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Boundedness of Vector Valued Bilinear Calderón-Zygmund Operators on Products of Weighted Herz-Morrey Spaces with Variable Exponents*

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Abstract In this paper, the authors obtain the boundedness of vector valued bilinear Calderón-Zygmund operators on products of weighted Herz-Morrey spaces with variable exponents.

Keywords Bilinear Calderón-Zygmund operator, Vector valued inequality, Muckenhoupt weight, Variable exponent, Herz-Morrey space
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1 Introduction

As a generalization of Calderón-Zygmund singular integral operators, the theory of multilinear Calderón-Zygmund singular integral operators began with the work of Coifman and Meyer [3] in 1975. After that, the multilinear Calderón-Zygmund singular integral operators have attracted more and more attention from many authors, see [2, 9–12, 15, 24, 31–32]. In recent years, Hu and Meng [13] established the boundedness of multilinear Calderón-Zygmund operators on products of Hardy spaces H^p. Lu and Zhu [21] obtained the boundedness of multilinear Calderón-Zygmund operators on Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents. Shen and the second author of this paper [27] obtained a vector valued inequality of multilinear Calderón-Zygmund operators on products of Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents. Wang and Liu [28] studied the boundedness of multilinear singular integral operators on the product of generalized Morrey spaces $(L^p(w), L^q)^\alpha$. Wang and the second author of this paper obtained Multilinear Calderón-Zygmund operators and their commutators with BMO functions in variable exponent Morrey spaces in [30]. Hu and the second author of this paper obtained the boundedness of Multilinear Calderón-Zygmund operators and their commutators with BMO functions in Herz-Morrey spaces with variable exponents

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$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ in [14]. In [29], we obtained a weighted norm inequality of bilinear Calderón-Zygmund operators in weighted Herz-Morrey spaces with variable exponents and weight in the variable Muckenhoupt class.

In this paper, as a continuation of [29], we will consider the boundedness of vector valued bilinear Calderón-Zygmund operators on products of weighted Herz-Morrey spaces with variable exponents. The plan of the paper is as follows. In Section 2, we collect some notations and state the main result. The proof of the main result will be given in Section 3.

2 Notations and Main Result

In this section, we firstly recall some definitions and notations, then we state our result. Let Ω be a positive measurable subset of $\mathbb{R}^n$, $p(\cdot)$ be a measurable function on Ω taking values in $[1,\infty)$, then the Lebesgue space with variable exoponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \Big\{f \text{ is measurable} : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0\Big\}.$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ becomes a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x \le 1\Big\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for all compact subsets $K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$. We denote $p_- := \operatorname*{essinf}_{x\in\mathbb{R}^n} p(x)$, $p_+ := \operatorname*{esssup}_{x\in\mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, which means $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a nonnegative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable functions f such that $fw \in L^{p(\cdot)}$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}}.$$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\mathrm{d}y, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls containing x in $\mathbb{R}^n$. In general, the Hardy-Littlewood maximal operator is not bounded on weighted variable Lebesgue spaces. But if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuous and $w \in A_{p(\cdot)}$, then M is bounded on $L^{p(\cdot)}(w)$.

Definition 2.1 *Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.*

(i) *The function $\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant C_1 such that*

$$|\alpha(x) - \alpha(y)| \le \frac{C_1}{\log\big(\mathrm{e} + \frac{1}{|x-y|}\big)}, \quad x, y \in \mathbb{R}^n, \ |x - y| < \frac{1}{2}.$$

(ii) *The function $\alpha(\cdot)$ is log-Hölder continuous at the origin if there exists a constant C_2 such that*

$$|\alpha(x) - \alpha(0)| \le \frac{C_2}{\log\big(\mathrm{e} + \frac{1}{|x|}\big)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.

(iii) *The function $\alpha(\cdot)$ is log-Hölder continuous at the infinity if there exists $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that*

$$|\alpha(x) - \alpha_\infty| \le \frac{C_3}{\log(\mathrm{e} + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) *The function $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.*

Definition 2.2 *Fix $p \in (1, \infty)$. A positive measurable function w is said to be in the Muckenhoupt class A_p, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$\Big(\frac{1}{B}\int_B w(x)\mathrm{d}x\Big)\Big(\frac{1}{B}\int_B w(x)^{1-p'}\mathrm{d}x\Big)^{p-1} \le C.$$

We say $w \in A_1$, if $Mw(x) \le Cw(x)$ for a.e. x. If $1 \le p < q < \infty$, then $A_p \subset A_q$. We denote $A_\infty = \bigcup\limits_{p>1} A_p$. The Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ was firstly proposed by Muckenhoupt in [22]. In [5], Cruz-Uribe, Fiorenza and Neugebauer introduced the variable Muckenhoupt $A_{p(\cdot)}$. For more details, see [4–5, 8, 16–17].

Definition 2.3 *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A positive measurable function w is said to be in $A_{p(\cdot)}$, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$\frac{1}{|B|}\|w\chi_B\|_{L^{p(\cdot)}}\|w^{-1}\chi_B\|_{L^{p'(\cdot)}} \le C.$$

Remark 2.1 It is easy to see that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$.

Lemma 2.1 (see [5, Theorem 1.5]) *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there is a positive constant C such that for each $f \in L^{p(\cdot)}(w)$,*

$$\|(Mf)w\|_{L^{p(\cdot)}} \le C\|fw\|_{L^{p(\cdot)}}.$$

To give the definitions of the weighted Herz space and Herz-Morrey space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$, we define $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $D_k := B_k \backslash B_{k-1}$, $\chi_k := \chi_{D_k}$, $\widetilde{\chi}_m = \chi_m$, $m \ge 1$, $\widetilde{\chi}_0 = \chi_{B_0}$. We also need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$, which was firstly defined by Almeida and Hästö in

[1]. Let w be a nonnegative measurable function. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, we define the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}((f_j)_j) := \sum_{j\in\mathbb{Z}} \inf\Big\{\lambda_j : \int_{\mathbb{R}^n} \Big(\frac{|f_j(x)w(x)|}{\lambda_j^{\frac{1}{q(x)}}}\Big)^{p(x)} \mathrm{d}x \le 1\Big\},$$

where $\lambda^{\frac{1}{\infty}} = 1$. If $q^+ < \infty$ or $q(\cdot) \le p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}((f_j)_j) = \sum_{j\in\mathbb{Z}} \||f_j w|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\|(f_j)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf\Big\{\mu > 0 :\ \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\Big(\Big(\frac{f_j}{\mu}\Big)_j\Big) \le 1\Big\}.$$

Definition 2.4 *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)$ and non-homogeneous weighted Herz space $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)$ are defined respectively by*

$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\omega)} < \infty\},$$

and

$$K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n, \omega) : \|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} := \|(2^{j\alpha(\cdot)} f\chi_j)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

and

$$K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\omega) :\ \|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} = \|(2^{j\alpha(\cdot)} f\chi_j)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} < \infty\}.$$

For any quantities A and B, if there exists a constant $C > 0$ such that $A \le CB$, we write $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

The following lemma is a corollary of [20, Theorem 3].

