Chin. Ann. Math. Ser. B
42(4), 2021, 625–642
DOI: 10.1007/s11401-021-0283-4

Chinese Annals of Mathematics, Series B

Perfect State Transfer on Weighted Abelian Cayley Graphs*

Xiwang CAO[1] Keqin FENG[2] Ying-Ying TAN[3]

Abstract Recently, there are extensive studies on perfect state transfer (PST for short) on graphs due to their significant applications in quantum information processing and quantum computations. However, there is not any general characterization of graphs that have PST in literature. In this paper, the authors present a depiction on weighted abelian Cayley graphs having PST. They give a unified approach to describe the periodicity and the existence of PST on some specific graphs.

Keywords: Perfect state transfer, Cayley graph, Eigenvalues of a graph, Weighted graph, Random walk
2000 MR Subject Classification: 05C25, 81P45, 81Q35

1 Introduction

Throughout this paper, we use $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ to stand for the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively.

A weighted graph Γ is a triple system $(V, E; \alpha)$, where $V = \{v_1, \cdots, v_n\}$ is a finite set, E is a subset of $V \times V$, and α is a complex-valued function, called a weight function, on E. The adjacency matrix of Γ is defined as

$$A = [a_{ij}]_{i,j=1}^n, \quad \text{where } a_{ij} = \alpha(v_i, v_j).$$

The eigenvalues of A will be referred to as the eigenvalues or the spectra of the graph Γ. A graph is named an integral graph if all its eigenvalues are integers.

Suppose that G is a finite group. A weighted Cayley graph $\Gamma = \mathrm{Cay}(G; \alpha)$ is just a triple system $(G, E; \alpha)$, where $E \subseteq G \times G$ and α is a complex-valued function such that the weight function, which is also denoted by α, satisfies

$$\alpha(g, h) = \alpha(g^{-1}h), \quad \forall g, h \in G.$$

We assume that there is no edge from g to gh if $\alpha(h)$ is zero. If the value set of the weight function α is $\{0, 1\}$ and the support set of α, i.e., $S = \{g \in G \mid \alpha(g) = 1\}$, generates G, then

Manuscript received May 14, 2019. Revised October 7, 2020.
[1]Corresponding author. Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China. E-mail: xwcao@nuaa.edu.cn
[2]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail: fengkq@mail.tsinghua.edu.cn
[3]School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China.
E-mail: tansusan1@ahjzu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11771007, 11601003, 11801007, 12031011) and Natural Science Foundation of Anhui Province (No. 1808085MA17).

Γ is a Cayley digraph, denoted by $\mathrm{Cay}(G, S)$. Particularly, if S is symmetric, i.e, $S = S^{-1} := \{s^{-1} \mid s \in S\}$ and S does not contain the identity element of G, then Γ is an undirected graph and is the usual Cayley graph. All the graphs considered in this paper are simple graphs, i.e, undirected connected graphs without loops.

A continuous random walk on Γ is determined by a family of matrices of the form $M(t)$, indexed by the vertices of Γ and parameterized by a real positive time t. The (u, v)-entry of $M(t)$ represents the probability of starting at vertex u and reaching vertex v at time t. Define a continuous random walk on Γ by setting

$$M(t) = \exp(\imath t(D - A)),$$

where $\imath = \sqrt{-1}$ and D is a diagonal matrix. Then each column of $M(t)$ corresponds to a probability density of a walk whose initial state is the vertex indexing the column.

For quantum computations, Fahri and Gutmann [10] proposed an analogue continuous quantum walk. For a connected simple graph Γ with adjacency matrix A, they define the transfer matrix of Γ as the following $n \times n$ matrix:

$$H(t) := \exp(\imath t A) = \sum_{s=0}^{+\infty} \frac{(\imath t A)^s}{s!} = (H_{g,h}(t))_{g,h \in V}, \quad t \in \mathbb{R},$$

where $n = |V|$ is the number of vertices in Γ.

Definition 1.1 *Let Γ be a graph. For $u, v \in V$, we say that Γ exhibits perfect state transfer* (PST *for short*) *from u to v at a time t (> 0) if the (u, v)-entry of $H(t)$, denoted by $H(t)_{uv}$, has absolute value* 1. *Further, when $|H(t)_{uu}| = 1$, we say that Γ is periodic at u with period t. If Γ is periodic with period t at every point, then Γ is named periodic.*

We say that Γ admits PST if there are two vertices u and v such that Γ has PST from u to v at some time $t > 0$.

Since $H(t)$ is a unitary matrix, if PST happens in the graph from u to v, then the entries in the u-th row and the entries in the v-th column of $H(t)$ are all zero except for the (u, v)-th entry. That is, the probability starting from u to v is absolutely 1, which is an idea model of state transferring. In other words, quantum walks on finite graphs provide useful simple models for quantum state transport. This phenomenon was first discovered by Bose [4] and was applied to spin chains for communication links in quantum computing. Some new quantum algorithmic computing techniques in this aspect were provided by Childs [7] and Farhi et al [10] around the same time. These algorithms are remarkable since they provably beat the corresponding classical resource bounds. For more background of applications of PST, we refer the readers to [4, 9] and the references therein.

Quantum walks on weighted graphs have been proposed as an efficient way to transfer quantum states (and therefore quantum information) with perfect fidelity without requiring external control (see [8]). Casaccino et al. [5] noticed that it is possible to achieve PST by using suitable energy shifts (by adding weighted self-loops) on two vertices of complete graphs, or on complete graphs with a missing link (even though there is no PST in certain unweighted cases).

In a previous paper [18], we presented a characterization on connected simple Cayley graphs $\Gamma = \mathrm{Cay}(G, S)$ having PST. We gave a unified interpretation of many previously known results. We provided several new results including the answers to the questions raised in [2, 11–12].

However, even though there are a lot of researches on PST, there is no general characterization on graphs which exhibit PST in literature. In this paper, we extend the results in [18] to weighted abelian Cayley graphs. We give a unified characterization of weighted abelian Cayley graphs having PST. Since weighted graphs and Cayley graphs are special kinds of weighted Cayley graphs, we can use our main results (Theorems 2.1–2.2) to explain many prior results on the existence of PST on circulant graphs (the underlying group is a cyclic group) and cubelike graphs (the underlying group is the addition group of a finite field of characteristic two). In [18], we proved that if $\Gamma = \mathrm{Cay}(G, S)$ is a connected simple abelian Cayley graph with $4 < |G| \equiv 2 \pmod 4$, then G cannot have PST between two distinct vertices. As an application of Theorem 2.2, we show that the same conclusion holds for integral weighted abelian Cayley graphs under certain conditions (Theorem 2.4). Conversely, if we assign the weight function properly, then we can get a connected simple weighted graph $\Gamma = \mathrm{Cay}(G; \alpha)$ having PST even if the order of G is not doubly even (Theorem 2.5). We provide a lower bound on the minimum time t at which a weighted Cayley graph has PST between two distinct vertices (Theorem 3.1) and show that this bound is tight (Theorem 3.2).

2 A Characterization on Weighted Abelian Cayley Graphs Having PST

Note that in this paper, from now on, all the groups are abelian (additive) and the identity element of the concerned group is denoted as 0. "gcd$(\cdots)$" stands for the greatest common divisor of some integers. In order to compute the transfer matrix of the weighted Cayley graph, we need to diagonalize the adjacency matrix A. Before doing that, we need some preliminaries on the dual group of an abelian group. Assume that an abelian group G has the following decomposition

$$G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \quad (n_s \geq 2),$$

where $\mathbb{Z}_m = (\mathbb{Z}/m\mathbb{Z}, +)$ is a cyclic group of order m. For every $x = (x_1, \cdots, x_r) \in G$, the mapping

$$\chi_x : G \to \mathbb{C}, \quad \chi_x(g) = \prod_{s=1}^{r} \omega_{n_s}^{x_s g_s} \quad \text{for } g = (g_1, \cdots, g_r) \in G$$

is a character of G, where $\omega_{n_s} = \exp\left(\frac{2\pi\imath}{n_s}\right)$ is a primitive n_s-th root of unity in $\mathbb{C}$. Obviously, we have $\chi_x(g) = \chi_g(x)$ for all $x, g \in G$.

For spectrum of the weighted Cayley graph $\Gamma = \mathrm{Cay}(G; \alpha)$, we have the following result.

