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Two Commuting Involutions Fixing $RP_1(2m+1)\bigcup RP_2(2m+1)^*$

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Abstract Let Z_2 denote a cyclic group of 2 order and $Z_2^2 = Z_2 \times Z_2$ the direct product of groups. Suppose that (M, Φ) is a closed and smooth manifold M with a smooth Z_2^2-action whose fixed point set is the disjoint union of two real projective spaces with the same dimension. In this paper, the authors give a sufficient condition on the fixed data of the action for (M, Φ) bounding equivariantly.

Keywords Z_2^2-action, Fixed data, Characteristic number, Simultaneous bordism, Stiefel-Whitney class
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1 Introduction

Let M be a smooth, closed manifold and $T : M \to M$ a smooth involution defined on M (i.e., Z_2-action, where Z_2 denotes a cyclic group of 2 order). It is well known that the fixed point set $F = \{x \in M \mid T(x) = x\}$ of the involution T is a finite and disjoint union of closed submanifolds of M. In this setting, for a given F, one naturally considers to classify the pairs (M, T) for which the fixed point set of T is F up to equivariant bordism. Let ν denote the normal bundle of F in M. It is known that the equivariant bordism class of (M, T) is determined by the bordism class of the bundle (F, ν), and the bordism class of the bundle (F, ν) is determined by its characteristic numbers (see [2]). For the vector bundle $\nu \to F$, there are the associated sphere bundle $S(\nu) \to F$ and a fibre preserving fixed point free involution $(S(\nu), T)$ which on each fibre agree with the antipodal map of sphere. The bundle $S(\nu)/T \to F$ is denoted by $RP(\nu) \to F$; that is the real projective space bundle associated to vector bundle ν. Further, the real projective space bundle $RP(\nu)$ bounds in the bordism of the classifying space $RP(\infty)$ for Z_2, where the map into $RP(\infty)$ classifies the double cover of $RP(\nu)$ by the sphere bundle $S(\nu)$ (see [2, p.88]). Conversely, being given a vector bundle ξ over F for which $RP(\xi)$ bounds in the sense just described, there is an involution fixing F with normal bundle $\nu = \xi$. Using the above results, in [4], Kosniowski and Stong gave a formula to express

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relationship between Stiefel-Whitney numbers of M and that of the bundle (F, ν), and proved the following result: If M^m is a closed and smooth m-dimensional manifold with a smooth involution $T : M^m \to M^m$ such that the fixed point set F of T has constant dimension n with $m > 2n$, then (M^m, T) bounds equivariantly. For the fixed point set F being the disjoint union of some spaces and product spaces such as the disjoint union of projective spaces and the product spaces of projective spaces, by computing characteristic numbers and using the formula in [4], one has given equivariant bordism classification of (M, T) with a given F (see [3, 6–10, 17–18, 20–21]).

For $k > 1$, let Z_2^k denote the direct product of k groups Z_2. Z_2^k is often considered as the group generated by k smooth commuting involutions $T_1, T_2, \cdots, T_k$ on M^m. The k commuting involutions determine a smooth Z_2^k-action $\Phi : Z_2^k \times M^m \to M^m$. The fixed data of Z_2^k-action Φ consists of $\eta = \bigoplus_\rho \varepsilon_\rho \to F_\Phi$, where $F_\Phi = \{x \in M^m \mid T_i(x) = x, i = 1, 2, \cdots, k\}$ is the fixed point set of Z_2^k-action and $\eta = \bigoplus_\rho \varepsilon_\rho$ is the normal bundle of F_Φ in M decomposed into eigenbundles ε_ρ with ρ running through the $2^k - 1$ nontrivial irreducible representations of Z_2^k (see [16]). An equivariant bordism classification of M^m with Z_2^k-actions is closely related to the fixed data of Z_2^k-action (see [19]). In a series of papers, the equivariant bordism classification of (M^m, Φ) with a given condition on the fixed data of Φ has been studied (see [5, 11–16, 19]). For $k > 1$ and $F_\Phi = RP(l) \cup RP(n)$, where $RP(\cdot)$ denotes a real projective space, the classifications in cases $(l, n) = (0, \text{odd}), (0, \text{even})$ were completely solved in [11–13]. In [15], Pergher, Ramos and Oliveira solved the case $(l, n) = (2, n)$ (n is even), where $n \geq 4$. Later, in [16], Pergher and Ramos solved the case $(l, n) = (2^s, n)$ (n is even), where $s \geq 1$ and $n \geq 2^{s+1}$, which extended the previous case ($s = 1$).

The purpose of this paper is to extend above results for Z_2^2-actions. Let $\Phi : Z_2^2 \times M \to M$ be a smooth action of the group $Z_2^2 = \{T_1, T_2 \mid T_i^2 = 1, i = 1, 2,\ T_1T_2 = T_2T_1\}$ on a smooth closed manifold M. Let $T_3 = T_1T_2$. The fixed data of Φ is $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $F_\Phi = \{x \in M \mid T_i(x) = x, i = 1, 2, 3\}$ is the fixed point set of Z_2^2-action on M, ε_i $(i = 1, 2, 3)$ is the normal bundle of F_Φ in $F_{T_i} = \{x \in M \mid T_i(x) = x\}$. We have the following result.

Theorem 1.1 *Let (M, Φ) be a closed and smooth manifold M with a smooth Z_2^2-action whose fixed point set is the disjoint union of two real projective spaces with dimension $2m+1$, that is, $F_\Phi = RP_1(2m+1) \cup RP_2(2m+1)$, where $RP_i(2m+1)$ denotes the i-th copy. Let*

$$(RP_1(2m+1); \mu_1, \mu_2, \mu_3) \cup (RP_2(2m+1); \nu_1, \nu_2, \nu_3)$$

be the fixed data of Φ, where μ_i and ν_i denote the normal bundle of components of F_Φ respectively. If at least two μ_i's have dimension greater than $2m+1$, and at least one ν_i has dimension greater than $2m+1$, then (M, Φ) bounds equivariantly.