Lemma 2.2 *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and w be a weight. If $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at infinity, then*

$$K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w) = K^{\alpha_\infty,q_\infty}_{p(\cdot)}(w).$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at the origin, then

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} \approx \Big(\sum_{k\le 0} \|2^{k\alpha(0)} f\chi_k\|^{q(0)}_{L^{p(\cdot)}(w)}\Big)^{\frac{1}{q(0)}} + \Big(\sum_{k>0} \|2^{k\alpha_\infty} f\chi_k\|^{q_\infty}_{L^{p(\cdot)}(w)}\Big)^{\frac{1}{q_\infty}}.$$

Definition 2.5 *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\lambda \in [0,\infty)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz-Morrey space $M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)$ and non-homogeneous weighted Herz-Morrey space $MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)$ are defined respectively by*

$$M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w) := \{f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\}, w) : \|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} < \infty\},$$

and

$$MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w) := \{f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n, w) : \|f\|_{MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\|(2^{\alpha(\cdot)k} f\chi_k)_{k\le L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

and

$$\|f\|_{MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda}\|(2^{\alpha(\cdot)k} f\widetilde{\chi}_k)_{k=0}^{L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

Proposition 2.1 *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, w be a weight, $\lambda \in [0,\infty)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.*

(i) *If $\alpha(\cdot)$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then for any $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n\backslash\{0\}, w)$,*

$$\begin{aligned}&\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)}\\ &\approx \max\Big\{\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\|(2^{k\alpha(0)} f\chi_k)_{k\le L}\|_{\ell^{q_0}(L^{p(\cdot)}(w))},\\ &\quad\sup_{L>0, L\in\mathbb{Z}}[2^{-L\lambda}\|(2^{k\alpha(0)} f\chi_k)_{k<0}\|_{\ell^{q_0}(L^{p(\cdot)}(w))} + 2^{-L\lambda}\|(2^{k\alpha_\infty} f\chi_k)_{k=0}^{L}\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}]\Big\},\end{aligned}$$

where and hereafter, $q_0 := q(0)$.

(ii) *If $\alpha(\cdot)$, $q(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then*

$$MK^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w) = MK^{\alpha_\infty,q_\infty}_{p(\cdot),\lambda}(w).$$

Proof Obviously,

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} = \max\Big\{&\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\|(2^{k\alpha(\cdot)} f\chi_k)_{k\le L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))},\\ &\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda}\|(2^{k\alpha(\cdot)} f\chi_k)_{k\le L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)(w)})}\Big\}.\end{aligned}$$

When $L \le 0$, from Lemma 2.2 we know that

$$\|(2^{k\alpha(\cdot)} f\chi_k)_{k\le L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \approx \|(2^{k\alpha(0)} f\chi_k)_{k\le L}\|_{\ell^{q_0}(L^{p(\cdot)}(w))}.$$

When $L > 0$, from Lemma 2.2 again we also obtain

$$\begin{aligned}\|(2^{k\alpha(\cdot)} f\chi_k)_{k<L}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \approx\ &\|(2^{k\alpha(0)} f\chi_k)_{k<0}\|_{\ell^{q_0}(L^{p(\cdot)}(w))}\\ &+ \|(2^{k\alpha_\infty} f\chi_k)_{k=0}^{L}\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}.\end{aligned}$$

Thus we obtain (i). Similarly, we can obtain (ii).

The following Lemma 2.3 has been proved by Izuki and Noi in [18–19].

Lemma 2.3 *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there exist constants δ_1, $\delta_2 \in (0,1)$ and $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}, \tag{2.1}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-1})}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2}. \tag{2.2}$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$. Denote by $L^\infty_C(\mathbb{R}^n)$ the space of compactly supported bounded functions, and $\operatorname{supp}(f)$ the support set of function f. Let T be a bilinear operator, which is originally defined on the 2-fold of Schwartz function space $\mathcal{S}(\mathbb{R}^n)$, and its value belongs to $\mathcal{S}'(\mathbb{R}^n)$:

$$T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

T is called a bilinear Calderón-Zygmund operator, if it extends to a bounded bilinear operator from $L^{p_1} \times L^{p_2}$ to L^p for some p_1, $p_2 \in (1,\infty)$ and p such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and for f_1, $f_2 \in L^\infty_C(\mathbb{R}^n)$, $x \notin \operatorname{supp}(f_1) \cap \operatorname{supp}(f_2)$,

$$T(f_1,f_2)(x) := \int_{\mathbb{R}^{2n}} K(x,y_1,y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2,$$

where the kernel K is a function in $\mathbb{R}^{3n}$ off from the diagonal $x = y_1 = y_2$ and there exist positive constants ε, A such that

$$|K(x,y_1,y_2)| \le \frac{A}{(|x-y_1| + |x-y_2| + |y_1-y_2|)^{2n}}$$

and

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \le \frac{A|x-x'|^\varepsilon}{(|x-y_1| + |x-y_2| + |y_1-y_2|)^{2n+\varepsilon}},$$

whenever $|x - x'| \le \frac{1}{2}\max\{|x-y_1|, |x-y_2|\}$, and the two analogous difference estimates with respect to the variables y_1 and y_2 hold.

Our main result is as follows.

Theorem 2.1 *Assume that T is a bilinear Calderón-Zygmund operator, $p_1(\cdot)$ and $p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfying $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ and $p(\cdot) \in \mathcal{P}_0$ such that there exists $s \le p_-$ such that $w^s \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(w^{-s})$, where $w = w_1 w_2$ and $w_i \in A_{p_i(\cdot)}$, $i = 1,2$. Suppose that $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)}$, $\frac{1}{q_\infty} = \frac{1}{q_{1\infty}} + \frac{1}{q_{2\infty}}$, $\lambda = \lambda_1 + \lambda_2$, $0 \le \lambda_i < \infty$, δ_{i1}, $\delta_{i2} \in (0,1)$ are the constants in Lemma 2.3 for exponents $p_i(\cdot)$ and weights w_i, $i = 1,2$. Let $r_i \in (1,\infty)$ for $i = 1,2$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. If $\lambda_i - n\delta_{i1} < \alpha_{i\infty}, \alpha_i(0) < n\delta_{i2}$ for $i = 1,2$, then there exists a positive constant C such that*

$$\Big\|\Big(\sum_{j=1}^\infty |T(f_1^j, f_2^j)|^r\Big)^{\frac{1}{r}}\Big\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} \le C \prod_{i=1}^2 \Big\|\Big(\sum_{j=1}^\infty |f_i^j|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}$$

for all $f_i^j \in M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)$, $j \in \mathbb{N}$, $i = 1,2$.