Lemma 2.1 (see [15]) *Let G be an abelian group of order n and $\{\lambda_g \mid g \in G\}$ be the set of spectra of the weighted* Cayley *graph $\Gamma = \mathrm{Cay}(G; \alpha)$. Then we have*

$$\lambda_g = \sum_{h \in G} \alpha(h) \chi_h(g), \quad g \in G,$$

where $\widehat{G} := \{\chi_h \mid h \in G\}$ is the dual group of G consisting of the characters of G.

Consider the following $n \times n$ matrix

$$P := \frac{1}{\sqrt{n}} (\gamma_{g,h})_{g,h \in G}, \quad \gamma_{g,h} := \chi_g(h), \ n = |G|.$$

By the orthogonal relation of characters, we know that P is a unitary matrix, i.e., $PP^* = I_n = P^*P$, where P^* means the conjugate transpose of P.

Let D be the following diagonal matrix

$$D = \mathrm{diag}(\lambda_g : g \in G) = (d_{g,h}), \quad d_{g,h} = \lambda_g \delta_{g,h},$$

where $\delta_{g,h} = 1$ if $g = h$ and 0 otherwise. Let A be the adjacency matrix of $\Gamma = \mathrm{Cay}(G; \alpha)$ and let $AP = (\eta_{g,h}), PD = (\nu_{g,h})$. Then for every $g, h \in G$,

$$\eta_{g,h} = \frac{1}{\sqrt{n}} \sum_{k \in G} \alpha(g,k)\gamma_{k,h} = \frac{\chi_h(g)}{\sqrt{n}} \sum_{k \in G} \alpha(k)\chi_h(k) = \frac{\chi_h(g)}{\sqrt{n}}\lambda_h.$$

Note that the last equality follows from Lemma 2.1. Moreover,

$$\nu_{g,h} = \frac{1}{\sqrt{n}} \sum_{k \in G} \gamma_{g,k} d_{k,h} = \frac{1}{\sqrt{n}} \sum_{k \in G} \chi_g(k)\lambda_k \delta_{k,h} = \frac{\chi_h(g)}{\sqrt{n}}\lambda_h.$$

Thus $P^*AP = D$ and

$$H(t) = \exp(\imath tA) = P \exp(-\imath tD)P^* = P \cdot \mathrm{diag}(\exp(-\imath t\lambda_g) : g \in G) \cdot P^* = (H_{g,h}(t))_{g,h \in G},$$

where

$$\begin{aligned} H_{g,h}(t) &= \frac{1}{n} \sum_{x,y \in G} \gamma_{g,x} \exp(-\imath t\lambda_x)\delta_{x,y}\overline{\gamma_{h,y}} \\ &= \frac{1}{n} \sum_{x \in G} \exp(-\imath t\lambda_x)\chi_g(x)\overline{\chi_h(x)} \\ &= \frac{1}{n} \sum_{x \in G} \exp(-\imath t\lambda_x)\chi_a(x), \end{aligned}$$

where $a = g - h$. Therefore,

$$|H_{g,h}(t)| = 1 \text{ if and only if } \Big| \sum_{x \in G} \exp(-\imath t\lambda_x)\chi_a(x) \Big| = n.$$

If we further assume that $\overline{\alpha(g)} = \alpha(-g)$ for all $g \in G$, where the "$\overline{\cdot}$" means the conjugate of a complex number, then for every $h \in G$, $\overline{\lambda_h} = \sum_{g \in G} \overline{\alpha(g)\chi_h(g)} = \sum_{g \in G} \alpha(-g)\chi_h(-g) = \sum_{g \in G} \alpha(g)\chi_h(g) = \lambda_h$, and then λ_x is a real number. Therefore $|\exp(-\imath t\lambda_x)\chi_a(x)| = 1$ for all $x \in G$. Then $|H_{g,h}(t)| = 1$ if and only if all $\exp(-\imath t\lambda_x)\chi_a(x)$ $(x \in G)$ are the same number. For $x = 0$, we have $\lambda_0 = \sum_{g \in G} \alpha(g)$. Thus we obtain the following preliminary result.

Lemma 2.2 *Let G be an abelian group and $\{\lambda_g \mid g \in G\}$ be the set of spectra of the weighted* Cayley *graph $\Gamma = \mathrm{Cay}(G; \alpha)$. Assume that for every $z \in G$, $\overline{\alpha(z)} = \alpha(-z)$. For $g, h \in G$, let $a = g - h$. Then the following statements are equivalent*:

(1) Γ *has* PST *between vertices g and h at time* $t > 0$;

(2) *for any* $x (\neq 0) \in G$, $\chi_a(x) = \exp(\imath t(\lambda_0 - \lambda_x))$.

As a consequence, we have the following simple corollaries.

Corollary 2.1 *Let G be an abelian group of order n. Let α be a weight function satisfying $\overline{\alpha(z)} = \alpha(-z)$ for every $z \in G$ and $\Gamma = \mathrm{Cay}(G;\alpha)$ be the corresponding weighted abelian* Cayley *graph. Then for $g, h \in G$, Γ has* PST *between g and h if and only if Γ has* PST *between $g+z$ and $h+z$ for all $z \in G$.*

Proof It follows directly from Lemma 2.2 (2).

Corollary 2.2 *Let G be an abelian group of order n and $\Gamma = \mathrm{Cay}(G;\alpha)$ be a weighted abelian* Cayley *graph with the weight function satisfying $\overline{\alpha(z)} = \alpha(-z)$ for every $z \in G$. Let $\Gamma' = \Gamma'(G;\alpha')$ be another weighted graph, where the weight function α' is defined by $\alpha'(z) = \alpha(z) + s$ for all $z \in G$, s is a real number. Suppose that $\frac{nst}{2\pi} \in \mathbb{Z}$. Then for $g, h \in G$, the following statements are equivalent*:

(1) Γ *has* PST *between g and h at time $t > 0$;

(2) Γ' *has* PST *between g and h at time $t > 0$.*

Proof Assume that Γ has PST between g and h at a time $t > 0$. Then by Lemma 2.2 (2), for any $x(\neq 0) \in G$, $\chi_a(x) = \exp(\imath t(\lambda_0 - \lambda_x))$, where $a = g - h$. Now, $\overline{\alpha'(z)} = \overline{\alpha(z)} + s = \alpha(-z) + s = \alpha'(-z)$ for all $z \in G$ and

$$\begin{aligned}\lambda'_x &= \sum_{z\in G}\alpha'(z)\chi_z(x) = \sum_{z\in G}(\alpha(z)+s)\chi_z(x)\\ &= \sum_{z\in G}\alpha(z)\chi_x(z) + s\sum_{z\in G}\chi_z(x) = \sum_{z\in G}\alpha(z)\chi_x(z) = \lambda_x,\end{aligned}$$

and

$$\lambda'_0 = \sum_{z\in G}\alpha'(z) = \sum_{z\in G}(\alpha(z)+s) = \sum_{z\in G}\alpha(z) + ns = \lambda_0 + ns.$$

Thus

$$\exp(\imath t(\lambda'_0 - \lambda'_x)) = \exp(2\pi\imath T(\lambda_0 - \lambda_x + ns)) = \exp(2\pi\imath T(\lambda_0 - \lambda_x)) = \chi_a(x).$$

By Lemma 2.2 (2) again, Γ' has PST between g and h at time t. By symmetry, the stated equivalence follows. This completes the proof.

Moreover, we have the following result on weighted abelain Cayley graphs having PST.

