

# Boundedness of Iterated Spherical Average on $\alpha$ -Modulation Spaces\*

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**Abstract** For the iteration of spherical average  $(A_1)^N$  and the Laplace operator  $\Delta$ , we consider the boundedness of the operator  $\Delta(A_1)^N$  on the  $\alpha$ -modulation spaces  $M_{p,q}^{s,\alpha}$ . The authors obtain some sufficient and necessary conditions to ensure the boundedness on the  $\alpha$ -modulation spaces. The main theorems significantly improve some known results.

**Keywords** Spherical average,  $\alpha$ -Modulation spaces, Bessel functions

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## 1 Introduction

Let  $\mathbb{S}^{n-1}$  be the unit sphere in the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ . We define the average operator of functions  $f$  on the unit sphere as

$$A_1(f)(x) = \int_{\mathbb{S}^{n-1}} f(x - y') d\sigma(y'),$$

where  $d\sigma(y')$  is the normalized surface Lebesgue measure.

This operator has a profound background in harmonic analysis, dating back to early 1970's (see [21–22]). About 120 years ago, Pearson (see [18]) first used it to study random walks in high dimensional spaces. An  $N$ -steps uniform walk in  $\mathbb{R}^n$  starts at the origin and consists of  $N$  independent steps of length 1, each of which is taken into a uniformly random direction. The probability density function  $p_N(\frac{n-2}{2}, x)$  of such a random walk is the Fourier inverse of  $(A_1)^N$  (see [4]), where  $(A_1)^N$  denotes the  $N$  iteration of  $A_1$ .

The operator  $A_1$  also plays a significant role in the approximation theory (see [1]). In order to obtain some equivalent forms of the K-functional in  $L^p(\mathbb{R}^n)$  spaces, Belinsky, Dai and Ditzian [1] studied the iterates  $\Delta(A_1)^N$  for positive integers  $N$ , where  $\Delta$  is the Laplacian. They obtained the following result.

**Theorem A** (see [1]) *Let  $1 \leq p \leq \infty$ ,  $n \geq 2$  and  $N > \frac{2(n+2)}{n-1}$ . The inequality*

$$\|\Delta(A_1)^N(f)\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$$

*holds for all  $f \in L^p(\mathbb{R}^n)$ .*

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Naturally, we may ask that what is the smallest positive integer  $N$  to guarantee the inequality

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}.$$

Fan and Zhao in [7] answered this question by using the well known estimates of wave operators (see [16, 19]). Recently, Fan, Lou and Wang [6] obtained the sufficient and necessary conditions for the smallest positive integer  $N$ , which completely solved the above question. We state their theorem as follows.

**Theorem B** (see [6]) *Let  $n \neq 3, 5$ , and  $N$  be positive integers. The inequality*

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$$

*holds if and only if  $N > \frac{n+3}{n-1}$ .*

*Let  $n = 3, 5$ , and  $N$  be positive integers. The inequality*

$$\|\Delta(A_1)^N(f)\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$$

*holds if and only if  $N \geq \frac{n+3}{n-1}$ .*

Later, in [13], Huang extended the boundedness of  $\Delta(A_1)^N$  on the modulation spaces  $M_{p,q}^s$  on full ranges of  $1 \leq p_1, p_2, q_1, q_2 \leq \infty$  and  $s_1, s_2 \in \mathbb{R}$ . They obtained the sufficiency and necessity for the boundedness of  $\Delta(A_1)^N$  from  $M_{p_1,q_1}^{s_1}$  to  $M_{p_2,q_2}^{s_2}$ . Moreover, he found the smallest iterate step  $N$  which ensures that  $\Delta(A_1)^N$  is bounded on modulation spaces  $M_{p,q}^s(\mathbb{R}^n)$  for all  $(p, q, s) \in [1, +\infty) \times [1, +\infty) \times \mathbb{R}$  is  $\frac{4}{n-1}$ , which is smaller than that in  $L^1(\mathbb{R}^n)$  spaces (see Theorem B). Precisely, the related result is stated as follows.

**Theorem C** (see [13]) *Let  $\sigma = 2 - \frac{n-1}{2}N$  and  $1 \leq p_i, q_i \leq \infty$ ,  $s_i \in \mathbb{R}$  for  $i = 1, 2$ . When  $q_1 \leq q_2$ , the iterated spherical average  $\Delta(A_1)^N$  is bounded from  $M_{p_1,q_1}^{s_1}(\mathbb{R}^n)$  to  $M_{p_2,q_2}^{s_2}(\mathbb{R}^n)$  if and only if*

$$p_1 \leq p_2, \quad s_1 \geq s_2 + \sigma.$$

*When  $q_1 > q_2$ , the iterated spherical average  $\Delta(A_1)^N$  is bounded from  $M_{p_1,q_1}^{s_1}(\mathbb{R}^n)$  to  $M_{p_2,q_2}^{s_2}(\mathbb{R}^n)$  if and only if*

$$p_1 \leq p_2, \quad s_1 + \frac{n}{q_1} > s_2 + \frac{n}{q_2} + \sigma.$$

The modulation space  $M_{p,q}^s$  was introduced by Feichtinger [8] in order to measure smoothness of a function or distribution in a way different from  $L^p$  spaces. Nowadays, spaces  $M_{p,q}^s$  are recognized as a useful tool for studying functional analysis, pseudo-differential operators and certain Cauchy problems of nonlinear partial differential equations (see [2, 5, 11, 15, 17, 20, 23, 24]). In addition, Gröbner in his unpublished thesis (see [9]) extended modulation space to  $\alpha$ -modulation space  $M_{p,q}^{s,\alpha}$  by using the  $\alpha$ -decomposition on the frequency space. Their definitions will be represented in Section 2. Here we first point out one of their significant properties: The parameter  $\alpha \in [0, 1)$  determines a segmentation of the frequency spaces. When  $\alpha = 0$ ,  $M_{p,q}^{s,0}$  is equivalent to the classical modulation space  $M_{p,q}^s$ . When  $\alpha \rightarrow 1$ ,  $M_{p,q}^{s,1}$  is considered equivalent to the classical Besov space. Obviously, it is proposed to be an intermediate function space between Besov space and modulation space. Hence, it is very important to study some analysis

and PDE's problems in  $\alpha$ -modulation space. Among numerous research papers, the reader may refer to [3, 10, 12, 14, 26] and the references therein.

Motivated by the above works, in this paper, we consider the boundedness of  $\Delta(A_1)^N$  on  $\alpha$ -modulation spaces and give some sufficient and necessary conditions on the boundedness of  $\Delta(A_1)^N$  from  $M_{p_1, q_1}^{s_1, \alpha}$  to  $M_{p_2, q_2}^{s_2, \alpha}$ . We state our main results as follows.