3 Proof of Theorem 2.1

To prove Theorem 2.1, we need a series of lemmas.

For $\delta > 0$, we denote $M(|f|^{\delta})^{\frac{1}{\delta}}$ by M_{δ}. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the sharp maximal function is defined by

$$M^{\sharp} f(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| \mathrm{d}y,$$

where the supremum is taken over all the cubes Q containing the point x, and where as usual f_Q denotes the average of f on Q. We denote $M(|f|^{\delta})^{\frac{1}{\delta}}$ by $M^{\sharp}_{\delta}$.

Lemma 3.1 (see [25, Theorem 2.1]) *Let $0 < p, \delta < \infty$ and $w \in A_{\infty}$. There exists a positive constant C such that*

$$\int_{\mathbb{R}^n} M_{\delta} f(x)^p w(x) \mathrm{d}x \le C \int_{\mathbb{R}^n} M^{\sharp}_{\delta} f(x)^p w(x) \mathrm{d}x$$

for every function f such that the left hand side is finite.

Lemma 3.2 (see [23, Theorem 2.1]) *Let T be a bilinear Calderón-Zygmund operator and let $0 < \delta < \frac{1}{2}$. Then, there exists a constant $C > 0$ such that*

$$M^{\sharp}_{\delta} T(f_1, f_2)(x) \le C \prod_{j=1}^{2} M f_j(x)$$

for any functions $f_1,\ f_2 \in L^{\infty}_C(\mathbb{R}^n)$.

Lemma 3.3 (see [15, Theorem 2.3]) *Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ for $x \in \mathbb{R}^n$. Then there exists a constant C_{p,p_1} independent of functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}} \le C_{p,p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$. If $p \in \mathcal{P}(\mathbb{R}^n)$, w is a weight with $w = w_1 \times w_2$, then

$$\|fg\|_{L^{p(\cdot)}(w)} \le C_{p,p_1} \|f\|_{L^{p_1(\cdot)}(w_1)} \|g\|_{L^{p_2(\cdot)}(w_2)}$$

Lemma 3.4 (see [26, Proposition 1.2]) *Let $0 < p \le \infty$, $\delta > 0$. Then there is a positive constant C such that*

$$\Big(\sum_{j=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k \Big)^p \Big)^{\frac{1}{p}} \le C \Big(\sum_{j=-\infty}^{\infty} a_j^p \Big)^{\frac{1}{p}} \tag{3.1}$$

for non-negative sequence $\{a_j\}_{j=-\infty}^{\infty}$. Here, when $p = \infty$, (3.1) stands for

$$\sup_{j \in \mathbb{Z}} \Big(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k \Big) \le C \sup_{j \in \mathbb{Z}} a_j.$$

Lemma 3.5 (see [7, Proposition 3.20]) *The following are equivalent*:

(a) *For every $p \in (0,\infty)$ and every $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} f(x)^p w(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} g(x)^p w(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}.$$

(b) *There exists $p_0 > 0$ such that for every p, $0 < p < p_0$, and every $w \in A_1$,*

$$\int_{\mathbb{R}^n} f(x)^p w(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} g(x)^p w(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}.$$

Let $L^p(w)$ be the weighted Lebesgue space with respect to the measure $w(x)\mathrm{d}x$.

Lemma 3.6 (see [6, Theorem 2.1]) *Assume that for some $p_0 \in (0,\infty)$ and every $w_0 \in A_\infty$, let $\mathcal{F}$ be a family of pairs of non-negative functions such that*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w_0(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}. \tag{3.2}$$

Then for all $0 < p < \infty$ and $w_0 \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^p w_0(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} g(x)^p w_0(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}. \tag{3.3}$$

Furthermore, for every p, $q \in (0,\infty)$, $w_0 \in A_\infty$, and sequence $\{(f_j,g_j)\}_j \subset \mathcal{F}$,

$$\Big\|\Big(\sum_{j=1}^\infty (f_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w_0)} \le C\Big\|\Big(\sum_{j=1}^\infty (g_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^p(w_0)}. \tag{3.4}$$

Lemma 3.7 (see [8, Theorem 2]) *Assume that for some $p_0 \in (0,\infty)$, every $w_0 \in A_1$, and let $\mathcal{F}$ be a family of pairs of non-negative functions,*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w_0(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}.$$

Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. If there exists $p_0 \le p_-$ such that $w^{p_0} \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{p_0})'}(w^{-p_0})$. Then for all $(f,g) \in \mathcal{F}$ with $f \in L^{p(\cdot)}(w)$,

$$\|f\|_{L^{p(\cdot)}(w)} \le \|g\|_{L^{p(\cdot)}(w)}.$$

Lemma 3.8 *Assume that for some p_0, every $w_0 \in A_\infty$, and let $\mathcal{F}$ be a family of pairs of non-negative functions such that (3.2) holds. Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. If there exists $s \le p_-$ such that $w^s \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(w^{-s})$. Then for every $q \in (1,\infty)$ and sequence $\{(f_j,g_j)\}_{j\in\mathbb{N}} \subset \mathcal{F}$,*

$$\Big\|\Big(\sum_{j=1}^\infty (f_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} \le C\Big\|\Big(\sum_{j=1}^\infty (g_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)}.$$

Proof Fix $q \in (1,\infty)$, we define the new family $\mathcal{F}_q$ to consist of the pair (F_q,G_q), where

$$F_q(x) = \Big(\sum_{j=1}^\infty f_j(x)^q\Big)^{\frac{1}{q}}, \quad G_q(x) = \Big(\sum_{j=1}^\infty f_j(x)^q\Big)^{\frac{1}{q}}, \quad \{(f_j,g_j)\}_{j=1}^N \subset \mathcal{F}. \tag{3.5}$$

Since (3.2) holds we have (3.4). Thus, by (3.4) and (3.5), we obtain

$$\int_{\mathbb{R}^n} (F_q(x))^p w_0(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} (G_q(x))^p w_0(x)\mathrm{d}x, \quad (F_q, G_q) \in \mathcal{F}_q.$$

By Lemmas 3.5 and 3.7, we have

$$\Big\|\Big(\sum_{j=1}^{\infty}(f_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} \le C\Big\|\Big(\sum_{j=1}^{\infty}(g_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)}.$$

This completes the proof.

