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Weighted Moore-Penrose Inverses and Weighted Core Inverses in Rings with Involution*

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Abstract In this paper, the authors derive the existence criteria and the formulae of the weighted Moore-Penrose inverse, the e-core inverse and the f-dual core inverse in rings. Also, new characterizations between weighted Moore-Penrose inverses and one-sided inverses along an element are given.

Keywords Weighted Moore-Penrose inverses, One-sided inverses along an element, Inverses along an element, e-Core inverses, f-Dual core inverses
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1 Introduction

Suppose that R is a unital $*$-ring, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$ for all $a, b \in R$.

Throughout this paper, we assume that R is a unital $*$-ring. Recall that an element $a \in R$ is called (von Neumann) regular if there exists some $x \in R$ such that $a = axa$. Such an x is called an inner inverse or $\{1\}$-inverse of a, and is denoted by a^-. An element $x \in R$ is called Hermitian if $x = x^*$. In what follows, let $e, f \in R$ be invertible Hermitian elements.

We say that $a \in R$ has a weighted Moore-Penrose inverse with weights e, f if there exists $x \in R$ such that

$$\text{(i) } axa = a, \quad \text{(ii) } xax = x, \quad \text{(iii) } (eax)^* = eax, \quad \text{(iv) } (fxa)^* = fxa,$$

where x is called a weighted Moore-Penrose inverse of a with weights e, f (abbr. weighted Moore-Penrose inverse). It is unique if it exists, and is denoted by $a^{\dagger}_{e,f}$. More generally, if a and x satisfy (i) $axa = a$ and (iii) $(eax)^* = eax$, then x is called an $\{e, 1, 3\}$-inverse of a, and is denoted by $a_e^{(1,3)}$. Similarly, if a and x satisfy (i) $axa = a$ and (iv) $(fxa)^* = fxa$, then x is called an $\{f, 1, 4\}$-inverse of a, and is denoted by $a_f^{(1,4)}$. As usual, we denote by $R^{\dagger}_{e,f}$, $R_e^{(1,3)}$ and $R_f^{(1,4)}$ the sets of all weighted Moore-Penrose invertible, $\{e, 1, 3\}$-invertible and $\{f, 1, 4\}$-invertible elements in R, respectively.

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Recently, Mosić et al. [5] introduced and investigated e-core inverses and f-dual core inverses, extending the notions of core inverses and dual core inverses in rings (see [6]). An element $a \in R$ is e-core invertible (see [5]) if there exists $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*e$. Such x is unique if it exists, and is denoted by $a^{e,\oplus}$. Dually, a is called f-dual core invertible if there exists $x \in R$ such that $axa = a$, $Rx = Ra$ and $fxR = a^*R$. The unique f-dual core inverse x of a, when exists, is denoted by $a_{f,\oplus}$. We denote by $R^{e,\oplus}$ and $R_{f,\oplus}$ the sets of all e-core invertible and f-dual core invertible elements in R. Further results on e-core inverses and f-dual core inverses can be referred to [10].

In this paper, we mainly investigate weighted Moore-Penrose inverses, e-core inverses, f-dual core inverses and one-sided inverses along an element in rings. The paper is organized as follows. In Section 2, several characterizations and expressions for $\{e, 1, 3\}$-inverses and $\{f, 1, 4\}$-inverses of elements are derived. Also, the existence criterion of the weighted Moore-Penrose inverse is given. Moreover, it is proved that $a \in R$ is weighted Moore-Penrose invertible if and only if it is both $\{e, 1, 3\}$-invertible and $\{f, 1, 4\}$-invertible. In Section 3, we present the existence criterion of both e-core invertible and f-dual core invertible elements. In Section 4, it is shown that $a \in R$ is weighted Moore-Penrose invertible if and only if $a \in R$ is left invertible along $f^{-1}a^*e$ if and only if $a \in R$ is right invertible along $f^{-1}a^*e$, extending [1, Theorem 3.2]. Also, it is proved that $a \in R$ is weighted Moore-Penrose invertible if and only if $f^{-1}a^*e$ is left invertible along a if and only if $f^{-1}a^*e$ is right invertible along a. Under the assumption $a \in R^\dagger_{e,f}$, we further prove that $a \in R$ is e-core invertible if and only if it is invertible along $af^{-1}a^*e$, and a is f-dual core invertible if and only if it is invertible along $f^{-1}a^*ea$.

2 Characterizations for Weighted Moore-Penrose Inverses

We begin this section with several characterizations for $\{e, 1, 3\}$-inverses and $\{f, 1, 4\}$-inverses of an element in a ring.

Proposition 2.1 *Let $a \in R$ and let $e \in R$ be an invertible* Hermitian *element. Then a is $\{e, 1, 3\}$-invertible if and only if $a \in Ra^*ea$. Moreover, if $a = xa^*ea$ for some $x \in R$, then x^*e is an $\{e, 1, 3\}$-inverse of a.*

Proof Suppose that a is $\{e, 1, 3\}$-invertible. Then we have $a = aa_e^{(1,3)}a = e^{-1}(eaa_e^{(1,3)})^*a = e^{-1}(a_e^{(1,3)})^*a^*ea \in Ra^*ea$.

Conversely, if $a \in Ra^*ea$, then $a = xa^*ea$ for some $x \in R$, and hence $ax^* = xa^*eax^* = xa^*e(xa^*)^*$. So, ax^* is Hermitian.