Example 1.1 Let $M = RP(4m+3) \times RP(4m+3)$. Considering the involution $T_1 : RP(4m+3) \to RP(4m+3)$ on the $(4m+3)$-dimensional real projective space $RP(4m+3)$ given by

$$T_1[x_0, x_1, \cdots, x_{4m+3}] = [-x_0, -x_1, \cdots, -x_{2m+1}, x_{2m+2}, \cdots, x_{4m+3}].$$

T_1 fixes the disjoint union $RP_1(2m+1)\cup RP_2(2m+1)$.

A Z_2^2-action Φ is defined by $(T_1\times T_1, S)$, where $S(x,y)=(y,x)$. The fixed data of Φ is $(RP_1(2m+1);\mu^{2m+2},\mu^{2m+2},\tau(RP_1(2m+1)))\cup(RP_2(2m+1);\nu^{2m+2},\nu^{2m+2},\tau(RP_2(2m+1)))$, where $\tau(RP_1(2m+1)$ means the tangent bundle. (M,Φ) satisfies the hypothesis of the theorem and (M,Φ) bounds equivariantly.

2 Preliminaries

Let Φ be a smooth Z_2^2-action on M with fixed data $(F_\Phi;\varepsilon_1,\varepsilon_2,\varepsilon_3)$. Each s-dimensional component of $(F_\Phi;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ can be considered as an element of $\mathscr{N}_s(\mathrm{BO}(s_1)\times\mathrm{BO}(s_2)\times\mathrm{BO}(s_3))$, the bordism of s-dimensional manifolds with a map into $\mathrm{BO}(s_1)\times\mathrm{BO}(s_2)\times\mathrm{BO}(s_3)$, where s_i is the dimension of ε_i over the component and $\mathrm{BO}(s_i)$ is the classifying space for s_i-dimensional vector bundles (this is the simultaneous bordism between lists of vector bundles: Two lists of vector bundles over closed n-dimensional manifolds, $(F^n;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ and $(V^n;\mu_1,\mu_2,\mu_3)$, are simultaneously bordant if there exists an $(n+1)$-dimensional manifold W^{n+1} with boundary $\partial(W^{n+1})=F^n\cup V^n$ (disjoint union) and a list of vector bundles over W^{n+1}, $(W^{n+1};\eta_1,\eta_2,\eta_3)$, so that η_ρ $(\rho=1,2,3)$ restricted to $F^n\cup V^n$ is equivalent to $\varepsilon_\rho\cup\mu_\rho$) (see [14]).

According to [19], the equivariant bordism class of (M,Φ) is determined by the simultaneous bordism class of $(F;\varepsilon_1,\varepsilon_2,\varepsilon_3)$. Also, if (M,Φ) has the fixed data $(F_\Phi;\varepsilon_1,\varepsilon_2,\varepsilon_3)$, which is simultaneously bordant to $(V;\mu_1,\mu_2,\mu_3)$, then there exists (N,Ψ) with fixed data $(V;\mu_1,\mu_2,\mu_3)$, hence (N,Ψ) is equivariantly bordant to (M,Φ). On the other hand, as in the case $k=1$, the simultaneous bordism class of $(F;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is determined by its characteristic numbers: write $W(F)=1+\omega_1+\omega_2+\cdots$ for the Stiefel-Whitney classes of the tangent bundle of F, and $W(\varepsilon_\rho)=1+v_1^\rho+v_2^\rho+\cdots$ for the Stiefel-Whitney classes of the bundles ε_ρ $(\rho=1,2,3)$. Then a characteristic number of $(F;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is an evaluation of the form $K[F]$, where K is a product of ω_i's and v_j^ρ's, $\rho\in\{1,2,3\}$, and $[F]$ is the fundamental Z_2-homology class of F; again, as in the case $k=1$, $K[F]$ must be understood as a sum $\sum_s K_s[F^s]$, where F^s is the union of the s-dimensional components of F, and K_s is the part of K with degree s. If every characteristic number of $(F;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is zero, then $(F;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ bounds simultaneously, hence (M,Φ) bounds equivariantly.

Now let $F_0\subset M$ be any component of F_{T_1}. Write $l=\dim(F_0)$, and denote by $F_0^i\subset F_0$, $0\le i<l$, the union of the i-dimensional components of F_Φ that are contained in F_0. Then, for each $0\le i<l$, one has that $\dim(\varepsilon_2)+\dim(\varepsilon_3)$ is equal to $\dim(M)-l$ over F_0^i. Let $r=\dim(M)-l$. Consider $RP(\varepsilon_1)\to F_0^i$, which is the real projective space bundle associated to $\varepsilon_1\to F_0^i$, and denote by $\xi\to RP(\varepsilon_1)$ line bundle of the double cover $S(\varepsilon_1)\to RP(\varepsilon_1)$, where $S(\varepsilon_1)$ is the sphere bundle of ε_1. Then, for each $0\le i<l$, one has the object

$$(RP(\varepsilon_1);\xi,\varepsilon_2\oplus(\varepsilon_3\otimes\xi)),$$

where $\varepsilon_1,\varepsilon_2$ and ε_3 are considered as bundles over F_0^i or the pull back over $RP(\varepsilon_1)$. This object represents an element in the bordism group $\mathscr{N}_{l-1}(\mathrm{BO}(1)\times\mathrm{BO}(r))$. For our purpose, let us recall the following lemmas.