Lemma 3.9 (see [8, Corollary 3.2]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. If the maximal operator M is bounded both on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$, $q \in (1, \infty)$, then*

$$\Big\|\Big(\sum_{j=1}^{\infty}(Mf_j)^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} \le C\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)}.$$

Lemma 3.10 *Let T be a bilinear Calderón-Zygmund operator and $p(\cdot) \in \mathcal{P}_0$ such that there exists $s \le p_-$ such that $w^s \in A_{\frac{p(\cdot)}{s}}$ and M is bounded on $L^{(\frac{p(\cdot)}{s})'}(w^{-s})$. Suppose that $w = w_1w_2$ and $w_i \in A_{p_i(\cdot)}$, $i = 1, 2$. If $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ for $i = 1, 2$ satisfying $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ for $x \in \mathbb{R}^n$, $i = 1, 2$, then there exists a constant C independent of the compactly supported bounded functions f_1^j, $f_2^j \in L^{p_0}(\mathbb{R}^n)$, $j \in \mathbb{N}$ such that*

$$\Big\|\Big(\sum_{j=1}^{\infty}|T(f_1^j, f_2^j)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} \le C\prod_{i=1}^{2}\Big\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(w_i)},$$

where $q_i \in (1, \infty)$ for $i = 1, 2$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Proof Since f_1^j, f_2^j are bounded functions with compact support, $T(f_1^j, f_2^j) \in L^p(\mathbb{R}^n)$ for every $0 < p < \infty$. With Lemmas 3.1–3.2, Pérez and Torres [23] showed that for all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |T(f_1, f_2)(x)|^p w(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} (Mf_1(x)Mf_2(x))^p w(x)\mathrm{d}x.$$

Therefore, by Lemmas 3.6 and 3.8, we have

$$\Big\|\Big(\sum_{j=1}^{\infty}|T(f_1^j, f_2^j)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} \lesssim \Big\|\Big(\sum_{j=1}^{\infty}|Mf_1^j(x)Mf_2^j(x)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)}.$$

Since $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$ and $w = w_1w_2$, by Hölder's inequality and Lemma 3.9, we have

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{\infty}|Mf_1^j(x)Mf_2^j(x)|^q\Big)^{\frac{1}{q}}\Big\|_{L^{p(\cdot)}(w)} &\lesssim \prod_{i=1}^{2}\Big\|\Big(\sum_{j=1}^{\infty}|Mf_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(w_i)} \\ &\lesssim \prod_{i=1}^{2}\Big\|\Big(\sum_{j=1}^{\infty}|f_i^j|^{q_i}\Big)^{\frac{1}{q_i}}\Big\|_{L^{p_i(\cdot)}(w_i)}.\end{aligned}$$

This completes the proof.

Proof of Theorem 2.1 Since the set of all bounded compactly supported functions is dense in weighted variable Lebesgue spaces (see [8, Lemma 3.1]), we only consider bounded compact supported functions. Let f_1^v and f_2^v be bounded functions with compact support for $v \in \mathbb{N}$ and write

$$f_i^v = \sum_{l=-\infty}^{\infty} f_{il}^v \chi_l =: \sum_{l=-\infty}^{\infty} f_{il}^v, \quad i=1,2,\ v \in \mathbb{N}.$$

By Proposition 2.1, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\Big\|_{M\dot K^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)}\\
&\approx \max\Big\{\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k\le L}\Big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
&\quad\sup_{L>0,L\in\mathbb{Z}}\Big[2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k<0}\Big\|_{l^{q_0}(L^{p(\cdot)}(w))}\\
&\qquad + 2^{-L\lambda}\Big\|\Big(2^{k\alpha_\infty}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k=0}^{L}\Big\|_{l^{q_\infty}(L^{p(\cdot)}(w))}\Big]\Big\}\\
&:= \max\{E,F\},
\end{aligned}$$

where

$$\begin{aligned}
E &:= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k\le L}\Big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
F &:= \sup_{L>0,L\in\mathbb{Z}}\{G+H\},\\
G &:= 2^{-L\lambda}\Big\|\Big(2^{k\alpha(0)}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k<0}\Big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
H &:= 2^{-L\lambda}\Big\|\Big(2^{k\alpha_\infty}\Big(\sum_{v=1}^{\infty}|T(f_1^v,f_2^v)|^r\Big)^{\frac1r}\chi_k\Big)_{k=0}^{L}\Big\|_{l^{q_\infty}(L^{p(\cdot)}(w))}.
\end{aligned}$$

Since to estimate G is essentially similar to estimate E, it is suffice to obtain that E and H are bounded in Herz-Morrey space with variable exponents. It is easy to see that

$$E \le C\sum_{i=i}^{9} E_i, \quad H \le C\sum_{i=i}^{9} H_i,$$

where

$$\begin{aligned}
E_1 &:= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}T(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac1r}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_2 &:= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}T(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac1r}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},\\
E_3 &:= \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty}T(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac1r}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},
\end{aligned}$$

$$E_4 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$E_5 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$E_6 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$E_7 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$E_8 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$E_9 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q(0)}\Big)^{\frac{1}{q(0)}},$$

$$H_1 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_2 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_3 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_4 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_5 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_6 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_7 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_8 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}},$$

$$H_9 := 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}.$$

We shall use the following estimates. If $l \le k-1$, then by Hölder's inequality, Lemma 2.3 and Definition 2.3, we have

$$\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}$$

$$\begin{aligned}
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\le C2^{-kn}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p_i'(\cdot)}(w_i^{-1})}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\le C2^{(l-k)n\delta_{2i}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}. \qquad (3.6)
\end{aligned}$$

If $l = k$, then

$$\begin{aligned}
&\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\le \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}. \qquad (3.7)
\end{aligned}$$

If $l \ge k+1$, then

$$\begin{aligned}
&\Big\|2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}^v|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}w_i\chi_l\Big\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}}\\
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|^{-1}_{L^{p_i(\cdot)}(w_i)}\\
&\quad\times\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&\le C2^{(l-k)n(1-\delta_{1i})}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}. \qquad (3.8)
\end{aligned}$$

By the interchange of f_1 and f_2, we see that the estimates of E_2, E_3 and E_6 are similar to those of E_4, E_7 and E_8, respectively. Thus we are only necessary to estimate E_1, E_2, E_3, E_5, E_6 and E_9.