It follows $ax^*ea = (ax^*)^*ea = xa^*ea = a$ and $(eax^*e)^* = exa^*e = e(ax^*)^*e = eax^*e$ that x^*e is an $\{e, 1, 3\}$-inverse of a.

Proposition 2.2 *Let $a \in R$ and let $f \in R$ be an invertible* Hermitian *element. Then a is $\{f, 1, 4\}$-invertible if and only if $a \in af^{-1}a^*R$. Moreover, if $a = af^{-1}a^*y$ for some $y \in R$, then $f^{-1}y^*$ is an $\{f, 1, 4\}$-inverse of a.*

It is well known that $a \in R^\dagger$ if and only if $a \in aa^*R \cap Ra^*a$. Motivated by this, we derive the characterization of the weighted Moore-Penrose inverse.

Theorem 2.1 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Then $a \in R^\dagger_{e,f}$ if and only if $a \in af^{-1}a^*R \cap Ra^*ea$. Moreover, if $a = xa^*ea = af^{-1}a^*y$ for some $x, y \in R$,* *then* $a^{\dagger}_{e,f} = f^{-1}y^*ax^*e$.

Proof Applying Propositions 2.1–2.2, it is obvious that $a \in R^{\dagger}_{e,f}$ implies $a \in af^{-1}a^*R \cap Ra^*ea$.

Suppose that $a = xa^*ea = af^{-1}a^*y$ for some $x, y \in R$. We next show that $z = f^{-1}y^*ax^*e$ is the weighted Moore-Penrose inverse of a.

Note that $f^{-1}y^*$ and x^*e are inner inverses of a. Then $af^{-1}y^*a = a = ax^*ea$, and consequently $aza = af^{-1}y^*ax^*ea = a$ and $zaz = z$.

Also, $eaz = eaf^{-1}y^*ax^*e = eax^*e = eaa^{(1,3)}_e$, which implies $eaz = (eaz)^*$.

Analogously, $fza = ff^{-1}y^*ax^*ea = y^*ax^*ea = y^*a$. As $y^*a = y^*af^{-1}a^*y$, we get $fza = (fza)^*$.

Thus, $a \in R^{\dagger}_{e,f}$ with $a^{\dagger}_{e,f} = f^{-1}y^*ax^*e$.

We next characterize the weighted Moore-Penrose inverse by ideals. Herein, a lemma is given.

Lemma 2.1 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. We have*

(i) *If $a = af^{-1}a^*eax$ for some $x \in R$, then $f^{-1}(eax)^*$ is both an $\{e, 1, 3\}$-inverse and an $\{f, 1, 4\}$-inverse of a.*

(ii) *If $a = yaf^{-1}a^*ea$ for some $y \in R$, then $(yaf^{-1})^*e$ is both an $\{e, 1, 3\}$-inverse and an $\{f, 1, 4\}$-inverse of a.*

Proof (i) By Proposition 2.2, we know that $f^{-1}(eax)^*$ is an $\{f, 1, 4\}$-inverse of a. To show that $f^{-1}(eax)^*$ is also an $\{e, 1, 3\}$-inverse of a, it is sufficient to prove that $eaf^{-1}(eax)^*$ is Hermitian.

By calculations, we have

$$\begin{aligned} eaf^{-1}(eax)^* &= eaf^{-1}x^*a^*e = eaf^{-1}x^*(af^{-1}a^*eax)^*e \\ &= eaf^{-1}(x^*)^2a^*ea(eaf^{-1})^* \\ &= eaf^{-1}(x^*)^2a^*e(af^{-1}a^*eax)(eaf^{-1})^* \\ &= eaf^{-1}(x^*)^2a^*eaf^{-1}a^*e(af^{-1}a^*eax)x(eaf^{-1})^* \\ &= eaf^{-1}(x^*)^2a^*eaf^{-1}a^*eaf^{-1}a^*eax^2(eaf^{-1})^* \\ &= eaf^{-1}(x^*)^2a^*eaf^{-1}a^*e(af^{-1}a^*ea)x^2(eaf^{-1})^* \\ &= eaf^{-1}(x^*)^2a^*eaf^{-1}a^*e(a^*eaf^{-1}a^*)^*x^2(eaf^{-1})^*. \end{aligned}$$

Hence, $f^{-1}(eax)^*$ is an $\{e, 1, 3\}$-inverse of a.

(ii) It can be proved similarly.

Theorem 2.2 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}_{e,f}$;

(ii) $a \in af^{-1}a^*eaR$;

(iii) $a \in Raf^{-1}a^*ea$.

In this case, $a^{\dagger}_{e,f} = f^{-1}(eax)^* = (yaf^{-1})^*e$, where $x, y \in R$ satisfy $a = af^{-1}a^*eax = yaf^{-1}a^*ea$.

Proof (i) $\Rightarrow$ (ii) Let $a \in R^{\dagger}_{e,f}$. Then

$$\begin{aligned} a &= af^{-1}(fa^{\dagger}_{e,f}a)^* \\ &= af^{-1}a^*(a^{\dagger}_{e,f})^*f \\ &= af^{-1}a^*(a^{\dagger}_{e,f}e^{-1}eaa^{\dagger}_{e,f})^*f \\ &= af^{-1}a^*(eaa^{\dagger}_{e,f})^*(a^{\dagger}_{e,f}e^{-1})^*f \\ &= af^{-1}a^*eaa^{\dagger}_{e,f}(a^{\dagger}_{e,f}e^{-1})^*f. \end{aligned}$$

Hence, $a \in af^{-1}a^*eaR$.

