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On Descriptions of Products of Simplices*

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Abstract The authors give several new criteria to judge whether a simple convex polytope in a Euclidean space is combinatorially equivalent to a product of simplices. These criteria are mixtures of combinatorial, geometrical and topological conditions that are inspired by the ideas from toric topology. In addition, they give a shorter proof of a well known criterion on this subject.

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1 Background

A convex polytope P is the convex hull of a finite set of points in a Euclidean space $\mathbb{R}^d$. The dimension of P is the dimension of the affine hull of these points. Any codimension-one face of P is called a facet of P. We call an n-dimensional convex polytope P simple if each vertex of P is the intersection of exactly n different facets of P. Two convex polytopes are combinatorially equivalent if their face lattices are isomorphic. Topologically, combinatorial equivalence corresponds to the existence of a (piecewise linear) homeomorphism between the two polytopes that restricts to homeomorphisms between their facets, and hence all their faces (see [20, Chapter 2.2]).

If $P_1 \subset \mathbb{R}^{n_1}$ and $P_2 \subset \mathbb{R}^{n_2}$ are two convex polytopes, then their product $P_1 \times P_2$ is a convex polytope in $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Products of simplices are special type of simple polytopes with very delicate combinatorial structures. They play an important role in Coxeter's famous work [10] on the discrete reflection groups in Euclidean spaces and also appear in many different researches in combinatorics (see [1, 12–13]). In this paper, we give several new criteria to judge whether a convex polytope is combinatorially equivalent to a product of simplices (Theorems 2.2–2.3) and at the same time, list some known ones (Proposition 2.1). Some of these criteria are purely combinatorial, while others are phrased in geometrical or topological terms. Since some of our new criteria are inspired from the ideas in toric topology, we first explain some basic constructions and facts in toric topology that are relevant to our discussion.

An abstract simplicial complex on a set $[m] = \{v_1, \cdots, v_m\}$ is a collection K of subsets

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$\sigma \subseteq [m]$ such that if $\sigma \in K$, then any subset of σ also belongs to K. We always assume that the empty set belongs to K and refer to $\sigma \in K$ as a simplex of K. In particular, one-element simplices are called vertices of K. If K contains all one-element subsets of $[m]$, then we say that K is a simplicial complex on the vertex set $[m]$. To avoid ambiguity in our argument, we also use $V(K)$ and $V(\sigma)$ to refer to the vertex sets of K and any simplex σ in K, respectively. For any subset $\omega \subseteq [m]$, we call $K|_\omega = \{\sigma \in K \,|\, \sigma \subseteq \omega\}$ the full subcomplex of K by restricting to ω.

Any abstract simplicial complex K admits a geometric realization in some Euclidean space. Also sometimes we use K to denote its geometric realization when the meaning is clear in the context.

Given a finite abstract simplicial complex K on a set $[m]$ and a pair of spaces (X, A) with $A \subset X$, we can construct a topological space $(X, A)^K$ by:

$$(X,A)^K = \bigcup_{\sigma \in K} (X,A)^\sigma, \quad \text{where } (X,A)^\sigma = \prod_{v_j \in \sigma} X \times \prod_{v_j \notin \sigma} A. \tag{1.1}$$

Here $\prod$ means Cartesian product. So $(X, A)^K$ is a subspace of the Cartesian product of m copies of X. It is called the polyhedral product or the generalized moment-angle complex of K and (X, A). In particular, $\mathcal{Z}_K = (D^2, S^1)^K$ and $\mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K$ are called the moment-angle complex and the real moment-angle complex of K, respectively (see [4, Section 4.1]). The natural actions of $(\mathbb{Z}_2)^m$ on $(D^1)^m$ and $(S^1)^m$ on $(D^2)^m$ induce canonical actions of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_K$ and $(S^1)^m$ on $\mathcal{Z}_K$, respectively.

When K is the boundary of the dual of a simple convex polytope P, the $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ are closed manifolds, also denoted by $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ respectively. In this case, $\mathcal{Z}_P$ and $\mathbb{R}\mathcal{Z}_P$ are called the moment-angle manifold and the real moment-angle manifold of P, respectively (see [3, Section 6.1]). These manifolds can be constructed in another way as described below (see [11, Construction 4.1]).

Let P^n be an n-dimensional simple convex polytope. Let $\mathcal{F}(P^n) = \{F_1, \cdots, F_m\}$ be the set of facets of P^n. Let $\{e_1, \cdots, e_m\}$ be a basis of $(\mathbb{Z}_2)^m$ and define a map $\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^m$ by $\lambda(F_i) = e_i$. Then we can construct a space

$$M(P^n, \lambda) := P^n \times (\mathbb{Z}_2)^m / \sim, \tag{1.2}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_p^\lambda$, where G_p^λ is the subgroup of $(\mathbb{Z}_2)^m$ generated by the set $\{\lambda(F_i) \,|\, p \in F_i\}$. Let $\pi_\lambda : M(P^n, \lambda) \to P^n$ be the quotient map. One can show that $\mathbb{R}\mathcal{Z}_{P^n}$ is homeomorphic to $M(P^n, \lambda)$ and the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_{P^n}$ can be written on $M(P^n, \lambda)$ as:

$$g' \cdot [(p, g)] = [(p, g' + g)], \quad p \in P^n, \; g, g' \in (\mathbb{Z}_2)^m. \tag{1.3}$$

The moment-angle manifold $\mathcal{Z}_{P^n}$ can be similarly constructed from P^n and a map $\Lambda : \mathcal{F}(P^n) \to \mathbb{Z}^m$, where $\{\Lambda(F_1), \cdots, \Lambda(F_m)\}$ is a unimodular basis of $\mathbb{Z}^m$. Indeed, if we identify the torus $(S^1)^m = \mathbb{R}^m / Z^m$, then we have

$$\mathcal{Z}_{P^n} \cong P^n \times (S^1)^m / \sim, \tag{1.4}$$

where $(p,g) \sim (p',g')$ if and only if $p = p'$ and $g^{-1}g' \in T_p^\lambda$, where T_p^λ is the subtorus of $(S^1)^m$ determined by the linear subspace of $\mathbb{R}^m$ spanned by the set $\{\Lambda(F_i) \,|\, p \in F_i\}$.

