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Coburn Type Operators and Compact Perturbations

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Abstract A bounded linear operator T acting on a Hilbert space is called Coburn operator if $\ker(T-\lambda)=\{0\}$ or $\ker(T-\lambda)^*=\{0\}$ for each $\lambda\in\mathbb{C}$. In this paper, the authors define other Coburn type properties for Hilbert space operators and investigate the compact perturbations of operators with Coburn type properties. They characterize those operators for which has arbitrarily small compact perturbation to have some fixed Coburn property. Moreover, they study the stability of these properties under small compact perturbations.

Keywords Coburn type properties, Compact perturbations
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1 Introduction

This paper is inspired by [1–5, 13–15, 18–19], where some special properties of Hilbert space operators under compact perturbations are studied. The purpose of this paper is to investigate the small compact perturbations of Coburn type properties.

Throughout this paper, $\mathbb{C}$ denotes the set of complex numbers, $\mathcal{H}$ will always denote a complex separable infinite dimensional Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$, and let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

Let $T\in\mathcal{B}(\mathcal{H})$. Denote by $\sigma(T)$ and $\sigma_p(T)$, the spectrum of T and the point spectrum of T, respectively. Denote by $\ker T$ and $\operatorname{ran} T$ the kernel of T and the range of T respectively.

A celebrate theorem of Coburn asserts that a nonzero Toeplitz operator on the Hardy space of the unit disk is injective or its adjoint operator is injective. This result is contained in the proof of [7, Theorem 4.1] (see [8, 17] also). Inspired by this fact, we give the following concepts.

Definition 1.1 *$T\in\mathcal{B}(\mathcal{H})$ is called a Coburn operator if* $\ker(T-\lambda)=\{0\}$ *or* $\ker(T-\lambda)^*=\{0\}$ *for each $\lambda\in\mathbb{C}$, denoted by $T\in(C)$.*

Definition 1.2 *$T\in\mathcal{B}(\mathcal{H})$ is called a generalized Coburn operator if the subspaces* $\ker(T-\lambda)$ *and* $\ker(T-\lambda)^*$ *are orthogonal for each $\lambda\in\mathbb{C}$, denoted by $T\in(gC)$.*

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There is a lot of work on Coburn theorem, some of which attempted to extend this result to other spaces such as Dirichlet space or the Hardy space of the bidisk (see [6, 12]). In order to state our main results, we first introduce some notations and terminologies.

Let $T \in \mathcal{B}(\mathcal{H})$. T is called a semi-Fredholm operator, if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T$ or $\operatorname{nul} T^*$ is finite, where $\operatorname{nul} T \triangleq \dim\ker T$ and $\operatorname{nul} T^* \triangleq \dim\ker T^*$; in this case, $\operatorname{ind} T \triangleq \operatorname{nul} T - \operatorname{nul} T^*$ is called the index of T. In particular, if $-\infty < \operatorname{ind} T < +\infty$, then T is called a Fredholm operator. The Wolf spectrum $\sigma_{lre}(T)$ and the essential spectrum $\sigma_e(T)$ are defined by

$$\sigma_{lre}(T) \triangleq \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}$$

and

$$\sigma_e(T) \triangleq \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

respectively. $\rho_{s-F}(T) \triangleq \mathbb{C} \setminus \sigma_{lre}(T)$ is called the semi-Fredholm domain of T. We denote

$$\rho_{s-F}^{+}(T) \triangleq \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) > 0\}$$

and

$$\rho_{s-F}^{-}(T) \triangleq \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) < 0\},$$

respectively.

For $\lambda_0 \in \mathbb{C}$ and $\delta > 0$, we denote $B_\delta(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$.

Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subset \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. Let $E(\sigma; T)$ denote the Riesz idempotent of T corresponding to σ, that is

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote $\mathcal{H}(\sigma; T) = \operatorname{ran} E(\sigma; T)$. Obviously, $\mathcal{H}(\sigma; T)$ is an invariant subspace of T. If $\lambda \in \operatorname{iso} \sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $\mathcal{H}(\lambda; T)$ instead of $\mathcal{H}(\{\lambda\}; T)$; if, in addition, $\dim \mathcal{H}(\lambda; T) < \infty$, then λ is called a normal eigenvalue of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$. Obviously, $\sigma_0(T)$ consists of at most countable many points of $\sigma(T)$.

The main results of this paper are listed as follows.

Theorem 1.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then for each $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (C)$ if and only if $\sigma_0(T) = \emptyset$.*

Theorem 1.2 *Given $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent*:

(1) *For each $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (gC)$.*

(2) *For each $\lambda \in \sigma_0(T)$, there exist unit vectors $e_\lambda \in \ker(\lambda - T)$ and $f_\lambda \in \ker(\lambda - T)^*$ such that $(e_\lambda, f_\lambda) = 0$.*

Recall that two operators $A, B \in \mathcal{B}(\mathcal{H})$ are similar (denoted by $A \sim B$) if there exists an invertible operator $X \in \mathcal{B}(\mathcal{H})$ such that $AX = XB$. Given $T \in \mathcal{B}(\mathcal{H})$, the similarity orbit

$\mathcal{S}(T)$ of T is the set $\{X \in \mathcal{B}(\mathcal{H}) : X \sim T\}$. We denote by $\mathcal{S}(gC)$ the set $\{X \in \mathcal{B}(\mathcal{H}) : X \sim T \text{ for some } T \in (gC)\}$.

It is easy to see that Coburn property is invariant under similarity.

Theorem 1.3 *Given $T \in \mathcal{B}(\mathcal{H})$. Then for each $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{S}(gC)$ if and only if $\dim \mathcal{H}(\lambda; T) \geq 2$ for each $\lambda \in \sigma_0(T)$.*

Theorem 1.4 *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin \mathcal{S}(gC)$.*

Remark 1.1 Obviously, $T \in (C)$ implies that $T \in (gC)$. By Theorem 1.4, for any $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (gC)$ and $T + K \notin (C)$.

The rest part of this paper is organized as follows. In Section 2, we make some preparations. In Section 3, we give the proofs of Theorems 1.1–1.4. In Section 4, we define other Coburn properties and study their compact perturbations.

2 Preparations

In this part, we list some useful results. The following is the famous decomposition theorem due to Riesz.

Lemma 2.1 (see [16, Theorem 2.10]) *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_i $(i = 1, 2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then $\mathcal{H}(\sigma_1; T) + \mathcal{H}(\sigma_2; T) = \mathcal{H}, \mathcal{H}(\sigma_1; T) \cap \mathcal{H}(\sigma_2; T) = \{0\}$ and T admits the following matrix representation*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T) \end{matrix},$$

where $\sigma(T_i) = \sigma_i$ $(i = 1, 2)$.

Lemma 2.2 (see [9, Corollary 3.22]) *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that T admits the following representation*

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where $\sigma(A) \cap \sigma(B) = \emptyset$. Then $T \sim A \oplus B$.

Using the above lemmas, we can obtain the following result and leave the proof to the reader.

Corollary 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that σ is a clopen subset of $\sigma(T)$. Then*

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp} \end{matrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp} \end{matrix},$$

where $\sigma(A) = \sigma$ and $\sigma(B) = \sigma(T) \setminus \sigma$.

Lemma 2.3 (see [11, Lemma 3.2.6]) *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\emptyset \neq \Gamma \subset \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where

(1) N *is a diagonal normal operator of uniformly infinite multiplicity,* $\sigma(N) = \sigma_{lre}(N) = \overline{\Gamma}$,
(2) $\sigma(T) = \sigma(A)$, $\sigma_{lre}(T) = \sigma_{lre}(A)$ *and* $\operatorname{ind}(T-\lambda) = \operatorname{ind}(A-\lambda)$ *for all* $\lambda \in \rho_{s-F}(T)$.