(ii) $\Leftrightarrow$ (iii) Assume that $a \in af^{-1}a^*eaR$. Then there exists $x \in R$ such that $a = af^{-1}a^*eax$, and hence $a^* = x^*a^*eaf^{-1}a^*$. Also, we have $(eax)^*a = (eax)^*af^{-1}a^*eax$, which implies that $(eax)^*a$ is Hermitian.

We obtain

$$\begin{aligned} a &= af^{-1}a^*eax = af^{-1}(eax)^*a = af^{-1}x^*a^*ea \\ &= af^{-1}x^*(x^*a^*eaf^{-1}a^*)ea \\ &= (af^{-1}x^*x^*a^*e)af^{-1}a^*ea. \end{aligned}$$

Thus, $a \in Raf^{-1}a^*ea$.

Conversely, if $a \in Raf^{-1}a^*ea$, then we can similarly obtain $a \in af^{-1}a^*eaR$.

(iii) $\Rightarrow$ (i) As $a \in Raf^{-1}a^*ea$, and consequently $a \in af^{-1}a^*eaR$, then $a \in af^{-1}a^*R \cap Ra^*ea$. It follows from Theorem 2.1 that $a \in R^{\dagger}_{e,f}$.

By Lemma 2.1, we get that $f^{-1}(eax)^*$ is both an $\{e,1,3\}$-inverse and an $\{f,1,4\}$-inverse of a.

Applying Theorem 2.1, we have

$$\begin{aligned} a^{\dagger}_{e,f} &= f^{-1}(eax)^*af^{-1}(eax)^* = f^{-1}(ax)^*eaf^{-1}(eax)^* \\ &= f^{-1}(ax)^*[eaf^{-1}(eax)^*]^* = f^{-1}(ax)^*(eax)f^{-1}a^*e \\ &= f^{-1}x^*a^*(eax)f^{-1}a^*e = f^{-1}x^*[af^{-1}(eax)^*a]^*e \\ &= f^{-1}x^*a^*e \\ &= f^{-1}(eax)^*. \end{aligned}$$

Dually, we can prove that $a^{\dagger}_{e,f} = (yaf^{-1})^*e$.

Set $e = f = 1$ in Theorem 2.2, then we get the characterization for the Moore-Penrose inverse.

Corollary 2.1 (see [9, Theorem 2.16] *Let $a \in R$. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$;

(ii) $a \in aa^*aR$;

(iii) $a \in Raa^*a$.

In this case, $a^{\dagger} = (ax)^* = (ya)^*$, *where* $x, y \in R$ *satisfy* $a = aa^*ax = yaa^*a$.

In 2017, Benítez and Boasso [1] characterized the weighted Moore-Penrose inverse of regular elements by the invertibility of certain elements. Inspired by this, we consider to characterize the weighted Moore-Penrose inverse of regular elements by one-sided invertibilities of some elements. Herein, a lemma is given.

Lemma 2.2 *Let $a, b \in R$.*

(i) *If there exists $c \in R$ such that $(1+ab)c = 1$, then $(1+ba)(1-bca) = 1$.*

(ii) *If there exists $d \in R$ such that $d(1+ab) = 1$, then $(1-bda)(1+ba) = 1$.*

It follows from Lemma 2.2 that $1+ab$ is (left, right) invertible if and only if $1+ba$ is (left, right) invertible. Moreover, $(1+ba)^{-1} = 1 - b(1+ab)^{-1}a$. The formula above is known as Jacobson's lemma.

Theorem 2.3 *Let $a \in R$ be regular and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}_{e,f}$;

(ii) $u = af^{-1}a^*e + 1 - aa^-$ *is left invertible*;

(iii) $v = f^{-1}a^*ea + 1 - a^-a$ *is left invertible*;

(iv) $u = af^{-1}a^*e + 1 - aa^-$ *is right invertible*;

(v) $v = f^{-1}a^*ea + 1 - a^-a$ *is right invertible.*

In this case, $a^{\dagger}_{e,f} = (u_l^{-1}af^{-1})^*e = f^{-1}(eav_r^{-1})^*$, where u_l^{-1} and v_r^{-1} denote a left inverse of u and a right inverse of v, respectively.

Proof (i) $\Rightarrow$ (ii) Suppose that $a \in R^{\dagger}_{e,f}$. Then, by Theorem 2.2, there exists some $y \in R$ such that $a = yaf^{-1}a^*ea$. Write $s = a^-ya + 1 - a^-a$, by a direct check, $s(a^-af^{-1}a^*ea + 1 - a^-a) = 1$. Note that $a^-af^{-1}a^*ea + 1 - a^-a = 1 + (a^-af^{-1}a^*e - a^-)a$. Then, from Lemma 2.2, it follows that $1 + a(a^-af^{-1}a^*e - a^-) = 1 + af^{-1}a^*e - aa^- = u$ is left invertible.

(ii) $\Leftrightarrow$ (iii) It follows from Lemma 2.2.