Proposition 2.1 *Let $\Gamma = \mathrm{Cay}(G;\alpha)$ be a weighted abelian* Cayley *graph with $\overline{\alpha(z)} = \alpha(-z) \in \mathbb{Z}$ for every $z \in G$ and $n = |G| \geq 3$. Assume that Γ has* PST *between a pair (g, h) of vertices. Then*

(1) Γ *is an integral graph. Namely, $\lambda_x \in \mathbb{Z}$ for all $x \in G$.*

(2) *If $a = g - h \neq 0$, then the order of a is two. Consequently, $|G| = n$ is even.*

Proof (1) Suppose that Γ has PST between g and $h \in G$. By Lemma 2.2, the equality

$$\chi_a(x) = \exp(\imath t(\lambda_0 - \lambda_x))$$

holds for every $x \in G$. Let m be the order of $a = g - h \in G$. Since $a \mapsto \chi_a$ gives an isomorphism of G and $\widehat{G}$, the order of χ_a is also m. Thus we can write

$$\chi_a(x) = \omega_m^{i_a(x)}, \quad \text{where } \omega_m = \exp\Big(\frac{2\pi\imath}{m}\Big),\ i_a(x) \in \mathbb{Z}_m.$$

Then the condition (2) of Lemma 2.2 becomes

$$\exp(\imath t(\lambda_0-\lambda_x))=\exp\Big(\frac{2\pi\imath i_a(x)}{m}\Big). \tag{2.1}$$

Denote $t=2\pi T$. From (2.1), we get

$$M_x:=T(\lambda_0-\lambda_x)-\frac{i_a(x)}{m}\in\mathbb{Z}\quad\text{for any }(0\neq)x\in G. \tag{2.2}$$

Thus the number $M:=\sum\limits_{0\neq x\in G}M_x\in\mathbb{Z}$. On the other hand,

$$M=\sum_{0\neq x\in G}\Big(T(\lambda_0-\lambda_x)-\frac{i_a(x)}{m}\Big)=(n-1)T\lambda_0-T\sum_{0\neq x\in G}\lambda_x-\frac{1}{m}\sum_{0\neq x\in G}i_a(x),$$

and

$$\begin{aligned}\sum_{0\neq x\in G}\lambda_x&=\sum_{0\neq x\in G}\sum_{z\in G}\alpha(z)\chi_x(z)=\sum_{z\in G}\alpha(z)\sum_{0\neq x\in G}\chi_z(x)\\&=(n-1)\alpha(0)-\sum_{0\neq z\in G}\alpha(z)=n\alpha(0)-\lambda_0.\end{aligned}$$

Thus $M=n(\lambda_0-\alpha(0))T-\frac{1}{m}\sum\limits_{0\neq x\in G}i_a(x)\in\mathbb{Z}$. Since $\lambda_0=\sum\limits_{g\in G}\alpha(g)\in\mathbb{Z}$, we know that $T\in\mathbb{Q}$. Then by (2.2) we get that $\lambda_x\in\mathbb{Q}$ for all $x\in G$. Now, $\alpha(g)\in\mathbb{Z}$ for all $g\in G$ implies that $\lambda_x=\sum\limits_{g\in G}\alpha(g)\chi_x(g)$ is an integral combinatorial of algebraic integers and thus an algebraic integer. It follows that $\lambda_x\in\mathbb{Z}$ for all $x\in G$.

(2) Suppose that the order of $a=g-h(\neq 0)$ is m and so is the order of χ_a. Then there exists an element $x\in G$ such that $\chi_a(x)=\omega_m^{i_a(x)}$ with $\gcd(i_a(x),m)=1$. Obviously, x should be non-zero. By (2.2), we have

$$T(\lambda_0-\lambda_x)-\frac{i_a(x)}{m}\in\mathbb{Z}. \tag{2.3}$$

Now, we consider λ_{-x}. By (1), $\lambda_x\in\mathbb{Z}$, thus

$$\lambda_{-x}=\sum_{g\in G}\alpha(g)\chi_{-x}(g)=\sum_{g\in G}\alpha(g)\chi_g(-x)=\sum_{g\in G}\overline{\alpha(g)\chi_g(x)}=\overline{\lambda_x}=\lambda_x,$$

and

$$\omega_m^{i_a(-x)}=\chi_a(-x)=\overline{\chi_a(x)}=\omega_m^{-i_a(x)}.$$

Thus, $i_a(-x)\equiv -i_a(x)\pmod m$. By (2.3), we have

$$T(\lambda_0-\lambda_x)+\frac{i_a(x)}{m}\in\mathbb{Z}. \tag{2.4}$$

Combining (2.3) and (2.4) together, we have $\frac{2i_a(x)}{m}\in\mathbb{Z}$. Since $\gcd(i_a(x),m)=1$, we get that $m=2$.

Next, we discuss the periodicity of a simple weighted ableian Cayley graph $\Gamma=\mathrm{Cay}(G;\alpha)$. By Corollary 2.2, we may assume that $\alpha(g)\geq 0$ for all $g\in G$. In order to get integral graphs, we need further assume that $\alpha(z)\in\mathbb{Z}$ for all $z\in G$. Based on these assumptions, we can state the following result.

Theorem 2.1 *Let G be a finite abelian group. Let α be a function on G satisfying $0 \leq \alpha(z) = \alpha(-z) \in \mathbb{Z}$ for all $z \in G$. Let $\Gamma = \mathrm{Cay}(G;\alpha)$ be the corresponding weighted abelian* Cayley *graph. Let λ_0 and $\cdots, \lambda_{n-1}$ be the eigenvalues of Γ and $n(\geq 3)$ be the order of G. If Γ is an integral graph, then for every $g \in G$, Γ is periodic at vertex g and the set*

$$\{t > 0 \mid \Gamma \textit{ is periodic at } g \textit{ with period } t\}$$

is $\{\frac{2\pi l}{N} \mid l = 1, 2, \cdots\}$, where $N = \gcd(\lambda_0 - \lambda_x : 0 \neq x \in G)$.

Proof Firstly, since Γ is integral, the number $N = \gcd(\lambda_0 - \lambda_x : 0 \neq x \in G)$ is well-defined. Secondly, from the proof of Proposition 2.1, we know that Γ has PST at the vertex g if and only if

$$M_x = T(\lambda_0 - \lambda_x) - \frac{i_a(x)}{m} \in \mathbb{Z}$$

for any $0 \neq x \in G$, where $a = g - g = 0$ and thus $i_a(x) = 0$. It is easy to see that $M_x = T(\lambda_0 - \lambda_x) \in \mathbb{Z}$ for all $0 \neq x \in G$ if and only if $TN \in \mathbb{Z}$.

Now, we consider those integral weighted abelian Cayley graphs which admit PST between two distinct vertices g and h. Denote $a = g - h$. By Proposition 2.1, the order of a is two and so is the order of χ_a. Therefore for every $x \in G$, we have $\chi_a(x) = \pm 1$. Define two subsets of G by

$$\Omega_+ := \{x \in G \mid \chi_a(x) = 1\}, \quad \Omega_- := \{x \in G \mid \chi_a(x) = -1\}. \tag{2.5}$$

It is easy to see that Ω_+ is a subgroup of G and G is a disjoint union of Ω_+ and Ω_-. Moreover $|\Omega_+| = |\Omega_-| = \frac{|G|}{2}$. Denote

$$N_0 = \gcd(\lambda_0 - \lambda_x : x \in \Omega_+), \quad N_1 = \gcd(\lambda_0 - \lambda_x : x \in \Omega_-). \tag{2.6}$$

Obviously, N_0 and N_1 are well-defined and $N = \gcd(\lambda_0 - \lambda_x : 0 \neq x \in G) = \gcd(N_0, N_1)$.

Recall that the 2-adic exponential valuation of rational numbers is defined by

$$v_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, \quad v_2(0) = \infty, \quad v_2\Big(2^\ell \frac{a}{b}\Big) = \ell, \quad \text{where } a, b, \ell \in \mathbb{Z} \text{ and } 2 \nmid ab.$$

The evaluation v_2 has the following properties. For $\beta, \beta' \in \mathbb{Q}$,

(P1) $v_2(\beta\beta') = v_2(\beta) + v_2(\beta')$;

(P2) $v_2(\beta + \beta') \geq \min(v_2(\beta), v_2(\beta'))$, and equality holds if $v_2(\beta) \neq v_2(\beta')$.

After the above preparation, we present our main result as follows.

Theorem 2.2 *Let G be an abelian group of order n and α be a function on G satisfying $\alpha(z) = \alpha(-z) \in \mathbb{Z}$ for all $z \in G$. Assume that $\Gamma = \mathrm{Cay}(G;\alpha)$ is the associated* Cayley *graph. Then for $g, h \in G, a = g - h \neq 0$, Γ has* PST *between g and h if and only if the following three conditions hold*:

(1) Γ *is an integral graph, i.e., the eigenvalues of Γ are all integers*;

(2) *the order of a is two*;

(3) *for all $x \in \Omega_-$, the 2-adic valuations of the numbers $\lambda_0 - \lambda_x$ are equal, say ρ, and $v_2(N_0) \geq \rho + 1$, where N_0 is defined by* (2.6).

Moreover, if the conditions (1)–(3) *are satisfied, then the set*

$$\{t > 0 \mid \ \Gamma \textit{ has } \mathrm{PST} \textit{ between } g \textit{ and } h \textit{ at time } t\}$$

is $\left\{\frac{\pi}{N}+\frac{2\pi}{N}\ell : \ell = 0,1,2,\cdots\right\}$, *where* $N=\gcd(\lambda_0-\lambda_x : 0\neq x\in G)$.