If $\sigma \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$, then we denote $\operatorname{dist}(\lambda, \sigma) = \inf\{|\lambda - \mu| : \mu \in \sigma\}$.

Lemma 2.4 (see [19, Corollary 2.9]) *Given* $T \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$. *There exists* $K \in \mathcal{K}(\mathcal{H})$ *with*

$$\|K\| < \varepsilon + \max\{\operatorname{dist}(\lambda, \partial\rho_{s-F}(T)) : \lambda \in \sigma_0(T)\},$$

such that $\sigma_p(T+K) = \rho^+_{s-F}(T)$ *and* $\sigma_p((T+K)^*) = \overline{\rho^-_{s-F}(T)}$.

This result was proved in [19] and plays a key role in this work. To make this paper more self-contained, we outline the sketch of its proof here. Given an operator $T \in \mathcal{B}(\mathcal{H})$, the essential minimum modulus $m_e(T)$ of T is defined by

$$m_e(T) = \min\{\lambda \in \sigma_e((T^*T)^{\frac{1}{2}})\}.$$

For $\gamma > 0$, define $\Delta_\gamma(T) = \{\mu \in \mathbb{C} : m_e(\mu - T) \leq \gamma\}$. Define

$$m_e(T; \lambda) = \min\{\gamma \geq 0 : \operatorname{dist}(\lambda, \Delta_\gamma(T)) \leq \gamma\}, \quad \lambda \in \mathbb{C}.$$

By using the property of $m_e(T)$ and the continuity of $m_e(T;\lambda)$, one can show that $m_e(T;\lambda) \leq \frac{1}{2}\operatorname{dist}(\lambda, \partial\sigma_e(T))$, $\forall \lambda \in \mathbb{C}$. By the following result, Lemma 2.4 is clear.

Lemma 2.5 (see [10, Proposition 3.4]) *Let* $T \in \mathcal{B}(\mathcal{H})$. *Given* $\varepsilon > 0$, *there exists* $K \in \mathcal{K}(\mathcal{H})$ *with*

$$\|K\| < \varepsilon + \max\{m_e(T;\lambda) : \lambda \in \sigma_0(T)\},$$

such that $\sigma_p(T+K) = \rho^+_{s-F}(T)$ *and* $\sigma_p((T+K)^*) = \overline{\rho^-_{s-F}(T)}$.

The following result characterizes the upper semi-continuity of separate parts of the spectrum.

Lemma 2.6 (see [9, Theorem 1.1] *Let* a *and* b *be two elements of the Banach algebra* $\mathcal{A}$ *with identity. Assume that the spectrum* $\sigma(a)$ *is the disjoint union of two compact subsets* σ_1 *and* σ_2 *such that* $\sigma_1 \neq \emptyset$, *and let* Ω *be a Cauchy domain such that* $\sigma_1 \subset \Omega$ *and* $\sigma_2 \cap \overline{\Omega} = \emptyset$. *If* $\|a - b\| < \min\{\|(\lambda - a)^{-1}\|^{-1} : \lambda \in \partial\Omega\}$, *then* $\sigma(b) \cap \Omega \neq \emptyset$ *and* $\sigma(b) \cap \partial\Omega = \emptyset$.

Lemma 2.7 (see [9, Proposition 1.7] *Let* a *be an element of the Banach algebra* $\mathcal{A}$ *with identity and let* f *be an analytic function defined in a neighborhood* Ω *of* $\sigma(a)$. *Given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $f(b)$ *is well-defined for all* b *in* $\mathcal{A}$ *satisfying* $\|b - a\| < \delta$ *and, moreover,*

$$\|f(b) - f(a)\| < \varepsilon.$$

For $\lambda \in \rho_{s-F}(T)$, the minimal index of $\lambda - T$ is defined as

$$\min \cdot\operatorname{ind}(\lambda - T) = \min\{\operatorname{nul}(\lambda - T), \operatorname{nul}(\lambda - T)^*\}.$$

Lemma 2.8 (see [9, Corollary 1.14]) *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then the function* $\lambda \mapsto \min \cdot \mathrm{ind}(\lambda - T)$ *is constant on every component of* $\rho_{s-F}(T)$ *except for an at most denumerable subset* $\rho^s_{s-F}(T)$ *of* $\rho_{s-F}(T)$ *without limit points in* $\rho_{s-F}(T)$*. Furthermore, if* $\mu \in \rho^s_{s-F}(T)$ *and* λ *is a point of* $\rho_{s-F}(T)$ *in the same component as* μ *but* $\lambda \notin \rho^s_{s-F}(T)$*, then*

$$\min \cdot \mathrm{ind}(\lambda - T) < \min \cdot \mathrm{ind}(\mu - T).$$

Remark 2.1 $\rho^s_{s-F}(T)$ is called the set of singular points of the semi-Fredholm domain $\rho_{s-F}(T)$ of T. $\rho^r_{s-F}(T) = \rho_{s-F}(T) \setminus \rho^s_{s-F}(T)$ is the set of regular points. If $\lambda \in \rho_{s-F}(T)$ and $\mathrm{nul}(\lambda - T) < \infty$ ($\mathrm{nul}(\lambda - T)^* < \infty$), then $\lambda \in \rho^r_{s-F}(T)$ if and only if $\ker(\lambda - T) \subset \bigcap_{n=1}^{\infty} \mathrm{ran}(\lambda - T)^n$ ($\ker(\lambda - T)^* \subset \bigcap_{n=1}^{\infty} \mathrm{ran}((\lambda - T)^*)^n$). Obviously, we have $\sigma_0(T) \subset \rho^s_{s-F}(T)$.

3 Proof of Main Theorems

Now we are going to prove Theorem 1.1.

Proof of Theorem 1.1 "$\Longleftarrow$" Suppose $\sigma_0(T) = \emptyset$. For any $\varepsilon > 0$, by Lemma 2.4, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that

$$\sigma_p(T + K) = \rho^+_{s-F}(T), \quad \sigma_p((T + K)^*) = \overline{\rho^-_{s-F}(T)}.$$

It is easy to check that $\sigma(T + K) = \rho^+_{s-F}(T) \cup \rho^-_{s-F}(T) \cup \sigma_{lre}(T + K)$. For $\lambda \in \rho^+_{s-F}(T)$, we have $\ker(T + K - \lambda)^* = \{0\}$. For $\lambda \in \rho^-_{s-F}(T)$, we have $\ker(T + K - \lambda) = \{0\}$. For $\lambda \in \sigma_{lre}(T)$, we have $\ker(T + K - \lambda) = \ker(T + K - \lambda)^* = \{0\}$. It follows that $T + K \in (C)$.