(iii) $\Rightarrow$ (i) As v, and hence u are both left invertible, then $ua = af^{-1}a^*ea$, and consequently $a = u_l^{-1}af^{-1}a^*ea \in Raf^{-1}a^*ea$. Therefore, by Theorem 2.2, $a \in R^{\dagger}_{e,f}$ and $a^{\dagger}_{e,f} = (u_l^{-1}af^{-1})^*e$.

Analogously, we can prove (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) and $a^{\dagger}_{e,f} = f^{-1}(eav_r^{-1})^*$.

Corollary 2.2 *Let $a \in R$ be regular and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}_{e,f}$;

(ii) $u = af^{-1}a^*e + 1 - aa^-$ *is invertible*;

(iii) $v = f^{-1}a^*ea + 1 - a^-a$ *is invertible.*

In this case, $a^{\dagger}_{e,f} = (u^{-1}af^{-1})^*e = f^{-1}(eav^{-1})^*$.

Corollary 2.3 (see [10, Theorem 3.3]) *Let $a \in R$ be regular and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$;

(ii) $u = aa^* + 1 - aa^-$ *is invertible*;

(iii) $v = a^*a + 1 - a^-a$ *is invertible.*

In this case, $a^{\dagger} = (u^{-1}a)^* = (av^{-1})^*$.

3 Characterizations of e-Core Inverses and f-Dual Core Inverses

Recall that an element $a \in R$ is group invertible if there exists $b \in R$ such that $aba = a$, $bab = b$ and $ab = ba$. Such a b is called a group inverse of a. It is unique if it exists, and is denoted by $a^{\#}$. By $R^{\#}$ we denote the set of all group invertible elements in R. It is well known that $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$ if and only if $a \in a^nR \cap Ra^n$ for any integer $n \geq 2$. In particular, if $a = a^2x = ya^2$ for some $x, y \in R$, then $a^{\#} = yax = y^2a = ax^2$.

In [5], Mosić et al. derived characterizations of e-core inverses by group inverses and $\{e, 1, 3\}$-inverses, and f-dual core inverses by group inverses and $\{f, 1, 4\}$-inverses in rings.

Next, we mainly investigate e-core inverses and f-dual core inverses by the intersection of ideals and units.

Lemma 3.1 *Let $a \in R$ be regular. Then the following conditions are equivalent*:

(i) $a \in R^{\#}$;

(ii) $a + 1 - aa^-$ *is invertible*;

(iii) $a + 1 - a^-a$ *is invertible.*

In this case, $a^{\#} = (a + 1 - aa^-)^{-2}a = a(a + 1 - a^-a)^{-2}$.

Lemma 3.2 (see [6, Theorem 2.1]) *Let $a \in R$ and let $e \in R$ be an invertible* Hermitian *element. Then the following conditions are equivalent*:

(i) *a is e-core invertible*;

(ii) $a \in R^{\#} \cap R_e^{(1,3)}$;

(iii) *there exists $x \in R$ such that* $(eax)^* = eax$, $xa^2 = a$ *and* $ax^2 = x$;

(iv) *there exists $x \in R$ such that* $(eax)^* = eax$, $xa^2 = a$, $ax^2 = x$, $xax = x$ *and* $axa = a$.

In this case, $a^{e,\oplus} = a^{\#}aa_e^{(1,3)}$.

Lemma 3.3 (see [6, Theorem 2.2]) *Let $a \in R$ and let $f \in R$ be an invertible* Hermitian *element. Then the following conditions are equivalent*:

(i) *a is f-dual core invertible*;

(ii) $a \in R^{\#} \cap R_f^{(1,4)}$;

(iii) *there exists $x \in R$ such that* $(fxa)^* = fxa$, $a^2x = a$ *and* $x^2a = x$;

(iv) *there exists $x \in R$ such that* $(fxa)^* = fxa$, $a^2x = a$, $x^2a = x$, $axa = a$ *and* $xax = x$.

In this case, $a_{f,\oplus} = a_f^{(1,4)}aa^{\#}$.

It is known from Theorem 2.1 that $a \in R_{e,f}^{\dagger}$ if and only if $a \in af^{-1}a^*R \cap Ra^*ea$. We next show that if the index n of a^* is no less than 2, then it is the characterization of both e-core invertible and f-dual core invertible elements. More precisely, $a \in R^{e,\oplus} \cap R_{f,\oplus}$ if and only if $a \in af^{-1}(a^n)^*R \cap R(a^n)^*ea$. First, a lemma is given.

Lemma 3.4 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Suppose that $n \geq 2$ is an integer. Then*

(i) $a \in af^{-1}a^*R \cap Ra^n$ *if and only if* $a \in af^{-1}(a^*)^nR$.

(ii) $a \in Ra^*ea \cap a^nR$ *if and only if* $a \in R(a^*)^nea$.

Proof (i) "$\Rightarrow$" If $a \in af^{-1}a^*R \cap Ra^n$, then there exist some $s, t \in R$ such that $a = af^{-1}a^*s = ta^n$, and hence $a = af^{-1}(ta^n)^*s = af^{-1}(a^*)^nt^*s \in af^{-1}(a^*)^nR$.