To estimate E_1, since $l,\ j \le k-2$, we deduce that for $i = 1, 2$,

$$|x-y_i| \ge |x| - |y_i| > 2^{k-1} - 2^{\min\{l,j\}} \ge 2^{k-2}, \quad x \in D_k,\ y_1 \in D_l,\ y_2 \in D_j.$$

Therefore, for $x \in D_k$, we have

$$|K(x,y_1,y_2)| \le C(|x-y_1| + |x-y_2|)^{-2n} \le C2^{-2kn}.$$

Thus, for any $x \in D_k$ and $l,\ j \le k-2$, we have

$$|T(f_{1l}^v, f_{2j}^v)(x)| \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2$$

$$\lesssim 2^{-2kn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.$$

Therefore, by Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}T(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim&\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
\lesssim&\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}\\
\lesssim&\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\times\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}\tag{3.9}$$

Since $\frac{1}{q(0)}=\frac{1}{q_1(0)}+\frac{1}{q_2(0)}$ and $\lambda=\lambda_1+\lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}
E_1\lesssim&\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\\
&\times\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}\\
\lesssim&\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_1}\\
&\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\times\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_2}\\
&\times\Big(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
:=&E_{1,1}\times E_{1,2},
\end{aligned}$$

where

$$\begin{aligned}
E_{1,i}:=&\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_i}\\
&\times\Big\{\sum_{k=-\infty}^{L}2^{k\alpha_i(0)q_i(0)}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}(y_i)|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}}.
\end{aligned}$$

Since $n\delta_{i2}-\alpha_i(0)>0$, by (3.6) and Lemma 3.4 we obtain

$$\begin{aligned}
E_{1,i} &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i(0)} \\
&\quad\times\Big(\sum_{l=-\infty}^{k-2} 2^{(l-k)n\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big)^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}} \\
&= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i} \\
&\quad\times\Big\{\sum_{k=-\infty}^{L}\Big(\sum_{l=-\infty}^{k-2} 2^{l\alpha_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)(n\delta_{i2}-\alpha_i(0))}\Big)^{q_i(0)}\Big\}^{\frac{1}{q_i(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{L-2} 2^{l\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Big)^{\frac{1}{q_i(0)}} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)},
\end{aligned}$$

where we write $2^{-|k-l|(n\delta_{i2}-\alpha_i(0))}=2^{-|k-l|\varepsilon_i}$ for $\varepsilon_i=n\delta_{i2}-\alpha_i(0)>0$. Thus, we obtain

$$E_1\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_2, since $l\le k-2$ and $k-1\le j\le k+1$, then we have

$$|x-y_2|\ge|x-y_1|\ge|x|-|y_1|\ge 2^{k-2},\quad x\in D_k,\ y_1\in D_l,\ y_2\in D_j.$$

Therefore, for $x\in D_k$, we have

$$|K(x,y_1,y_2)|\le C(|x-y_1|+|x-y_2|)^{-2n}\le C2^{-2kn}.$$

Thus, for any $x\in D_k$, $l\le k-2$, $k-1\le j\le k+1$, we have

$$\begin{aligned}
|T(f_{1l}^v,f_{2j}^v)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim 2^{-2kn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}$$

Therefore, by Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1}T(f_{1l}^v,f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\
&\quad\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}
\end{aligned}$$

$$\lesssim \Big\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{1l}^v(y_1)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_1 \chi_k \Big\|_{L^{p_1(\cdot)}(w_1)}$$
$$\times \Big\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{2j}^v(y_2)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_2 \chi_k \Big\|_{L^{p_2(\cdot)}(w_2)}. \tag{3.10}$$

Since $\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)}$ and $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned} E_2 \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \Big\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{1l}^v(y_1)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_1 \chi_k \Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ & \times \Big\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{2j}^v(y_2)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_2 \chi_k \Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \Big)^{\frac{1}{q(0)}} \\ \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ & \times \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \Big\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{1l}^v(y_1)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_1 \chi_k \Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \Big)^{\frac{1}{q_1(0)}} \\ & \times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\ & \times \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \Big\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \Big(\sum_{v=1}^{\infty} |f_{2j}^v(y_2)|^r \Big)^{\frac{1}{r}} \mathrm{d}y_2 \chi_k \Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \Big)^{\frac{1}{q_2(0)}} \\ := & E_{2,1} \times E_{2,2}. \end{aligned}$$

It is obvious that

$$E_{2,1} = E_{1,1} \lesssim \Big\| \Big(\sum_{v=1}^{\infty} |f_1^v|^{r_1} \Big)^{\frac{1}{r_1}} \Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Now we estimate $E_{2,2}$. Taking (3.6)–(3.8) together, we have

$$\begin{aligned} E_{2,2} \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \Big\| \sum_{j=k-1}^{k+1} 2^{(j-k)n} \Big(\sum_{v=1}^{\infty} |f_2^v|^{r_2} \Big)^{\frac{1}{r_2}} \chi_j \Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \Big)^{\frac{1}{q_2(0)}} \\ \lesssim & \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \Big(\sum_{k=-\infty}^{L+1} 2^{k\alpha_2(0)q_2(0)} \Big\| \Big(\sum_{v=1}^{\infty} |f_2^v|^{r_2} \Big)^{\frac{1}{r_2}} \chi_k \Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \Big)^{\frac{1}{q_2(0)}} \\ \lesssim & \Big\| \Big(\sum_{v=1}^{\infty} |f_2^v|^{r_2} \Big)^{\frac{1}{r_2}} \Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}, \end{aligned}$$

where we use $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n} < 2^{2n}$, $j \in \{k-1, k, k+1\}$ for (3.6) and (3.8) respectively. Thus, we obtain

$$E_2 \lesssim \Big\| \Big(\sum_{v=1}^{\infty} |f_1^v|^{r_1} \Big)^{\frac{1}{r_1}} \Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \Big\| \Big(\sum_{v=1}^{\infty} |f_2^v|^{r_2} \Big)^{\frac{1}{r_2}} \Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To estimate E_3, since $l \le k-2$ and $j \ge k+2$, we have

$$|x - y_1| \ge |x| - |y_1| \ge 2^{k-2}, \quad |x - y_2| \ge |y_2| - |x| > 2^{j-2}, \quad x \in D_k,\ y_1 \in D_l,\ y_2 \in D_j.$$

Therefore, fir any $x \in D_k$, $l \le k-2$, $j \ge k+2$, we get

$$|T(f_{1l}, f_{2j})(x)| \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} \mathrm{d}y_1\mathrm{d}y_2$$
$$\lesssim 2^{-kn}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.$$

Thus, by Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}\\
&\lesssim \Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned}\tag{3.11}$$