**Theorem 1.1** *Let  $\sigma = 2 - \frac{n-1}{2}N + \frac{n\alpha}{2}$  and  $1 \leq p_i, q_i \leq \infty$ ,  $s_i \in \mathbb{R}$  for  $i = 1, 2$ ,  $\alpha \in [0, 1)$ . When*

$$q_1 \leq q_2, \quad p_1 \leq p_2, \quad s_1 - \frac{n\alpha}{p_1} \geq s_2 - \frac{n\alpha}{p_2} + \sigma$$

or

$$q_1 > q_2, \quad p_1 \leq p_2, \quad s_1 - \frac{n\alpha}{p_1} + \frac{n(1-\alpha)}{q_1} > s_2 - \frac{n\alpha}{p_2} + \frac{n(1-\alpha)}{q_2} + \sigma,$$

the iterated spherical average  $\Delta(A_1)^N$  is bounded from  $M_{p_1, q_1}^{s_1, \alpha}(\mathbb{R}^n)$  to  $M_{p_2, q_2}^{s_2, \alpha}(\mathbb{R}^n)$ .

**Theorem 1.2** *Let  $1 \leq p_i, q_i \leq \infty$ ,  $s_i \in \mathbb{R}$  for  $i = 1, 2$ ,  $\alpha \in [0, 1)$ . If the iterated spherical average  $\Delta(A_1)^N$  is bounded from  $M_{p_1, q_1}^{s_1, \alpha}(\mathbb{R}^n)$  to  $M_{p_2, q_2}^{s_2, \alpha}(\mathbb{R}^n)$ , then the following conditions must hold*

$$p_1 \leq p_2, \quad s_1 \geq s_2 + 2 - \frac{n-1}{2}N, \quad \text{when } q_1 \leq q_2$$

or

$$p_1 \leq p_2, \quad s_1 + \frac{n(1-\alpha)}{q_1} > s_2 + \frac{n(1-\alpha)}{q_2} + 2 - \frac{n-1}{2}N, \quad \text{when } q_1 > q_2.$$

**Remark 1.1** When  $\alpha = 0$ , we can see that the above results are sharp and coincide with that in modulation spaces which was obtained in [13]. But for the case  $\alpha \in (0, 1)$ , our results are not sharp for some technical problems. Essentially, these difficulties are due to two properties of Bessel function. The first one is that  $V_\delta(r)$  and  $\frac{d}{dr}V_\delta(r)$  share the same upper bound as  $r \rightarrow \infty$ . The other fact is that the first term of asymptotic expansion of the Bessel function is  $\sqrt{\frac{2}{\pi r}}\cos(r - \frac{\delta\pi}{2})$ , which exists root in every interval  $[2k\pi, 2(k+1)\pi]$  for  $k \in \mathbb{Z}$ .

The proof of Theorem 1.1 is somewhat routine with the help of Bernstein's multiplier theorem. However the proof of Theorem 1.2 is quite involved. Based on the structure of  $M_{p, q}^{s, \alpha}$  and the asymptotic form of the Fourier transform of  $\Delta(A_1)^N$ , we construct a sequence of functions  $\{f_{k_j, \lambda}\}$  to achieve the necessary conditions.

This paper is organized as follows. In Section 2, we will introduce some preliminary knowledge which includes some properties of  $\alpha$ -modulation spaces and some useful lemmas. The proofs of main results will be presented in Section 3.

Throughout this paper, we use the inequality  $A \preceq B$  to mean that there is a positive number  $C$  independent of all main variables such that  $A \leq CB$ , and use the notation  $A \simeq B$  to mean  $A \preceq B$  and  $B \preceq A$ .

## 2 Preliminaries and Lemmas

In this section, we give the definition and discuss some basic properties of  $\alpha$ -modulation spaces. Also, we will state some estimates and lemmas which will be used in our proofs.

**Definition 2.1** ( $\alpha$ -Modulation Space) *Let  $\rho$  be a nonnegative smooth radial bump function supported in  $B(0, 2)$ , satisfying  $\rho(\xi) = 1$  for  $|\xi| < 1$  and  $\rho(\xi) = 0$  for  $|\xi| \geq 2$ . For any  $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ , we set*

$$\rho_k^\alpha(\xi) = \rho\left(\frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k \rangle^{\frac{\alpha}{1-\alpha}}}\right)$$

and

$$\eta_k^\alpha = \frac{\rho_k^\alpha(\xi)}{\sum_{l \in \mathbb{Z}^n} \rho_l^\alpha(\xi)}.$$

We define the ball

$$B_k^r := \{\xi \in \mathbb{R}^n : |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| < r \langle k \rangle^{\frac{\alpha}{1-\alpha}}\}.$$

It is easy to check that  $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$  satisfy

$$\begin{aligned} \text{supp } \eta_k^\alpha &\subset B_k^2, \\ \eta_k^\alpha(\xi) &= 1, \quad \forall \xi \in B_k^1, \\ \sum_{k \in \mathbb{Z}^n} \eta_k^\alpha(\xi) &\equiv 1, \quad \xi \in \mathbb{R}^n \end{aligned}$$

and

$$|\partial^\gamma \eta_k^\alpha(\xi)| \leq C_{|\gamma|} \langle k \rangle^{-\frac{\alpha|\gamma|}{1-\alpha}}, \quad \forall \xi \in \mathbb{R}^n, \gamma \in \mathbb{N}^n. \tag{2.1}$$

This type of decomposition on frequency space is called  $\alpha$ -decomposition which is a generalization of the uniform decomposition and the dyadic decomposition. Corresponding to the above sequence  $\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}$ , we can construct an operator sequence  $\{\square_k^\alpha\}_{k \in \mathbb{Z}^n}$  by

$$\square_k^\alpha = \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F},$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the standard Fourier transform, and inverse Fourier transform respectively. For  $\alpha \in [0, 1), 0 < p, q \leq \infty, s \in \mathbb{R}$ , using this  $\alpha$ -decomposition, we define  $\alpha$ -modulation space as

$$M_{p,q}^{s,\alpha} = \{f \in \mathcal{S}' : \|f\|_{M_{p,q}^{s,\alpha}} < \infty\},$$

where

$$\|f\|_{M_{p,q}^{s,\alpha}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{sq}{1-\alpha}} \|\square_k^\alpha f\|_{L^p}^q \right)^{\frac{1}{q}}$$

and  $\langle k \rangle = \sqrt{1 + |k|^2}$ . See [12] for details.

We now list some basic properties about  $\alpha$ -modulation spaces.

**Proposition 2.1** (Almost Orthogonality) (see [12]) *For any  $k \in \mathbb{Z}^n$ , we define*

$$\Lambda_k^\alpha = \{l \in \mathbb{Z}^n : \text{supp } \eta_l^\alpha \cap \text{supp } \eta_k^\alpha \neq \emptyset\}.$$

Then the cardinality of  $\Lambda_k^\alpha$  is uniformly finite for all  $k \in \mathbb{Z}^n$ .

**Proposition 2.2** (Isomorphism) (see [12]) *Let  $0 < p, q \leq \infty, s, \tau \in \mathbb{R}$ .*

$$J_\sigma = (I - \Delta)^{\frac{\sigma}{2}} : M_{p,q}^{s,\alpha} \rightarrow M_{p,q}^{s-\tau,\alpha}$$

is an isomorphic mapping, where  $I$  is the identity mapping and  $\Delta$  is the Laplacian.