"$\Longrightarrow$" If $\sigma_0(T) \neq \emptyset$, then we arbitrarily fix $\lambda_0 \in \sigma_0(T)$. Choose $\delta > 0$ such that $B_\delta(\lambda_0) \cap [\sigma(T) \setminus \{\lambda_0\}] = \emptyset$. Let

$$\varepsilon_0 = \min\{\|(\mu - T)^{-1}\|^{-1} : \mu \in \partial B_\delta(\lambda_0)\}.$$

Then $\varepsilon_0 > 0$. For any $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon_0$, by Lemma 2.6, we have $\sigma(T + K) \cap B_\delta(\lambda_0) \neq \emptyset$ and $\sigma(T + K) \cap \partial B_\delta(\lambda_0) = \emptyset$. Since

$$\sigma(T) \cap B_\delta(\lambda_0) = \{\lambda_0\} \subset \sigma_0(T),$$

we have $\sigma_{lre}(T) \cap B_\delta(\lambda_0) = \emptyset$. Hence $\sigma_{lre}(T + K) \cap B_\delta(\lambda_0) = \emptyset$. It follows that $\sigma(T + K) \cap B_\delta(\lambda_0)$ consists of finite many normal eigenvalues of $T + K$. Choose $\mu_0 \in \sigma(T + K) \cap B_\delta(\lambda_0)$, we have $\mu_0 \in \sigma_0(T + K)$ and $\overline{\mu}_0 \in \sigma_0(T + K)^*$. It follows that $\ker(T + K - \mu_0) \neq \{0\}$ and $\ker(T + K - \mu_0)^* \neq \{0\}$. So we have $T + K \notin (C)$.

Lemma 3.1 (see [14, Theorem 3.1]) *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $0 \in \sigma_0(T)$*. If* $\{K_n\}_{n=1}^{\infty} \subset \mathcal{K}(\mathcal{H})$ *and* $\|K_n\| \to 0$*, then there exists a subsequence* $\{K_{n_k}\}_{k=1}^{\infty}$ *of* $\{K_n\}_{n=1}^{\infty}$*,* $\lambda_k \to 0$ *and* $x_k \in \ker(T + K_{n_k} - \lambda_k)$ *with* $\|x_k\| = 1$ *for* $k \geq 1$ *such that* x_k *converges to some* $x \in \ker(T)$ *in the norm topology.*

Corollary 3.1 *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $0 \in \sigma_0(T)$*. If* $\{K_n\}_{n=1}^{\infty} \subset \mathcal{K}(\mathcal{H})$ *and* $\|K_n\| \to 0$*, then there exists a subsequence* $\{K_{n_k}\}_{k=1}^{\infty}$ *of* $\{K_n\}_{n=1}^{\infty}$*,* $\lambda_k \to 0$*,* $x_k \in \ker(T + K_{n_k} - \lambda_k)$ *with* $\|x_k\| = 1$

and $y_k \in \ker(T+K_{n_k}-\lambda_k)^*$ *with* $\|y_k\|=1$ *for* $k\ge 1$ *such that* x_k *converges to some* $x\in\ker(T)$ *and* y_k *converges to some* $y\in\ker(T^*)$ *in the norm topology.*

Proof of Theorem 1.2 "(2)$\Longrightarrow$(1)". For any $\varepsilon>0$, let

$$\sigma=\Big\{\lambda\in\sigma_0(T):\operatorname{dist}(\lambda,\partial\rho_{s-F}(T))\ge\frac{\varepsilon}{2}\Big\}.$$

Then σ is a finite clopen subset of $\sigma(T)$. By Corollary 2.1, T can be written as

$$T=\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}\begin{matrix}\mathcal{H}(\sigma;T)\\ \mathcal{H}(\sigma;T)^{\perp}\end{matrix},$$

where $\mathcal{H}(\sigma;T)$ is a finite dimensional space, $\sigma(A)=\sigma$ and $\sigma(C)=\sigma(T)\setminus\sigma$. Also, we have

$$\operatorname{dist}(\lambda,\partial\rho_{s-F}(C))<\frac{\varepsilon}{2}$$

for each $\lambda\in\sigma_0(C)$. By Lemma 2.4, there exists a compact operator $\overline{K_0}$ on $\mathcal{H}(\sigma;T)^{\perp}$ with $\|\overline{K_0}\|<\frac{\varepsilon}{2}$ such that

$$\sigma_p(C+\overline{K_0})=\rho^{+}_{s-F}(C)=\rho^{+}_{s-F}(T) \tag{3.1}$$

and

$$\sigma_p((C+\overline{K_0})^*)=\overline{\rho^{-}_{s-F}(C)}=\overline{\rho^{-}_{s-F}(T)}. \tag{3.2}$$

We list σ as follows

$$\sigma=\{\lambda_1,\lambda_2,\cdots,\lambda_k\}.$$

Also by Corollary 2.1, $T|_{\mathcal{H}(\sigma;T)}$ admits the following representation

$$T|_{\mathcal{H}(\sigma;T)}=\begin{bmatrix}A_1 & * & * & *\\ & A_2 & * & *\\ & & \ddots & \vdots\\ & & & A_k\end{bmatrix}\begin{matrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \vdots\\ \mathcal{H}_k\end{matrix},$$

where $\sigma(A_i)=\{\lambda_i\}$, $\mathcal{H}_i$ is a finite dimensional Hilbert space for each $1\le i\le k$. In fact, $\{\mathcal{H}_1,\mathcal{H}_2,\cdots,\mathcal{H}_k\}$ are mutually orthogonal and defined such that $\bigoplus_{i=1}^{m}\mathcal{H}_i=\mathcal{H}(\{\lambda_1,\cdots,\lambda_m\};T)$ for each $1\le m\le k$. By condition (2), there exist unit vectors $e_i\in\ker(\lambda_i-T)$ and $f_i\in\ker(\lambda_i-T)^*$ such that $e_i\perp f_i$ for $1\le i\le k$. It follows that $P_{\mathcal{H}_i}e_i\ne 0$, $P_{\mathcal{H}_i}f_i\ne 0$ and $P_{\mathcal{H}_i}e_i\perp P_{\mathcal{H}_i}f_i$ for each $1\le i\le k$. A direct calculation shows that $(\lambda_i-A_i)\big(\frac{P_{\mathcal{H}_i}e_i}{\|P_{\mathcal{H}_i}e_i\|}\big)=0$ and $(\lambda_i-A_i)^*\big(\frac{P_{\mathcal{H}_i}f_i}{\|P_{\mathcal{H}_i}f_i\|}\big)=0$. Applying Gramm-Schimidt process, we can choose an orthornormal basis $\{h_j^{(i)}\}_{j=1}^{n_i}$ of $\mathcal{H}_i$ with $h_1^{(i)}=\frac{P_{\mathcal{H}_i}e_i}{\|P_{\mathcal{H}_i}e_i\|}$, $h_{n_i}^{(i)}=\frac{P_{\mathcal{H}_i}f_i}{\|P_{\mathcal{H}_i}f_i\|}$ such that A_i admits the following representation

$$A_i=\begin{bmatrix}\lambda_i & * & * & * & *\\ & \lambda_i & * & * & *\\ & & \ddots & \ddots & \vdots\\ & & & \lambda_i & *\\ & & & & \lambda_i\end{bmatrix}\begin{matrix}h_1^{(i)}\\ h_2^{(i)}\\ \vdots\\ h_{n_i-1}^{(i)}\\ h_{n_i}^{(i)}\end{matrix}.$$

Choose an operator $\overline{K_i}$ on $\mathcal{H}_i$ with $\|\overline{K_i}\| < \frac{\varepsilon}{2}$ such that

$$A_i + \overline{K_i} = \begin{bmatrix} \lambda_i & \mu_1 & * & * & * \\ & \lambda_i & \mu_2 & * & * \\ & & \ddots & \ddots & \vdots \\ & & & \lambda_i & \mu_{n_i-1} \\ & & & & \lambda_i \end{bmatrix} \begin{matrix} h_1^{(i)} \\ h_2^{(i)} \\ \vdots \\ h_{n_i-1}^{(i)} \\ h_{n_i}^{(i)} \end{matrix},$$

where $\mu_j \neq 0$ for all $1 \leq j \leq n_i - 1$.