Lemma 2.1 (see [14]) *The object* $\bigcup_{i=0}^{l-1}(RP(\varepsilon_1);\xi,\varepsilon_2\oplus(\varepsilon_3\otimes\xi))$ *bounds as an element of* $\mathcal{N}_{l-1}(\mathrm{BO}(1)\times\mathrm{BO}(r))$.

Remark 2.1 If $(F_\Phi;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is the fixed data of a Z_2^2-action, then the same is true for $(F_\Phi;\varepsilon_i,\varepsilon_j,\varepsilon_k)$, where (i,j,k) is any permutation of $(1,2,3)$. Then, in the above lemma, $\bigcup_{i=0}^{l-1}(RP(\varepsilon_1);\xi,\ \varepsilon_2\oplus(\varepsilon_3\otimes\xi))$ can be replaced by $\bigcup_{i=0}^{l-1}(RP(\varepsilon_i);\xi,\varepsilon_j\oplus(\varepsilon_k\otimes\xi))$ for any permutation (i,j,k) of $(1,2,3)$.

Lemma 2.2 (see [14]) *Let* (M,Φ) *be a* Z_2^2*-action with fixed data* $(F_\Phi;\varepsilon_1,\varepsilon_2,\varepsilon_3)$. *Suppose that* $V\subset M$ *is an* h*-dimensional component of* F_Φ. *Let* P *be the component of* F_{T_1} *that contains* V. *Suppose that* P *satisfies the following conditions*:

(1) $\dim(P)>2h$;

(2) V *is the unique component of* F_Φ *contained in* P.

Then $(V;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ *bounds simultaneously.*

3 Proof of Theorem

We review some general background information.

Let

$$W(RP_1(2m+1))=1+w_1+w_2+\cdots+w_{2m+1}=(1+\alpha)^{2m+2}$$

be the Stiefel-Whitney class of $RP_1(2m+1)$ and $\lambda_1\to RP_1(2m+1)$ the canonical real line bundle over $RP_1(2m+1)$. From the structure of Grothendieck ring $KO(RP_1(2m+1))$, one has that any bundle $\mu_i\to RP_1(2m+1)$ $(i=1,2,3)$ is stably equivalent to $l_i\lambda_1\to RP_1(2m+1)$ for some $l_i\geq 0$, which implies that

$$\begin{aligned}W(\mu_i)&=1+\mu_1^i+\mu_2^i+\cdots+\mu_{m_i}^i=(1+\alpha)^{l_i}\\&=1+\binom{l_i}{1}\alpha+\binom{l_i}{2}\alpha^2+\cdots+\binom{l_i}{m_i}\alpha^{m_i}\end{aligned}$$

is the Stiefel-Whitney class of $\mu_i\to RP_1(2m+1)$, $i=1,2,3$, where α is the generator of $H^1(RP_1(2m+1);Z_2)$, $m_i=\dim(\mu_i)$. If 2^a is the greatest power of 2 of the 2-adic expansion of $2m+1$, and $l_i\equiv p \pmod{2^{a+1}}$, then $(1+\alpha)^{l_i}=(1+\alpha)^p$. So we could assume $l_i\leq 2^{a+1}-1$. Throughout the paper, we use the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of b is a subset of the 2-adic expansion of a.

Let $c\in H^1(RP_1(\mu_1);Z_2)$ be the first Stiefel-Whitney class of the line bundle $\xi\to RP_1(\mu_1)$ for the double cover $S(\mu_1)\to RP(\mu_1)$. From [2, p.75] , one knows that the Stiefel-Whitney class of $RP_1(\mu_1)$ is

$$\begin{aligned}W(RP_1(\mu_1))&=(1+w_1+\cdots+w_{2m+1})\{(1+c)^{m_1}+\mu_1^1(1+c)^{m_1-1}+\cdots+\mu_{m_1}^1\}\\&=(1+\alpha)^{2m+2}\Big\{(1+c)^{m_1}+\binom{l_1}{1}\alpha(1+c)^{m_1-1}+\cdots+\binom{l_1}{m_1}\alpha^{m_1}\Big\}\end{aligned}$$

with a relation

$$c^{m_1}+\mu_1^1c^{m_1-1}+\mu_2^1c^{m_1-2}+\cdots+\mu_{m_1}^1=0.$$

The Stiefel-Whitney class of ξ is

$$W(\xi) = 1 + c,$$

and the Stiefel-Whitney class of the bundle $\mu_2 \oplus (\mu_3 \otimes \xi)$ is

$$\begin{aligned} W(\mu_2 \oplus (\mu_3 \otimes \xi)) &= (1 + \mu_1^2 + \cdots + \mu_{m_2}^2)\{(1+c)^{m_3} + \mu_1^3(1+c)^{m_3-1} + \cdots + \mu_{m_3}^3\} \\ &= (1+\alpha)^{l_2}\Big\{(1+c)^{m_3} + \binom{l_3}{1}\alpha(1+c)^{m_3-1} + \cdots + \binom{l_3}{m_3}\alpha^{m_3}\Big\}. \end{aligned}$$

On the component $RP_2(2m+1)$, we write

$$\begin{aligned} W(RP_2(2m+1)) &= 1 + v_1 + v_2 + \cdots + v_{2m+1} = (1+\beta)^{2m+2}, \\ W(\nu_i) &= 1 + \nu_1^i + \nu_2^i + \cdots + \nu_{n_i}^i = (1+\beta)^{t_i} \\ &= 1 + \binom{t_i}{1}\beta + \binom{t_i}{2}\beta^2 + \cdots + \binom{t_i}{n_i}\beta^{n_i}, \end{aligned}$$

where $\beta \in H^1(RP_2(2m+1); Z_2)$ is the generator, $n_i = \dim(\nu_i)$ for $i = 1, 2, 3$. If 2^a is the greatest power of 2 of the 2-adic expansion of $2m+1$, and $t_i \equiv q \pmod{2^{a+1}}$, then $(1+\beta)^{t_i} = (1+\beta)^q$. We can assume $t_i \le 2^{a+1} - 1$.