**Proposition 2.3** (Embedding) (see [12]) *Suppose that  $0 < p_1 \leq p_2 \leq \infty$ ,  $0 < q_1, q_2 \leq \infty$  and if*

$$q_1 \leq q_2, \quad s_1 - \frac{n\alpha}{p} \geq s_2 - \frac{n\alpha}{p_2}$$

or

$$q_1 > q_2, \quad s_1 - \frac{\alpha n}{p_1} + \frac{(1-\alpha)n}{q_1} > s_2 - \frac{\alpha n}{p_2} + \frac{(1-\alpha)n}{q_2},$$

we have

$$M_{p_1, q_1}^{s_1, \alpha} \subset M_{p_2, q_2}^{s_2, \alpha}.$$

The Fourier multiplier  $m(D)$  is a linear operator whose action on a test function  $f$  is formally defined by

$$\widehat{m(D)f}(\xi) = m(\xi)\widehat{f}(\xi).$$

The function  $m(\xi)$  is called the symbol or multiplier of  $m(D)$ . Up to a constant multiple,  $m(D)$  is a convolution operator with the kernel. In the sense of distribution, it is defined as

$$K(x) = (m(\cdot))^\vee(x) = \int_{\mathbb{R}^n} m(\xi)e^{i\xi \cdot x} d\xi.$$

By the Young inequality, if  $\|m^\vee\| \in L^1$  and  $f \in L^p$ , then we have

$$\|m(D)f\|_{L^p} \preceq \|m^\vee\|_{L^1} \|f\|_{L^p}$$

for any  $1 \leq p \leq \infty$ . We will use the following Bernstein multiplier theorem to estimate  $\|m^\vee\|_{L^1}$ .

**Lemma 2.1** (Bernstein’s multiplier Theorem) (see [25]) *Assume that  $0 < p \leq 2$  and  $\partial^\gamma m(\xi) \in L^2$  for all multi-indices  $\gamma$  with  $|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1$ . We have*

$$\|m^\vee\|_{L^p} \preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \|\partial^\gamma m\|_{L^2}.$$

By checking the Fourier transform (see [22]), we have that

$$\mathcal{F}(\Delta(A_1)^N f)(\xi) \simeq |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N \widehat{f}(\xi),$$

where

$$V_\delta(r) = \frac{J_\delta(r)}{r^\delta}$$

and  $J_\delta(r)$  is the Bessel function of order  $\delta$  which is defined as

$$J_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{irs\sin\theta} e^{-i\delta\theta} d\theta.$$

Here we state some basic properties about the Bessel function which will be used in our proofs.

**Lemma 2.2** (see [22]) *For any  $\delta > -\frac{1}{2}$ , we have*

$$\frac{dV_\delta(r)}{dr} = -rV_{\delta+1}(r), \tag{2.2}$$

$$V_\delta(r) = O(1), \quad \text{if } |r| \leq 1. \tag{2.3}$$

**Lemma 2.3** (see [22]) *Let  $r > 1$  and  $\delta > -\frac{1}{2}$ . For any positive integer  $L$  and  $r \in [1, \infty)$ , we have*

$$J_\delta(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\delta\pi}{2} - \frac{\pi}{4}\right) + \sum_{j=1}^L a_j e^{ir} r^{-\frac{1}{2}-j} + \sum_{j=1}^L b_j e^{-ir} r^{-\frac{1}{2}-j} + E(r),$$

where  $a_j$  and  $b_j$  are constants for all  $j$ , and  $E(r)$  is a  $C^\infty$  function satisfying

$$|E^{(k)}(r)| \leq r^{-\frac{1}{2}-L-1}$$

for any  $k = 0, 1, 2, \dots$ .

### 3 Proofs of Main Results

Now we show the proof of Theorem 1.1. By the definition of  $\alpha$ -modulation spaces, we need to estimate  $\|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}(\mathbb{R}^n)}$ . First, we obtain the following lemma.

**Lemma 3.1** *Let  $1 \leq p_2 \leq \infty$  and  $\sigma = 2 - \frac{n-1}{2}N + \frac{n\alpha}{2}$ . Then*

$$\|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}(\mathbb{R}^n)} \leq \langle k \rangle^{\frac{\sigma}{1-\alpha}} \|\square_k^\alpha f\|_{L^{p_2}(\mathbb{R}^n)}.$$

**Proof** For any  $k \in \mathbb{Z}^n$ ,  $\square_k^\alpha \Delta(A_1)^N f$  is a Fourier multiplier  $m_k^\alpha(D)(f) = \Omega_k^\alpha(x) * f$ , where

$$\Omega_k^\alpha(x) = \int_{\mathbb{R}^n} \eta_k^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N e^{i\xi \cdot x} d\xi. \tag{3.1}$$

By the almost orthogonality of  $\alpha$ -decomposition (Proposition 2.1), there exists an integer  $k_0(n)$  which depends only on  $n$ , such that  $\eta_l^\alpha(\xi)\eta_k^\alpha(\xi) = 0$  when  $|l - k| \geq k_0(n)$ . By the definition of  $\alpha$ -decomposition, we have

$$\sum_{k \in \mathbb{Z}^n} \square_k^\alpha = I,$$

where  $I$  is the identity operator. Then, Young’s inequality and Minkowski’s inequality yield

$$\begin{aligned} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}} &\leq \sum_{l \in \mathbb{Z}^n, |l-k| \leq k_0(n)} \|\square_l^\alpha \Delta(A_1)^N \square_k^\alpha f\|_{L^{p_2}} \\ &\leq \sum_{l \in \mathbb{Z}^n, |l-k| \leq k_0(n)} \|(\eta_l^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1} \|\square_k^\alpha f\|_{L^{p_2}}. \end{aligned}$$

Thus, we only need to estimate

$$\sum_{l \in \mathbb{Z}^n, |l-k| \leq k_0(n)} \|(\eta_l^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1}$$

for every  $k \in \mathbb{Z}^n$ . By Proposition 2.1, it suffices to estimate

$$\|(\eta_l^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1} = \|\Omega_l^\alpha(x)\|_{L^1}$$

for  $\langle l \rangle \simeq \langle k \rangle$ .

When  $|k| < 100$ , by (2.3) in Lemma 2.2 and  $\text{supp } \eta_l^\alpha(\xi)$ , we have that  $|\Omega_l^\alpha(x)| \leq 1$  for  $|x| < 100$ .

On the other hand, when  $|k| < 100$  and  $|x| \geq 100$ , without loss of generality, we may assume  $|x_1| \geq \frac{|x|}{n}$ . By the derivative formula of  $V_\delta(r)$ (see (2.2) in Lemma 2.2) and taking integration by part on  $\xi_1$  variable in (3.1), we obtain that

$$|\Omega_l^\alpha(x)| \leq \frac{1}{|x_1|^{n+1}} \leq \frac{1}{|x|^{n+1}}$$

for  $|x| \geq 100$ . This estimate implies that  $\|\Omega_l^\alpha(x)\|_{L^1} \leq 1$  when  $|k| < 100$ , since  $\langle l \rangle \simeq \langle k \rangle$ .