In addition, $\mathbb{R}\mathcal{Z}_{P^n}$ and $\mathcal{Z}_{P^n}$ are smooth manifolds. In fact, there exists an equivariant smooth structure on $\mathbb{R}\mathcal{Z}_{P^n}$ (or $\mathcal{Z}_{P^n}$) with respect to the canonical $(\mathbb{Z}_2)^m$-action (or $(S^1)^m$-action). The reader is referred to [3, Chapter 6] or [4, Chapter 6] for the discussion of smooth structures on (real) moment-angle manifolds. Moreover, for any proper face f of P^n, $\pi_\lambda^{-1}(f)$ is an embedded closed smooth submanifold of $\mathbb{R}\mathcal{Z}_{P^n}$ which is the fixed point set of the subgroup of $(\mathbb{Z}_2)^m$ generated by $\{\lambda(F_i) \,|\, f \in F_i\}$ under the canonical $(\mathbb{Z}_2)^m$-action.

2 Descriptions of Products of Simplices

For any $k \in \mathbb{N}$, let Δ^k denote the standard k-dimensional simplex, which is

$$\Delta^k = \{(x_1, \cdots, x_k, x_{k+1}) \in \mathbb{R}^{k+1} \,|\, x_1 + \cdots + x_{k+1} = 1, x_1, \cdots, x_{k+1} \geq 0\}.$$

For any $n_1, \cdots, n_q \in \mathbb{N}$, consider $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$ as a product of $\Delta^{n_1}, \cdots, \Delta^{n_q}$ in the Cartesian product $\mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_q+1}$.

We first list some descriptions of products of simplices appearing in Wiemeler's paper [19].

Theorem 2.1 (see [19]) *Let P^n be an n-dimensional simple convex polytope with m facets, $n \geq 3$. Then the following statements are equivalent*:

(a) *P^n is combinatorially equivalent to a product of simplices.*

(b) *Any 2-dimensional face of P^n is either a 3-gon or a 4-gon.*

(c) *There exists a quasitoric manifold M^{2n} over P^n which admits a nonnegatively curved Riemannian metric that is invariant under the canonical $(S^1)^n$-action on M^{2n}.*

A quasitoric manifold M^{2n} over P^n is the quotient space of $\mathcal{Z}_{P^n}$ under a free action of a rank $m-n$ toral subgroup of $(S^1)^m$ (see [11]). There is a canonical $(S^1)^n$-action on M^{2n} induced from the canonical action of $(S^1)^m$ on $\mathcal{Z}_{P^n}$, which makes M^n a torus manifold (see [14]).

The equivalence of Theorem 2.1 (a) and (b) is a corollary of [19, Proposition 4.5]. The equivalence of Theorem 2.1 (a) and (c) is a corollary of [19, Lemma 4.2]. Note that Theorem 2.1(b) is a particularly useful description of products of simplices. Indeed, the proofs of many other descriptions of products of simplices in this paper boil down to this one first. But the proof of [19, Proposition 4.5] is a little long and not particularly easy to follow. We will give a shorter proof of the equivalence of Theorem 2.1 (a) and (b) in the appendix to make our paper more self-contained.

Remark 2.1 The equivalence of Theorem 2.1 (a) and (b) also implies that a simple convex polytope is combinatorially equivalent to a product of simplices if and only if every facet of the polytope is combinatorially equivalent to a product of simplices. In fact this statement appeared in [10, Lemma 2.7] where a product of simplices is called a "simplicial prism". But the proof of [10, Lemma 2.7] in [10] is a bit vague in the final step.

Next, we give more descriptions of products of simplices from combinatorial and topological viewpoints. For convenience, let us introduce some notations first.

• For any topological space X and any field $\mathbf{k}$, let

$$\mathrm{hrk}(X;\mathbf{k}) = \sum_{i=0}^{\infty} \dim_{\mathbf{k}} H^i(X;\mathbf{k}),$$

where $H^i(X;\mathbf{k})$ is the singular cohomology of X with coefficient in $\mathbf{k}$.

• For any vertex v in a simplicial complex K, we denote by $\mathrm{link}_K v$ the link of v in K. We denote a simplex spanned by vertices $v_0, v_1, \cdots, v_p$ in K by $[v_0, v_1, \cdots, v_p]$ and its boundary complex by $\partial[v_0, v_1, \cdots, v_p]$.

In addition, for a simplicial complex K on the vertex set $[m] = \{v_1, \cdots, v_m\}$, we can define a new simplicial complex $L(K)$ from K, called the double of K, where $L(K)$ is a simplicial complex on the vertex set $[2m] = \{v_1, v_1', \cdots, v_m, v_m'\}$ determined by the following condition: $\omega \subset [2m]$ is a minimal (by inclusion) missing simplex of $L(K)$ if and only if ω is of the form $\{v_{i_1}, v_{i_1}', \cdots, v_{i_k}, v_{i_k}'\}$, where $\{v_{i_1}, \cdots, v_{i_k}\}$ is a minimal missing simplex of K. Note that any minimal missing simplex in $L(K)$ must have even number of vertices. The double of K is a special case of iterated simplicial wedge construction (also called simplicial wedge J-construction). Indeed, by the notation introduced in [2], $L(K) = K(2, \cdots, 2)$.

The following are some basic facts about $L(K)$ (see [18–19]).

• $\dim(L(K)) = m + \dim(K)$ (see [18, Lemma 1.2]).

• $L(K_1 * K_2) = L(K_1) * L(K_2)$ (here $*$ is the join of two simplicial complexes).

• If $K = \partial P^*$, where P^* is the simplicial polytope dual to a simple convex polytope P, then $L(K) = \partial L(P)^*$, where $L(P)$ is a simple convex polytope called the double of P (see [17] for the construction of $L(P)$).

• $L(\partial\Delta^k) = \partial\Delta^{2k+1}$.

The following are some easy or well known facts on products of simplices. We list them here and give a simple proof for reference.