Let

$$K = \begin{bmatrix} \overline{K_1} & & & & \\ & \overline{K_2} & & & \\ & & \ddots & & \\ & & & \overline{K_k} & \\ & & & & \overline{K_0} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_k \\ \mathcal{H}(\sigma; T)^{\perp} \end{matrix}.$$

Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. It is easy to check that

$$\ker(T + K - \lambda_i) \subset \Big(\bigoplus_{j=1}^{i-1} \mathcal{H}_j \Big) \oplus \bigvee \{h_1^{(i)}\} \tag{3.3}$$

and

$$\ker(T + K - \lambda_i)^* \subset \bigvee \{h_{n_i}^{(i)}\} \oplus \Big(\bigoplus_{j=i+1}^{k} \mathcal{H}_j \Big) \oplus \mathcal{H}(\sigma; T)^{\perp} \tag{3.4}$$

for each $1 \leq i \leq k$. It follows that $\ker(T + K - \lambda_i) \perp \ker(T + K - \lambda_i)^*$ for each $1 \leq i \leq k$. Obviously, we have $\sigma_p(T+K) \subset \{\lambda_i\}_{i=1}^k \cup \sigma_p(C+\overline{K_0})$ and $\sigma_p(T+K)^* \subset \{\overline{\lambda_i}\}_{i=1}^k \cup \sigma_p(C+\overline{K_0})^*$. For fixed $\lambda \in \sigma_p(T+K)$, we shall show that $\ker(T+K-\lambda)$ and $\ker(T+K-\lambda)^*$ are orthogonal. If $\lambda = \lambda_i$ for some $1 \leq i \leq k$, by (3.3) and (3.4), $\ker(T+K-\lambda)$ and $\ker(T+K-\lambda)^*$ are orthogonal. If $\lambda \in \sigma_p(C + \overline{K_0})$, by (3.1) and (3.2), we have $\overline{\lambda} \notin \sigma_p(T + K)^*$. Hence $\ker(T + K - \lambda)^* = \{0\}$. So we have $T + K \in (gC)$.

"(1)$\Longrightarrow$(2)". We directly assume that $\sigma_0(T)$ consists of infinitely many points. The finite case is much easier to prove. Let $\sigma_0(T) = \{\lambda_k\}_{k=1}^{\infty}$ be an enumeration of $\sigma_0(T)$. Since (1) holds for T, there exists $\{K_n\}_{n=1}^{\infty} \subset \mathcal{K}(\mathcal{H})$ with $\|K_n\| \to 0$ such that $T + K_n \in (gC)$ for each $n \geq 1$. By Corollary 3.1 and diagonal process, we can find $\{K_{n_j}\}_{j=1}^{\infty}$ and $\{\mu_j^{(k)}\}_{j=1}^{\infty}$ with $\mu_j^{(k)} \to \lambda_k$ as $j \to \infty$, $x_j^{(k)} \in \ker(T + K_{n_j} - \mu_j^{(k)})$ with $\|x_j^{(k)}\| = 1$ and $y_j^{(k)} \in \ker(T + K_{n_j} - \mu_j^{(k)})^*$ with $\|y_j^{(k)}\| = 1$ such that $x_j^{(k)}$ converges to some $x_k \in \ker(T - \lambda_k)$ and $y_j^{(k)}$ converges to some $y_k \in \ker(T - \lambda_k)^*$ as $j \to \infty$ for each $k \geq 1$. Obviously, we have $\|x_k\| = \|y_k\| = 1$ for each $k \geq 1$. Since $T + K_{n_j} \in (gC)$ for each $j \geq 1$, we have $(x_j^{(k)}, y_j^{(k)}) = 0$ for each $j \geq 1$. It follows that $(x_k, y_k) = 0$ for each $k \geq 1$.

Proof of Theorem 1.3 "$\Longrightarrow$". We assume that there exists a sequence of compact operators $\{K_n\}_{n=1}^{\infty}$ with $\|K_n\| \to 0$ such that $T + K_n \in \mathcal{S}(gC)$. Then we shall show that $\dim \mathcal{H}(\lambda; T) \geq 2$ for each $\lambda \in \sigma_0(T)$. Otherwise, there exists some $\lambda_0 \in \sigma_0(T)$ such that $\dim \mathcal{H}(\lambda_0; T) = 1$. Then we can choose a $\delta > 0$ such that $B_\delta(\lambda_0) \cap [\sigma(T) \setminus \{\lambda_0\}] = \emptyset$. By Lemma 2.6, there exists n_0 such that $\sigma(T+K_n) \cap B_\delta(\lambda_0) \neq \emptyset$ and $\sigma(T+K_n) \cap \partial B_\delta(\lambda_0) = \emptyset$ for each

$n \geq n_0$. Applying Lemma 2.7, we can choose n_1 large enough such that $E(B_\delta(\lambda_0); T + K_{n_1})$ and $E(B_\delta(\lambda_0); T)$ are similar. Notice that $\operatorname{ran} E(B_\delta(\lambda_0); T) = \operatorname{ran} E(\lambda_0; T) = \mathcal{H}(\lambda_0; T)$, we have

$$\dim \operatorname{ran} E(B_\delta(\lambda_0); T + K_{n_1}) = 1.$$

Since $T + K_{n_1} \in \mathcal{S}(gC)$, there exists $T_0 \in (gC)$ such that T_0 and $T + K_{n_1}$ are similar.

This means that $B_\delta(\lambda_0) \cap \sigma(T_0)$ consists of only one point. We denote it by $\{\mu\}$. Also we have $\dim \mathcal{H}(\mu; T_0) = 1$. By Corollary 2.1, T_0 can be written as

$$T_0 = \begin{bmatrix} \mu & B \\ 0 & C \end{bmatrix} \begin{matrix} \mathcal{H}(\mu; T_0) \\ \mathcal{H}(\mu; T_0)^\perp \end{matrix},$$

where $\mu \notin \sigma(C)$. Arbitrarily choose a unit vector $e \in \mathcal{H}(\mu; T_0)$. We have $\mathcal{H}(\mu; T_0) = \bigvee\{e\}$. A direct calculation shows that $\ker(T_0 - \mu) = \bigvee\{e\}$. Since $T_0 \in (gC)$, we have $\ker(T_0 - \mu)^* \subset \mathcal{H}(\mu; T_0)^\perp$. Notice that $\overline{\mu} \notin \sigma(C^*)$, we have $\ker(T_0 - \mu)^* = \{0\}$. This is contradict to the fact that $\overline{\mu} \in \sigma_0(T_0^*)$.

"$\Longleftarrow$". Apply the same argument as "(2) $\Longrightarrow$ (1)" in the proof of Theorem 1.2. For each $\varepsilon > 0$, we can choose $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ can be written as

$$T + K = \begin{bmatrix} A_1 + \overline{K_1} & * & * & * & * \\ & A_2 + \overline{K_2} & * & * & * \\ & & \ddots & \ddots & \vdots \\ & & & A_k + \overline{K_k} & * \\ & & & & C + \overline{K_0} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_k \\ \mathcal{H}(\sigma; T)^\perp \end{matrix},$$

where $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_k$, $\mathcal{H}(\sigma; T)^\perp$ are mutually orthogonal and $\dim \mathcal{H}_i \geq 2$ for each $1 \leq i \leq k$. Also, we have

$$\sigma_p(C + \overline{K_0}) = \rho_{s-F}^+(T) \tag{3.5}$$

and

$$\sigma_p((C + \overline{K_0})^*) = \overline{\rho_{s-F}^-(T)}. \tag{3.6}$$

By Corollary 2.1 and the Jordan standard theorem, $T + K$ can be similar to the following operator T_1,

$$T_1 = \begin{bmatrix} \overline{A_1} & & & & \\ & \overline{A_2} & & & \\ & & \ddots & & \\ & & & \overline{A_k} & \\ & & & & C + \overline{K_0} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_k \\ \mathcal{H}(\sigma; T)^\perp \end{matrix},$$

where

$$\overline{A_i} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \begin{matrix} e_1^{(i)} \\ e_2^{(i)} \\ \vdots \\ e_{n_i-1}^{(i)} \\ e_{n_i}^{(i)} \end{matrix},$$

$n_i \geq 2$, $\{e^{(i)}\}_{j=1}^{n_i}$ is an orthonormal basis of $\mathcal{H}_i$ for each $1 \leq i \leq k$.