Next, we consider the case  $|k| \geq 100$ . Choosing  $L = 1$  in Lemma 2.3, we have the following asymptotic form of  $V_\delta(r)$

$$V_\delta(r) = r^{-\delta-\frac{1}{2}} \left( \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{\delta\pi}{2} - \frac{\pi}{4}\right) \right) + O(r^{-\delta-\frac{3}{2}})$$

for  $|r| > 1$ .

By the definition of  $\{\eta_k^\alpha\}$  and noticing that  $\alpha \in [0, 1)$ , we have that

$$|\xi| \sim \langle k \rangle^{\frac{1}{1-\alpha}}$$

for  $\xi \in \text{supp } \eta_k^\alpha$ . Therefore, when  $|k| > 100$  and  $\langle l \rangle \simeq \langle k \rangle$ , we have

$$|V_\delta(|\xi|)^N| \leq \langle l \rangle^{\frac{(-\delta-\frac{1}{2})N}{1-\alpha}} \simeq \langle k \rangle^{\frac{(-\delta-\frac{1}{2})N}{1-\alpha}}$$

for  $\xi \in \text{supp } \eta_l^\alpha(\xi)$ . Now, by the chain rule and the derivative formula of  $V_\delta(r)$ (see (2.2) in Lemma 2.2), we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N &= -N(V_\delta(|\xi|))^{N-1} |\xi| \cdot V_{\delta+1}(|\xi|) \cdot \frac{\xi_i}{|\xi|} \\ &= -N(V_\delta(|\xi|))^{N-1} V_{\delta+1}(|\xi|) \cdot \xi_i. \end{aligned}$$

By the asymptotic form of  $V_\delta(r)$ , we obtain that

$$\left| \frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N \right| \leq |\xi|^{(-\delta-\frac{1}{2})(N-1)} |\xi|^{-\delta-\frac{3}{2}} |\xi| \leq \langle l \rangle^{\frac{(-\delta-\frac{1}{2})N}{1-\alpha}} \simeq \langle k \rangle^{\frac{(-\delta-\frac{1}{2})N}{1-\alpha}}$$

for  $\xi \in \text{supp } \eta_l^\alpha(\xi)$ .

Thus,  $V_\delta(|\xi|)^N$  and  $\frac{\partial}{\partial \xi_i} (V_\delta(|\xi|))^N$  share the same upper bound which is  $\langle l \rangle^{\frac{(-\delta-\frac{1}{2})N}{1-\alpha}}$ , for any  $\delta > -\frac{1}{2}$  and  $\xi \in \text{supp } \eta_l^\alpha(\xi)$ . By the fact

$$\partial^\gamma (|\xi|^2) \begin{cases} \leq |\xi|^{2-|\gamma|}, & |\gamma| \leq 2, \\ = 0, & |\gamma| > 2 \end{cases}$$

and (2.1), using Bernstein's multiplier Theorem (Lemma 2.1), we can obtain that

$$\begin{aligned} \|\Omega_l^\alpha(x)\|_{L^1} &= \|(\eta_l^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1} \\ &\leq \sum_{|\gamma| \leq [\frac{n}{2}] + 1} \|\partial^\gamma (\eta_l^\alpha(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)\|_{L^2} \end{aligned}$$

$$\begin{aligned} &\preceq \sum_{|\gamma| \leq [\frac{n}{2}] + 1} \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} \|\partial^{\gamma_1} \eta_l^\alpha(\xi) \cdot \partial^{\gamma_2} |\xi|^2 \cdot \partial^{\gamma_3} (V_{\frac{n-2}{2}}(|\xi|))^N\|_{L^2(\text{supp } \eta_l^\alpha(\xi))} \\ &\preceq \sum_{|\gamma| \leq [\frac{n}{2}] + 1} \|\eta_l^\alpha(\xi) \cdot |\xi|^2 \cdot \partial^\gamma (V_{\frac{n-2}{2}}(|\xi|))^N\|_{L^2(\text{supp } \eta_l^\alpha(\xi))} \\ &\preceq |l|^{\frac{2 - \frac{n-1}{2}N}{1-\alpha}} \langle l \rangle^{\frac{n\alpha}{2(1-\alpha)}} \simeq \langle k \rangle^{\frac{2 - \frac{n-1}{2}N + \frac{n\alpha}{2}}{1-\alpha}} \end{aligned}$$

for  $l \in \Lambda_k$ .

Combining all the above estimates of  $\|\Omega_l^\alpha(x)\|_{L^1}$ , we have finished the proof of Lemma 3.1.

Next, we turn to prove Theorem 1.1. By the definition of  $\alpha$ -modulation spaces and Lemma 3.1, we have that

$$\begin{aligned} &\|\Delta(A_1)^N f\|_{M_{p_2, q_2}^{s_2, \alpha}} \\ &= \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &= \left( \sum_{|k| < 100} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}^{q_2}}^{q_2} + \sum_{|k| \geq 100} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &\preceq \left( \sum_{|k| < 100} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} + \left( \sum_{|k| \geq 100} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &\preceq \left( \sum_{|k| < 100} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} + \left( \sum_{|k| \geq 100} \langle k \rangle^{\frac{(s_2 + 2 - \frac{n-1}{2}N + \frac{n\alpha}{2})q_2}{1-\alpha}} \|\square_k^\alpha f\|_{L^{p_2}^{q_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &\preceq \|f\|_{M_{p_2, q_2}^{s_2 + 2 - \frac{n-1}{2}N + \frac{n\alpha}{2}}} = \|f\|_{M_{p_2, q_2}^{s_2 + \sigma}}. \end{aligned}$$

By the embedding properties of  $\alpha$ -modulation spaces (Proposition 2.3), we can easily obtain that

$$\|\Delta(A_1)^N f\|_{M_{p_2, q_2}^{s_2, \alpha}} \preceq \|f\|_{M_{p_2, q_2}^{s_2 + \sigma, \alpha}} \preceq \|f\|_{M_{p_1, q_1}^{s_1, \alpha}}.$$

When

$$q_1 \leq q_2, \quad p_1 \leq p_2, \quad s_1 - \frac{n\alpha}{p_1} \geq s_2 - \frac{n\alpha}{p_2} + \sigma$$

or

$$q_1 > q_2, \quad p_1 \leq p_2, \quad s_1 - \frac{n\alpha}{p_1} + \frac{n(1-\alpha)}{q_1} > s_2 - \frac{n\alpha}{p_2} + \frac{n(1-\alpha)}{q_2} + \sigma.$$

Thus, we have completed the proof of Theorem 1.1.

Now, we turn to prove Theorem 1.2. For this purpose, we need to establish the following two lemmas (Lemma 3.2 and Lemma 3.3). The idea of Lemma 3.3 is derived from [13], and Lemma 3.2 has been proved in [13]. For the sake of completeness, we will show all details in the following text.