Since $\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)}$ and $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}
E_3 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\\
&\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\\
&\quad\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}}\\
&\quad\times\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\\
&\quad\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&:= E_{3,1}\times E_{3,2}.
\end{aligned}$$

It is obvious that

$$E_{3,1} = E_{1,1} \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Since $n\delta_{21}+\alpha_2(0)>0$, by (3.8), we obtain

$$\begin{aligned}
E_{3,2} &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \\
&\quad\times\Big(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{21}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times\Big(\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L} 2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\quad+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times\Big(\sum_{k=-\infty}^{L}\Big(2^{k\alpha_2(0)}\sum_{j=L+1}^{0}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)n\delta_{21}}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\quad+\sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times\Big(\sum_{k=-\infty}^{L}\Big(2^{k\alpha_2(0)}\sum_{j=1}^{\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)n\delta_{21}}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&:= I_1+I_2+I_3.
\end{aligned}$$

First, we consider I_1. By Lemma 3.4, we have

$$\begin{aligned}
I_1 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times\Big(\sum_{k=-\infty}^{L}\Big(\sum_{j=k+2}^{L} 2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{(k-j)(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{j=-\infty}^{L+2} 2^{j\alpha_2(0)q_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},
\end{aligned}$$

where we write $2^{-|k-j|(n\delta_{21}+\alpha_2(0))}=2^{-|k-j|\eta_2}$ for $\eta_2=n\delta_{21}+\alpha_2(0)>0$.

Next, we consider I_2. Since $n\delta_{21}+\alpha_2(0)-\lambda_2>0$, we obtain

$$\begin{aligned}
I_2 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}} 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0} 2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} \\
&\quad\times 2^{-j(n\delta_{21}+\alpha_2(0))}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}}\sup_{j\le 0} 2^{-j\lambda_2}2^{j\alpha_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} \\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=L+1}^{0} 2^{-j(n\delta_{21}+\alpha_2(0)-\lambda_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}
\end{aligned}$$

$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}\sup_{L\le 0,L\in\mathbb{Z}}2^{L(-n\delta_{21}-\alpha_2(0))}\Big(\sum_{k=-\infty}^{L}2^{k(n\delta_{21}+\alpha_2(0))q_2(0)}\Big)^{\frac{1}{q_2(0)}}$$

$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

Then, we consider I_3. Since $n\delta_{21}+\alpha_2(0)-\lambda_2>0$, we obtain

$$\begin{aligned}
I_3 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\\
&\quad\times\sum_{j=1}^{\infty}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}2^{-j(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}}\sup_{j\ge 1}2^{-j\lambda_2}2^{j\alpha_\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}\Big(2^{k(n\delta_{21}+\alpha_2(0))}\sum_{j=1}^{\infty}2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}\Big)^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}\sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}2^{k(n\delta_{21}+\alpha_2(0))q_2(0)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}\sup_{L\le 0,L\in\mathbb{Z}}2^{L(-\lambda_2+n\delta_{21}+\alpha_2(0))}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}$$

Thus, we have

$$E_3\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_5, using Hölder's inequality and Lemma 3.10, we have

$$\begin{aligned}
E_5 &\lesssim \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\Big\|\Big(\sum_{v=1}^{\infty}|T(f_{1l},f_{2j})|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|^{q(0)}_{L^{p(\cdot)}(w)}\Big)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda}\Big(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\Big(\Big\|\Big(\sum_{v=1}^{\infty}|f_{1l}^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\Big(\sum_{v=1}^{\infty}|f_{2j}^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\le 0,L\in\mathbb{Z}}2^{-L\lambda_1}\Big(\sum_{k=-\infty}^{L}2^{k\alpha_1(0)q_1(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|^{q_1(0)}_{L^{p_1(\cdot)}(w_1)}\Big)^{\frac{1}{q_1(0)}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=-\infty}^{L}2^{k\alpha_2(0)q_2(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|^{q_2(0)}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_2(0)}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}$$

To estimate E_6, since $k-1 \le l \le k+1$ and $j \ge k+2$, we obtain

$$|x-y_1| > 2^{k-2}, \quad |x-y_2| > 2^{j-2}, \quad x \in D_k,\ y_1 \in D_l,\ y_2 \in D_j.$$

Thus, for any $x \in D_k$, $k-1 \le l \le k+1$ and $j \ge k+2$, we obtain

$$\begin{aligned}|T(f_{1l}^v, f_{2j}^v)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim 2^{-kn}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

Therefore, by Hölder's inequality and Minkowski's inequality, we obtain

$$\begin{aligned}&\Big\|\Big(\sum_{v=1}^{\infty}\Big|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1 \sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ &\lesssim \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad\times \Big\|\Big(\sum_{v=1}^{\infty}\Big(\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)} \\ &\lesssim \Big\|\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad\times \Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{3.12}$$

Since $\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)}$ and $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}E_6 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad\times\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ &\quad\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}} \\ &\quad\times\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\ &\quad\times\Big(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^{\infty}2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}} \\ &:= E_{6,1}\times E_{6,2}.\end{aligned}$$

By the interchange of f_1 and f_2, we see that the estimate of $E_{6,1}$ is similar to that of $E_{2,2}$ and $E_{6,2} = E_{3,2}$.

To estimate E_9, since $l,\ j \geq k+2$, we get

$$|x - y_i| > 2^{k-2}, \quad x \in D_k,\ y_1 \in D_l,\ y_2 \in D_j.$$

Therefore, for any $x \in D_k$, $l,\ j \geq k+2$, we have

$$\begin{aligned}|T(f_{1l}^v, f_{2j}^v)(x)| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}^v(y_1)||f_{2j}^v(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim 2^{-ln}2^{-jn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)||f_{2j}^v(y_2)|\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

Thus, by Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned}&\Big\|\Big(\sum_{v=1}^\infty\Big|\sum_{l=k+2}^\infty\sum_{j=k+2}^\infty T(f_{1l}^v, f_{2j}^v)\Big|^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ &\lesssim \Big\|\Big(\sum_{v=1}^\infty\Big(\sum_{l=k+2}^\infty 2^{-ln}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\sum_{j=k+2}^\infty 2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^r\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)} \\ &\lesssim \Big\|\Big(\sum_{v=1}^\infty\Big(\sum_{l=k+2}^\infty 2^{-ln}\int_{\mathbb{R}^n}|f_{1l}^v(y_1)|\mathrm{d}y_1\Big)^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad\times\Big\|\Big(\sum_{v=1}^\infty\Big(\sum_{j=k+2}^\infty 2^{-jn}\int_{\mathbb{R}^n}|f_{2j}^v(y_2)|\mathrm{d}y_2\Big)^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)} \\ &\lesssim \Big\|\sum_{l=k+2}^\infty 2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad\times\Big\|\sum_{j=k+2}^\infty 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}.\end{aligned}\tag{3.13}$$