Proposition 2.1 *Let P be an n-dimensional simple polytope with m facets and let K be the boundary of the simplicial polytope dual to P. Then the following statements are all equivalent*:

(a) *P is combinatorially equivalent to a product of simplices.*

(b) *K is simplicially isomorphic to $\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ for some $n_1 \cdots, n_q \in \mathbb{N}$.*

(c) *The vertex sets of all the minimal missing faces of K form a partition of $V(K)$.*

(d) *$L(K)$ is simplicially isomorphic to $\partial\Delta^{l_1} * \cdots * \partial\Delta^{l_q}$ for some $l_1 \cdots, l_q \in \mathbb{N}$.*

(e) *There exists some field $\mathbf{k}$ so that $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_K;\mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_P;\mathbf{k}) = 2^{m-n}$.*

(f) *There exists some field $\mathbf{k}$ so that $\mathrm{hrk}(\mathcal{Z}_K;\mathbf{k}) = 2^{m-\dim(K)-1}$, or equivalently $\mathrm{hrk}(\mathcal{Z}_P;\mathbf{k}) = 2^{m-n}$.*

Proof The equivalences of (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) are easy to see.

(b) $\Rightarrow$ (d) If $K = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$, then

$$L(K) = L(\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}) = L(\partial\Delta^{n_1}) * \cdots * L(\partial\Delta^{n_q}) = \partial\Delta^{2n_1+1} * \cdots * \partial\Delta^{2n_q+1}.$$

(d) $\Rightarrow$ (c) Suppose $L(K) = \partial\Delta^{l_1} * \cdots * \partial\Delta^{l_q}$. Notice that for each $1 \le j \le q$, Δ^{l_j} is a minimal missing simplex of $L(K)$. So Δ^{l_j} must have even number of vertices, which implies that l_j is an odd integer. Then by (b) $\Leftrightarrow$ (c), the vertex sets of all the minimal missing faces of $L(K)$ form a partition of $V(L(K))$. This forces the vertex sets of all the minimal missing faces of K to form a partition of $V(K)$ as well, which is (c).

(a) $\Rightarrow$ (e) and (f) If $P = \Delta^{n_1} \times \cdots \times \Delta^{n_q}$, $n_1 + \cdots + n_q = n$, then

$$\mathcal{Z}_P = S^{2n_1+1} \times \cdots \times S^{2n_q+1}, \quad \mathbb{R}\mathcal{Z}_P = S^{n_1} \times \cdots \times S^{n_q}.$$

The number of facets of P is $m = n + q$. It is clear that for any field $\mathbf{k}$,

$$\mathrm{hrk}(\mathcal{Z}_P; \mathbf{k}) = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_P; \mathbf{k}) = 2^q = 2^{m-n}.$$

(e) $\Rightarrow$ (a) For any vertex v of K, let m_v be the number of vertices in $\mathrm{link}_K v$. According to the proof of [18, Theorem 3.2] (note that the argument there works for any coefficient), there is a subspace X of $\mathbb{R}\mathcal{Z}_K$ so that

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_P; \mathbf{k}) = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_K; \mathbf{k}) \ge \mathrm{hrk}(X; \mathbf{k}),$$

where X is the disjoint union of 2^{m-m_v-1} copies of $\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v}$. So we have

$$2^{m-n} = \mathrm{hrk}(\mathbb{R}\mathcal{Z}_K; \mathbf{k}) \ge 2^{m-m_v-1}\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v}; \mathbf{k}).$$

Then $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v}; \mathbf{k}) \le 2^{m_v-n+1}$. On the other hand, [18, Theorem 3.2] tells us that $\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v}; \mathbf{k}) \ge 2^{m_v-n+1}$ (since $\dim(\mathrm{link}_K v) = n-2$). So we obtain

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v}; \mathbf{k}) = 2^{m_v-n+1}.$$

Note that if v is the vertex corresponding to a facet F of P, then $\mathbb{R}\mathcal{Z}_{\mathrm{link}_K v} = \mathbb{R}\mathcal{Z}_F$. Therefore, we have shown that if the condition (e) holds for P, it should hold for any facet of P as well.

By iterating the above argument, we deduce that the condition (e) holds for all the two dimensional faces of P. It is easy to show that the real moment-angle manifold of a k-gon is a closed connected orientable surface with genus $1 + (k-4)2^{k-3}$ (see [4, Proposition 4.1.8]). So any 2-dimensional face of P is either a 3-gon or a 4-gon. Then by Theorem 2.1(b), the polytope P is combinatorially equivalent to a product of simplices.

(f) $\Rightarrow$ (a) First of all, [18, Lemma 2.2] says that there is a homeomorphism $\mathcal{Z}_K \cong \mathbb{R}\mathcal{Z}_{L(K)}$. Since K has m vertices, $\dim(L(K)) = m + \dim(K) = m + n - 1$. If $\mathrm{hrk}(\mathcal{Z}_K; \mathbf{k}) = 2^{m-n}$, then we have

$$\mathrm{hrk}(\mathbb{R}\mathcal{Z}_{L(K)}; \mathbf{k}) = 2^{m-n} = 2^{2m-(m+n-1)-1} = 2^{2m-\dim(L(K))-1}.$$

So (e) holds for $L(K)$. Since we have already shown (e) $\Rightarrow$ (a) $\Leftrightarrow$ (b), $L(K)$ is simplicially isomorphic to $\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ for some $n_1 \cdots, n_q \in \mathbb{N}$. Then we finish the proof by the equivalence of (d) and (a).

Remark 2.2 The equivalences of (b), (e) and (f) in Proposition 2.1 are stated in [4, Section 4.8] as an exercise.

Moreover, we can judge whether a simple polytope P is combinatorially equivalent to a specific product of simplices via some combinatorial invariants called bigraded Betti numbers, which are derived from the Stanley-Reisner ring of P (see [4, Section 3.2] for the definition). Indeed, it is shown in [9] that a simple polytope P is combinatorially equivalent to $\Delta^{n_1}\times\cdots\times\Delta^{n_q}$ if and only if P has the same bigraded Betti numbers as $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$. Simple polytopes with this kind of property are called combinatorially rigid (see [6, 8]).

Next, we give a new combinatorial criterion to judge whether a simple polytope is combinatorialy equivalent to a product of simplices.