Proof Conditions (1), (2) follow directly from Proposition 2.1. Thus Γ has PST between g and h at the time $t := 2\pi T$ if and only if the following two conditions hold:

(i) $T(\lambda_0-\lambda_x)\in\mathbb{Z}$ for all $x\in\Omega_+$;

(ii) $T(\lambda_0-\lambda_x)-\frac{1}{2}\in\mathbb{Z}$ for all $x\in\Omega_-$.

Condition (i) means that $T\in\frac{1}{N_0}\mathbb{Z}=\left\{\frac{\ell}{N_0}\mid \ell\in\mathbb{Z}\right\}$. Now, we consider the condition (ii). Suppose that $x,x'\in\Omega_-$. Then $T(\lambda_0-\lambda_x), T(\lambda_0-\lambda_{x'})\in\frac{1}{2}+\mathbb{Z}$, and then $T\in\mathbb{Q}\setminus\{0\}$ and $v_2(T(\lambda_0-\lambda_x))=v_2(T(\lambda_0-\lambda_{x'}))=-1$. Therefore we have $v_2(\lambda_0-\lambda_x)=v_2(\lambda_0-\lambda_{x'})=-1-v_2(T)$. Hence, for all $x\in\Omega_-$, $v_2(\lambda_0-\lambda_x)$ is a constant, say ρ. Moreover, if $v_2(\lambda_0-\lambda_x)=\rho$ for all $x\in\Omega_-$, then $v_2(N_1)=\rho$ and $v_2(T)=v_2(T(\lambda_0-\lambda_x))-v_2(\lambda_0-\lambda_x)=-(\rho+1)$. Thus condition (ii) means that $T\in\frac{1}{N_1}\left(\frac{1}{2}+\mathbb{Z}\right)=\left\{\frac{1}{N_1}\left(\frac{1}{2}+\ell\right):\ell\in\mathbb{Z}\right\}$. This completes the proof.

Next, we present an example to illustrate our results.

Example 2.1 Let $G=\mathbb{Z}/6\mathbb{Z}$ be a cyclic group of order 6 and the weight function is defined by $\alpha(0)=\alpha(3)=0$, $\alpha(1)=\alpha(5)=1$, $\alpha(2)=\alpha(4)=2$. Let $\Gamma=\mathrm{Cay}(G;\alpha)$ be the corresponding weighted Cayley graph. Then the adjacency matrix of Γ is

$$A=\begin{pmatrix} 0&1&2&0&2&1\\ 1&0&1&2&0&2\\ 2&1&0&1&2&0\\ 0&2&1&0&1&2\\ 2&0&2&1&0&1\\ 1&2&0&2&1&0 \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_0=6,\quad \lambda_1=\lambda_5=-1,\quad \lambda_2=\lambda_4=-3,\quad \lambda_3=2.$$

In fact, let $P=\frac{1}{\sqrt{6}}(\gamma_{ij})_{0\le i,j\le 5}$, where $\gamma_{ij}=\omega_6^{i\cdot j}$. Then P is a unitary matrix and

$$P^*AP=\mathrm{diag}(\lambda_0,\cdots,\lambda_5).$$

A direct computation shows that the transfer matrix $H(t)=(h_{ij}(t))_{1\le i,j\le 6}$ satisfies

$$h_{ii}(t)=\frac{1}{6}(\exp(6t\imath)+2\exp(-t\imath)+2\exp(-3t\imath)+\exp(2t\imath)),\quad i=1,2,\cdots,6$$

and

$$h_{i,i+3}(t)=\frac{1}{6}(\exp(6t\imath)-2\exp(-t\imath)+2\exp(-3t\imath)-\exp(2t\imath)),\quad i=1,2,3.$$

The other entries (except for $h_{ii}(t), i=1,\cdots,6$ and $h_{i,i+3}(t), h_{i+3,i}(t), i=1,2,3$) have the form $\frac{1}{6}(\exp(6t\imath)+s_1\exp(-t\imath)+s_2\exp(-3t\imath)+s_3\exp(2t\imath))$, where $s_1,s_2,s_3\in\{-1,1\}$, and thus their absolute value are less than 1 for every real number t. Thus Γ cannot have PST between vertices g and h when $g-h\neq 0$. But Γ is periodic at any vertex $g\in G$. These results are consistent with Theorem 2.2. Indeed, $x\in\Omega_-$ if and only if $x\in\{1,3,5\}$. Now, $v_2(\lambda_0-\lambda_1)=0$, $v_2(\lambda_0-\lambda_3)=2$. Thus the condition (3) in Theorem 2.2 does not hold and thus Γ cannot have PST between vertices g and h when $g-h\neq 0$.

In view of Theorem 2.2, we need to investigate integral weighted abelian Cayley graphs. Note that, for abelian Cayley graph, this topic has been discussed by many authors, see for example [1, 11, 13, 19] and the references therein.

We consider integral weighted abelian Cayley graph $\Gamma = \mathrm{Cay}(G;\alpha)$ with the weight function α satisfying $\alpha(z) = \alpha(-z) \in \mathbb{Z}$ for all $z \in G$. Let $e = \exp(G)$ be the least common multiple of the order of the elements in G. Since the eigenvalues of $\Gamma = \Gamma(G;\alpha)$ are $\{\lambda_x : x \in G\}$, they are contained in the cyclotomic field $\mathbb{Q}(\omega_e)$, here ω_e is a primitive e-th root of unity in $\mathbb{C}$. It is well-known that $\mathbb{Q}(\omega_e)/\mathbb{Q}$ is a Galois extension and the Galois group of this extension is

$$\mathrm{Gal}(\mathbb{Q}(\omega_e)/\mathbb{Q}) = \{\sigma_\ell : \ell \in \mathbb{Z}_e^*\},$$

where $\mathbb{Z}_{\mathrm{e}}^* = \{1 \le \ell \le e : \gcd(\ell, e) = 1\}$ and σ_ℓ is defined as $\omega_e \mapsto \omega_{\mathrm{e}}^\ell$. Therefore, Γ is integral if and only if $\sigma_\ell(\lambda_x) = \lambda_x$ for all $x \in G$ and $\ell \in \mathbb{Z}$ with $\gcd(\ell, e) = 1$. Now,

$$\sigma_\ell(\lambda_x) = \sigma_\ell\Big(\sum_{g\in G} \alpha(g)\chi_x(g)\Big) = \sum_{g\in G} \alpha(g)\chi_x(g)^\ell = \sum_{g\in G} \alpha(g)\chi_{\ell x}(g) = \lambda_{\ell x}.$$

Meanwhile,

$$\sigma_\ell(\lambda_x) = \sigma_\ell\Big(\sum_{g\in G} \alpha(g)\chi_x(g)\Big) = \sum_{g\in G} \alpha(g)\chi_x(\ell g) = \sum_{g\in G} \alpha(\ell^{-1}g)\chi_x(g).$$

Thus, $\lambda_x \in \mathbb{Z}$ for all $x \in G$ if and only if $\sum\limits_{g\in G}(\alpha(\ell^{-1}g) - \alpha(g))\chi_x(g) = 0$ for all $\ell \in \mathbb{Z}$ with $\gcd(\ell, e) = 1$ and all $x \in G$. By the orthogonality of characters, $\lambda_x \in \mathbb{Z}$ for all $x \in G$ if and only if $\alpha(\ell g) = \alpha(g)$ for all $g \in G$ and $\ell \in \mathbb{Z}$ with $\gcd(\ell, e) = 1$.

We define an equivalent relation "$\sim$" on G by setting $g \sim h$ if and only if there exists an element $\ell \in \mathbb{Z}_{\mathrm{e}}^*$ such that $g = \ell h$. The equivalent class containing g is denoted by $[g]$. A function f is called a c-function if it is a constant on each equivalent class. That is, if f is a c-function, and $g \sim h$, then $f(g) = f(h)$. Using this notation, we have the following result.