Since $\frac{1}{q(0)} = \frac{1}{q_1(0)} + \frac{1}{q_2(0)}$ and $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}E_9 &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^L 2^{k\alpha(0)q(0)}\Big\|\sum_{l=k+2}^\infty 2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad\times\Big\|\sum_{j=k+2}^\infty 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Big)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ &\quad\times\Big(\sum_{k=-\infty}^L 2^{k\alpha_1(0)q_1(0)}\Big\|\sum_{l=k+2}^\infty 2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Big)^{\frac{1}{q_1(0)}} \\ &\quad\times\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\ &\quad\times\Big(\sum_{k=-\infty}^L 2^{k\alpha_2(0)q_2(0)}\Big\|\sum_{j=k+2}^\infty 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^\infty|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Big)^{\frac{1}{q_2(0)}}\end{aligned}$$

$$:= E_{9,1} \times E_{9,2}.$$

Obviously, the estimates of $E_{9,i}$ are similar to those of $E_{3,2}$ for $i = 1, 2$, respectively.

Taking all estimates for E_i together, $i = 1, 2, \cdots, 9$, we obtain

$$E \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

Finally, we estimate H. By the interchange of f_1 and f_2, we see that the estimates of H_2, H_3 and H_6 are similar to those of H_4, H_7 and H_8, respectively. Thus we are only necessary to estimate H_1, H_2, H_3, H_5, H_6 and H_9.

To go on, we need further preparation.

If $l < 0$, by Proposition 2.1, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\\
&= 2^{-l\alpha_i(0)}\Big(2^{l\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_i(0)}_{L^{p_i(\cdot)}(w_i)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{-l\alpha_i(0)}\Big(\sum_{t=-\infty}^{l}2^{t\alpha_i(0)q_i(0)}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_t\Big\|^{q_i(0)}_{L^{p_i(\cdot)}(w_i)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda-\alpha_i(0))}2^{-l\lambda}\Big(\sum_{t=-\infty}^{l}\Big\|2^{t\alpha_i(0)}\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_t\Big\|^{q_i(0)}_{L^{p_i(\cdot)}(w_i)}\Big)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda-\alpha_i(0))}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.
\end{aligned}\tag{3.14}$$

To estimate H_1, since l, $j \le k-2$, $\frac{1}{q_\infty} = \frac{1}{q_{1\infty}} + \frac{1}{q_{2\infty}}$ and $\lambda = \lambda_1 + \lambda_2$, by (3.9) and Hölder's inequality, we have

$$\begin{aligned}
H_1 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|^{q_\infty}_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|^{q_\infty}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)}\Big)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_{2\infty}}}\\
&:= H_{1,1}\times H_{1,2},
\end{aligned}$$

where

$$H_{1,i} := 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{il}(y_i)|^{r_i}\Big)^{\frac{1}{r_i}}\mathrm{d}y_i\chi_k\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}\Big\}^{\frac{1}{q_{i\infty}}}.$$

By (3.6), we obtain

$$\begin{aligned}
H_{1,i} &\lesssim 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{k-2}2^{(l-k)n\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\\
&\quad+\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\quad+2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&=: I_4+I_5.
\end{aligned}$$

If $q_{i\infty}\ge 1$, since $n\delta_{i2}-\alpha_{i\infty}>0$ and $n\delta_{i2}-\alpha_i(0)>0$, by the Minkowski's inequality and (3.14), we obtain

$$\begin{aligned}
I_4 &= 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big\{\sum_{k=0}^{L}\Big(2^{k\alpha_{i\infty}}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\sum_{l=-\infty}^{-1}2^{ln\delta_{i2}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}\Big\{\sum_{k=0}^{L}2^{-k(n\delta_{i2}-\alpha_{i\infty})q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}2^{-L\lambda_i}\sum_{l=-\infty}^{-1}2^{l(n\delta_{i2}+\lambda_i-\alpha_i(0))}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.
\end{aligned}$$

If $q_{i\infty}<1$, since $n\delta_{i2}-\alpha_{i\infty}>0$ and $n\delta_{i2}-\alpha_i(0)>0$, by (3.14), we have

$$\begin{aligned}
I_4 &\lesssim 2^{-L\lambda_i}\Big(\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&= 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}2^{ln\delta_{i2}q_{i\infty}}\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}2^{-kn\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&= 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}2^{ln\delta_{i2}q_{i\infty}}\sum_{k=0}^{L}2^{-k(n\delta_{i2}-\alpha_{i\infty})q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}2^{ln\delta_{i2}q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}
\end{aligned}$$

$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)} 2^{-L\lambda_i}\Big(\sum_{l=-\infty}^{-1}2^{l(n\delta_{i2}+\lambda_i-\alpha_i(0))q_{i\infty}}\Big)^{\frac{1}{q_{i\infty}}}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.$$

We consider I_5. Since $n\delta_{i2}-\alpha_{i\infty}>0$, by Lemma 3.4, we have

$$\begin{aligned}
I_5 &= 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}2^{k\alpha_{i\infty}q_{i\infty}}\Big(\sum_{l=0}^{k}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)n\delta_{i2}}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&= 2^{-L\lambda_i}\Big\{\sum_{k=0}^{L}\Big(\sum_{l=0}^{k}2^{l\alpha_{i\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|_{L^{p_i(\cdot)}(w_i)}2^{(l-k)(n\delta_{i2}-\alpha_{i\infty})}\Big)^{q_{i\infty}}\Big\}^{\frac{1}{q_{i\infty}}}\\
&\lesssim 2^{-L\lambda_i}\Big(\sum_{l=0}^{k}2^{l\alpha_{i\infty}q_{i\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\chi_l\Big\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}\Big)^{\frac{1}{q_{i\infty}}}\\
&\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_i^v|^{r_i}\Big)^{\frac{1}{r_i}}\Big\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)},
\end{aligned}$$

where we write $2^{-|k-l|(n\delta_{i2}-\alpha_{i\infty})}\lesssim 2^{-|k-l|\eta_i}$ for $\eta_i=n\delta_{i2}-\alpha_{i\infty}$.