Theorem 2.2 *Let K be the boundary of the simplicial polytope dual to a simple polytope P. Then P is combinatorially equivalent to a product of simplices if and only if the following conditions hold for K: For any maximal simplex σ in K and any vertex v of σ, the full subcomplex of K by restricting to $V(K) - V(\sigma)$ is a simplex of K, denoted by ξ_σ, and moreover the intersection of ξ_σ and* $\mathrm{link}_K v$ *is a simplex (could be empty) as well.*

Proof Suppose that P is a product of simplices. Then $K = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ for some $n_1, \cdots, n_q \in \mathbb{N}$. Denote the vertices of $\partial\Delta^{n_k}$ by $v_0^k, v_1^k, \cdots, v_{n_k}^k$ for each $k = 1, \cdots, q$. Then for a maximal simplex σ in K, there exists $0 \le l_k \le n_k$, $k = 1, \cdots, q$, so that

$$\sigma = [v_0^1, \cdots, \widehat{v}_{l_1}^1, \cdots, v_{n_1}^1] * [v_0^2, \cdots, \widehat{v}_{l_2}^2, \cdots, v_{n_2}^2] * \cdots * [v_0^q, \cdots, \widehat{v}_{l_q}^q, \cdots, v_{n_q}^q],$$

where $[v_0^k, \cdots, \widehat{v}_{l_k}^k, \cdots, v_{n_k}^k]$ is the simplex spanned by all the vertices of $\partial\Delta^{n_k}$ except $v_{l_k}^k$ for each $1 \le k \le q$. It is easy to see that the full subcomplex of K by restricting to $V(K) - V(\sigma)$ is just the simplex $[v_{l_1}^1, v_{l_2}^2, \cdots, v_{l_q}^q] = v_{l_1}^1 * v_{l_2}^2 * \cdots * v_{l_q}^q$. All the vertices of σ are $\{v_{i_k}^k\,;\, 0 \le i_k \ne l_k \le n_k, 1 \le k \le q\}$. And we have

$$\mathrm{link}_K\, v_{i_k}^k = \partial\Delta^{n_1} * \cdots * \partial[v_0^k, \cdots, \widehat{v}_{i_k}^k, \cdots, v_{n_k}^k] * \cdots * \partial\Delta^{n_q}.$$

Note that when $n_k = 1$, $\partial[v_0^k, \cdots, \widehat{v}_{i_k}^k, \cdots, v_{n_k}^k]$ is empty. Then the intersection of $[v_{l_1}^1, v_{l_2}^2, \cdots, v_{l_q}^q]$ and $\mathrm{link}_K\, v_i^k$ is exactly the simplex $[v_{l_1}^1, v_{l_2}^2, \cdots, v_{l_q}^q]$ if $n_k > 1$, and is $[v_{l_1}^1, \cdots, \widehat{v}_{l_k}^k, \cdots, v_{l_q}^q]$ if $n_k = 1$. The necessity of these conditions is proved.

For the sufficiency, we first show that if these conditions hold for K, then they also hold for the link of any vertex of K. When $\dim(K) \le 1$, the theorem is obviously true. So we assume $\dim(K) \ge 2$ below. Let u be an arbitrary vertex of K. Let σ be a maximal simplex of K containing u and let v be an arbitrary vertex of σ different from u. By our assumption, the intersection $\xi_\sigma \cap \mathrm{link}_K\, u$ and $\xi_\sigma \cap \mathrm{link}_K\, v$ are both simplices. Let τ be the simplex with $V(\tau) = V(\sigma) - \{u\}$. Then τ is a maximal simplex in $\mathrm{link}_K\, u$. Since $V(\xi_\sigma) = V(K) - V(\sigma)$, we have

$$V(\mathrm{link}_K\, u) - V(\tau) = V(\xi_\sigma) \cap V(\mathrm{link}_K\, u).$$

Since $\xi_\sigma \cap \mathrm{link}_K\, u$ is a simplex, the full subcomplex of $\mathrm{link}_K\, u$ by restricting to $V(\mathrm{link}_K\, u) - V(\tau)$ must agree with $\xi_\sigma \cap \mathrm{link}_K\, u$. Moreover, since v could be any vertex of τ, we need to show that the intersection of the simplex $\xi_\sigma \cap \mathrm{link}_K\, u$ with $\mathrm{link}_{\mathrm{link}_K\, u}\, v$ is also a simplex. Observe that $\mathrm{link}_{\mathrm{link}_K\, u}\, v = \mathrm{link}_K\, u \cap \mathrm{link}_K\, v$. So we have

$$\begin{aligned}(\xi_\sigma \cap \mathrm{link}_K\, u) \cap \mathrm{link}_{\mathrm{link}_K\, u}\, v &= (\xi_\sigma \cap \mathrm{link}_K\, u) \cap (\mathrm{link}_K\, u \cap \mathrm{link}_K\, v)\\ &= (\xi_\sigma \cap \mathrm{link}_K\, u) \cap (\xi_\sigma \cap \mathrm{link}_K\, v).\end{aligned}$$

The intersection of the two simplices $\xi_\sigma \cap \mathrm{link}_K\, u$ and $\xi_\sigma \cap \mathrm{link}_K\, v$ has to be a simplex (could be empty) by the definition of simplicial complex. Moreover, when σ ranges over all the maximal simplices of K containing u, the vertex v will range over all the vertices in $\mathrm{link}_K\, u$. So our argument shows that these conditions hold for $\mathrm{link}_K\, u$.

By iterating the above argument, we can prove that for any codimension-two simplex η of K, the link of η in K is a simplicial circle which satisfies the conditions. This forces the link of η to be either $\partial\Delta^2$ or $\partial\Delta^1 * \partial\Delta^1$. Dually it means that any 2-dimensional face of P is either a 3-gon or a 4-gon. Then by Theorem 2.1(b), the polytope P is combinatorially equivalent to a product of simplices.

Next, we give some new descriptions of products of simplices in terms of geometric conditions on real moment-angle manifolds of simple convex polytopes. We first recall a concept in metric geometry (see [5, Definition 3.1.12]).

Definition 2.1 (Quotient Metric Space) *Let (X, d) be a metric space and let $\mathcal{R}$ be an equivalence relation on X. The quotient semi-metric $d_{\mathcal{R}}$ is defined as*

$$d_{\mathcal{R}}(x, y) = \inf\Big\{ \sum_{i=1}^{k} d(p_i, q_i) \ : \ p_1 = x,\ q_k = y,\ k \in \mathbb{N} \Big\},$$

where the infimum is taken over all choices of $\{p_i\}$ and $\{q_i\}$ such that the point q_i is $\mathcal{R}$-equivalent to p_{i+1} for all $i = 1, \cdots, k-1$. Moreover, by identifying points with zero $d_{\mathcal{R}}$-distance, we obtain a metric space $(X/\mathcal{R}, d)$ called the quotient metric space of (X, d).

