2}{p}+1}}{(1-\overline{z}\omega)^{\frac{2(\alpha+2)}{p}+1}}\Big), \quad \omega \in \mathbb{D}.$$

Thus, we have $f_z(z) = 0$ and $f_z'(z) = \frac{-\overline{z}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}$. Therefore,

$$\frac{|\overline{z}g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} = |g'(z)f_z(z) + g(z)f_z'(z)| = |(gf_z)'(z)| \lesssim h(z) \quad \text{a.e. } [A_\beta]\,.$$

Hence, for $|z| > \frac{1}{2}$, it holds that

$$\frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \lesssim h(z) \quad \text{a.e. } [A_\beta]\,.$$

For $|z| \le \frac{1}{2}$, it follows from the continuity of the function $\frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}}}$ that

$$\frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \lesssim 1\,.$$

Now, by taking the constant function 1 and the monomial z as the test function in D^p_α, we get that $|g'(z)| \lesssim h(z)$ a.e. $[A_\beta]$, and $|g'(z)z + g(z)| \lesssim h(z)$ a.e. $[A_\beta]$. Thus, for $|z| \le \frac{1}{2}$, it also holds that

$$\frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \lesssim h(z) \quad \text{a.e. } [A_\beta]\,.$$

In conclusion, for all $z \in \mathbb{D}$,

$$\frac{|g(z)|}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \lesssim h(z) \quad \text{a.e. } [A_\beta]\,,$$

which implies that

$$\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty\,.$$

The proofs of (2) and (3) are similar to that of (1) by some minor modifications. For example, in (2), we take the test function

$$f_z(\omega) = \frac{\log(\frac{2}{1-\overline{z}\omega})}{\log(\frac{2}{1-|z|^2})^{\frac{1}{p}}} - \frac{\big(\log\big(\frac{2}{1-\overline{z}\omega}\big)\big)^2}{\log(\frac{2}{1-|z|^2})^{\frac{1}{p}+1}}\,, \quad \omega \in \mathbb{D}\,.$$

Thus the proof is complete.

By Theorems 4.1–4.3, we obtain the following corollary.

Corollary 4.2 *Let* $-1 < \alpha, \beta$, $g \in H(\mathbb{D})$ *and* $0 < p < \alpha+2$, $0 < q < \infty$. *Then the following statements are equivalent*:

(1) $S_g : D^p_\alpha \to D^q_\beta$ *is order bounded*;

(2) $M_g : D^p_\alpha \to D^q_\beta$ *is order bounded*;

(3) $\int_{\mathbb{D}} \frac{|g(z)|^q}{(1-|z|^2)^{\frac{q(\alpha+2)}{p}}} \mathrm{d}A_\beta < \infty$, *that is,* $g \in A^q_{\beta - \frac{q(\alpha+2)}{p}}$.

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References

[1] Aleman, A. and Cima, J., An integral operator on H^p and Hardy's inequality, *J. Anal. Math.*, **85**, 2001, 157–176.

[2] Aleman, A. and Siskakis, A., An integral operator on H^p, *Complex Variables Theory Appl.*, **28**(2), 1995, 149–158.

[3] Aleman, A. and Siskakis, A., Integration operators on Bergman spaces, *Indiana Univ. Math. J.*, **46**(2), 1997, 337–356.

[4] Carleson, L., An interpolation problem for bounded analytic functions, *Amer. J. Math.*, **80**, 1958, 921–930.

[5] Constantin, O. and Peláez, J., Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces, *J. Geom. Anal.*, **26**(2), 2016, 1109–1154.

[6] Čučković, Ž. and Zhao, R., Weighted composition operators between different weighted Bergman spaces and different Hardy spaces, *Illinois J. Math.*, **51**, 2007, 479–498.

[7] Duren, P., Extension of a theorem of Carleson, *Bull. Amer. Math. Soc.*, **75**, 1969, 143–146.

[8] Duren, P. and Schuster, A., Bergman Spaces, Math. Surveys Monogr., **100**, Amer. Math. Soc., Providence, RI, 2004.