**Lemma 3.2** For  $j \in \mathbb{N}^+$ , define

$$\Lambda_{1,j} := \{k \in \mathbb{Z}^n : |k| \in [j\pi + 0.07, (j+1)\pi - 0.07]\}$$

and

$$\Lambda_{0,j} := \{k \in \mathbb{Z}^n : |k| \in [j\pi, (j+1)\pi]\}.$$



When  $j$  is big enough, we have

$$|\Lambda_{1,j}| \geq C(n)|\Lambda_{0,j}|,$$

where  $C(n)$  is a positive constant depending only on  $n$ .

**Proof** The proofs for  $n = 2$  and  $n > 2$  share the same idea. We only prove the case  $n = 2$  explicitly and leave the proof of another case to the reader.

By symmetry, we only need to consider the case  $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ . For  $j \in \mathbb{Z}^+$ , we define

$$\begin{aligned} \text{I}_{x>y} &= \{(x, y) \in \mathbb{R}^2 : |(x, y)| \in [j\pi, j\pi + 0.07], x > y \geq 0\}, \\ \text{I}_{x\leq y} &= \{(x, y) \in \mathbb{R}^2 : |(x, y)| \in [j\pi, j\pi + 0.07], y \geq x \geq 0\}, \\ \text{II} &= \{(x, y) \in \mathbb{R}^2 : |(x, y)| \in [j\pi + 0.07, j\pi + \pi - 0.07], x, y \geq 0\}, \\ \text{III}_{x>y} &= \{(x, y) \in \mathbb{R}^2 : |(x, y)| \in [j\pi + \pi - 0.07, j\pi + \pi], x > y \geq 0\} \end{aligned}$$

and

$$\text{III}_{x\leq y} = \{(x, y) \in \mathbb{R}^2 : |(x, y)| \in [j\pi + \pi - 0.07, j\pi + \pi], y \geq x \geq 0\}.$$

Moreover, for  $r, a > 0, 0 \leq y \leq r$ , we define an auxiliary function

$$f_{r,a}(y) = \sqrt{(r+a)^2 - y^2} - \sqrt{r^2 - y^2}.$$

Taking derivative, we know that

$$f'_{r,a}(y) = y \left( \frac{1}{\sqrt{r^2 - y^2}} - \frac{1}{\sqrt{(r+a)^2 - y^2}} \right) \geq 0,$$

and  $f_{r,a}(y)$  is a monotone increasing function.

Then, for any  $(x_0, y_0) \in \text{I}_{x>y}$ , we have

$$|\{x : (x, y_0) \in \text{I}_{x>y}\}| = f_{j\pi, 0.07}(y_0).$$

Therefore,

$$\begin{aligned} \max_{(x_0, y_0) \in \text{I}_{x>y}} f_{j\pi, 0.07}(y_0) &= f_{j\pi, 0.07} \left( \frac{1}{\sqrt{2}}(j\pi + 0.07) \right) \\ &= \sqrt{(j\pi + 0.07)^2 - \frac{1}{2}(j\pi + 0.07)^2} - \sqrt{j\pi^2 - \frac{1}{2}(j\pi + 0.07)^2} \\ &= \frac{0.14j + 0.07^2}{\sqrt{(j\pi + 0.07)^2 - \frac{1}{2}(j\pi + 0.07)^2} + \sqrt{j\pi^2 - \frac{1}{2}(j\pi + 0.07)^2}}. \end{aligned}$$

It is obvious to see that  $\lim_{j \rightarrow +\infty} \max_{(x_0, y_0) \in \text{I}_{x>y}} f_{j\pi, 0.07}(y_0) = \frac{0.14}{\sqrt{2}} < 1$ . Thus, for any  $(x_0, y_0) \in \text{I}_{x>y}$ , we have

$$|\{x : (x, y_0) \in \text{I}_{x>y}\}| \leq \max_{(x, y) \in \text{I}_{x>y}} f_{j\pi, 0.07}(y) < 1,$$

when  $j$  is big enough.

On the other hand, for any  $(x_0, y_0) \in \text{II}$ , we have

$$|\{x : (x, y_0) \in \text{II}\}| = f_{j\pi+0.07, \pi-0.14}(y_0).$$

By monotonicity of  $f_{r,a}(y)$ ,

$$\min_{(x_0, y_0) \in \text{II}} |\{x : (x, y_0) \in \text{II}\}| = f_{j\pi+0.07, \pi-0.14}(0) = \pi - 0.14 > 3. \tag{3.2}$$

Thus, for every  $(x_0, y_0) \in \text{I}_{x>y} \cap \mathbb{Z}^2$ , we have

$$|\{(x, y_0) \in \mathbb{Z}^2 : (x, y_0) \in \text{I}_{x>y}\}| = |\{(x_0, y_0)\}| = 1$$

and

$$|\{(x, y_0) \in \mathbb{Z}^2 : (x, y_0) \in \text{II}\}| \geq 3.$$

Combining all above analysis, we have

$$|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in \text{II}\}| \geq 3|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in \text{I}_{x>y}\}|.$$

Now, we consider the domain  $\text{III}_{x>y}$ . By the same argument, for any  $(x_0, y_0) \in \text{III}_{x>y}$ , we have

$$\begin{aligned} |\{x : (x, y_0) \in \text{III}_{x>y}\}| &\leq \max_{(x_0, y_0) \in \text{III}_{x>y}} f_{j\pi+\pi-0.07, 0.07}(y_0) \\ &= f_{j\pi+\pi-0.07, 0.07}\left(\frac{1}{\sqrt{2}}(j\pi + \pi)\right) \\ &= \sqrt{(j\pi + \pi)^2 - \frac{1}{2}(j\pi + \pi)^2} - \sqrt{(j\pi + \pi - 0.07)^2 - \frac{1}{2}(j\pi + \pi)^2} \\ &= \frac{0.14j + 0.07(2\pi - 0.07)}{\sqrt{(j\pi + \pi)^2 - \frac{1}{2}(j\pi + \pi)^2} + \sqrt{(j\pi + \pi - 0.07)^2 - \frac{1}{2}(j\pi + \pi)^2}}. \end{aligned}$$

It is easy to see that

$$\lim_{j \rightarrow +\infty} \max_{(x_0, y_0) \in \text{III}_{x>y}} f_{j\pi+\pi-0.07, 0.07}(y_0) = \frac{0.14}{\sqrt{2}} < 1.$$

Thus, for any  $(x_0, y_0) \in \text{III}_{x>y}$ , we have

$$|\{x : (x, y_0) \in \text{III}_{x>y}\}| \leq \max_{(x, y) \in \text{I}_{x>y}} f_{j\pi+\pi-0.07, 0.07}(y) < 1,$$

when  $j$  is big enough. Moreover, it is obvious

$$\frac{1}{\sqrt{2}}(j\pi + \pi) < j\pi + 0.07,$$

when  $j \geq 3$ . So, for every  $(x_0, y_0) \in \text{III}_{x>y}$ ,

$$|\{x : (x, y_0) \in \text{II}\}| = f_{j\pi+0.07, \pi-0.14}(y_0),$$

when  $j \geq 3$ . By (3.2), we can also obtain

$$|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in \text{II}\}| \geq 3|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in \text{III}_{x>y}\}|.$$