Thus, we get

$$H_1\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate H_2, since $l\le k-2$, $k-1\le j\le k+1$, $\frac{1}{q_\infty}=\frac{1}{q_{1\infty}}+\frac{1}{q_{2\infty}}$ and $\lambda=\lambda_1+\lambda_2$, by (3.10) and Hölder's inequality, we have

$$\begin{aligned}
H_2 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_1\chi_k\Big\|^{q_\infty}_{L^{p_1(\cdot)}(w_1)}\\
&\quad\times\Big\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_2\chi_k\Big\|^{q_\infty}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_\infty}}\\
&\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_1\chi_k\Big\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)}\Big)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^r\Big)^{\frac{1}{r}}\mathrm{d}y_2\chi_k\Big\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_{2\infty}}}\\
&:= H_{2,1}\times H_{2,2}.
\end{aligned}$$

It is obvious that

$$H_{2,1}=H_{1,1}\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.$$

Now we estimate $H_{2,2}$. Combining (3.6)–(3.8), we have

$$H_{2,2}\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\sum_{j=k-1}^{k+1}2^{(j-k)n}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\Big)^{\frac{1}{q_{2\infty}}}$$

$$\lesssim 2^{-L\lambda_2}\Big(\sum_{k=-1}^{L+1} 2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)},$$

where we use $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{21})} < 2^{(j-k)n}$ for (3.6) and (3.8), respectively. Thus, we obtain

$$H_2 \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To estimate H_3, since $l \le k-2$, $j \ge k+2$, $\frac{1}{q_\infty} = \frac{1}{q_{1\infty}} + \frac{1}{q_{2\infty}}$ and $\lambda = \lambda_1 + \lambda_2$, by (3.11) and Hölder's inequality, we have

$$H_3 \lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}$$
$$\times\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}$$
$$\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}$$
$$\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$:= H_{3,1}\times H_{3,2}.$$

It is obvious that

$$H_{3,1} = H_{1,1} \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}.$$

Since $n\delta_{21} + \alpha_{2\infty} > 0$, by (3.8), we obtain

$$H_{3,2} \lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}}\Big(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{21}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(\sum_{j=k+2}^{L+2} 2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$+ 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k\alpha_{2\infty}}\sum_{j=L+3}^{\infty}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)n\delta_{21}}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$:= I_6 + I_7.$$

For I_6, by Lemma 3.4, we obtain

$$I_6 \lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(\sum_{j=k+2}^{L+2} 2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$

$$\lesssim 2^{-L\lambda_2}\Big(\sum_{j=0}^{L+2} 2^{j\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},$$

where we write $2^{-|k-j|(n\delta_{21}+\alpha_{2\infty})} = 2^{-|k-j|\vartheta_2}$ for $\vartheta_2 = n\delta_{21}+\alpha_{2\infty} > 0$.

For I_7, since $n\delta_{21}+\alpha_{2\infty}-\lambda_2 > 0$, we have

$$I_7 \lesssim 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k(n\delta_{21}+\alpha_{2\infty})}\sum_{j=L+3}^{\infty}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}$$
$$\times 2^{-j(n\delta_{21}+\alpha_{2\infty})}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim \sup_{j\ge 1} 2^{-j\lambda_2}2^{j\alpha_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_j\Big\|_{L^{p_2(\cdot)}(w_2)}$$
$$\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}\Big(2^{k(n\delta_{21}+\alpha_{2\infty})}\sum_{j=L+3}^{\infty}2^{-j(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}\Big)^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)} 2^{-L\lambda_2+(n\delta_{21}+\alpha_{2\infty})L-L(n\delta_{21}+\alpha_{2\infty}-\lambda_2)}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

Thus, we get

$$H_3 \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate H_5, using Hölder's inequality and Lemma 3.10, we have

$$H_5 \lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1}\Big\|\Big(\sum_{v=1}^{\infty}|T(f_{1l},f_{2j})|^{r}\Big)^{\frac{1}{r}}\chi_k\Big\|_{L^{p(\cdot)}(w)}^{q_\infty}\Big)^{\frac{1}{q_\infty}}$$
$$\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\Big(\Big\|\Big(\sum_{v=1}^{\infty}|f_{1l}^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{L^{p_1(\cdot)}(w_1)}$$
$$\times\Big\|\Big(\sum_{v=1}^{\infty}|f_{2j}^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{L^{p_2(\cdot)}(w_2)}\Big)^{q_\infty}\Big)^{\frac{1}{q_\infty}}$$
$$\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}}$$
$$\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}}$$
$$\lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate H_6, since $k-1 \le l \le k+1,\ j \ge k+2,\ \frac{1}{q_\infty} = \frac{1}{q_{1\infty}} + \frac{1}{q_{2\infty}}$ and $\lambda = \lambda_1 + \lambda_2$, by (3.12) and Hölder's inequality, we have

$$\begin{aligned} H_6 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\| \sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \\ &\quad\times \Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}} \\ &\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &:= H_{6,1}\times H_{6,2}. \end{aligned}$$

By the interchange of f_1 and f_2, we see that that of $H_{6,1}$ is similar to the estimate of $H_{2,2}$ and $H_{6,2} = H_{3,2}$.

To estimate H_9, since $l,\ j \ge k+2,\ \frac{1}{q_\infty} = \frac{1}{q_{1\infty}} + \frac{1}{q_{2\infty}}$ and $\lambda = \lambda_1 + \lambda_2$, by (3.13) and Hölder's inequality, we have

$$\begin{aligned} H_9 &\lesssim 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\Big\| \sum_{l=k+2}^{\infty} 2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \\ &\quad\times \Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\Big)^{\frac{1}{q_\infty}} \\ &\lesssim 2^{-L\lambda_1}\Big(\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}}\Big\|\sum_{l=k+2}^{\infty} 2^{-ln}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{1l}^v(y_1)|^{r_1}\Big)^{\frac{1}{r_1}}\mathrm{d}y_1\chi_k\Big\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\Big)^{\frac{1}{q_{1\infty}}} \\ &\quad\times 2^{-L\lambda_2}\Big(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}}\Big\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\Big(\sum_{v=1}^{\infty}|f_{2j}^v(y_2)|^{r_2}\Big)^{\frac{1}{r_2}}\mathrm{d}y_2\chi_k\Big\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\Big)^{\frac{1}{q_{2\infty}}} \\ &:= H_{9,1}\times H_{9,2}. \end{aligned}$$

Obviously, the estimates of $H_{9,i}$ are similar to those of $H_{3,2}$ for $i = 1, 2$, respectively.

Taking all estimates for H_i together, $i = 1, 2, \cdots, 9$, we obtain

$$H \lesssim \Big\|\Big(\sum_{v=1}^{\infty}|f_1^v|^{r_1}\Big)^{\frac{1}{r_1}}\Big\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\Big\|\Big(\sum_{v=1}^{\infty}|f_2^v|^{r_2}\Big)^{\frac{1}{r_2}}\Big\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

This completes the proof.

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