Suppose that P is a simple convex polytope in a Euclidean space $\mathbb{R}^d$. Consider P to be equipped with the intrinsic metric. More precisely, the intrinsic metric on P defines the distance between any two points x and y in P to be the infimum of lengths of piecewise smooth paths in P that connect x and y. Note that the intrinsic metric on P coincides with the subspace metric on P, since P is convex.

By the construction in (1.2), $\mathbb{R}\mathcal{Z}_P = M(P, \lambda)$ is a closed manifold obtained by gluing 2^m copies of P along their facets. We can assume that the 2^m copies of P are congruent convex polytopes inside the same Euclidean space and the gluings of their facets are all isometries. Then by Definition 2.1, we obtain a quotient metric on $\mathbb{R}\mathcal{Z}_P$, denoted by d_P. It is clear that the metric d_P is invariant with respect to the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_P$ (see (1.3)).

Remark 2.3 We can also call $(\mathbb{R}\mathcal{Z}_P, d_P)$ a Euclidean polyhedral space, which just means that it is built from Euclidean polyhedra (see [5, Definition 3.2.4]).

Note that if P' is another simple convex polytope combinatorially equivalent to P but not congruent to P, the two metric spaces $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ and $(\mathbb{R}\mathcal{Z}_P, d_P)$ are not isometric in general (though $\mathbb{R}\mathcal{Z}_{P'}$ is homeomorphic to $\mathbb{R}\mathcal{Z}_P$).

Theorem 2.3 *Let P be an n-dimensional simple convex polytope with m facets, $n \geq 2$. Then the following statements are all equivalent*:

(a) *P is combinatorially equivalent to a product of simplices.*

(b) *There exists a non-negatively curved Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ that is invariant under the canonical $(\mathbb{Z}_2)^m$-action on $\mathbb{R}\mathcal{Z}_P$.*

(c) *There exists a simple convex polytope P' combinatorially equivalent to P so that the metric space $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is non-negatively curved.*

(d) *There exists a simple convex polytope P' combinatorially equivalent to P so that all the dihedral angles of P' are non-obtuse.*

Note that a Riemannian metric on a manifold is non-negatively curved means that its sectional curvature is everywhere non-negative, while a metric space being non-negatively curved is defined via comparison of triangles (see [5, Section 4]).

Proof (a) $\Rightarrow$ (b) The real moment-angle manifold of a product of simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$ is diffeomorphic to a product of standard spheres $S^{n_1} \times \cdots \times S^{n_q}$, where $S^k = \{(x_1, \cdots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^2 + \cdots + x_{k+1}^2 = 1\}$ for any $k \in \mathbb{N}$. Let S^k be equipped with the induced Riemannian metric from $\mathbb{R}^{k+1}$. Then it is easy to check that $S^{n_1} \times \cdots \times S^{n_q}$ is a nonnegatively curved Riemannian manifold with respect to the product of the Riemannian metrics on $S^{n_1}, \cdots, S^{n_q}$.

(b) $\Rightarrow$ (a) Recall the definition of $\pi_\lambda : M(P, \lambda) = \mathbb{R}\mathcal{Z}_P \to P$ in (1.2). For any proper face f of P, let $M_f = \pi_\lambda^{-1}(f)$. It is easy to see the following.

• M_f is an embedded closed submanifold of $\mathbb{R}\mathcal{Z}_P$ which has $2^{m+\dim(f)-n-m_f}$ connected components, where m_f is the number of facets of f.

• Each connected component of M_f is diffeomorphic to $\mathbb{R}\mathcal{Z}_f$.

Note that M_f is the fixed point set of a rank $n - \dim(f)$ subgroup of $(\mathbb{Z}_2)^m$ under the canonical action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_P$. Morever, since the Riemannian metric is $(\mathbb{Z}_2)^m$-invariant, each component of M_f is a totally geodesic submanifold of $\mathbb{R}\mathcal{Z}_P$ (see [15, Theorem 5.1]), and so is non-negatively curved with respect to the induced Riemannian metric from $\mathbb{R}\mathcal{Z}_P$. This implies that the condition (b) holds for $\mathbb{R}\mathcal{Z}_f$ as well.

In particular when $\dim(f) = 2$, the $\mathbb{R}\mathcal{Z}_f$ is a closed connected surface with non-negatively curved Riemannian metric. Then by Gauss-Bonnet theorem, the Euler characteristic $\chi(\mathbb{R}\mathcal{Z}_f) \geq 0$, which implies that f has to be a 3-gon or a 4-gon. Then by Theorem 2.1(b), the polytope P is combinatorially equivalent to a product of simplices.

(a) $\Rightarrow$ (c) Suppose that P is combinatorially equivalent to $\Delta^{n_1} \times \cdots \times \Delta^{n_q}$, where $n_1 + \cdots + n_q = n$. Consider the standard simplex Δ^k as a metric subspace of $\mathbb{R}^{k+1}$ with the intrinsic metric. Let $P' = \Delta^{n_1} \times \cdots \times \Delta^{n_q}$ be the product of the q metric spaces $\Delta^{n_1}, \cdots, \Delta^{n_q}$. For each $1 \le i \le q$, let $\{v^i_0, \cdots, v^i_{n_i}\}$ be the set of vertices of Δ^{n_i}. Then all the facets of P' are (see [7])

$$\{F^i_{k_i} = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_{k_i} \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_q} \mid 0 \le k_i \le n_i,\ i = 1, \cdots, q\},$$

where $f^i_{k_i}$ is the codimension-one face of the simplex Δ^{n_i}, which is opposite to the vertex $v^i_{k_i}$. The total number of facets of P' is $m = n + q$.

Claim As a metric space, $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is isometric to the product of the q metric spaces $(\mathbb{R}\mathcal{Z}_{\Delta^{n_1}}, d_{\Delta^{n_1}}), \cdots, (\mathbb{R}\mathcal{Z}_{\Delta^{n_q}}, d_{\Delta^{n_q}})$.