" $\Leftarrow$ " If $a \in af^{-1}(a^*)^nR$, then $a = af^{-1}(a^*)^n r$ for some $r \in R$. This implies $a \in af^{-1}a^*R$ and $a \in R_f^{(1,4)}$ by Proposition 2.2. Moreover, $f^{-1}((a^*)^{n-1}r)^*$ is an $\{f,1,4\}$-inverse of a. Hence, we have $a = aa_f^{(1,4)}a = af^{-1}((a^*)^{n-1}r)^*a = af^{-1}r^*a^n \in Ra^n$.

(ii) It can be proved by a similar way as (i).

Theorem 3.1 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Suppose that $n \geq 2$ is an integer. Then the following conditions are equivalent*:

(i) $a \in R^{e,\oplus} \cap R_{f,\oplus}$;

(ii) $a \in af^{-1}(a^n)^*R \cap R(a^n)^*ea$.

Proof From Theorem 2.1 and Lemmas 3.2–3.3, it is known that $a \in R^{e,\oplus} \cap R_{f,\oplus}$ if and only if $a \in R_{e,f}^{\dagger} \cap R^{\#}$ if and only if $a \in af^{-1}a^*R \cap Ra^*ea \cap a^nR \cap Ra^n$. Again by Lemma 3.4, $a \in R^{e,\oplus} \cap R_{f,\oplus}$ if and only if $a \in af^{-1}(a^n)^*R \cap R(a^n)^*ea$, as required.

The following result gives the characterization of both e-core invertible and f-dual core invertible elements by units in a ring R.

Theorem 3.2 *Let $a \in R$ be regular and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\#} \cap R_{e,f}^{\dagger}$;

(ii) $a \in R^{e,\oplus} \cap R_{f,\oplus}$;

(iii) $u = af^{-1}a^*ea + 1 - aa^-$ *is invertible*;

(iv) $v = f^{-1}a^*ea^2 + 1 - a^-a$ *is invertible*;

(v) $s = af^{-1}a^*ea + 1 - a^-a$ *is invertible*;

(vi) $t = a^2f^{-1}a^*e + 1 - aa^-$ *is invertible.*

In this case, $a^{e,\oplus} = u^{-1}af^{-1}a^*e$ *and* $a_{f,\oplus} = f^{-1}a^*eas^{-1}$.

Proof (i) $\Leftrightarrow$ (ii) It follows from Theorem 2.1 and Lemmas 3.2–3.3.

(ii) $\Rightarrow$ (iii) As $a \in R^{e,\oplus} \cap R_{f,\oplus}$, and hence $a \in R^{\#} \cap R_{e,f}^{\dagger}$. By Lemma 3.1, $a \in R^{\#}$ implies that $a + 1 - aa^-$ is invertible. Also, $a \in R_{e,f}^{\dagger}$ guarantees that $af^{-1}a^*e + 1 - aa^-$ is invertible by Corollary 2.2, and hence $af^{-1}a^*eaa^- + 1 - aa^-$ is invertible by Lemma 2.2. So, $(af^{-1}a^*eaa^- + 1 - aa^-)(a + 1 - aa^-) = af^{-1}a^*ea + 1 - aa^- = u$ is invertible.

(iii) $\Leftrightarrow$ (iv) By Lemma 2.2.

(iv) $\Rightarrow$ (i) Since $v = f^{-1}a^*ea^2 + 1 - a^-a$ is invertible, we have $av = af^{-1}a^*ea^2$ and hence $a = af^{-1}a^*ea^2v^{-1} \in af^{-1}a^*eaR$. Hence, $a \in R_{e,f}^{\dagger}$ and $a_{e,f}^{\dagger} = f^{-1}(ea^2v^{-1})^*$ by Theorem 2.2. Again, from Corollary 2.2 and Lemma 2.2, we obtain that $af^{-1}a^*eaa^- + 1 - aa^-$ is invertible, and consequently $a + 1 - aa^- = (af^{-1}a^*eaa^- + 1 - aa^-)^{-1}u$ is invertible, which gives $a \in R^{\#}$ by Lemma 3.1. Hence, $a \in R^{\#} \cap R_{e,f}^{\dagger}$.

Analogously, we can prove (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

As $a^{\#} = (u^{-1}af^{-1}a^*e)^2a$ and $a = u^{-1}af^{-1}a^*ea^2$, by applying Lemma 3.2, we get

$$\begin{aligned} a^{e,\oplus} &= a^{\#}aa_{e,f}^{\dagger} \\ &= (u^{-1}af^{-1}a^*e)^2a^2a_{e,f}^{\dagger} \\ &= u^{-1}af^{-1}a^*e(u^{-1}af^{-1}a^*ea^2)a_{e,f}^{\dagger} \\ &= u^{-1}af^{-1}a^*eaa_{e,f}^{\dagger} \\ &= u^{-1}af^{-1}a^*(eaa_{e,f}^{\dagger})^* \end{aligned}$$

$$= u^{-1}af^{-1}(eaa^{\dagger}_{e,f}a)^*$$
$$= u^{-1}af^{-1}(ea)^*$$
$$= u^{-1}af^{-1}a^*e.$$

Also, as $a = a^2(f^{-1}a^*eas^{-1})$, we obtain $a^{\#} = a(f^{-1}a^*eas^{-1})^2$ and

$$\begin{aligned} a_{f,\oplus} &= a^{\dagger}_{e,f}aa^{\#} \\ &= a^{\dagger}_{e,f}a^2(f^{-1}a^*eas^{-1})^2 \\ &= a^{\dagger}_{e,f}(a^2f^{-1}a^*eas^{-1})f^{-1}a^*eas^{-1} \\ &= a^{\dagger}_{e,f}af^{-1}a^*eas^{-1} \\ &= f^{-1}fa^{\dagger}_{e,f}af^{-1}a^*eas^{-1} \\ &= f^{-1}(fa^{\dagger}_{e,f}a)^*(af^{-1})^*eas^{-1} \\ &= f^{-1}(af^{-1}fa^{\dagger}_{e,f}a)^*eas^{-1} \\ &= f^{-1}a^*eas^{-1}. \end{aligned}$$

The proof is completed.