It is easy to see that

$$\sigma_p(T_1) = \{\lambda_i\}_{i=1}^k \cup \sigma_p(C + \overline{K_0}) \tag{3.7}$$

and

$$\sigma_p(T_1^*) = \{\overline{\lambda_i}\}_{i=1}^k \cup \sigma_p(C + \overline{K_0})^*. \tag{3.8}$$

It suffices to show that $T_1 \in (gC)$. For any $\lambda \in \sigma_p(T_1)$, we shall show that $\ker(T_1 - \lambda)$ and $\ker(T_1 - \lambda)^*$ are orthogonal. If $\lambda = \lambda_i$ for some $1 \le i \le k$, we have $\ker(T_1 - \lambda_i) = \bigvee\{e_1^{(i)}\}$ and $\ker(T_1 - \lambda_i)^* = \bigvee\{e_{n_i}^{(i)}\}$. Then $\ker(T_1 - \lambda_i)$ and $\ker(T_1 - \lambda_i)^*$ are orthogonal. If $\lambda \in \sigma_p(C + \overline{K_0})$, by (3.5)–(3.6), we have $\overline{\lambda} \notin \sigma_p(T_1^*)$. Then $\ker(T_1 - \lambda)^* = \{0\}$ and hence $\ker(T_1 - \lambda)$ and $\ker(T_1 - \lambda)^*$ are orthogonal. It follows that $T_1 \in (gC)$.

Proof of Theorem 1.4 Arbitrarily fix $\varepsilon > 0$. Since $\partial\sigma(T) \cap \sigma_{lre}(T) \neq \emptyset$, we choose $\lambda_0 \in \partial\sigma(T) \cap \sigma_{lre}(T)$. By Lemma 2.3, there exists $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \frac{\varepsilon}{2}$ such that

$$T + K_1 = \begin{bmatrix} \lambda_0 & C \\ 0 & A \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix},$$

where e is a unit vector in $\mathcal{H}$ and $\sigma(A) = \sigma(T)$. We can choose $\mu_0 \notin \sigma(T)$ such that $|\mu_0 - \lambda_0| < \frac{\varepsilon}{2}$. Let

$$K_2 = \begin{bmatrix} \mu_0 - \lambda_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix}.$$

Then K_2 is a rank-one operator. and hence $K = K_1 + K_2$. We have $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. Moreover, we have

$$T + K = \begin{bmatrix} \mu_0 & C \\ 0 & A \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix}.$$

It suffices to show that $T + K \notin \mathcal{S}(gC)$. Otherwise, there exists $T_1 \in (gC)$ such that T_1 and $T + K$ are similar. Since $\mu_0 \notin \sigma(T)$ and $\sigma(A) = \sigma(T)$, μ_0 is an isolated point in $\sigma(T + K)$. It follows that $\mu_0 \in \sigma_0(T + K)$ and hence $\mu_0 \in \sigma_0(T_1)$. Obviously, we have $\mathcal{H}(\mu_0; T + K) = 1$ and hence $\mathcal{H}(\mu_0; T_1) = 1$. Then T_1 can be written as

$$T_1 = \begin{bmatrix} \mu_0 & E \\ 0 & F \end{bmatrix} \begin{matrix} \mathcal{H}(\mu_0; T_1) \\ \mathcal{H}(\mu_0; T_1)^\perp \end{matrix},$$

where $\mu_0 \notin \sigma(F)$.

It is easy to see that $\ker(T_1 - \mu_0) = \mathcal{H}(\mu_0; T_1)$. Since $T_1 \in (gC)$, we have $\ker(T_1 - \mu_0) \perp \ker(T_1 - \mu_0)^*$ and hence $\ker(T_1 - \mu_0)^* \subset \mathcal{H}(\mu_0; T_1)^\perp$. Notice that $\mu_0 \notin \sigma(F)$, it follows that $\ker(T_1 - \mu_0)^* = \{0\}$. This is contradict to the fact that $\overline{\mu_0} \in \sigma_0(T_1^*)$.

4 Other Coburn Type Properties and Compact Perturbations

First, we give the following definitions.

Definition 4.1 *$T \in \mathcal{B}(\mathcal{H})$ is called a* GC *operator if the subspaces* $\ker(T - \lambda) \cap \ker(T - \lambda)^* = \{0\}$ *for each* $\lambda \in \mathbb{C}$, *denoted by* $T \in (GC)$.

Obviously, $T \in (gC)$ implies that $T \in (GC)$.

Definition 4.2 *$T \in \mathcal{B}(\mathcal{H})$ is said to have property (Re), denoted by $T \in (Re)$, if* $\ker(T-\lambda) \subset \bigcap_{n=0}^{\infty} \operatorname{ran}(T-\lambda)^n$ *for each $\lambda \in \mathbb{C}$.*

Definition 4.3 *$T \in \mathcal{B}(\mathcal{H})$ is said to have property (P), denoted by $T \in (P)$, if* $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ *for each $\lambda \in \mathbb{C}$.*

In this part, we get the following results.

Theorem 4.1 *Given $T \in \mathcal{B}(\mathcal{H})$. Then for each $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (GC)$.*

Theorem 4.2 *Given $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent*:

(1) *For any $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (Re)$;*

(2) $\sigma_0(T) = \emptyset$.

Theorem 4.3 *Given $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent*:

(1) *For any $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (P)$.*

(2) *$\rho_{s-F}^{+}(T) = \emptyset$ and $\mathcal{H}(\lambda; T)$ reduces T, $T|_{\mathcal{H}(\lambda;T)}$ is normal for each $\lambda \in \sigma_0(T)$.*

Theorem 4.4 *Given $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent*:

(1) *For each $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (GC)$.*

(2) *There exists a sequence of unit vectors $\{e_n\}_{n=1}^{\infty}$ such that*

$$\|Te_n\|^2 - |(Te_n, e_n)|^2 + \|T^*e_n\|^2 - |(T^*e_n, e_n)|^2 \to 0$$

as $n \to +\infty$.

Theorem 4.5 *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (Re)$ and $T + K \notin (P)$.*

Proof of Theorem 4.1 For any $\varepsilon > 0$, let

$$\sigma = \Big\{\lambda \in \sigma_0(T) : \operatorname{dist}(\lambda, \partial\rho_{s-F}(T)) \geq \frac{\varepsilon}{2}\Big\}.$$

Then σ is a finite clopen subset of $\sigma(T)$. By Corollary 2.1, T can be written as

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp} \end{matrix},$$

where $\mathcal{H}(\sigma; T)$ is a finite dimensional space, $\sigma(A) = \sigma$ and $\sigma(C) = \sigma(T) \setminus \sigma$. Also, we have

$$\operatorname{dist}(\lambda, \partial\rho_{s-F}(C)) < \frac{\varepsilon}{2}$$

for each $\lambda \in \sigma_0(C)$. By Lemma 2.4, there exists a compact operator $\overline{K_0}$ on $\mathcal{H}(\sigma; T)^{\perp}$ with $\|\overline{K_0}\| < \frac{\varepsilon}{2}$ such that

$$\sigma_p(C + \overline{K_0}) = \rho_{s-F}^{+}(C) = \rho_{s-F}^{+}(T) \tag{4.1}$$

and

$$\sigma_p((C + \overline{K_0})^*) = \overline{\rho_{s-F}^{-}(C)} = \overline{\rho_{s-F}^{-}(T)}. \tag{4.2}$$

Assume that $\dim \mathcal{H}(\sigma; T) = N$. We divide the proof into two cases.