Indeed if we glue two copies of P' along the facet $F^i_{k_i}$, we obtain

$$\Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times \Big(\Delta^{n_i} \bigcup_{f^i_{k_i}} \Delta^{n_i}\Big) \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_q}.$$

We can decompose the gluing procedure in the construction (1.2) for $\mathbb{R}\mathcal{Z}_{P'}$ into q steps. The i-th step only glues those facets of the form $\{F^i_{k_i}, 0 \le k_i \le n_i\}$ in the 2^m copies of P', which gives us the factor $(\mathbb{R}\mathcal{Z}_{\Delta^{n_i}}, d_{\Delta^{n_i}})$, while fixing all other factors in the product. After the first step, we obtain 2^{m-n_1-1} copies of $\mathbb{R}\mathcal{Z}_{\Delta^{n_1}} \times \Delta^{n_2} \times \cdots \times \Delta^{n_q}$. After the second step, we obtain $2^{m-n_1-n_2-2}$ copies of $\mathbb{R}\mathcal{Z}_{\Delta^{n_1}} \times \mathbb{R}\mathcal{Z}_{\Delta^{n_2}} \times \Delta^{n_3} \times \cdots \times \Delta^{n_q}$ and so on. Then our claim follows.

Moreover, observe that for any $k \in \mathbb{N}$, $(\mathbb{R}\mathcal{Z}_{\Delta^k}, d_{\Delta^k})$ is isometric to the boundary of the $(k+1)$-dimensional cross-polytope Q^{k+1} whose vertices are

$$\{(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0),\ (0, \cdots, 0, \overset{i}{-1}, 0, \cdots, 0)\ ;\ i = 1, \cdots, k+1\}.$$

Recall that the n-dimensional cross-polytope is the simplicial polytope dual to the n-dimensional cube (see Figure 1 for the cases $n = 2, 3$).

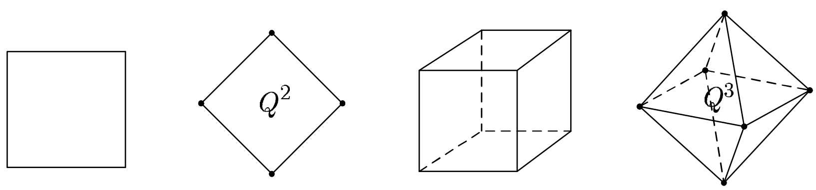

Figure 1 Cross-polytopes of dimension 2 and 3.

It is well known that the intrinsic metric on any convex hypersurface (i.e., the boundary of a compact convex set with nonempty interior) in a Euclidean space $\mathbb{R}^n$ $(n \ge 3)$ is non-negatively curved (see [5, p.359]). Since Q^{k+1} is a convex polytope in $\mathbb{R}^{k+1}$, $(\mathbb{R}\mathcal{Z}_{\Delta^k}, d_{\Delta^k})$ is non-negatively curved for any $k \ge 2$. When $k = 1$, the boundary of Q^2 is a piecewise smooth simple curve in $\mathbb{R}^2$. By definition (see [5, Definition 4.1.9]), the intrinsic metric on any piecewise smooth simple curve is non-negatively curved because any geodesic triangle on the curve is degenerate. Therefore, we can conclude that $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is non-negatively curved because the product of non-negatively curved Alexandrov spaces is still non-negatively curved (see [5, Chapter 10]).

(c) $\Rightarrow$ (d) If the metric $d_{P'}$ on $\mathbb{R}\mathcal{Z}_{P'}$ is non-negatively curved, we want to show that the dihedral angle between any two adjacent facets F_1 and F_2 of P' is non-obtuse. Otherwise, we assume that the dihedral angle θ between F_1 and F_2 is obtuse. Choose a point O in the relative interior of $F_1 \cap F_2$, a point $A \in F_1$ and $B \in F_2$ so that the line segments $\overline{OA}$ and $\overline{OB}$ are perpendicular to $F_1 \cap F_2$. Then $\angle AOB = \theta$. Suppose that the lengths of the line segments $\overline{OA}$, $\overline{OB}$ and $\overline{AB}$ are

$$|\overline{OA}| = |\overline{OB}| = a, \quad |\overline{AB}| = b.$$

In the gluing construction (1.2) for $\mathbb{R}\mathcal{Z}_{P'}$, consider two copies of P' glued along the facet F_1. We then have an isosceles triangle $\triangle AB_1B_2$ in $\mathbb{R}\mathcal{Z}_{P'}$ (see Figure 2). When a is small enough, the distance between B_1 and B_2 in $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$ is $2a$ by the definition of the quotient metric because $\overline{B_1O} \cup \overline{OB_2}$ is the shortest path between B_1 and B_2 in $(\mathbb{R}\mathcal{Z}_{P'}, d_{P'})$. Moreover, let $\triangle\overline{A}\,\overline{B}_1\overline{B}_2$ be a triangle in the Euclidean plane $\mathbb{R}^2$ which has the same lengths of sides as $\triangle AB_1B_2$. Since θ is obtuse, it is clear that $\triangle AB_1B_2$ is strictly thinner than $\triangle\overline{A}\,\overline{B}_1\overline{B}_2$, i.e.,

$$\angle AB_1B_2 < \angle\overline{A}\,\overline{B}_1\overline{B}_2, \quad \angle AB_2B_1 < \angle\overline{A}\,\overline{B}_2\overline{B}_1, \quad \angle B_1AB_2 < \angle\overline{B}_1\overline{A}\,\overline{B}_2.$$

But this contradicts our assumption that the metric $d_{P'}$ on $\mathbb{R}\mathcal{Z}_{P'}$ is non-negatively curved (see [5, Section 4.1.5]). Therefore, θ has to be non-obtuse.

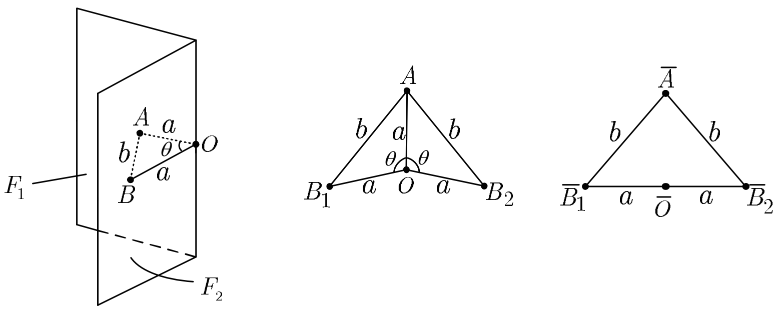

Figure 2 Comparison of triangles.