If $e = 1$, then the e-core inverse is just the core inverse. If $f = 1$, then the f-dual core inverse is the dual core inverse. By $R^{\oplus}$ and $R_{\oplus}$ we denote the sets of all core invertible and dual core invertible elements in R.

Corollary 3.1 (see [2, Theorem 5.6]) *Let $a \in R$ be regular. Then the following conditions are equivalent*:

(i) $a \in R^{\#} \cap R^{\dagger}$;
(ii) $a \in R^{\oplus} \cap R_{\oplus}$;
(iii) $u = aa^*a + 1 - aa^-$ *is invertible*;
(iv) $v = a^*a^2 + 1 - a^-a$ *is invertible*;
(v) $s = aa^*a + 1 - a^-a$ *is invertible*;
(vi) $t = a^2a^* + 1 - aa^-$ *is invertible.*

In this case, $a^{\oplus} = u^{-1}aa^*$ *and* $a_{\oplus} = a^*as^{-1}$.

By Corollary 3.1, we know that the core and dual core inverses of a are characterized by the invertibility of $a^2a^* + 1 - aa^-$. In [3, Theorem 4.1], Li and Chen proved that the result is true when the quadratic component a^2a^* in $a^2a^* + 1 - aa^-$ is changed to $a(a^*)^2$. More precisely, $a \in R^{\oplus} \cap R_{\oplus}$ if and only if $a(a^*)^2 + 1 - aa^-$ is invertible if and only if $(a^*)^2a + 1 - a^-a$ is invertible.

For the case of the e-core inverse and the f-dual core inverse, one can also get their similar characterizations.

Theorem 3.3 *Let $a \in R$ be regular and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{e,\oplus} \cap R_{f,\oplus}$;
(ii) $u = f^{-1}(a^2)^*ea + 1 - a^-a$ *is invertible*;
(iii) $v = af^{-1}(a^2)^*e + 1 - aa^-$ *is invertible.*

Proof It follows from Lemma 2.2 that (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (ii) Note that the equality $a^2_{f,\oplus}a = a_{f,\oplus}$. Then $a^2_{f,\oplus}a^2 = a_{f,\oplus}a$. Write $s = a^-aa^{e,\oplus}(a^2_{f,\oplus}e^{-1})^*f + 1 - a_{f,\oplus}a$, by a direct check, we get $us = 1$. Indeed,

$$\begin{aligned} us &= (f^{-1}(a^2)^*ea + 1 - a^-a)(a^-aa^{e,\oplus}(a^2_{f,\oplus}e^{-1})^*f + 1 - a_{f,\oplus}a) \\ &= f^{-1}(a^2)^*eaa^{e,\oplus}(a^2_{f,\oplus}e^{-1})^*f + 1 - a_{f,\oplus}a \\ &= f^{-1}(a^2)^*(eaa^{e,\oplus})^*(a^2_{f,\oplus}e^{-1})^*f + 1 - a_{f,\oplus}a \\ &= f^{-1}(a^2)^*(fa^2_{f,\oplus}e^{-1}eaa^{e,\oplus})^* + 1 - a_{f,\oplus}a \\ &= f^{-1}(fa^2_{f,\oplus}aa^{e,\oplus}a^2)^* + 1 - a_{f,\oplus}a \\ &= f^{-1}(fa^2_{f,\oplus}a^2)^* + 1 - a_{f,\oplus}a \\ &= f^{-1}(fa_{f,\oplus}a)^* + 1 - a_{f,\oplus}a \\ &= f^{-1}fa_{f,\oplus}a + 1 - a_{f,\oplus}a \\ &= a_{f,\oplus}a + 1 - a_{f,\oplus}a \\ &= 1. \end{aligned}$$

So, u is right invertible and s is a right inverse of u.

Similarly, set $t = e^{-1}((a^{e,\oplus})^2)^*fa_{f,\oplus}aa^- + 1 - aa^{e,\oplus}$, we can prove $tv = 1$. Hence, $v = af^{-1}(a^2)^*e + 1 - aa^-$ is left invertible, and consequently u is left invertible by Lemma 2.2.

(ii) $\Rightarrow$ (i) As u, and hence v are both invertible, also, we have $au = af^{-1}(a^2)^*ea = va$. Hence, $a = af^{-1}(a^2)^*eau^{-1} = v^{-1}af^{-1}(a^2)^*ea$, which implies $a \in af^{-1}(a^2)^*R \cap R(a^2)^*ea$, and by Theorem 3.1, $a \in R^{e,\oplus} \cap R_{f,\oplus}$.