Case 1, $N \geq 2$. There exists a compact operator $\overline{K_1}$ with $\|\overline{K_1}\| < \frac{\varepsilon}{2}$ acting on $\mathcal{H}(\sigma, T)$ such that

$$A + \overline{K_1} = \begin{bmatrix} \mu_1 & \nu_1 & * & \cdots & * \\ & \mu_2 & \nu_2 & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & \mu_{N-1} & \nu_{N-1} \\ & & & & \mu_N \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \\ e_N \end{matrix},$$

where $\mu_i \neq \mu_j$ for each $i \neq j$, $\mu_j \notin \sigma(C)$ for $1 \leq j \leq N$ and $\nu_j \neq 0$ for each $1 \leq j \leq N-1$, $\{e_j\}_{j=1}^N$ is an orthonormal basis of $\mathcal{H}(\sigma; T)$. Let

$$K_0 = \begin{bmatrix} 0 & \\ & \overline{K_0} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^\perp \end{matrix}$$

and

$$K_1 = \begin{bmatrix} \overline{K_1} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^\perp \end{matrix}.$$

Let $K = K_0 + K_1$. Then we have $K \in \mathcal{K}(\mathcal{H})$, $\|K\| < \varepsilon$ and

$$T + K = \begin{bmatrix} \mu_1 & \nu_1 & * & \cdots & * & * \\ & \mu_2 & \nu_2 & \cdots & * & * \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \mu_{N-1} & \nu_{N-1} & * \\ & & & & \mu_N & * \\ & & & & & C + \overline{K_0} \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \\ e_N \\ \mathcal{H}(\sigma; T)^\perp \end{matrix}.$$

For $1 \leq i \leq N$, by (4.1) and (4.2) we have

$$\ker(C + \overline{K_0} - \mu_i) = \ker(C + \overline{K_0} - \mu_i)^* = \{0\}.$$

It is easy to see that

$$\ker(T + K - \mu_i) \subset \bigvee_{j=1}^{i} \{e_j\}$$

and

$$\ker(T + K - \mu_i)^* \subset \left(\bigvee_{j=i}^{N} \{e_j\} \right) \oplus \mathcal{H}(\sigma; T)^\perp.$$

Hence, we have

$$\ker(T + K - \mu_i) \cap \ker(T + K - \mu_i)^* \subset \vee\{e_i\}.$$

To get $\ker(T + K - \mu_i) \cap \ker(T + K - \mu_i)^* = \{0\}$, it suffices to show that $e_i \notin \ker(T + K - \mu_i) \cap \ker(T + K - \mu_i)^*$. For $1 \leq i \leq N-1$, by $\nu_i \neq 0$, it is easy to see that $e_i \notin \ker(T + K - \mu_i)^*$. For $i = N$, by $\nu_{N-1} \neq 0$, we have $e_N \notin \ker(T + K - \mu_N)$. Hence

$$\ker(T + K - \mu_i) \cap \ker(T + K - \mu_i)^* = \{0\} \tag{4.3}$$

for each $1 \leq i \leq N$.

For any $\lambda \in \sigma_p(C + \overline{K_0})$, by (4.1) and (4.2), we have $\overline{\lambda} \notin \sigma_p(T + K)^*$. It follows that

$$\ker(T + K - \lambda)^* = \{0\}. \tag{4.4}$$

Since $\sigma_p(T + K) \subset \{\mu_i\}_{i=1}^N \cup \sigma_p(C + \overline{K_0})$, by (4.3) and (4.4), we have

$$\ker(T + K - \lambda) \cap \ker(T + K - \lambda)^* = \{0\}$$

for each $\lambda \in \sigma_p(T + K)$. Hence $T + K \in (GC)$.

Case 2, $N = 1$. Choose a unit vector e in $\mathcal{H}(\sigma; T)$. Then T can be rewritten as

$$T = \begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix} \begin{matrix} e \\ e^{\perp} \end{matrix}.$$

We also let

$$K_0 = \begin{bmatrix} 0 & \\ & \overline{K_0} \end{bmatrix} \begin{matrix} e \\ e^{\perp} \end{matrix}.$$

Then we have

$$T + K_0 = \begin{bmatrix} \lambda & B \\ 0 & C + \overline{K_0} \end{bmatrix} \begin{matrix} e \\ e^{\perp} \end{matrix}.$$

We can choose a rank-one operator K_1 with $\|K_1\| < \frac{\varepsilon}{2}$ such that

$$T + K_0 + K_1 = \begin{bmatrix} \lambda & \overline{B} \\ 0 & C + \overline{K_0} \end{bmatrix} \begin{matrix} e \\ e^{\perp} \end{matrix},$$

where $\overline{B} \neq 0$. We shall show that $T + K_0 + K_1 \in (GC)$. As the proof of Case 1, it suffices to show that

$$\ker(T + K_0 + K_1 - \lambda) \cap \ker(T + K_0 + K_1 - \lambda)^* = \{0\}. \tag{4.5}$$

Since $\lambda \notin \sigma(C)$, then $\mathrm{ind}(C + \overline{K_0} - \lambda) = 0$. By (4.1) and (4.2), we have $\ker(C + \overline{K_0} - \lambda) = \ker(C + \overline{K_0} - \lambda)^* = \{0\}$. It is easy to see that $\ker(T + K_0 + K_1 - \lambda) = \bigvee\{e\}$. If (4.5) does not hold, then we have $e \in \ker(T + K_0 + K_1 - \lambda)^*$. Since $\overline{B} \neq 0$, a direct calculation shows that $(T + K_0 + K_1 - \lambda)^* e \neq 0$. This leads a contradiction.

Proof of Theorem 4.2 "(1) $\Longrightarrow$ (2)". Assume that there exists a sequence of compact operators $\{K_n\}_{n=1}^{\infty}$ with $\|K_n\| \to 0$ such that $T + K_n \in (Re)$ for each $n \geq 1$. We shall show that $\sigma_0(T) = \emptyset$. Otherwise, arbitrarily choose $\lambda_0 \in \sigma_0(T)$. By Lemma 2.6, there exists n_0 such that $\sigma_0(T + K_{n_0}) \neq \emptyset$. Choose $\mu_0 \in \sigma_0(T + K_{n_0})$. By Lemma 2.1, $T + K_{n_0}$ can be written as

$$T + K_{n_0} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\mu_0; T + K_{n_0}) \\ \mathcal{H}(\sigma(T + K_{n_0}) \setminus \{\mu_0\}; T + K_{n_0}) \end{matrix}.$$

We denote $p = \dim \mathcal{H}(\mu_0; T + K_{n_0})$, then $1 \leq p < \infty$. It follows that $\ker(T + K_{n_0} - \mu_0) \neq \emptyset$ and $\ker(T + K_{n_0} - \mu_0) \subset \mathcal{H}(\mu_0; T + K_{n_0})$. On the other hand, we have $(A - \mu_0)^p = 0$ and $\mu_0 \notin \sigma(B)$. We have $\mathrm{ran}(T + K_{n_0} - \mu_0)^p = \mathcal{H}(\sigma(T + K_{n_0}) \setminus \{\mu_0\}; T + K_{n_0})$. So $T + K_{n_0} \notin (Re)$. This leads a contradiction.