On the other hand, for  $I_{y \geq x}$  and  $III_{y \geq x}$ , by the same method on the auxiliary function

$$g_{r,a}(x) = \sqrt{(r+a)^2 - x^2} - \sqrt{r^2 - x^2},$$

we can obtain that

$$|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in II\}| \geq 3|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in I_{x \leq y}\}|$$

and

$$|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in II\}| \geq 3|\{(x, y) \in \mathbb{Z}^2 : (x, y) \in III_{x \leq y}\}|.$$

Combining all above estimates, we have

$$|\Lambda_{1,j}| \geq \frac{3}{7}|\Lambda_{0,j}|.$$

**Lemma 3.3** *Let  $1 \leq p \leq \infty$ . There exists a constant  $\rho = \rho(n) > 0$  which only depends on  $n$  and a subsequence  $\{k_j\} \subseteq \mathbb{Z}^n$  such that*

$$\|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p} \simeq \langle k_j \rangle^{\frac{2-\frac{n-1}{2}N}{1-\alpha}} \|g_{k_j}\|_{L^p},$$

where  $\{g_{k_j}(x)\}$  is a sequence of Schwartz functions with  $\text{supp } \widehat{g_{k_j}}(\xi) \subset \{\xi \in \mathbb{R}^n : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}$ .

**Proof** Let  $\delta(\xi)$  be a smooth function with  $\text{supp } \delta(\xi) \subset \{\xi : |\xi| \leq 2\rho\}$  and  $\delta(\xi) \equiv 1$  for  $\xi \in \{\xi : |\xi| \leq \rho\}$ . We define

$$\delta_k(\xi) = \delta(\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k), \quad k \in \mathbb{Z}^n.$$

For any  $g_k$  with  $\text{supp } \widehat{g_k}(\xi) \subset \{\xi \in \mathbb{R}^n : |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| \leq \rho\}$ , we have

$$\delta_k(\xi) \widehat{g_k}(\xi) = \widehat{g_k}(\xi).$$

By the same method as in Lemma 3.1, it is easy to get

$$\begin{aligned} \|\square_k^\alpha \Delta(A_1)^N g_k\|_{L^p} &= \|(|\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N \eta_k^\alpha(\xi) \widehat{g_k}(\xi))^\vee\|_{L^p} \\ &= \|(|\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N \eta_k^\alpha(\xi) \delta_k(\xi) \widehat{g_k}(\xi))^\vee\|_{L^p} \\ &= \|(\delta_k(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1} \|\square_k^\alpha g_k\|_{L^p}. \end{aligned}$$

By the definition of  $\delta_k(\xi)$  and Bernstein's multiplier Theorem (Lemma 2.1), we have

$$\begin{aligned} &\|(\delta_k(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)^\vee\|_{L^1} \leq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \|\partial^\gamma (\delta_k(\xi) |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N)\|_{L^2} \\ &\preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} \|\partial^{\gamma_1} \delta_k(\xi) \cdot \partial^{\gamma_2} |\xi|^2 \cdot \partial^{\gamma_3} (V_{\frac{n-2}{2}}(|\xi|))^N\|_{L^2} \\ &\preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \sum_{\gamma_1 + \gamma_2 = \gamma} \| |\xi|^2 \cdot \partial^{\gamma_1} \delta_k(\xi) \cdot \partial^{\gamma_2} (V_{\frac{n-2}{2}}(|\xi|))^N \|_{L^2(\text{supp } \delta_k(\xi))} \\ &\preceq \langle k \rangle^{\frac{2-\frac{n-1}{2}N}{1-\alpha}} \end{aligned}$$

for all  $k \in \mathbb{Z}^n$ . Therefore, for  $\forall k \in \mathbb{Z}^n$ , we have

$$\|\square_k^\alpha \Delta(A_1)^N g_k\|_{L^p} \preceq \langle k \rangle^{\frac{2-n-1}{2}N} \|\square_k^\alpha g_k\|_{L^p}, \tag{3.3}$$

when  $\text{supp } \widehat{g}_k(\xi) \subset \{\xi \in \mathbb{R}^n : |\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}} k| \leq \rho\}$ .

Thus, we only need to prove the inverse inequality. By Lemma 2.3, we have

$$\begin{aligned} V_{\frac{n-2}{2}}(r) &= r^{-\frac{n-1}{2}} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{n\pi}{4} + \frac{\pi}{4}\right) + O(r^{-\frac{n+1}{2}}) \\ &= r^{-\frac{n-1}{2}} \sqrt{\frac{2}{\pi}} \sin\left(r - \frac{n\pi}{4} + \frac{3\pi}{4}\right) + O(r^{-\frac{n+1}{2}}) \end{aligned}$$

for  $|r| > 1$ . We consider

$$u(r) := \sin\left(r - \frac{n\pi}{4} + \frac{3\pi}{4}\right)$$

in every semiperiod  $r - \frac{n\pi}{4} + \frac{3\pi}{4} \in [j\pi, (j+1)\pi]$ ,  $j = 1, 2, \dots$ .

Choosing  $\varepsilon_0 = \sin(0.07)$ , we have

$$|u(r)| \geq \varepsilon_0$$

for  $r - \frac{n\pi}{4} + \frac{3\pi}{4} \in [j\pi + 0.07, j\pi + \pi - 0.07]$ , which is equivalent to

$$r \in \left[ j\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + 0.07, j\pi + \pi + \frac{n\pi}{4} - \frac{3\pi}{4} - 0.07 \right].$$

By Lemma 3.2, for every  $j \in \mathbb{N}^+$ , the set

$$\Lambda_{1,j} := \left\{ k \in \mathbb{Z}^n : |k| \in \left[ j\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + 0.07, j\pi + \pi + \frac{n\pi}{4} - \frac{3\pi}{4} - 0.07 \right] \right\}$$

is not empty. So, there exists a subsequence of integer  $\{k_j\}$ , such that  $k_j \in \Lambda_{1,j}$  and

$$|u(|k_j|)| \geq \varepsilon_0.$$

Moreover,

$$|u'(r)| = \left| \cos\left(r - \frac{n\pi}{4} + \frac{3\pi}{4}\right) \right| \leq 1,$$

which means that

$$|u(r)| \geq \frac{\varepsilon_0}{2}$$

for

$$r \in \left[ |k_j| - \frac{\varepsilon_0}{4}, |k_j| + \frac{\varepsilon_0}{4} \right]$$

and

$$|k_j| \in \left[ j\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + 0.07, j\pi + \pi + \frac{n\pi}{4} - \frac{3\pi}{4} - 0.07 \right].$$