(d) $\Rightarrow$ (a) Suppose that F_1, F_2 and F_3 are three facets of P' with $F_1 \cap F_2 \cap F_3 \neq \varnothing$. Then $F_1 \cap F_2$ and $F_1 \cap F_3$ are codimension-one faces of F_1. By our assumption, the dihedral angles of (F_1, F_2), (F_1, F_3) and (F_2, F_3) are all non-obtuse. We claim that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in F_1 is non-obtuse as well.

Indeed, we can assume that P' sits inside $\mathbb{R}^n$ and let $\eta_i \in \mathbb{R}^n$ $(i = 1, 2, 3)$ be a normal vector of F_i pointing to the interior of P (see Figure 3). By choosing a proper coordinate system of $\mathbb{R}^n$, we can assume that $\eta_1 = (0, \cdots, 0, 1) \in \mathbb{R}^n$ and F_1 lies in the coordinate hyperplane $\{x_n = 0\} \subset \mathbb{R}^n$. Let $\eta_2 = (a_1, \cdots, a_{n-1}, a_n)$, $\eta_3 = (b_1, \cdots, b_{n-1}, b_n)$. Since the dihedral angles of (F_1, F_2), (F_1, F_3) and (F_2, F_3) are all non-obtuse, the inner products of η_1, η_2, η_3 satisfy

$$\begin{aligned} &\eta_1 \cdot \eta_2 = a_n \leq 0, \quad \eta_1 \cdot \eta_3 = b_n \leq 0, \quad (\eta_2, \eta_3) = a_1b_1 + \cdots + a_{n-1}b_{n-1} + a_nb_n \leq 0 \\ &\quad \Rightarrow \quad a_1b_1 + \cdots + a_{n-1}b_{n-1} \leq 0. \end{aligned} \tag{2.1}$$

Note that $(a_1, \cdots, a_{n-1}, 0)$ and $(b_1, \cdots, b_{n-1}, 0)$ are normal vectors of $F_1 \cap F_2$ and $F_1 \cap F_3$ inside F_1 respectively. So (2.1) implies that the dihedral angle between $F_1 \cap F_2$ and $F_1 \cap F_3$ in F_1 is non-obtuse. Our claim is proved.

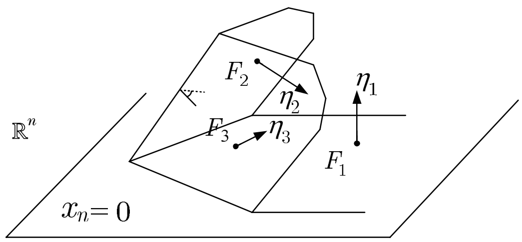

Figure 3 Dihedral angles of a simple convex polytope.

By iterating the above arguments, we can show that for any 2-dimensional face f of P', any interior angle of f is non-obtuse. Since f is a Euclidean polygon, it must be either a 3-gon or a 4-gon. Since P is combinatorially equivalent to P', any 2-face of P is either a 3-gon or a 4-gon, too. Then by Theorem 2.1(b), the polytope P is combinatorially equivalent to a product of simplices.

Remark 2.4 The equivalence of Theorem 2.3 (a) and (d) is also stated in [10, Lemma 2.8].

Remark 2.5 In the statement of Theorem 2.3(b), if we do not require the Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ to be $(\mathbb{Z}_2)^m$-invariant, it is still likely that P has to be combinatorially equivalent to a product of simplices (see [16, Section 5.2]). But we do not know how to prove this so far.

3 Appendix

Here we give another proof of Theorem 2.1 (a)$\Leftrightarrow$(b). For brevity, we say that a simplicial complex is a sphere join if it is isomorphic to $\partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$ for some $n_1, \cdots, n_q \in \mathbb{N}$. One dimensional sphere join is either $\partial\Delta^2$ (boundary of a triangle) or $\partial\Delta^1 * \partial\Delta^1$ (boundary of a square). Let us first prove the following theorem.

Theorem 3.1 *Let K be a simplicial complex of dimension n. Suppose that K satisfies the following two conditions*:

(a) *K is a pseudomanifold,*

(b) *the link of any vertex of K is a sphere join of dimension $n-1$,*

Then K is a sphere join.

Recall that K is an n-dimensional pseudomanifold if the following conditions hold:

(i) Every $(n-1)$-simplex of K is a face of exactly two n-simplices for $n > 1$.

(ii) For every pair of n-simplices σ and σ' in K, there exists a sequence of n-simplices $\sigma = \sigma_0, \sigma_1, \cdots, \sigma_k = \sigma'$ such that the intersection $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$-simplex for all i.

The condition (ii) means that K is a strongly connected simplicial complex.

Proof First of all, assumption (b) implies that the link of any k-simplex in K is a sphere join of dimension $n-k-1$. This is because for any k-simplex σ with a vertex w, the link of σ in K is the link of the $(k-1)$-simplex $\sigma \cap \operatorname{link}_K w$ in $\operatorname{link}_K w$. Then by the assumption that $\operatorname{link}_K w$ is a sphere join of dimension $n-1$, the link of any $(k-1)$-simplex in $\operatorname{link}_K w$ must be a sphere join of dimension $n-k-1$ (corresponding to the fact that any face of a product of simplices is also a product of simplices).

Let w be an arbitrary vertex of K. By assumption (b), the link $\operatorname{link}_K w$ is of the form $\operatorname{link}_K w = \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_q}$, where $n_1 + \cdots + n_q = n$. Denote the vertices of $\partial\Delta^{n_k}$ by $v_0^k, v_1^k, \cdots, v_{n_k}^k$ for $k = 1, 2, \cdots, q$, so that

$$\operatorname{link}_K w = \partial[v_0^1, v_1^1, \cdots, v_{n_1}^1] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q]. \tag{3.1}$$

Let I be the set of vertices $v_1^1, \cdots, v_{n_1}^1, \cdots, v_1^q, \cdots, v_{n_q}^q$. Then $[I]$ is a maximal simplex in $\operatorname{link}_K w$ and the simplex $[I, w]$ spanned by I and w is of dimension n. Since K is a pseudomanifold by assumption (a), there is a unique vertex v in K such that $[I, v] \cap [I, w] = [I]$. We have two cases below.