Also, we denote by $\lambda \to RP_2(\nu_1)$ the line bundle for the double cover $S(\nu_1) \to RP(\nu_1)$, and by

$$W(\lambda) = 1 + d$$

its Stiefel-Whitney class. One has

$$\begin{aligned} W(RP_2(\nu_1)) &= (1 + v_1 + v_2 + \cdots + v_{2m+1})\{(1+d)^{n_1} + \nu_1^1(1+d)^{n_1-1} + \cdots + \nu_{n_1}^1\} \\ &= (1+\beta)^{2m+2}\Big\{(1+d)^{n_1} + \binom{t_1}{1}\beta(1+d)^{n_1-1} + \cdots + \binom{t_1}{n_1}\beta^{n_1}\Big\} \end{aligned}$$

and

$$\begin{aligned} W(\nu_2 \oplus (\nu_3 \otimes \lambda)) &= (1 + \nu_1^2 + \nu_2^2 + \cdots + \nu_{n_2}^2)\{(1+d)^{n_3} + \nu_1^3(1+d)^{n_3-1} + \cdots + \nu_{n_3}^3\} \\ &= (1+\beta)^{t_2}\Big\{(1+d)^{n_3} + \binom{t_3}{1}\beta(1+d)^{n_3-1} + \cdots + \binom{t_3}{n_3}\beta^{n_3}\Big\}. \end{aligned}$$

For any integer r, we introduce the following characteristic classes which were initially introduced in [17],

$$W[r] = \frac{W(RP_1(\mu_1))}{(1+c)^{m_1-r}}$$

and

$$U[r] = \frac{W(\mu_2 \oplus (\mu_3 \otimes \xi))}{(1+c)^{m_3-r}}.$$

The classes $W[r]_t$ and $U[r]_l$ are polynomials in $W_i(RP_1(\mu_1)), c, W_j(\mu_2 \oplus (\mu_3 \otimes \xi))$, hence they can be used to give characteristic numbers. Also, these classes satisfy the following special properties:

$$W[r]_{2r-1} = w_{r-1}c^r + \text{terms with smaller } c \text{ powers},$$

$$W[r]_{2r} = w_r c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+1} = (w_{r+1} + \mu^1_{r+1})c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+2} = \mu^1_{r+1} c^{r+1} + \text{terms with smaller } c \text{ powers},$$

and in the same way,

$$U[r]_{2r-1} = \mu^2_{r-1} c^r + \text{terms with smaller } c \text{ powers},$$
$$U[r]_{2r} = \mu^2_r c^r + \text{terms with smaller } c \text{ powers},$$
$$U[r]_{2r+1} = (\mu^2_{r+1} + \mu^3_{r+1})c^r + \text{terms with smaller } c \text{ powers},$$
$$U[r]_{2r+2} = \mu^3_{r+1} c^{r+1} + \text{terms with smaller } c \text{ powers}.$$

Now we consider a Z_2^2-action (M, Φ) with fixed data

$$(RP_1(2m+1); \mu_1, \mu_2, \mu_3) \cup (RP_2(2m+1); \nu_1, \nu_2, \nu_3),$$

where at least two $\mu_i's$ have dimension greater than $2m+1$, and at least one ν_i has dimension greater than $2m+1$. In order to obtain our result, we only need to prove $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ $\cup(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$ bounds, that is, to show that every number of $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ is equal to every characteristic number of $(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$. For our purpose, we need the following notations: For a sequence $\omega = (j_1, j_2, \cdots, j_s)$ of natural numbers, one lets $|\omega| = j_1 + j_2 + \cdots + j_s$, and for $\mu = 1 + \mu_1 + \cdots + \mu_p$, let $\mu_\omega = \mu_{j_1}\mu_{j_2}\cdots\mu_{j_s}$ be the product of the classes μ_j.

Taking sequences $\omega = (j_1, j_2, \cdots, j_s)$ and $\omega_i = (j_1^i, j_2^i, \cdots, j_s^i)$ for $i = 1, 2, 3$, with $|\omega| + \sum_{i=1}^{3} |\omega_i| = 2m+1$, we consider the characteristic numbers

$$W(RP_1(2m+1))_\omega \prod_{i=1}^{3} W(\mu_i)_{\omega_i}[RP_1(2m+1)]$$

and

$$W(RP_2(2m+1))_\omega \prod_{i=1}^{3} W(\nu_i)_{\omega_i}[RP_2(2m+1)].$$

We will prove

$$\begin{aligned} &W(RP_1(2m+1))_\omega \prod_{i=1}^{3} W(\mu_i)_{\omega_i}[RP_1(2m+1)] \\ &= W(RP_2(2m+1))_\omega \prod_{i=1}^{3} W(\nu_i)_{\omega_i}[RP_2(2m+1)]. \end{aligned}$$

For $i = 1, 2, 3$, denote by P_i the component of F_{T_i} containing $RP_1(2m+1)$ and Q_i the component of F_{T_i} containing $RP_2(2m+1)$. Then either $P_i = Q_i$ or $P_i \cap Q_i = \varnothing$.