For the remainder  $O(r^{-\frac{n+1}{2}})$  in the expansion of  $V_{\frac{n-2}{2}}(r)$ , it is obvious that, when  $r$  is large enough,

$$O(r^{-\frac{n+1}{2}}) \leq \frac{\varepsilon_0}{4} r^{-\frac{n-1}{2}}.$$

Let  $\epsilon, \rho = \frac{\epsilon_0}{4}$ . We obtain that there exist some constants  $\epsilon, \rho > 0$  and a subsequence  $\{k_j\} \subseteq \mathbb{Z}^+$  such that

$$|V_{\frac{n-2}{2}}(|\xi|)| \geq \epsilon |\xi|^{-\frac{n-1}{2}} \tag{3.4}$$

for  $\xi \in \{\xi : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}$  when  $j$  is large enough. Moreover the subsequence  $\{k_j\} \subseteq \mathbb{Z}^n$  satisfies

$$|k_j| \in \left[ j\pi + \frac{n\pi}{4} - \frac{3\pi}{4} + 0.07, j\pi + \pi + \frac{n\pi}{4} - \frac{3\pi}{4} - 0.07 \right], \tag{3.5}$$

when the positive integer  $j$  is large enough.

Therefore, when  $\xi \in \{\xi : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}$  and  $N \in \mathbb{Z}^+$ , we have

$$|V_{\frac{n-2}{2}}(|\xi|)^{-N}| \preceq |\xi|^{(\frac{n-1}{2})N} \simeq \langle k_j \rangle^{\frac{(\frac{n-1}{2})N}{1-\alpha}}.$$

Using the chain rule and the derivative formula of  $V_\delta(t)$ ,

$$\frac{\partial}{\partial \xi_i} (V_{\frac{n-2}{2}}(|\xi|))^{-N} = -N (V_{\frac{n-2}{2}}(|\xi|))^{-(N+1)} V_{\frac{n-2}{2}+1}(|\xi|) \cdot \xi_i.$$

By the asymptotic form of  $V_\delta(r)$  and (3.4), we have

$$\left| \frac{\partial}{\partial \xi_i} (V_{\frac{n-2}{2}}(|\xi|))^{-N} \right| \preceq |\xi|^{(-\frac{n-1}{2})(-N-1)} |\xi|^{-\frac{n+1}{2}} |\xi| \simeq \langle k_j \rangle^{\frac{(\frac{n-1}{2})N}{1-\alpha}},$$

when  $\xi \in \{\xi : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}$ . As a result,  $V_{\frac{n-2}{2}}(|\xi|)^{-N}$  and  $\frac{\partial}{\partial \xi_i} (V_{\frac{n-2}{2}}(|\xi|))^{-N}$  share the same upper bound which is  $\langle k_j \rangle^{\frac{(\frac{n-1}{2})N}{1-\alpha}}$ , for  $\xi \in \{\xi : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}$ .

Moreover, since  $\rho = \frac{1}{4} \sin(0.07) < \frac{1}{4}$ , for the definition of  $\alpha$ -decomposition  $\{\eta_k^\alpha\}$  (see Definition 2.1), we have that

$$\eta_{k_j}^\alpha(\xi) \widehat{g_{k_j}}(\xi) = \widehat{g_{k_j}}(\xi),$$

with

$$\text{supp } \widehat{g_{k_j}}(\xi) \subset \{\xi : |\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j| \leq \rho\}.$$

Therefore, by the definition of  $\delta_k(\xi)$ , Bernstein multiplier theorem (Lemma 2.1) and (3.4), we obtain that

$$\begin{aligned} & \|g_{k_j}\|_{L^p} = \|(\widehat{g_{k_j}})^\vee\|_{L^p} \\ & = \|(\eta_{k_j}^\alpha(\xi) |\xi|^{-2} (V_{\frac{n-2}{2}}(|\xi|))^{-N} \cdot |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|))^N \delta_{k_j}(\xi) \widehat{g_{k_j}}(\xi))^\vee\|_{L^p} \\ & \preceq \|(\delta_{k_j}(\xi) |\xi|^{-2} (V_{\frac{n-2}{2}}(|\xi|))^{-N})^\vee\|_{L^1} \|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p} \\ & \preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \|\partial^\gamma (\delta_{k_j}(\xi) |\xi|^{-2} (V_{\frac{n-2}{2}}(|\xi|))^{-N})\|_{L^2} \|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p} \\ & \preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \sum_{\gamma_1 + \gamma_2 + \gamma_3 = \gamma} \|\partial^{\gamma_1} \delta_{k_j}(\xi) \cdot \partial^{\gamma_2} |\xi|^{-2} \cdot \partial^{\gamma_3} (V_{\frac{n-2}{2}}(|\xi|))^{-N}\|_{L^2} \|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p} \\ & \preceq \sum_{|\gamma| \leq [n(\frac{1}{p} - \frac{1}{2})] + 1} \sum_{\gamma_1 + \gamma_2 = \gamma} \| |\xi|^{-2} \cdot \partial^{\gamma_1} \delta_{k_j}(\xi) \cdot \partial^{\gamma_2} (V_{\frac{n-2}{2}}(|\xi|))^{-N} \|_{L^2} \|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p} \end{aligned}$$

$$\preceq \langle k_j \rangle^{-\frac{(2-\frac{n-1}{2}N)}{1-\alpha}} \|\square_{k_j}^\alpha \Delta(A_1)^N g_{k_j}\|_{L^p}.$$

Combining the above estimate with (3.3), Lemma 3.2 is proved.

Now, we continue to prove Theorem 1.2. We first consider the case  $q_1 \leq q_2$ . Let  $f(x)$  be a nonzero Schwartz function with  $\text{supp } \widehat{f}(\xi) \subset \{\xi : |\xi| < \frac{1}{2}\}$ . Define

$$\widehat{f_{k_j,\lambda}}(\xi) = \widehat{f}\left(\frac{\xi - \langle k_j \rangle^{\frac{\alpha}{1-\alpha}} k_j}{\lambda}\right) \tag{3.6}$$

for  $\lambda \in (0, \rho]$ , where  $\rho$  and  $\{k_j\}$  are defined in Lemma 3.3. By the definition of  $f_{k_j,\lambda}(x)$ , we have

$$\square_{k_j}^\alpha f_{k_j,\lambda} = f_{k_j,\lambda} \tag{3.7}$$

and

$$\square_i^\alpha f_{k_j,\lambda}(x) = 0, \quad \text{if } i \neq k_j. \tag{3.8}$$