Theorem 2.3 *Assume that* $\Gamma = \mathrm{Cay}(G;\alpha)$ *is a weighted abelian* Cayley *graph, where the weight function* α *satisfies* $\alpha(z) = \alpha(-z) \in \mathbb{Z}$ *for all* $z \in G$. *Then the following statements are equivalent*:

(1) Γ *is integral, i.e.,* $\lambda_x = \sum\limits_{g\in G} \alpha(g)\chi_x(g)$ *is an integer for all* $x \in G$;

(2) $\lambda_x = \lambda_{\ell x}$ *for all* $x \in G$ *and* $\ell \in \mathbb{Z}_{\mathrm{e}}^*$, *that is,* λ_x *is a c-function defined on* G;

(3) $\alpha(g) = \alpha(\ell g)$ *for all* $g \in G$ *and* $\ell \in \mathbb{Z}_{\mathrm{e}}^*$, *that is,* $\alpha(g)$ *is a c-function defined on* G.

We note that, when Γ is an abelian Cayley graph $\mathrm{Cay}(G;S)$, then Theorem 2.3 is reduced to the following result obtained independently by Bridge [3] and Klotz [14].

Corollary 2.3 (see [3, 14]) *Let* G *be a finite abelian group,* $S \subseteq G$. *Then the* Cayley *graph* $\Gamma = \mathrm{Cay}(G;S)$ *is integral if and only if* S *is a disjoint union of several equivalent classes of* G.

We have shown that if a weighted abelian Cayley graph $\Gamma = \mathrm{Cay}(G;\alpha)$ has PST between two distinct vertices g and $g+a$ in G, then the order of a should be two and then the order of G is even. For integral circulant Cayley graphs $\mathrm{Cay}(G,S)$ having PST, Petkovic [17] proved that the order of G should be doubly even, i.e., $4||G|$. In [18], we proved that if $\Gamma = \mathrm{Cay}(G,S)$ is a connected simple abelian Cayley graph with $4 \le |G| \equiv 2 \pmod 4$, then G cannot have PST between two distinct vertices. In other words, we generalized Petkovic's result (see [17]) to abelian Cayley graphs. As another application of Theorem 2.2, we can show that the same conclusion holds for integral weighted abelian Cayley graphs under certain conditions. Conversely, if we assign the weight function suitably, then we can get a simple weighted graph $\Gamma = \mathrm{Cay}(G;\alpha)$ having PST even if the order of G is not doubly even.

Theorem 2.4 *Let $\Gamma = \Gamma(G;\alpha)$ be a connected weighted integral abelian* Cayley (*simple*) *graph and $4 < n = |G| \equiv 2 \pmod 4$. Let $a \in G$ be the unique element of order 2. Then Γ has no* PST *between distinct vertices if there is an element $g(\neq a) \in G$ such that $v_2(\alpha(g)) \le v_2(\alpha(a))$.*

Proof We use Theorem 2.2 to prove this result. Since $4 < n = |G| \equiv 2 \pmod 4$, we can write $G = \mathbb{Z}_2 \bigoplus H$, where H is an abelian group, and $|H| := m$ is odd. Then $a = (1,0)$. If Γ has PST between g and $g + a'$, then a' is of order two, and thus $a' = a = (1,0) \in G$. The character group of G is $\widehat{G} = \{\eta^i\chi_h \mid i = 0,1, \text{ and } h \in H\}$, where $\langle\eta\rangle = \widehat{\mathbb{Z}_2}$, $\widehat{H} = \{\chi_h : h \in H\}$. For any element $(x,y) \in G$,

$$(\eta^i\chi_h)((x,y)) = \begin{cases} \chi_h(y), & \text{if } i = 0, \\ (-1)^x\chi_h(y), & \text{otherwise.} \end{cases}$$

Thus, by (2.5), we know that

$$\Omega_+ = \{(0,h) \mid h \in H\}, \quad \Omega_- = \{(1,h) \mid h \in H\}.$$

By Theorems 2.2–2.3, we can obtain $v_2(\lambda_x - \lambda_y) = \rho$ for all $x \in \Omega_+, y \in \Omega_-$ and the weight function α is a class function. Letting $x = (0,h) \in \Omega_+, y = (1,h) \in \Omega_-$, we get

$$\lambda_x = \sum_{g\in G}\alpha(g)\chi_x(g) = \sum_{g=(0,g')\in\Omega_+}\alpha(g)\chi_h(g') + \sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g')$$

and

$$\lambda_y = \sum_{g\in G}\alpha(g)\chi_x(g) = \sum_{g=(0,g')\in\Omega_+}\alpha(g)\chi_h(g') - \sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g').$$

Therefore, we have

$$\lambda_x - \lambda_y = 2\sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g')$$

and

$$v_2\Big(\sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g')\Big) = \rho - 1 \quad \text{for any } h \in H.$$

Particularly, taking $h = 0$, we get

$$v_2\Big(\sum_{g\in\Omega_-}\alpha(g)\Big) = \rho - 1.$$

Since for every $\ell \in \mathbb{Z}_{2m}^*$, $\ell\Omega_+ = \{\ell z : z \in \Omega_+\} = \Omega_+$, $\ell\Omega_- = \{\ell z : z \in \Omega_-\} = \Omega_-$, we have

$$\sigma_\ell\Big(\sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g')\Big) = \sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g')^\ell = \sum_{g=(1,g')\in\Omega_-}\alpha(g)\chi_h(g'), \quad \text{for all } h \in H.$$

Thus $\sum_{g=(1,g')\in\Omega_-} \alpha(g)\chi_h(g') \in \mathbb{Z}$. Similarly, we have $\sum_{g=(0,g')\in\Omega_+} \alpha(g)\chi_h(g') \in \mathbb{Z}$. From (**??**), it follows that

$$\sum_{g=(1,g')\in\Omega_-} \alpha(g)\chi_h(g') \equiv 2^{\rho-1} \pmod{2^\rho} \quad \text{for all } h \in H. \tag{2.7}$$

Since $a = (1, 0)$ is the unique element in Ω_- whose equivalent class has odd size (if $z \in \Omega_-$, then the size of $[z]$ is $\varphi(\mathrm{ord}(z))$, here φ is the Euler phi-function). Since Ω_- is a union of some equivalent classes, we obtain that $|\Omega_-|$ is odd. Moreover, α is a c-function on G, we know that $\sum_{g\in\Omega_-} \alpha(g) \equiv \alpha(a) \pmod 2$. We define a function f on H as follows:

$$f : H \to \mathbb{Z}, \quad f(z) = \begin{cases} \alpha((1, z)), & \text{if } (1, z) \in \Omega_-, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transformation of $f(z)$ is

$$F(h) = \sum_{z\in H} f(z)\chi_z(h) = \sum_{g=(1,z)\in\Omega_-} \alpha(g)\chi_h(z) \equiv 2^{\rho-1} \pmod{2^\rho} \quad \text{for all } h \in H.$$

By inverse Fourier transformation,

$$mf(z) = \sum_{h\in H} F(h)\overline{\chi_h(z)} \equiv 2^{\rho-1} \sum_{h\in H} \overline{\chi_h(z)} \pmod{2^\rho}.$$

If $z \neq 0$, then $\sum_{h\in H} \overline{\chi_h(z)} = 0$ and $f(z) \equiv 0 \pmod{2^\rho}$ since m is odd. Thus for every $(1, z) \in \Omega_-$, and $z \neq 0$, one has that $f(z) = \alpha((1, z)) \equiv 0 \pmod{2^\rho}$. By (2.7), we have

$$\alpha(a) \equiv 2^{\rho-1} \pmod{2^\rho} \quad \text{and} \quad \alpha((1, h)) \equiv 0 \pmod{2^\rho} \quad \text{for all } 0 \neq h \in H. \tag{2.8}$$

By Theorem 2.2, for any $x = (0, h) \in \Omega_+$, $y = (1, h) \in \Omega_-$, we have $v_2(\lambda_0 - \lambda_x) \geq \rho + 1$, $v_2(\lambda_0 - \lambda_y) = \rho$. By Property (P2), it follows that

$$\rho = v_2(2\lambda_0 - (\lambda_x + \lambda_y)) = v_2\Big(2\lambda_0 - 2 \sum_{g=(0,h)\in\Omega_+} \alpha(g)\chi_h(h)\Big).$$

Thus

$$v_2\Big(\lambda_0 - \sum_{g=(0,z)\in\Omega_+} \alpha(g)\chi_h(z)\Big) = \rho - 1.$$

Therefore,

$$\sum_{g=(0,z)\in\Omega_+} \alpha(g)\chi_h(z) \equiv 2^{\rho-1} - \lambda_0 \pmod{2^\rho} \quad \text{for all } h \in H. \tag{2.9}$$