Suppose that $P_i \cap Q_i = \varnothing$ holds for some $i \in \{1, 2, 3\}$. Let us suppose first that this number is 2 or 3. Because of the hypothesis concerning the number of bundles with dimension greater than $2m+1$, there exists $i \in \{1, 2, 3\}$ such that $P_i \cap Q_i = \varnothing$ and $\dim(\mu_i) > 2m+1$. By applying Lemma 2.2 on the component $RP_1(2m+1) \subset P_i$, one concludes that $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ bounds simultaneously, thus it can be equivariantly removed to give a Z_2^2-action (N, Ψ), equivariantly bordant to (M, Φ), and with fixed data $(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$. Since at least one ν_j has $\dim(\nu_j) > 2m+1$, using Lemma 2.2 on the component $RP_2(2m+1) \subset Q_j$, one concludes that $(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$ bounds simultaneously. It follows that (N, Ψ) (and thus (M, Φ)) bounds equivariantly, and the theorem is proved.

In this way, we could suppose that there exists a unique $i \in \{1, 2, 3\}$ such that $P_i \cap Q_i = \varnothing$ or $P_i = Q_i$.

By making permutation on $i \in \{1, 2, 3\}$ if necessary, we can suppose with out loss that

$$P_1 = Q_1, \quad P_2 \cap Q_2 = \varnothing \text{ (or } P_2 = Q_2), \quad P_3 = Q_3, \quad m_1 > 2m+1,\ m_3 > 2m+1.$$

Since $P_1 = Q_1$ and $P_3 = Q_3$, one has $\dim(\nu_1) = m_1$ and $\dim(\nu_3) = m_3$. Now

$$2m + 1 + m_1 + m_2 + m_3 = 2m + 1 + m_1 + \dim(\nu_2) + m_3,$$

thus $\dim(\nu_2) = m_2$.

From Lemma 2.1 $\left(\text{with } F_0 = P_1 \text{ and } \bigcup_{i=0}^{l-1} F_0^i = RP_1(2m+1) \cup RP_2(2m+1)\right)$, one has that

$$(RP(\mu_1); \xi, \mu_2 \oplus (\mu_3 \otimes \xi))$$

is bordant to

$$(RP(\nu_1); \lambda, \nu_2 \oplus (\nu_3 \otimes \lambda))$$

in the bordism group

$$N_{2m+m_1}(\mathrm{BO}(1) \times \mathrm{BO}(m_2 + m_3)).$$

Then any class of dimension $2m + m_1$ given by a product of the class

$$W_i(RP(\mu_1)), c, W_j(\mu_2 \oplus (\mu_3 \otimes \xi))$$

evaluated on $[RP(\mu_1)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes

$$W_i(RP(\nu_1)), d, W_j(\nu_2 \oplus (\nu_3 \otimes \lambda))$$

evaluated on $[RP(\nu_1)]$. To find the value of such numbers, we have a formula of Conner (see [1, Lemma 3.1]),

$$\alpha^i c^j [RP(\mu_1)] = \begin{cases} \alpha^i \overline{W}_{j-m_1+1}(\mu_1)[RP_1(2m+1)], & j \geq m_1 - 1, \\ 0, & j < m_1 - 1, \end{cases}$$

where $i + j = 2m + m_1$ and $\overline{W}(\mu_1) = \frac{1}{W(\mu_1)}$ is the dual Stiefel-Whitney class of μ_1.

$$\beta^i d^j [RP(\nu_1)] = \begin{cases} \beta^i \overline{W}_{j-n_1+1}(\nu_1)[RP_2(2m+1)], & j \geq n_1 - 1, \\ 0, & j < n_1 - 1, \end{cases}$$

where $i + j = 2m + n_1$ and $\overline{W}(\nu_1) = \frac{1}{W(\nu_1)}$ is the dual Stiefel-Whitney class of ν_1. We apply the above equations to prove that

$$(RP_1(2m+1); \mu_1, \mu_2, \mu_3) \cup (RP_2(2m+1); \nu_1, \nu_2, \nu_3)$$

bounds.

We know $W(\mu_1) = (1+\alpha)^{l_1}, W(\mu_2) = (1+\alpha)^{l_2}, W(\mu_3) = (1+\alpha)^{l_3}$, $W(\nu_1) = (1+\beta)^{t_1}, W(\nu_2) = (1+\beta)^{t_2}, W(\nu_3) = (1+\beta)^{t_3}$. If t_1, t_2 and t_3 are even (or l_1, l_2 and l_3 are even), one concludes that $(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$ (or $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$) bounds simultaneously, thus it can be equivariantly removed to give a Z_2^2-action (N, Ψ), equivariantly bordant to (M, Φ), and with fixed data $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ (or $(RP_2(2m+1); \nu_1, \nu_2, \nu_3)$). Since at least two μ_i's (or one ν_i) have $\dim(\mu_i) > 2m+1$ (has $\dim(\nu_i) > 2m+1$), using Lemma 2.2 on the component $RP_1(2m+1) \subset P_i$ (or $RP_2(2m+1) \subset Q_i$), one concludes that $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ $((RP_2(2m+1); \nu_1, \nu_2, \nu_3))$ bounds simultaneously. It follows that (N, Ψ) (and thus (M, Φ)) bounds equivariantly, and the theorem is proved. So, we always suppose that not all l_1, l_2 and l_3 are even (or t_1, t_2 and t_3 are even). In this case, we only need to prove $l_1 = t_1, l_2 = t_2, l_3 = t_3$. The proof is divided into several cases.

Proposition 3.1 $l_1 = t_1$.

Proof If $l_1 \neq t_1$, we will prove that there does not exist Z_2^2-action (M, Φ). We divided the arguments into the following cases.

(1) l_1 is odd and t_1 is even. Then on $RP_1(2m+1)$,

$$W[0]_1 = \alpha.$$

On $RP_2(2m+1)$,

$$W[0]_1 = 0.$$

We form the class $(W[0]_1)^{2m+1} c^{m_1-1}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(W[0]_1)^{2m+1} d^{m_1-1}$, then

$$\begin{aligned}&(W[0]_1)^{2m+1} c^{m_1-1}[RP(\mu_1)] = \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)] = 1,\\&(W[0]_1)^{2m+1} d^{m_1-1}[RP(\nu_1)] = 0.\end{aligned}$$

In the same way, we can prove the case that l_1 is even and t_1 is odd.