4 Relations with (one-sided) Inverses along an Element

Given $a, d \in R$, a is left invertible along d (see [7]) if there exists $b \in R$ such that $bad = d$ and $b \in Rd$. Such b is called a left inverse of a along d, and is denoted by $a_l^{\|d}$. Dually, we call a is right invertible along d (see [7]) if there exists $b \in R$ satisfying $dab = b$ and $b \in dR$. A right inverse of a along d is denoted by $a_r^{\|d}$.

Lemma 4.1 (see [9, Theorems 2.3–2.4]) *Let $a, d \in R$. Then*

(i) *a is left invertible along d if and only if $d \in Rdad$.*

(ii) *a is right invertible along d if and only if $d \in dadR$.*

An element $a \in R$ is called invertible along d if there exists $b \in R$ such that $bad = d = dab$ and $b \in dR \cap Rd$. The inverse of a along d is unique if it exists, and is denoted by $a^{\|d}$. Hence, if a is both left and right invertible along d, then a is invertible along d and $a^{\|d} = a_l^{\|d} = a_r^{\|d}$. Also, it follows from Lemma 4.1 that a is invertible along d if and only if $d \in dadR \cap Rdad$. More results on the inverse along an element can be referred to [9–11].

Recently, Benítez and Boasso derived the equivalence between $a^\dagger_{e,f}$ and $a^{\|f^{-1}a^*e}$ (see [1, Theorem 3.2]). We next consider to characterize $a^\dagger_{e,f}$ by one-sided inverse of a along $f^{-1}a^*e$.

Theorem 4.1 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^\dagger_{e,f}$;

(ii) *a is left invertible along $f^{-1}a^*e$;*

(iii) *a is right invertible along $f^{-1}a^*e$.*
In this case, $a^{\dagger}_{e,f} = a_l^{\| f^{-1}a^*e} = a_r^{\| f^{-1}a^*e}$.

Proof (i) $\Leftrightarrow$ (ii) Suppose that $a^{\dagger}_{e,f}$ exists. Then, by Theorem 2.2, $a \in af^{-1}a^*eaR$, which yields $a^* \in Ra^*eaf^{-1}a^*$. As f is invertible, $a^* \in Rf^{-1}a^*eaf^{-1}a^*$ and hence $a^*e \in Rf^{-1}a^*eaf^{-1}a^*e$. Thus, we get $f^{-1}a^*e \in f^{-1}Rf^{-1}a^*eaf^{-1}a^*e = Rf^{-1}a^*eaf^{-1}a^*e$. It follows from Lemma 4.1 that a is left invertible along $f^{-1}a^*e$.

Conversely, as a is left invertible along $f^{-1}a^*e$, by Lemma 4.1, $f^{-1}a^*e \in Rf^{-1}a^*eaf^{-1}a^*e$, and consequently $a^* \in Rf^{-1}a^*eaf^{-1}a^*$. So, $a \in af^{-1}a^*eaf^{-1}R = af^{-1}a^*eaR$. Again, applying Theorem 2.2, $a \in R^{\dagger}_{e,f}$.

(i) $\Leftrightarrow$ (iii) It can be proved analogously.

Let $a_l^{\| f^{-1}a^*e} = b$. Then there exists some $x \in R$ such that $b = xf^{-1}a^*e$. Since $f^{-1}a^*e = baf^{-1}a^*e = (xf^{-1}a^*e)af^{-1}a^*e$, multiplying the above equality by f on the left and e^{-1} on the right gives $a^* = fxf^{-1}a^*eaf^{-1}a^*$ and hence $a = af^{-1}a^*eaf^{-1}x^*f$. We obtain $a^{\dagger}_{e,f} = f^{-1}(eaf^{-1}x^*f)^* = xf^{-1}a^*e = a_l^{\| f^{-1}a^*e}$ by Theorem 2.2.

Dually, we can get $a^{\dagger}_{e,f} = a_r^{\| f^{-1}a^*e}$.

The following result shows that $a \in R^{\dagger}_{e,f}$ if and only if $f^{-1}a^*e$ is left (resp. right) invertible along a, whose proof is essentially the same as Theorem 4.1 above.

Theorem 4.2 *Let $a \in R$ and let $e, f \in R$ be invertible* Hermitian *elements. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}_{e,f}$;

(ii) *$f^{-1}a^*e$ is left invertible along a*;

(iii) *$f^{-1}a^*e$ is right invertible along a.*
In this case, $a^{\dagger}_{e,f} = (f^{-1}a^*e)_l^{\| a} = (f^{-1}a^*e)_r^{\| a}$.

Mary and Patrício [4] derived the characterization for the existence of $a^{\| d}$, i.e., a is invertible along d if and only if $u = da + 1 - dd^-$ is invertible, provided that d is regular. Hence, $af^{-1}a^*e$ is invertible along a if and only if $a^2f^{-1}a^*e + 1 - aa^-$ is invertible.

It follows from Theorem 3.2 that $a \in R^{e,\oplus} \cap R_{f,\oplus}$ if and only if $a^2f^{-1}a^*e + 1 - aa^-$ is invertible. Hence, a is both e-core and f-dual core invertible if and only if $af^{-1}a^*e$ is invertible along a.

It is natural to consider whether we can characterize the e-core inverse (resp. the f-dual core inverse) by the inverse of an element. We next show the fact that a is e-core invertible if and only if it is invertible along $af^{-1}a^*e$, and a is f-dual core invertible if and only if it is invertible along $f^{-1}a^*ea$, under the assumption $a \in R^{\dagger}_{e,f}$.