We define a function g on H as follows:

$$g : H \to \mathbb{Z}, \quad g(z) = \begin{cases} \alpha((0, z)), & \text{if } (0, z) \in \Omega_+, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transformation of $g(z)$ is

$$G(h) = \sum_{z\in H} g(z)\chi_z(h) = \sum_{g=(0,z)\in\Omega_+} \alpha(g)\chi_h(z) \equiv 2^{\rho-1} - \lambda_0 \pmod{2^\rho} \quad \text{for all } h \in H.$$

By inverse Fourier transformation again,

$$mg(z) = \sum_{h\in H} G(h)\overline{\chi_h(z)} \equiv (2^{\rho-1} - \lambda_0) \sum_{h\in H} \overline{\chi_h(z)} \pmod{2^\rho}.$$

For $z(\neq 0) \in H$, we have $g(z) = \alpha((0, z)) \equiv 0 \pmod{2^\rho}$. By (2.9), we have

$$\alpha((0,0)) \equiv 2^{\rho-1} - \lambda_0 \pmod{2^\rho} \quad \text{and} \quad \alpha((0,h)) \equiv 0 \pmod{2^\rho} \quad \text{for all } (0 \neq)h \in H. \tag{2.10}$$

Since Γ is a simple graph, $\alpha((0, 0)) = 0$, we get that $\lambda_0 \equiv 2^{\rho-1} \pmod{2^\rho}$.

Combining (2.8) and (2.10), we get

$$\alpha(a) = \alpha((1,0)) \equiv 2^{\rho-1} \pmod{2^\rho} \quad \text{and} \quad \alpha(g) \equiv 0 \pmod{2^\rho} \quad \text{for all } (a \neq)g \in G.$$

Thus we get a contradiction with the assumption that there exists an element $g \in G$ such that $v_2(\alpha(g)) \le v_2(a)$.

For the converse of Theorem 2.4, we show that for some abelian groups of order $2m$ with odd integer m, there exists a weight function α such that $\Gamma = \mathrm{Cay}(G; \alpha)$ has PST. More specifically, we have the following result.

Theorem 2.5 *Let $G = (\mathbb{Z}_{2m}, +)$ be an abelian group, where $m > 1$ is an odd integer. Let $\Gamma = \Gamma(G; \alpha)$ be a weighted (simple) graph, where the weight function α is defined by $\alpha(0) = 0, \alpha(m) = 1, \alpha(g) = 2$ for all $g \in G$ and $g \neq 0, m$. Then for every $g \in G$, Γ has* PST *between g and $g + m$ at time $\frac{\pi}{2}$.*

Proof Since G is in fact a cyclic group, the dual group of G is also cyclic. By Lemma 2.1, a direct calculation shows that the eigenvalues of Γ are

$$\begin{aligned} \lambda_0 &= 4m - 3, \\ \lambda_{2k-1} &= -1, \quad 1 \le k \le m, \\ \lambda_{2k} &= -3, \quad 1 \le k \le m-1. \end{aligned}$$

And it is easy to see that

$$\Omega_+ = \{2k-1 : 1 \le k \le m\}, \quad \Omega_- = \{2k : 1 \le k \le m-1\}.$$

Therefore,

$$\begin{aligned} v_2(\lambda_0 - \lambda_x) &= v_2(4m) = 2 \quad \text{for all } 0 \neq x \in \Omega_+, \text{ and} \\ v_2(\lambda_0 - \lambda_y) &= v_2(2(2m-1)) = 1 \quad \text{for all } y \in \Omega_-. \end{aligned}$$

By Theorem 2.2, we know that Γ has PST between two distinct vertices at time $\frac{\pi}{2}$.

For the graph in the Example 2.1, if we change the weight function according to Theorem 2.5, then we get the following weighted graph which has PST.

Example 2.2 Let $G = \mathbb{Z}/6\mathbb{Z}$ be a cyclic group of order 6 and the weight function is defined by $\alpha(0) = 0, \alpha(3) = 1, \alpha(1) = \alpha(5) = \alpha(2) = \alpha(4) = 2$. Let $\Gamma = \mathrm{Cay}(G; \alpha)$ be the corresponding weighted Cayley graph. Then the adjacency matrix of Γ is

$$A = \begin{pmatrix} 0 & 2 & 2 & 1 & 2 & 2 \\ 2 & 0 & 2 & 2 & 1 & 2 \\ 2 & 2 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 2 & 1 & 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 2 & 2 & 0 \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_0 = 9, \quad \lambda_1 = \lambda_3 = \lambda_5 = -1, \quad \lambda_2 = \lambda_4 = -3.$$

Note that

$$v_2(\lambda_0 - \lambda_x) = v_2(12) = 2 \quad \text{for all } x \in \Omega_+ \setminus \{0\} = \{2, 4\},$$

and

$$v_2(\lambda_0 - \lambda_y) = v_2(10) = 1 \quad \text{for all } y \in \Omega_- = \{1, 3, 5\}.$$

By Theorem 2.2, there exists PST between g and $g + 3$ for every $g \in G$. Indeed, let $P = \frac{1}{\sqrt{6}}(\gamma_{ij})_{0 \le i,j \le 5}$, where $\gamma_{ij} = \omega_6^{i \cdot j}$. Then P is a unitary matrix and

$$A = P\mathrm{diag}(\lambda_0, \cdots, \lambda_5)P^*.$$

A direct computation shows that the transfer matrix $H(t) = (h_{ij}(t))_{1 \le i,j \le 6}$ satisfies

$$h_{i,i+3}(t) = \frac{1}{6}(\mathrm{e}^{9\imath t} - 3\mathrm{e}^{-\imath t} + 2\mathrm{e}^{-3\imath t}), \quad i = 0, 1, 2,$$

and $\left|h_{i,i+3}\left(\frac{\pi}{2}\right)\right| = |\imath| = 1$. Therefore, Γ has PST between vertices i and $i + 3$ for $i = 0, 1, 2$.

In [18], we showed that if $\Gamma = \mathrm{Cay}(G, S)$ is a cubelike abelian Cayley graph with $|S| \ge 1$, then $N = \gcd(\lambda_0 - \lambda_x, x \in G)$ is a power of two. Next, we show that this result can be extended to weighted Cayley graphs. We believe that the following result has its own independent interest in graph theory.

Lemma 2.3 *Let G be an abelian group and α be a weight function from G to $\mathbb{Z}$. Assume that* $\gcd(\alpha(z) : z \in G) = 1$ *and let* $\Gamma = \mathrm{Cay}(G; \alpha)$ *be a simple weighted* Cayley *graph. Then* $N = \gcd(\lambda_0 - \lambda_x, x \in G)$ *is a divisor of $|G|$. Consequently, if G is a p-group, then N is a power of p.*

Proof By definition, we know that for any $x \in G$, $N \mid (\lambda_0 - \lambda_x)$. Assume that $\lambda_x = \lambda_0 - N\theta(x)$, $\theta(x) \in \mathbb{Z}$. Noticing that

$$\lambda_z = \sum_{g \in G} \alpha(g)\chi_z(g)$$

is the Fourier transform of $\alpha(z)$, by the inverse transform formula, we get

$$|G|\alpha(z) = \sum_{g \in G} \lambda_g \overline{\chi_g(z)} = \sum_{g \in G} (\lambda_0 - N\theta(g))\overline{\chi_g(z)} = -N \sum_{g \in G} \theta(g)\overline{\chi_g(z)}. \tag{2.11}$$

Due to $\gcd(\alpha(z) : z \in G) = 1$, there exist $|G|$ integers $\ell(z), z \in G$ such that $\sum_{z\in G} \ell(z)\alpha(z) = 1$. From (2.11), we get

$$\frac{|G|}{N} = -\sum_{z\in G}\sum_{g\in G} \ell(z)\theta(g)\overline{\chi_g(z)}. \tag{2.12}$$

The right hand side of (2.12) is an algebraic integer, and the left hand side of (2.12) is a rational number. Thus both of them are integers. This completes the proof.

The following corollary follows immediately.

Corollary 2.4 *Suppose that* $\Gamma = \mathrm{Cay}(G, S)$ *is an integral abelian* Cayley *graph with* $|S| = s$ *and* $\lambda_0(= s), \lambda_1, \cdots, \lambda_r$ *are the eigenvalues of* Γ. *Then* $N = \gcd(s - \lambda_i : 1 \le i \le r)$ *is a divisor of* $|G|$.