This is a contradiction.

(2) l_1 is odd and t_1 is odd and $l_1 \neq t_1$. Let

$$2^s = \min\left\{2^x \,\middle|\, \binom{l_1}{2^x} \neq \binom{t_1}{2^x}\right\}.$$

We suppose $\binom{l_1}{2^s} = 1$, $\binom{t_1}{2^s} = 0$. Then on $RP_1(2m+1)$,

$$\begin{aligned}&W[0]_1 = \alpha,\\&W[2^s-1]_{2(2^s-1)+2} = \binom{l_1}{2^s}\alpha^{2^s}c^{2^s} + \text{terms with smaller } c \text{ powers}\end{aligned}$$

$$= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers.}$$

On $RP_2(2m+1)$,

$$W[0]_1 = \beta,$$
$$W[2^s-1]_{2(2^s-1)+2} = \binom{t_1}{2^s} \beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers}$$
$$= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers.}$$

We form the class $(W[0]_1)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-1-2^s}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(W[0]_1)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-1-2^s}$. Then

$$\begin{aligned}
&(W[0]_1)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-1-2^s}[RP(\mu_1)] \\
&= \alpha^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-1-2^s}[RP(\mu_1)] \\
&= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)] \\
&= 1, \\
&(W[0]_1)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-1-2^s}[RP(\nu_1)] = 0.
\end{aligned}$$

This is a contradiction.

(3) l_1 and t_1 are even and $l_1 \neq t_1$. Let

$$2^s = \min\left\{2^x \mid \binom{l_1}{2^x} \neq \binom{t_1}{2^x}\right\}.$$

We suppose $\binom{l_1}{2^s} = 1$, $\binom{t_1}{2^s} = 0$.

Since not all l_1, l_2 and l_3 are even, one of l_2 and l_3 must be odd. If l_2 is odd, then on $RP_1(2m+1)$,

$$W[2^s-1]_{2(2^s-1)+2} = \binom{l_1}{2^s} \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers}$$
$$= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers,}$$
$$U[1]_2 = \alpha c + \text{terms with smaller } c \text{ powers.}$$

On $RP_2(2m+1)$,

$$W[2^s-1]_{2(2^s-1)+2} = \binom{t_1}{2^s} \beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers}$$
$$= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers,}$$
$$U[1]_2 = \binom{t_2}{1} \beta d + \text{terms with smaller } d \text{ powers.}$$

We form the class $(U[1]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[1]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}$. Then

$$(U[1]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}[RP(\mu_1)]$$

$$= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)]$$

$$= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)]$$

$$= 1,$$

$$(U[1]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}[RP(\nu_1)] = 0.$$

This is a contradiction.

If l_3 is odd, on $RP_1(2m+1)$,

$$W[2^s-1]_{2(2^s-1)+2} = \binom{l_1}{2^s} \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers}$$
$$= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers},$$
$$U[0]_2 = \alpha c + \text{terms with smaller } c \text{ powers}.$$

On $RP_2(2m+1)$,

$$W[2^s-1]_{2(2^s-1)+2} = \binom{t_1}{2^s} \beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers}$$
$$= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers},$$
$$U[0]_2 = \binom{t_3}{1} \beta d + \text{terms with smaller } d \text{ powers}.$$

We form the class $(U[0]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[0]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}$. Then

$$(U[0]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}[RP(\mu_1)]$$

$$= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)]$$

$$= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)]$$

$$= 1,$$

$$(U[0]_2)^{2m+1-2^s} W[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}[RP(\nu_1)] = 0.$$

This is a contradiction.

Proposition 3.1 holds.

Proposition 3.2 $l_2 = t_2$.

Proof If $l_2 \neq t_2$, we will prove that there does not exist Z_2^2-action (M, Φ). We divided the arguments into the following cases.

(1) l_2 is odd and t_2 is even. Then on $RP_1(2m+1)$,

$$U[1]_2 = \alpha c + \text{terms with smaller } c \text{ powers}.$$

On $RP_2(2m+1)$,

$$U[1]_2 = 0\beta d + \text{terms with smaller } d \text{ powers}.$$

We form the class $(U[1]_2)^{2m+1}c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[1]_2)^{2m+1}d^{m_1-2m-2}$. Then

$$(U[1]_2)^{2m+1}c^{m_1-2m-2}[RP(\mu_1)] = \alpha^{2m+1}c^{m_1-1}[RP(\mu_1)] = 1,$$
$$(U[1]_2)^{2m+1}d^{m_1-2m-2}[RP(\nu_1)] = 0.$$

In the same way, we can prove the case that l_2 is even and t_2 is odd.

This is a contradiction.

(2) l_2 is odd and t_2 is odd and $l_2 \neq t_2$. Let

$$2^s = \min\left\{2^x \middle| \binom{l_2}{2^s} \neq \binom{t_2}{2^x}\right\}.$$

We suppose $\binom{l_2}{2^s} = 1$, $\binom{t_2}{2^s} = 0$.