Then, by Lemma 3.3, we have

$$\begin{aligned} \|\Delta(A_1)^N f_{k_j,\lambda}\|_{M_{p_2,q_2}^{s_2,\alpha}} &= \langle k_j \rangle^{\frac{s_2}{1-\alpha}} \|\square_{k_j}^\alpha \Delta(A_1)^N f_{k_j,\lambda}\|_{L^{p_2}} \\ &\simeq \langle k_j \rangle^{\frac{s_2+2-\frac{n-1}{2}N}{1-\alpha}} \|f_{k_j,\lambda}\|_{L^{p_2}} \\ &\simeq \langle k_j \rangle^{\frac{s_2+2-\frac{n-1}{2}N}{1-\alpha}} \lambda^{n(1-\frac{1}{p_2})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f_{k_j,\lambda}\|_{M_{p_1,q_1}^{s_1,\alpha}} &= \langle k_j \rangle^{\frac{s_1}{1-\alpha}} \|\square_{k_j}^\alpha f_{k_j,\lambda}\|_{L^{p_1}} \\ &\simeq \langle k_j \rangle^{\frac{s_1}{1-\alpha}} \|f_{k_j,\lambda}\|_{L^{p_1}} \\ &\simeq \langle k_j \rangle^{\frac{s_1}{1-\alpha}} \lambda^{n(1-\frac{1}{p_1})}. \end{aligned}$$

By the assumption that  $\Delta(A_1)^N$  is bounded from  $M_{p_1,q_1}^{s_1,\alpha}$  to  $M_{p_2,q_2}^{s_2,\alpha}$ , we have that

$$\langle k_j \rangle^{s_2+2-\frac{n-1}{2}N} \lambda^{n(1-\frac{1}{p_2})} \preceq \langle k_j \rangle^{s_1} \lambda^{n(1-\frac{1}{p_1})}$$

for all  $|k_j|$  sufficiently large and  $0 < \lambda \leq \rho$ . Fix  $k_j$  and let  $\lambda \rightarrow 0$ . We have

$$\lambda^{n(1-\frac{1}{p_2})} \preceq \lambda^{n(1-\frac{1}{p_1})} \quad \text{for } 0 < \lambda \leq \rho.$$

Thus, the condition  $p_1 \leq p_2$  must be held. Moreover, when  $\lambda$  is fixed and  $k_j$  goes to infinity, we have

$$\langle k_j \rangle^{s_2+2-\frac{n-1}{2}N} \preceq \langle k_j \rangle^{s_1}, \quad \text{as } k_j \rightarrow +\infty,$$

which yields  $s_2 + 2 - \frac{n-1}{2}N \leq s_1$ .

Next, we consider the case  $q_1 > q_2$ . Let  $M$  be a large positive number. Define

$$F_M(x) = \sum_{100 < |k_j| < M} a_j f_{k_j,\rho}(x),$$

where  $a_j > 0$  are constants to be chosen later and  $f_{k_j, \rho}(x)$  are defined in (3.6) with all  $k_j$  satisfying

$$|k_j| \in [L\pi + 0.07, L\pi + \pi - 0.07]$$

for some  $L \in \mathbb{N}^+$ .

By (3.7)–(3.8) and the almost orthogonality of  $\{\eta_k^\alpha\}$ , we have

$$\begin{aligned} \|\Delta(A_1)^N F_M\|_{M_{p_2, q_2}^{s_2, \alpha}} &= \left( \sum_{k \in \mathbb{Z}^n} a_j^{q_2} \langle k \rangle^{\frac{s_2 q_2}{1-\alpha}} \|\square_k^\alpha \Delta(A_1)^N F_M\|_{L^{p_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &\simeq \left( \sum_{100 < |k_j| < M} a_j^{q_2} \langle k_j \rangle^{\frac{s_2 q_2}{1-\alpha}} \langle k_j \rangle^{\frac{(2-\frac{n-1}{2}N)q_2}{1-\alpha}} \|f_{k_j, \rho}(x)\|_{L^{p_2}}^{q_2} \right)^{\frac{1}{q_2}} \\ &\simeq \left( \sum_{100 < |k_j| < M} a_j^{q_2} \langle k_j \rangle^{\frac{s_2 q_2 + (2-\frac{n-1}{2}N)q_2}{1-\alpha}} \right)^{\frac{1}{q_2}} \end{aligned}$$

and

$$\begin{aligned} \|F_M\|_{M_{p_1, q_1}^{s_1, \alpha}} &= \left( \sum_{k \in \mathbb{Z}^n} a_j^{q_1} \langle k \rangle^{\frac{s_1 q_1}{1-\alpha}} \|\square_k^\alpha F_M\|_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\ &\simeq \left( \sum_{100 < |k_j| < M} a_j^{q_1} \langle k_j \rangle^{\frac{s_1 q_1}{1-\alpha}} \|f_{j, \rho}(x)\|_{L^{p_1}}^{q_1} \right)^{\frac{1}{q_1}} \\ &\simeq \left( \sum_{100 < |k_j| < M} a_j^{q_1} \langle k_j \rangle^{\frac{s_1 q_1}{1-\alpha}} \right)^{\frac{1}{q_1}}. \end{aligned}$$

By the assumption that  $\Delta(A_1)^N$  is bounded from  $M_{p_1, q_1}^{s_1, \alpha}$  to  $M_{p_2, q_2}^{s_2, \alpha}$ , we have

$$\left( \sum_{100 < |k_j| < M} a_j^{q_2} \langle k_j \rangle^{s_2 q_2 + (2-\frac{n-1}{2}N)q_2} \right)^{\frac{1}{q_2}} \preceq \left( \sum_{100 < |k_j| < M} a_j^{q_1} \langle k_j \rangle^{s_1 q_1} \right)^{\frac{1}{q_1}}.$$

By choosing  $a_j = \langle k_j \rangle^{\frac{s_1 q_1 - (s_2 + (2-\frac{n-1}{2}N)q_2)}{(1-\alpha)(q_1 - q_2)}}$ , we obtain

$$\left( \sum_{100 < |k_j| < M} \langle k_j \rangle^{\frac{[s_1 - (s_2 + 2 - \frac{n-1}{2}N)]q_1 q_2}{(1-\alpha)(q_2 - q_1)}} \right)^{\frac{1}{q_2}} \preceq \left( \sum_{100 < |k_j| < M} \langle k_j \rangle^{\frac{[s_1 - (s_2 + 2 - \frac{n-1}{2}N)]q_1 q_2}{(1-\alpha)(q_2 - q_1)}} \right)^{\frac{1}{q_1}}.$$

By the assumption  $q_1 > q_2$ , the above series must be convergent as  $M \rightarrow +\infty$ . By Lemma 3.2, we have

$$\sum_{100 < |k_j| < M} \langle k_j \rangle^{\frac{[s_1 - (s_2 + 2 - \frac{n-1}{2}N)]q_1 q_2}{(1-\alpha)(q_2 - q_1)}} \simeq \sum_{100 < |k| < M} \langle k \rangle^{\frac{[s_1 - (s_2 + 2 - \frac{n-1}{2}N)]q_1 q_2}{(1-\alpha)(q_2 - q_1)}}.$$

Therefore, it must yield

$$\frac{[s_1 - (s_2 + 2 - \frac{n-1}{2}N)]q_1 q_2}{(1-\alpha)(q_2 - q_1)} < -n,$$

which is equivalent to  $s_1 + \frac{n(1-\alpha)}{q_1} > s_2 + \frac{n(1-\alpha)}{q_2} + 2 - \frac{n-1}{2}N$ . Theorem 1.2 is proved.

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