"(2) $\Longrightarrow$ (1)". Assume that $\sigma_0(T) = \emptyset$. For each $\varepsilon > 0$, we shall show that there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (Re)$. By Lemma 2.5, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that

$$\sigma_p(T + K) = \rho_{s-F}^{+}(T), \quad \sigma_p((T + K)^*) = \overline{\rho_{s-F}^{-}(T)}. \tag{4.6}$$

It suffices to show that $T+K\in(Re)$. Given $\lambda\in\mathbb{C}$, if $\lambda\in\sigma_{lre}(T)$, by (4.6), we have $\ker(T+K-\lambda)=\{0\}$. If $\lambda\in\rho_{s-F}(T)$, we divide the proof into three cases. When $\operatorname{ind}(T+K-\lambda)=0$, by (4.6), we have $\ker(T+K-\lambda)=\{0\}$. When $\operatorname{ind}(T+K-\lambda)>0$, also by (4.6), we have $\ker(T+K-\lambda)^*=\{0\}$ and hence $\ker((T+K-\lambda)^n)^*=\{0\}$ for each $n\in\{0\}\cup\mathbb{N}$. It follows that $\operatorname{ran}(T+K-\lambda)^n=\mathcal{H}$ and hence $\bigcap_{n=0}^{\infty}\operatorname{ran}(T+K-\lambda)^n=\mathcal{H}$. When $\operatorname{ind}(T+K-\lambda)<0$, by (4.6), we have $\ker(T+K-\lambda)=\{0\}$. In any case, we have

$$\ker(T+K-\lambda)\subset\bigcap_{n=0}^{\infty}\operatorname{ran}((T+K-\lambda)^n)$$

for each $\lambda\in\mathbb{C}$. Hence $T+K\in(Re)$.

Proof of Theorem 4.3 "(1) $\Longrightarrow$ (2)". Assume (1) holds for T. First, we shall show $\rho^+_{s-F}(T)=\emptyset$. Otherwise, we can choose $\lambda_0\in\rho^+_{s-F}(T)$. For any $K\in\mathcal{K}(\mathcal{H})$, we have $\operatorname{ind}(T+K-\lambda_0)>0$. It follows that $\dim\ker(T+K-\lambda_0)>\dim(T+K-\lambda_0)^*$. This means that $T+K\notin(P)$. This leads a contradiction.

Second, we shall show $\mathcal{H}(\lambda_0;T)$ reduces T for each $\lambda_0\in\sigma_0(T)$. Let $\lambda_0\in\sigma_0(T)$. Then we can choose a $\delta>0$ such that $B_\delta(\lambda_0)\cap[\sigma(T)\setminus\{\lambda_0\}]=\emptyset$. By Lemma 2.6, there exists n_0 such that $\sigma(T+K_n)\cap B_\delta(\lambda_0)\neq\emptyset$ and $\sigma(T+K_n)\cap\partial B_\delta(\lambda_0)=\emptyset$ for each $n\geq n_0$. By Lemma 2.7, we have

$$\|E(B_\delta(\lambda_0);T+K_n)-E(B_\delta(\lambda_0);T)\|\to 0 \tag{4.7}$$

as $n\to+\infty$. Since $T+K_n\in(P)$, it is easy to see that $\mathcal{H}(B_\delta(\lambda_0);T+K_n)$ reduces $T+K_n$ for $n\geq n_0$. This means that $E(B_\delta(\lambda_0);T+K_n)=E(B_\delta(\lambda_0);T+K_n)^*$ and

$$E(B_\delta(\lambda_0);T+K_n)(T+K_n)=(T+K_n)E(B_\delta(\lambda_0);T+K_n).$$

As $n\to+\infty$, we have

$$E(B_\delta(\lambda_0);T)T=TE(B_\delta(\lambda_0);T).$$

By (4.7), we also have $E(B_\delta(\lambda_0);T)=E(B_\delta(\lambda_0);T)^*$. Since $B_\delta(\lambda_0)\cap\sigma(T)=\{\lambda_0\}$, so we have got

$$E(\lambda_0;T)T=TE(\lambda_0;T),\quad E(\lambda_0;T)=E(\lambda_0;T)^*.$$

Hence $\mathcal{H}(\lambda_0;T)$ reduces T.

Third, we shall show that $T|_{\mathcal{H}(\lambda_0;T)}$ is normal for each $\lambda\in\sigma_0(T)$. Assume that there exists $\{K_n\}_{n=1}^{\infty}\subset\mathcal{K}(\mathcal{H})$ with $\|K_n\|\to 0$ such that $T+K_n\in(P)$ for each $n\geq 1$. Fix $\lambda_0\in\sigma_0(T)$. Choose $\delta>0$ such that $B_\delta(\lambda_0)\cap\ [\sigma(T)\setminus\{\lambda_0\}]=\emptyset$. By Lemma 2.6, there exists n_0 such that $\sigma(T+K_n)\cap B_\delta(\lambda_0)\neq\emptyset$ and $\sigma(T+K_n)\cap\partial B_\delta(\lambda_0)=\emptyset$ for each $n\geq n_0$. We denote

$$f(z)=\begin{cases} z, & z\in B_\delta(\lambda_0),\\ 0, & z\notin \overline{B_\delta(\lambda_0)}.\end{cases}$$

By Lemma 2.7, we have $\|f(T+K_n)-f(T)\|\to 0$ as $n\to\infty$. In fact, we have that $f(T+K_n)|_{\mathcal{H}(B_\delta(\lambda_0);T+K_n)}=(T+K_n)|_{\mathcal{H}(B_\delta(\lambda_0);T+K_n)}$ and its spectrum consists of finite many normal eigenvalues of $T+K_n$. We denote $\sigma((T+K_n)|_{\mathcal{H}(B_\delta(\lambda_0);T+K_n)})=\{\mu_1,\mu_2,\cdots,\mu_{k_n}\}$. Since

$T+K_n \in (P)$, we have $\ker(T+K_n-\mu_i) \subset \ker(T+K_n-\mu_i)^*$ for $1 \le i \le k_n$. It follows that $f(T+K_n)$ is normal and hence $f(T)$ is normal. Hence $T|_{\mathcal{H}(\lambda_0;T)} = f(T)|_{\mathcal{H}(\lambda_0;T)}$ is normal.

"(2) $\Longrightarrow$ (1)". For any $\varepsilon > 0$, let

$$\sigma = \{\lambda \in \sigma_0(T) : \mathrm{dist}(\lambda, \partial\rho_{s-F}(T)) \ge \varepsilon\}.$$

Then σ is a finite clopen subset of $\sigma(T)$. By Corollary 2.1 and condition (2), T can be written as

$$T = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp \end{matrix},$$

where A is a normal operator acting on the finite dimensional space $\mathcal{H}(\sigma;T)$, $\sigma(A) = \sigma$ and $\sigma(C) = \sigma(T) \setminus \sigma$. Then it follows that

$$\mathrm{dist}(\lambda, \partial\rho_{s-F}(C)) < \varepsilon$$

for each $\lambda \in \sigma_0(C)$. By Lemma 2.4, there exists a compact operator $\overline{K}$ on $\mathcal{H}(\sigma;T)^\perp$ with $\|\overline{K}\| < \varepsilon$ such that $\sigma_p(C+\overline{K}) = \rho_{s-F}^+(T) = \emptyset$. Let

$$K = \begin{bmatrix} 0 & 0 \\ 0 & \overline{K} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp \end{matrix}.$$

Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. Also, we have

$$T+K = \begin{bmatrix} A & 0 \\ 0 & C+\overline{K} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp \end{matrix}.$$

So we have $\sigma_p(T+K) = \sigma$. Since A is normal on finite dimensional space, we have $\ker(A-\lambda) = \ker(A-\lambda)^*$ for each $\lambda \in \sigma$. It follows that

$$\ker(T-\lambda) \subset \ker(T-\lambda)^*$$

for each $\lambda \in \mathbb{C}$. This finishes the proof.