On $RP_1(2m+1)$,

$$\begin{aligned} U[1]_2 &= \alpha c + \text{terms with smaller } c \text{ powers}, \\ U[2^s]_{2^{s+1}} &= \binom{l_2}{2^s}\alpha^{2^s}c^{2^s} + \text{terms with smaller } c \text{ powers} \\ &= \alpha^{2^s}c^{2^s} + \text{terms with smaller } c \text{ powers}. \end{aligned}$$

On $RP_2(2m+1)$,

$$\begin{aligned} U[1]_2 &= \beta d + \text{terms with smaller } d \text{ powers}, \\ U[2^s]_{2^{s+1}} &= \binom{t_2}{2^s}\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers} \\ &= 0\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers}. \end{aligned}$$

We form the class $U[2^s]_{2^{s+1}}(U[1]_2)^{2m+1-2^s}c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $U[2^s]_{2^{s+1}}(U[1]_2)^{2m+1-2^s}d^{m_1-2m-2}$. Then

$$\begin{aligned} &U[2^s]_{2^{s+1}}(U[1]_2)^{2m+1-2^s}c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2^s}c^{2^s}\alpha^{2m+1-2^s}c^{2m+1-2^s}c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1}c^{m_1-1}[RP(\mu_1)] \\ &= 1, \\ &U[2^s]_{2^{s+1}}(U[1]_2)^{2m+1-2^s}d^{m_1-2m-2}[RP(\nu_1)] = 0. \end{aligned}$$

This is a contradiction.

(3) l_2 and t_2 are even and $l_2 \neq t_2$. Let

$$2^s = \min\left\{2^x \middle| \binom{l_2}{2^x} \neq \binom{t_2}{2^x}\right\}.$$

We suppose $\binom{l_2}{2^s} = 1$, $\binom{t_2}{2^s} = 0$.

Since not all l_1, l_2 and l_3 are even, one of l_1 and l_3 must be odd. If l_1 is odd, then on $RP_1(2m+1)$,

$$\begin{aligned} W[0]_2 &= \alpha c + \text{terms with smaller } c \text{ powers}, \\ U[2^s]_{2^{s+1}} &= \binom{l_2}{2^s} \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers} \\ &= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers}. \end{aligned}$$

On $RP_2(2m+1)$, since $l_1 = t_1$, so

$$\begin{aligned} W[0]_2 &= \beta d + \text{terms with smaller } d \text{ powers}, \\ U[2^s]_{2^{s+1}} &= \binom{t_2}{2^s} \beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers} \\ &= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers}. \end{aligned}$$

We form the class $(W[0]_2)^{2m+1-2^s} U[2^s]_{2^{s+1}} c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(W[0]_2)^{2m+1-2^s} U[2^s]_{2^{s+1}} d^{m_1-2m-2}$. Then

$$\begin{aligned} &(W[0]_2)^{2m+1-2^s} U[2^s]_{2^{s+1}} c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1} c^{m_1-1}[RP(2m+1)] \\ &= 1, \\ &(W[0]_2)^{2m+1-2^s} U[2^s]_{2^{s+1}} d^{m_1-2m-2}[RP(\nu_1)] = 0. \end{aligned}$$

This is a contradiction.

If l_3 is odd, on $RP_1(2m+1)$,

$$\begin{aligned} U[0]_2 &= \alpha c + \text{terms with smaller } c \text{ powers}, \\ U[2^s]_{2^{s+1}} &= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers}. \end{aligned}$$

On $RP_2(2m+1)$,

$$\begin{aligned} U[0]_2 &= \binom{t_3}{1} \beta d + \text{terms with smaller } d \text{ powers}, \\ U[2^s]_{2^{s+1}} &= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers}. \end{aligned}$$

We form the class $(U[0]_2)^{2m+1-2^s} U[2^s]_{2^s+1} c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[0]_2)^{2m+1-2^s} U[2^s]_{2^s+1} d^{m_1-2m-2}$. Then

$$\begin{aligned} &(U[0]_2)^{2m+1-2^s} U[2^s]_{2^s+1} c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)] \end{aligned}$$

$$= 1,$$
$$(U[0]_2)^{2m+1-2^s}U[2^s]_{2^s+1}d^{m_1-2m-2}[RP(\nu_1)] = 0.$$

This is a contradiction.

Proposition 3.2 holds.

Proposition 3.3 $l_3 = t_3$.

Proof If $l_3 \neq t_3$, we will prove there does not exist Z_2^2-action (M,Φ). We divided the arguments into the following cases.

(1) l_3 is odd and t_3 is even. Then on $RP_1(2m+1)$,

$$U[0]_2 = \alpha c + \text{terms with smaller } c \text{ powers.}$$

On $RP_2(2m+1)$,

$$U[0]_2 = 0\beta d + \text{terms with smaller } d \text{ powers.}$$

We form the class $(U[0]_2)^{2m+1}c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[0]_2)^{2m+1}d^{m_1-2m-2}$. Then

$$(U[0]_2)^{2m+1}c^{m_1-2m-2}[RP(\mu_1)] = \alpha^{2m+1}c^{m_1-1}[RP(\mu_1)] = 1,$$
$$(U[0]_2)^{2m+1}d^{m_1-2m-2}[RP(\nu_1)] = 0.$$

In the same way, we can prove the case that l_3 is even and t_3 is odd.

This is a contradiction.

(2) l_3 is odd and t_3 is odd and $l_3 \neq t_3$. Let

$$2^s = \min\left\{2^x \,\middle|\, \binom{l_3}{2^x} \neq \binom{t_3}{2^x}\right\}.$$

We suppose $\binom{l_3}{2^s} = 1$, $\binom{t_3}{2^s} = 0$. Then on $RP_1(2m+1)$,

$$U[0]_2 = \alpha c + \text{terms with smaller } c \text{ powers,}$$
$$U[2^s-1]_{2(2^s-1)+2} = \binom{l_3}{2^s}\alpha^{2^s}c^{2^s} + \text{terms with smaller } c \text{ powers}$$
$$= \alpha^{2^s}c^{2^s} + \text{terms with smaller } c \text{ powers.}$$

On $RP_2(2m+1)$,

$$U[0]_2 = \beta d + \text{terms with smaller } d \text{ powers,}$$
$$U[2^s-1]_{2(2^s-1)+2} = \binom{t_3}{2^s}\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers}$$
$$= 0\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers.}$$

We form the class $(U[0]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[0]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}d^{m_1-2m-2}$. Then

$$(U[0]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}c^{m_1-2m-2}[RP(\mu_1)]$$

$$
\begin{aligned}
&= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)] \\
&= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)] \\
&= 1, \\
&\quad (U[0]_2)^{2m+1-2^s} U[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}[RP(\nu_1)] = 0.
\end{aligned}
$$

This is a contradiction.