[9] Galanopoulos, P., Girela, D. and Peláez, J., Multipliers and integration operators on Dirichlet spaces, *Trans. Amer. Math. Soc.*, **363**(4), 2011, 1855–1886.

[10] Gao, Y., Kumar, S. and Zhou, Z., Order bounded weighted composition operators mapping into the Dirichlet type spaces, *Chin. Ann. Math. Ser. B*, **37**(4), 2016, 585–594.

[11] Girela, D. and Peláez, J., Carleson measures, multipliers and integration operators for spaces of Dirichlet type, *J. Funct. Anal.*, **241**(1), 2006, 334–358.

[12] Hedenmalm, H., Korenblum, B. and Zhu, K., Theory of Bergman Spaces, Grad. Texts in Math., **199**, Springer-Verlag, New York, 2000.

[13] Hibschweiler, R., Order Bounded Weighted Composition Operators, Contemp. Math., **454**, Amer. Math. Soc., Providence, RI, 2008.

[14] Hunziker, H. and Jarchow, H., Composition operators which improve integrability, *Math. Nachr.*, **152**, 1991, 83–99.

[15] Kumar, S., Weighted composition operators between spaces of Dirichlet type, *Rev. Mat. Complut.*, **22**(2), 2009, 469–488.

[16] Laitila, J., Miihkinen, S. and Nieminen, P., Essential norms and weak compactness of integration operators, *Arch. Math.*, **97**(1), 2011, 39–48.

[17] Li, P., Liu, J. and Lou, Z., Integral operators on analytic Morrey spaces, *Sci. China Math.*, **57**(9), 2014, 1961–1974.

[18] Li, S. and Stević, S., Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, **338**(2), 2008, 1282–1295.

[19] Li, S. and Stević, S., Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.*, **345**(1), 2008, 40–52.

[20] Lin, Q., Volterra type operators between Bloch type spaces and weighted Banach spaces, *Integral Equations Operator Theory*, **91**(2), 2019, 91:13.

[21] Lin, Q., Liu, J. and Wu, Y., Volterra type operators on $S^p(\mathbb{D})$ spaces, *J. Math. Anal. Appl.*, **461**, 2018, 1100–1114.

[22] Lin, Q., Liu J. and Wu, Y., Strict singularity of Volterra type operators on Hardy spaces, *J. Math. Anal. Appl.*, **492**(1), 2020, 124438, 9 pages.

[23] Luecking, D., Forward and reverse inequalities for functions in Bergman spaces and their derivatives, *Amer. J. Math.*, **107**, 1985, 85–111.

[24] Mengestie, T., Product of Volterra type integral and composition operators on weighted Fock spaces, *J. Geom. Anal.*, **24**(2), 2014, 740–755.

[25] Mengestie, T., Path connected components of the space of Volterra-type integral operators, *Arch. Math.*, **111**(4), 2018, 389–398.

[26] Pau, J. and Zhao, R., Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces, *Integr. Equ. Oper. Theory*, **78**, 2014, 483–514.

[27] Pommerenke, Ch., Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation, *Comment. Math. Helv.* (*German*), **52**(4), 1977, 591–602.

[28] Sharma, A., On order bounded weighted composition operators between Dirichlet spaces, *Positivity*, **21**(3), 2017, 1213–1221.

[29] Ueki, S., Order bounded weighted composition operators mapping into the Bergman space, *Complex Anal. Oper. Theory*, **6**(2), 2012, 549–560.

[30] Wang, S., Wang, M. and Guo, X., Differences of Stević-Sharma operators, *Banach J. Math. Anal.*, **14**(3), 2020, 1019–1054.

[31] Wu, Z., Carleson measures and multipliers for Dirichlet spaces, *J. Funct. Anal.*, **169**, 1999, 148–163.

[32] Xiao, J., The Q_p Carleson measure problem, *Adv. Math.*, **217**(5), 2008, 2075–2088.

[33] Zhao, R., Pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces, *Ann. Acad. Sci. Fenn. Math.*, **29**(1), 2004, 139–150.

[34] Zhu, K., Operator Theory in Function Spaces, 2nd ed., Mathematical Surveys and Monographs, **138**, Amer. Math. Soc., Providence, RI, 2007.