Case 1 The case where $v \notin \operatorname{link}_K w$. In this case, we claim $K = \partial[v, w] * \operatorname{link}_K w$. The proof is as follows. Choose an element from I arbitrarily, say v_j^i ($1 \le i \le q$, $1 \le j \le n_i$). Set $\overline{I} = (I \backslash \{v_j^i\}) \cup \{v_0^i\}$. Then $[\overline{I}]$ is an $(n-1)$-simplex of $\operatorname{link}_K w$ by (3.1), so there is a unique vertex $\overline{v}$ of K such that $[\overline{I}, \overline{v}] \cap [\overline{I}, w] = [\overline{I}]$ as before since K is a pseudomanifold. Now we shall observe the link of an $(n-2)$-simplex $[I \cap \overline{I}] = [I \backslash \{v_j^i\}]$ in K. By our construction, four n-simplices in K containing $[I \cap \overline{I}]$ are as follows:

$$[I \cap \overline{I}, v_j^i, w], \quad [I \cap \overline{I}, v_0^i, w], \quad [I \cap \overline{I}, v_j^i, v], \quad [I \cap \overline{I}, v_0^i, \overline{v}].$$

Therefore the vertices $v_j^i, w, v_0^i, v, \overline{v}$ are in the link of the $(n-2)$-simplex $[I \cap \overline{I}]$. But by assumption (b), this link is a sphere join of dimension one which can have at most four vertices. Note that v_j^i, w, v_0^i are mutually distinct and $v, \overline{v}$ are different from v_j^i, w, v_0^i. So we must have $\overline{v} = v$. Now let v_j^i run over all elements of I, then $\overline{I}$ runs over all $(n-1)$-simplices in $\operatorname{link}_K w$ that share a $(n-2)$-simplex with I. Moreover by (3.1), $\operatorname{link}_K w$ is a strongly connected simplicial complex. By applying our argument to $[I]$ and all other $(n-1)$-simplices in K, we can show that $\partial[v, w] * \operatorname{link}_K w$ is a subcomplex of K. However, $\partial[v, w] * \operatorname{link}_K w$ and K are both pseudomanifolds and have the same dimension, so they must agree. This proves the claim.

Case 2 The case where $v \in \operatorname{link}_K w$, so v is one of $v_0^1, v_0^2, \cdots, v_0^q$. We may assume $v = v_0^1$ without loss of generality. Then

$$[v, I] = [v_0^1, v_1^1, \cdots, v_{n_1}^1, v_1^2, \cdots, v_{n_2}^2, \cdots, v_1^q, \cdots, v_{n_q}^q] \text{ is an } n\text{-simplex in } K. \tag{3.2}$$

We look at $\operatorname{link}_K v$. Since $v = v_0^1$, it follows from (3.1) that $\operatorname{link}_K v$ contains

$$\partial[v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q] \tag{3.3}$$

as a subcomplex. This together with assumption (b) implies that there is a vertex w' different from vertices in (3.3) such that $\mathrm{link}_K\, v$ is one of the following:

$$\begin{aligned}&\partial[w', v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q],\\ &\partial[v_1^1, \cdots, v_{n_1}^1] * \partial[w', v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q],\\ &\qquad\vdots \qquad\qquad\qquad \vdots\\ &\partial[v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[w', v_0^q, v_1^q, \cdots, v_{n_q}^q].\end{aligned}$$

However, the fact (3.2) implies that none of the above occurs except the first one. So we have

$$\mathrm{link}_K\, v = \partial[w', v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q]. \tag{3.4}$$

The simplex $[I]$ is in $\mathrm{link}_K\, v$ by (3.2) and the n-simplices $[I, v]$ and $[I, w]$ share $[I]$. Note that $w \in \mathrm{link}_K\, v$ in this case but $w \neq v_j^i$ for all i and j. From (3.4), we can conclude $w = w'$. Then

$$\mathrm{link}_K\, v = \partial[w, v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q]. \tag{3.5}$$

Remember that $v = v_0^1$. We claim that K contains

$$\partial[w, v_0^1, v_1^1, \cdots, v_{n_1}^1] * \partial[v_0^2, v_1^2, \cdots, v_{n_2}^2] * \cdots * \partial[v_0^q, v_1^q, \cdots, v_{n_q}^q] \tag{3.6}$$

as a subcomplex. Indeed, any n-simplex in (3.6) is spanned by $n+1$ vertices which consist of n_1+1 vertices from $\partial[w, v_0^1, v_1^1, \cdots, v_{n_1}^1]$ and n_i vertices from $\partial[v_0^i, v_1^i, \cdots, v_{n_i}^i]$ for $i = 2, 3, \cdots, q$. Since $v_0^1 = v$, either w or v is in the $n_1 + 1$ vertices from $\partial[w, v_0^1, v_1^1, \cdots, v_{n_1}^1]$. If w (resp. v) is in the $n_1 + 1$ vertices from $\partial[w, v_0^1, v_1^1, \cdots, v_{n_1}^1]$, then any n-simplex formed this way is in K by (3.5) (resp. (3.1)). This proves the claim.

Finally, since K and the subcomplex (3.6) are both pseudomanifolds and have the same dimension, they must agree. So we finish the proof of the theorem.

Proof of Theorem 2.1 (a) $\Leftrightarrow$ (b) Suppose that any 2-dimensional face of P is either a 3-gon or a 4-gon. We want to show that P is combinatorially equivalent to a product of simplices, or equivalently ∂P^* is a sphere join. Let us do induction on the dimension of P. When $\dim P = 2$, the proof is trivial. If $\dim P \geq 3$, we can show that ∂P^* satisfies the two conditions in Theorem 3.1. Indeed, condition (a) is obvious. By induction assumption, all facets of P are product of simplices which means that ∂P^* satisfies condition (b). So we finish the induction by Theorem 3.1.

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