Proof of Theorem 4.4 "(1) $\Longrightarrow$ (2)". For any $\varepsilon > 0$, we shall choose a unit vector e such that $\|Te\|^2 - |(Te,e)|^2 + \|T^*e\|^2 - |(T^*e,e)|^2 < \varepsilon$. By (1), there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \sqrt{\frac{\varepsilon}{2}}$ such that $T+K \notin (GC)$. Hence there exists $\lambda_0 \in \mathbb{C}$ such that $\ker(T+K-\lambda_0) \cap \ker(T+K-\lambda_0)^* \ne \{0\}$. We can choose a unit vector $e \in \ker(T+K-\lambda_0) \cap \ker(T+K-\lambda_0)^*$. Then we have

$$T+K-\lambda_0 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix}.$$

We rewrite K as follows

$$K = \begin{bmatrix} \mu & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix}.$$

It follows that

$$T = \begin{bmatrix} \lambda_0 - \mu & -K_{12} \\ -K_{21} & \lambda_0 + A - K_{22} \end{bmatrix} \begin{matrix} e \\ e^\perp \end{matrix}.$$

Since $\|Te\|^2 - |(Te,e)|^2 = \|K_{21}e\|^2 \le \|K\|^2$ and $\|T^*e\|^2 - |(T^*e,e)|^2 = \|K_{12}^*e\|^2 \le \|K^*\|^2$, we have

$$\|Te\|^2 - |(Te,e)|^2 + \|T^*e\|^2 - |(T^*e,e)|^2 < \varepsilon.$$

"(2) $\Longrightarrow$ (1)". Assume (2) holds, for any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$\|Te_{n_0}\|^2 - |(Te_{n_0}, e_{n_0})|^2 + \|T^*e_{n_0}\|^2 - |(T^*e_{n_0}, e_{n_0})|^2 < \varepsilon^2. \tag{4.7}$$

Then T can be written as

$$T = \begin{bmatrix} \lambda & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{matrix} e_{n_0} \\ e_{n_0}^{\perp} \end{matrix}.$$

Let

$$K = \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix} \begin{matrix} e_{n_0} \\ e_{n_0}^{\perp} \end{matrix}.$$

Obviously, $K \in \mathcal{K}(\mathcal{H})$. By (4.7), it is easy to see that $\|K\| < \varepsilon$. Then we have

$$T - K = \begin{bmatrix} \lambda & 0 \\ 0 & T_{22} \end{bmatrix} \begin{matrix} e_{n_0} \\ e_{n_0}^{\perp} \end{matrix}.$$

It follows that $e_{n_0} \in \ker(T - K - \lambda) \cap \ker(T - K - \lambda)^*$ and hence $T - K \notin (GC)$.

Proof of Theorem 4.5 Similarly to the proof of Theorem 1.4, for each $\varepsilon > 0$, we can choose a compact operator K with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} \mu_0 & B \\ 0 & A \end{bmatrix} \begin{matrix} e \\ e^{\perp} \end{matrix},$$

where μ_0 is a normal eigenvalue of $T+K$ and $B \neq 0$. By Lemma 2.8 and Remark 2.1, it is easy to see that μ_0 is a singular point in $\rho_{s-F}(T+K)$. Hence "$\ker(T+K-\mu_0) \subset \bigcap_{n=0}^{\infty} \operatorname{ran}((T+K-\mu_0)^n)$" does not hold. Also, it is easy to see that $e \in \ker(T + K - \mu_0)$ and $e \notin \ker(T + K - \mu_0)^*$. Hence $T + K \notin (Re)$ and $T + K \notin (P)$.

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References

[1] Aiena, P., Property (w) and perturbations, II, *J. Math. Anal. Appl.,* **342**(2), 2008, 830–837.

[2] Aiena, P. and Aponte, E., Polaroid type operators under perturbations, *Studia Math.,* **214**(2), 2013, 121–136.

[3] Aiena, P., Aponte, E., Guillén, J. R. and Peña, P., Property (R) under perturbations, *Mediterr. J. Math.,* **10**(1), 2013, 367–382.

[4] Aiena, P., Biondi, M. T. and Villafañe, F., Property (w) and perturbations, III, *J. Math. Anal. Appl.,* **353**(1), 2009, 205–214.

[5] Aiena, P., Guillén, J. R. and Peña, P., Property (w) for perturbations of polaroid operators, *Linear Algebra Appl.,* **428**(8–9), 2008, 1791–1802.

[6] Chen, Y., Izuchi, K. J. and Young, Y. J., A Coburn type theorem on the Hardy space of the bidisk, *J. Math. Anal. Appl.,* **466**(1), 2018, 1043–1059.

[7] Coburn, L. A., Weyl's theorem for nonnormal operators, *Michigan Math. J.,* **13**, 1966, 285–288.

[8] Douglas, R. G., Banach Algebra Techniques in Operator Theory, Second edition, Graduate Texts in Mathematics, 179, Springer-Verlag, New York, 1998.

[9] Herrero, D. A., Approximation of Hilbert Space Operators, **1**, 2nd ed., Pitman Research Notes in Mathematics Series, **224**, Longman Scientific & Technical, Harlow, 1989.

[10] Herrero, D. A., Taylor, T. J. and Wang, Z. Y., Variation of the point spectrum under compact perturbations, Topics in Operator Theory, Oper. Theory Adv. Appl., **32**, Birkhäuser, Basel, 1988, 113–158.

[11] Jiang, C. L. and Wang, Z. Y., Structure of Hilbert Space Operators, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[12] Lee, Y. J., A Coburn type theorem for Toeplitz operators on the Dirichlet space, *J. Math. Anal. Appl.,* **414**(1), 2014, 237–242.

[13] Li, C. G. and Zhou, T. T., Polaroid type operators and compact perturbations, *Studia Mathematica*, **221**(2), 2014, 175–192.

[14] Li, C. G. and Zhou, T. T., Orthogonality property $\mathcal{O}$ and compact perturbations, *Complex Anal. Oper. Theory,* **10**(8), 2016, 1741–1755.

[15] Li, C. G., Zhu, S. and Feng, Y. L., Weyl's theorem for functions of operators and approximation, *Integral Equations Operator Theory,* **67**(4), 2010, 481–497.

[16] Radjavi, H. and Rosenthal, P., Invariant Subspaces, 2nd ed., Dover Publications Inc., Mineola, NY, 2003.

[17] Vukotić, D., A note on the range of Toeplitz operators, *Integral Equations Operator Theory*, **50**(4), 2004, 565–567.

[18] Zhou, T. T., Li, C. G. and Zhu, S., Generalized Weyl's theorem for functions of operators and compact perturbations, *Bull. Korean Math. Soc.,* **49**(5), 2012, 899–910.

[19] Zhu, S. and Li, C. G., SVEP and compact perturbations, *J. Math. Anal. Appl.,* **380**(1), 2011, 69–75.