3 PST on Weighted Cubelike Cayley Graphs

In this section, we let G be the additive group of the finite field $\mathbb{F}_q$, where $q = 2^n$. Let α be a weight function from G to $\mathbb{Z}$. We can view $\mathbb{F}_q$ as an n-dimensional vector space over $\mathbb{F}_2$. There are two ways to represent the additive characters of $\mathbb{F}_q$. The first one is

$$\widehat{G} = \{\chi_x : x \in \mathbb{F}_q\}, \quad \text{where } \chi_x(z) = (-1)^{x\cdot z} \text{ for all } x, z \in \mathbb{F}_2^n,$$

in which $x \cdot z$ is the usual inner product of x, $z \in \mathbb{F}_2^n$. The second one is

$$\widehat{G} = \{\chi_x : x \in \mathbb{F}_q\}, \quad \text{where } \chi_x(z) = (-1)^{\mathrm{tr}(xz)} \text{ for all } x, z \in \mathbb{F}_q,$$

here $\mathrm{tr}(\cdot)$ is the trace mapping.

In 2012, Godsil [12] raised a question: Are there some cubelike graphs that have PST at time t, which can be arbitrarily small? In 2013, Chan [6] gave a confirmative answer to this question by presenting some deterministic constructions of such graphs. She utilized some Hamming schemes to get an infinite family of graphs having PST at an arbitrarily small time. In this section, we also give positive answers to the above mentioned question. By Theorems 2.1–2.2, the minimum time t of PST in cubelike graph $\Gamma = \mathrm{Cay}(G, \alpha)$ is $\frac{\pi}{N}$, where $N = \gcd(\lambda_0 - \lambda_z : z(\neq 0) \in G)$. Firstly, we know that N should be a power of 2 (see Lemma 2.3). Then we provide a lower bound on the time t such that $\Gamma = \mathrm{Cay}(G, \alpha)$ has PST between two distinct vertices (see Theorem 3.1) and show that this lower bound is tight (see Theorem 3.2).

First of all, we have the following two lemmas.

Lemma 3.1 *Let* $G = (\mathbb{F}_q, +)$ *be the additive group of* $\mathbb{F}_q$, $q = 2^n$ *and* $\Gamma = \mathrm{Cay}(G; \alpha)$ *be a weighted abelian* Cayley *graph with* $\alpha(z) \in \mathbb{Z}$ *for every* $z \in G$. *Let* $c \in \mathbb{F}_q^*$ *and* $\Gamma' = \Gamma'(G; \alpha')$ *be another weighted graph, where the weight function* α' *is defined by* $\alpha'(z) = \alpha(cz)$ *for all* $z \in G$. *Then the following statements are equivalent*:

(1) Γ *has* PST *between* g *and* h *at time* $t > 0$;

(2) Γ' *has* PST *between* $c^{-1}g$ *and* $c^{-1}h$ *at time* $t > 0$.

Proof It is easy to see that the spectrum of Γ' is

$$\lambda'_x = \sum_{g\in G} \alpha'(g)\chi_x(g) = \sum_{g\in G} \alpha(g)\chi_x(c^{-1}g) = \sum_{g\in G} \alpha(g)\chi_{c^{-1}x}(g) = \lambda_{c^{-1}x}, \quad \text{for all } x \in G.$$

Thus by Lemma 2.2, Γ' has PST between $c^{-1}g$ and $c^{-1}h$ at time $t > 0$ if and only if for all $x \in G$, it holds that

$$\chi_{a'}(x) = \exp(\imath t(\lambda'_0 - \lambda'_x)),$$

where

$$a' = c^{-1}(g-h) = c^{-1}a, \quad \lambda'_0 = \lambda_0, \quad \lambda'_x = \lambda_{c^{-1}x};$$

if and only if

$$\chi_a(c^{-1}x) = \exp(\imath t(\lambda_0 - \lambda_{c^{-1}x})) \quad \text{for all } x \in G;$$

if and only if Γ has PST between g and h at the time $t > 0$.

Lemma 3.2 *Let $G = (\mathbb{F}_q, +)$ be the additive group of $\mathbb{F}_q$, $q = 2^n$ and $\Gamma = \mathrm{Cay}(G;\alpha)$ be a weighted abelian* Cayley *graph with $\alpha(z) \in \mathbb{Z}$ for every $z \in G$. Assume that* $\gcd(\alpha(z) : z \in G) = d$ *and let $\Gamma' = \mathrm{Cay}(G;\alpha')$ be a cubelike weighted* Cayley *graph, where the weight function α' is defined by $\alpha'(z) = \frac{\alpha(z)}{d}$. Then the following statements are equivalent*:

(1) Γ *has* PST *between g and h at time $t > 0$;*

(2) Γ' *has* PST *between g and h at time $dt > 0$.*

The proof of Lemma 3.2 is straightforward and thus is omitted. Thus, without loss of generality, we assume that $\gcd(\alpha(z) : z \in G) = 1$ in the following context.

Theorem 3.1 *Let $G = \mathbb{F}_2^n$ and α be a weight function from G to $\mathbb{Z}$ satisfying* $\gcd(\alpha(z) : z \in G) = 1$*. Let $\Gamma = \mathrm{Cay}(G;\alpha)$ be a connected simple weighted* Cayley *graph and $0 \neq a \in \mathbb{F}_2^n$. If Γ has* PST *between g and $g+a$ at time $t > 0$, then the minimum time t is $\frac{\pi}{N}$, where $N = 2^\ell$. Moreover, $\ell \leq \lfloor \log_2(2M\sqrt{L}) \rfloor$, where $M = \max\{|\alpha((1,z))| : z \in \mathbb{F}_2^{n-1}\}$, L is the number of elements $z \in \mathbb{F}_2^{n-1}$ such that $\alpha((1,z)) \neq 0$, and $\lfloor x \rfloor$ is the floor function which is defined as the least integer greater than or equal to x.*

Proof Without loss of generality, we can assume that $a = (1, 0, \cdots, 0) \in \mathbb{F}_2^n$ by Lemma 3.1. In this case, we can find that (see (2.5))

$$\Omega_+ = \{z \in \mathbb{F}_2^n : \chi_z(a) = 1\} = (0, \mathbb{F}_2^{n-1}),$$
$$\Omega_- = \{z \in \mathbb{F}_2^n : \chi_z(a) = -1\} = (1, \mathbb{F}_2^{n-1}).$$

If Γ has PST between g and $g+a$, then by Theorem 2.2, there is a nonnegative integer ρ such that $v_2(\lambda_0 - \lambda_y) = \rho$ for all $y \in \Omega_-$ and $v_2(\lambda_0 - \lambda_x) \geq \rho + 1$ for all $x \in \Omega_+$. Therefore $\min(v_2(\lambda_0 - \lambda_z), z \in G) = \rho$. By Lemma 2.3, we have $2^\ell = N = 2^\rho$ and thus $\rho = \ell$. By Theorems 2.2–2.3, we obtain that $v_2(\lambda_x - \lambda_y) = \rho$ for all $x \in \Omega_+, y \in \Omega_-$. Taking $x = (0,h) \in \Omega_+$ and $y = (1,h) \in \Omega_-$, we get

$$\lambda_x = \sum_{z \in G} \alpha(z)\chi_x(z) = \sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((0,z'))(-1)^{h \cdot z'} + \sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((1,z'))(-1)^{h \cdot z'}$$

and

$$\lambda_y = \sum_{z \in G} \alpha(z)\chi_y(z) = \sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((0,z'))(-1)^{h \cdot z'} - \sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((1,z'))(-1)^{h \cdot z'}.$$

Thus for all $h \in \mathbb{F}_2^{n-1}$, we have

$$v_2\Big(\sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((1,z'))(-1)^{h\cdot z'}\Big) = \rho - 1.$$

Thus there exists an odd integer $\theta(h)$ such that

$$\sum_{z' \in \mathbb{F}_2^{n-1}} \alpha((1,z'))(-1)^{h\cdot z'} = 2^{\rho-1}\theta(h).$$

By the inverse formula, we have

$$\alpha(1,z) = 2^{\rho-n} \sum_{h \in \mathbb{F}_2^{n-1}} \theta(h)(-1)^{h\cdot z}, \quad \forall z \in \mathbb{F}_2^{n-1}.$$

Therefore,

$$\begin{aligned}\sum_{z \in \mathbb{F}_2^{n-1}} \alpha((1,z))^2 &= 2^{2\rho-2n} \sum_{h_1,h_2 \in \mathbb{F}_2^{n-1}} \theta(h_1)\theta(h_2) \sum_{z \in \mathbb{F}_2^{n-1}} (-1)^{(h_1-h_2)\cdot z} \\ &= 2^{2\rho-n-1} \sum_{h \in \mathbb{F}_2^{n-1}} \theta(h)^2 \ \ (\text{and by (3), } 2 \nmid \theta(h)) \\ &\geq 2^{2\rho-2}.\end{aligned}$$

Thus

$$LM^2 \geq 2^{2\rho-2}.$$

That is, $\rho \leq \log_2(2M\sqrt{L})$.