Theorem 4.3 *Let $a \in R^{\dagger}_{e,f}$. Then a is e-core invertible if and only if it is invertible along $af^{-1}a^*e$. In this case,* $a^{e,\oplus} = a^{\| af^{-1}a^*e}$.

Proof Suppose that a is invertible along $af^{-1}a^*e$ with $x = a^{\| af^{-1}a^*e}$. Then, we have

$$xa^2f^{-1}a^*e = af^{-1}a^*e = af^{-1}a^*eax, \quad x \in af^{-1}a^*eR \cap Raf^{-1}a^*e.$$

By a direct calculation, it follows

$$eax = (eaa^{\dagger}_{e,f})^*ax = (a^{\dagger}_{e,f})^*a^*eax$$

$$
\begin{aligned}
&= (fa^{\dagger}_{e,f})^{*}f^{-1}a^{*}eax \\
&= (fa^{\dagger}_{e,f}aa^{\dagger}_{e,f})^{*}f^{-1}a^{*}eax \\
&= (a^{\dagger}_{e,f})^{*}fa^{\dagger}_{e,f}(af^{-1}a^{*}eax) \\
&= (a^{\dagger}_{e,f})^{*}fa^{\dagger}_{e,f}(af^{-1}a^{*}e) \\
&= (a^{\dagger}_{e,f})^{*}(f^{-1}fa^{\dagger}_{e,f}a)^{*}a^{*}e \\
&= (a^{\dagger}_{e,f}aa^{\dagger}_{e,f})^{*}a^{*}e \\
&= (a^{\dagger}_{e,f})^{*}a^{*}e \\
&= (eaa^{\dagger}_{e,f})^{*} \\
&= eaa^{\dagger}_{e,f},
\end{aligned}
$$

which implies $eax = (eax)^{*}$.

As e is an invertible Hermitian element, $ax = aa^{\dagger}_{e,f}$ and hence $axa = a$. Since $x \in af^{-1}a^{*}eR$, there exists some $y \in R$ such that $x = af^{-1}a^{*}ey = axaf^{-1}a^{*}ey = ax^{2}$.

Similarly, we get

$$
\begin{aligned}
xa^{2} &= xa^{2}f^{-1}fa^{\dagger}_{e,f}a \\
&= xa^{2}f^{-1}a^{*}(a^{\dagger}_{e,f})^{*}f \\
&= (xa^{2}f^{-1}a^{*}e)e^{-1}(a^{\dagger}_{e,f})^{*}f \\
&= af^{-1}a^{*}ee^{-1}(a^{\dagger}_{e,f})^{*}f \\
&= af^{-1}(fa^{\dagger}_{e,f}a)^{*} \\
&= a.
\end{aligned}
$$

Therefore, $x = a^{\parallel af^{-1}a^{*}e}$ is the e-core inverse of a.

Conversely, suppose that $a \in R^{e,\oplus}$ with $a^{e,\oplus} = z$. Then, by Lemma 3.2, $aza = a$, $zaz = a$, $az^{2} = z$, $za^{2} = a$ and $eaz = (eaz)^{*}$. To show that z is the inverse of a along $d = af^{-1}a^{*}e$, it is sufficient to prove $zad = d = daz$ and $z \in dR \cap Rd$.

We get $zad = zaaf^{-1}a^{*}e = za^{2}f^{-1}a^{*}e = af^{-1}a^{*}e$ and $daz = af^{-1}a^{*}eaz = af^{-1}a^{*}(eaz)^{*} = af^{-1}(eaza)^{*} = af^{-1}(ea)^{*} = af^{-1}a^{*}e = d$.

Since $az = aa^{(1,3)}_{e} = aa^{\dagger}_{e,f}$ and $z = az^{2}$, we have

$$
\begin{aligned}
z &= aa^{\dagger}_{e,f}z \\
&= af^{-1}(fa^{\dagger}_{e,f}a)^{*}a^{\dagger}_{e,f}z \\
&= af^{-1}a^{*}(a^{\dagger}_{e,f})^{*}fa^{\dagger}_{e,f}z \\
&= af^{-1}a^{*}(a^{\dagger}_{e,f}e^{-1}eaa^{\dagger}_{e,f})^{*}fa^{\dagger}_{e,f}z \\
&= af^{-1}a^{*}eaa^{\dagger}_{e,f}e^{-1}(a^{\dagger}_{e,f})^{*}fa^{\dagger}_{e,f}z,
\end{aligned}
$$

which gives $z \in dR$.

Also, note the equality $z = zaz$, we can obtain $z \in Rd$.

Therefore, a is invertible along $af^{-1}a^{*}e$.

Theorem 4.4 *Let $a \in R^{\dagger}_{e,f}$. Then a is f-dual core invertible if and only if it is invertible along $f^{-1}a^{*}ea$. In this case, $a_{f,\oplus} = a^{\parallel f^{-1}a^{*}ea}$.*

Corollary 4.1 (see [8, Theorem 4.3]) *Let* $a \in R^{\dagger}$*. Then*

(i) *a is core invertible if and only if it is invertible along* aa^**. In this case,* $a^{\oplus} = a^{\| aa^*}$*.*

(ii) *a is dual core invertible if and only if it is invertible along* a^*a*. In this case,* $a_{\oplus} = a^{\| a^*a}$*.*

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