(3) l_3 and t_3 are even and $l_3 \neq t_3$. Let

$$
2^s = \min\left\{ 2^x \middle| \binom{l_3}{2^x} \neq \binom{t_3}{2^x} \right\}.
$$

We suppose $\binom{l_3}{2^s} = 1$, $\binom{t_3}{2^s} = 0$.

Since not all l_1, l_2 and l_3 are even, one of l_1 and l_2 must be odd. If l_1 is odd, then on $RP_1(2m+1)$,

$$
\begin{aligned}
W[0]_2 &= \alpha c + \text{terms with smaller } c \text{ powers}, \\
U[2^s-1]_{2(2^s-1)+2} &= \binom{l_3}{2^s} \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers} \\
&= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers.}
\end{aligned}
$$

On $RP_2(2m+1)$,

$$
\begin{aligned}
W[0]_2 &= \beta d + \text{terms with smaller } c \text{ powers}, \\
U[2^s-1]_{2(2^s-1)+2} &= \binom{t_3}{2^s} \beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers} \\
&= 0\beta^{2^s} d^{2^s} + \text{terms with smaller } d \text{ powers.}
\end{aligned}
$$

We form the class $(W[0]_2)^{2m+1-2^s} U[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(W[0]_2)^{2m+1-2^s} U[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}$. Then

$$
\begin{aligned}
&\quad (W[0]_2)^{2m+1-2^s} U[2^s-1]_{2(2^s-1)+2} c^{m_1-2m-2}[RP(\mu_1)] \\
&= \alpha^{2m+1-2^s} c^{2m+1-2^s} \alpha^{2^s} c^{2^s} c^{m_1-2m-2}[RP(\mu_1)] \\
&= \alpha^{2m+1} c^{m_1-1}[RP(\mu_1)] \\
&= 1, \\
&\quad (W[0]_2)^{2m+1-2^s} U[2^s-1]_{2(2^s-1)+2} d^{m_1-2m-2}[RP(\nu_1)] = 0.
\end{aligned}
$$

This is a contradiction.

If l_2 is odd, on $RP_1(2m+1)$,

$$
\begin{aligned}
U[1]_2 &= \alpha c + \text{terms with smaller } c \text{ powers}, \\
U[2^s-1]_{2(2^s-1)+2} &= \binom{l_3}{2^s} \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers} \\
&= \alpha^{2^s} c^{2^s} + \text{terms with smaller } c \text{ powers.}
\end{aligned}
$$

On $RP_2(2m+1)$, since $l_2 = t_2$, so

$$\begin{aligned} U[1]_2 &= \beta d + \text{terms with smaller } d \text{ powers}, \\ U[2^s-1]_{2(2^s-1)+2} &= \binom{t_3}{2^s}\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers} \\ &= 0\beta^{2^s}d^{2^s} + \text{terms with smaller } d \text{ powers}. \end{aligned}$$

We form the class $(U[1]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}c^{m_1-2m-2}$ on $RP_1(2m+1)$, and the corresponding class on $RP_2(2m+1)$ is $(U[1]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}d^{m_1-2m-2}$. Then

$$\begin{aligned} &(U[1]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1-2^s}c^{2m+1-2^s}\alpha^{2^s}c^{2^s}c^{m_1-2m-2}[RP(\mu_1)] \\ &= \alpha^{2m+1}c^{m_1-1}[RP(\mu_1)] \\ &= 1, \\ &(U[1]_2)^{2m+1-2^s}U[2^s-1]_{2(2^s-1)+2}d^{m_1-2m-2}[RP(\nu_1)] = 0. \end{aligned}$$

This is a contradiction.

Proposition 3.3 holds.

From above discussions, we know that $l_1 = t_1, l_2 = t_2$ and $l_3 = t_3$. If we take any sequences $\omega = (j_1, j_2, \cdots, j_s)$ and $\omega_i = (j_1^i, j_2^i, \cdots, j_s^i)$ for $i = 1, 2, 3$ with $|\omega| + \sum_{i=1}^{3}|\omega_i| = 2m+1$, we have

$$\begin{aligned} &W(RP_1(2m+1))\omega\prod_{i=1}^{3}W(\mu_i)_{\omega_i}[RP_1(2m+1)] \\ &= W(RP_2(2m+1))\omega\prod_{i=1}^{3}W(\mu_i)_{\omega_i}[RP_2(2m+1)]. \end{aligned}$$

That is,

$$\begin{aligned} &W(RP_1(2m+1))\omega\prod_{i=1}^{3}W(\mu_i)_{\omega_i}[RP_1(2m+1)] \\ &+ W(RP_2(2m+1))\omega\prod_{i=1}^{3}W(\mu_i)_{\omega_i}[RP_2(2m+1)] = 0. \end{aligned}$$

We conclude that every characteristic number of $(RP_1(2m+1); \mu_1, \mu_2, \mu_3)$ is equal to the characteristic number of $(RP_2(2m+1), \nu_1, \nu_2, \nu_3)$. By [19] $(RP_1(2m+1), \mu_1, \mu_2, \mu_3) \cup (RP_2(2m+1), \nu_1, \nu_2, \nu_3)$ bounds simultaneously, (M, Φ) bounds equivariantly, and the theorem is proved.

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