Remark 3.1 When Γ is a cubelike Cayley graph, then the parameter M in Theorem 3.1 is 1, and then we have $\ell = \lfloor \frac{n}{2} \rfloor$. In [18], we provided some graphs which exhibit PST at the time meeting the lower bound in Theorem 3.1 when n is even. This means that the upper bound for ℓ in Theorem 3.1 is tight.

In what follows, we present a result which shows that the upper bound for ℓ in Theorem 3.1 is also tight when n is odd. Before doing that, we need some preparations.

Definition 3.1 *Let n be a positive integer, $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a* Boolean *function. The* Walsh *transformation of f is $W_f : \mathbb{F}_2^n \to \mathbb{Z}$ defined by*

$$W_f(y) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+x\cdot y} \quad \text{for all } y \in \mathbb{F}_2^n.$$

If $n = 2m$ and $|W_f(y)| = 2^m$ for all $y \in \mathbb{F}_2^n$, then f is called a bent function.

It is well-known that bent functions exist in $\mathbb{F}_2^{2m}$ for all $m \geq 1$ and many classes of bent functions have been constructed, see [16].

Theorem 3.2 *Let $m \geq 2$ be a positive integer and $n = 2m + 1$. Let f be a bent function mapping from $\mathbb{F}_2^{2m}$ to $\mathbb{F}_2$ and $f(0) = 0$. Suppose that $G = (\mathbb{F}_2^n, +)$ and α is the weight function defined by*

$$\alpha(0) = 0, \quad \text{and} \quad \alpha(0,z) = (-1)^{f(z)}, \quad \alpha(1,z) = (-1)^{f(z)+1} \quad \text{for all } z \in \mathbb{F}_2^{2m}.$$

Let $\Gamma = \mathrm{Cay}(G;\alpha)$ *be the cubelike weighted* Cayley *graph associated with the weight function* α. *Then*

(1) *the graph* Γ *is connected*;

(2) *for* $a = (1, 0, \cdots, 0) \in \mathbb{F}_2^n$, Γ *has* PST *between* g *and* $g + a$ *for any* $g \in \mathbb{F}_2^n$ *at time* $\frac{\pi}{2^{m+1}}$;

(3) *the minimum period of any vertex in* Γ *is* $\frac{\pi}{2^m}$.

Proof (1) Since Γ is a complete graph, it is obviously connected.

(2) By (2.5), it is easy to see that

$$\Omega_+ = (0, \mathbb{F}_2^{2m}), \quad \Omega_- = (1, \mathbb{F}_2^{2m}).$$

For $x = (0, x') \in \Omega_+, y = (1, y') \in \Omega_-$, we have

$$\begin{aligned}
\lambda_x &= \sum_{z\in G} \alpha(z)\chi_x(z) \\
&= \sum_{z\in\mathbb{F}_2^{2m}} \alpha((0,z))(-1)^{(0,z)\cdot(0,x')} + \sum_{z\in\mathbb{F}_2^{2m}} \alpha((1,z))(-1)^{(1,z)\cdot(0,x')} \\
&= \sum_{z\in\mathbb{F}_2^{2m}\setminus\{0\}} (-1)^{f(z)+z\cdot x'} + \sum_{z\in\mathbb{F}_2^{2m}} (-1)^{1+f(z)+z\cdot x'} \\
&= -1.
\end{aligned}$$

Particularly, $\lambda_0 = -1$. Meanwhile,

$$\begin{aligned}
\lambda_y &= \sum_{z\in G} \alpha(z)\chi_y(z) \\
&= \sum_{z\in\mathbb{F}_2^{2m}} \alpha((0,z))(-1)^{(0,z)\cdot(1,y')} + \sum_{z\in\mathbb{F}_2^{2m}} \alpha((1,z))(-1)^{(1,z)\cdot(1,y')} \\
&= \sum_{z\in\mathbb{F}_2^{2m}\setminus\{0\}} (-1)^{f(z)+z\cdot y'} + \sum_{z\in\mathbb{F}_2^{2m}} (-1)^{f(z)+z\cdot y'} \\
&= -1 + 2W_f(y').
\end{aligned}$$

Since f is a bent function, $W_f(y') = \pm 2^m$, thus we get that

$$\lambda_0 - \lambda_x = 0, \quad v_2(\lambda_0 - \lambda_y) = m + 1.$$

By Theorem 2.2 and Lemma 2.3, we can get the result in the item (2).

(3) It is a direct consequence of Theorem 2.1.

Acknowledgement The authors would like to express their grateful thankfulness to the editor for his/her valuable comments and suggestions which improve the quality and presentation of this paper.

References

[1] Balilńska, K., Cvetković, D., Radosavljević, Z., et al., A survey on integral graphs, *Publ. Elektrotehn. Fak. Ser. Mat.*, **13**, 2002, 42–65.

[2] Bašić, M., Petković, M. D. and Stevanovic, D., Perfect state transfer in integral circulant graphs, *Appl. Math. Lett.*, **22**(7), 2009, 1117–1121.

[3] Bridge, W. G. and Mena, R. A., Rational G-matrices with rational eigenvalues, *J. Combin. Theory Ser. A*, **32**, 1982, 264–280.

[4] Bose, S., Quantum communication through an unmodulated spin chain, *Phys. Rev. Lett.*, **91**(20), 2003, Article ID: 207901.

[5] Casaccino, A., Lloyd, S., Mancini, S. and Severini, S., Quantum state transfer through a qubit network with energy shifts and fluctuations, *Internation Journal of Quantum Information*, **7**, 2009, 1417–1427.

[6] Chan, A., Complex Hadamard matrices, instantaneous uniform mixing and cubes, 2013, arXiv:1305.5811v1.

[7] Childs, A., Cleve, R., Deotto, E., et. al., Exponential algorithmic speedup by a quantum walk, *Proc.* 35*th ACM Symp. Theory of Computing*, 2003, 59–68.

[8] Christandl, M., Datta, N., Dorlas, T., et. al., Perfect state transfer of arbitary state in quantum spin networks, *Phys. Rev. A*, **73**(3), 2005, Article ID: 032312.

[9] Coutinho, G., Quantum State Transfer in Graphs, PhD dissertation, University of Waterloo, 2014.

[10] Farhi, E., Goldstone, J. and Gutmann, S., A quantum algorithm for the Hamiltonian NAND tree, *Theory Compt.*, **4**(8), 2008, 169–190.

[11] Godsil, C., Periodic graphs, *Electronic J. Comb.,* **18**(1), 2011, ♯23.

[12] Godsil, C., State transfer on graphs, *Disc. Math.*, **312**(1), 2012, 129–147.

[13] Harary, F. and Schwenk, A. J., Which graphs have integral spectra? Graphs and Combinatorics (R. Bari and F. Harary, eds.), Springer-Verlag, Berlin, 1974, 45–51.

[14] Klotz, W. and Sander, T., Integral Cayley graphs over abelian groups, *Electronic J. Combin.*, **17**(1), 2010, R81.

[15] Lovasz, L., Spectra of graphs with transitive groups, *Period. Math. Hungar.*, **6**, 1975, 191–196.

[16] Mesnager, S., Bent Functions, Fundamentals and Results, Springer-Verlag, International Publishing, Switzerland, 2016.

[17] Petkovic, M. D. and Bašić, M., Further results on the perfect state transfer in integral circulant graphs, *Computers and Math. Appl.*, **61**(2), 2011, 300–312.

[18] Tan, Y. Y., Feng, K. Q. and Cao, X. W., Perfect state transfer on abelian Cayley graphs, *Linear Algebra Appl.,* **563**, 2019, 331–352.

[19] Zhan, H., An infinite family of circulant graphs with perfect state trnasfer in discrete quantum walks, 2017, arXiv: 1707.06703v1.