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Reduced Crossed Products of Pro-Banach Algebras*

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Abstract This paper gives the concept of the reduced pro-Banach algebra crossed product associated with inversely pro-Banach algebra dynamical system, and shows that the reduced crossed product is an inverse limit of an inverse system of Banach algebra crossed products. Also, the authors show that if the locally compact group is amenable, then the crossed product and the reduced crossed product are isometrically isomorphic.

Keywords Pro-Banach algebras dynamical systems, Reduced crossed products, Crossed products, Representations
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1 Introduction and Preliminaries

The crossed product algebras is on the main pillars of the theory of operator algebras, and it has many applications in mathematics and quantum physics. Over the past few decades, the crossed product algebras, including von Neumann algebras, C^*-algebras, Banach algebras etc, have attracted a great deal of attention, and a large part of the literature is concerned with crossed products (see [1–3, 12–15] and the literature therein). For a given C^*-algebra dynamical system, in addition to the full crossed products of C^*-algebra, there is another very important crossed product C^*-algebra, the reduced crossed products. A lot of work has been done on the connection between the crossed products and the reduced crossed products. Zeller-Meier [16] has firstly proved that the crossed product and the associated reduced crossed product are equal in the C^*-algebra dynamical system with the discrete group, and whereafter Takai [13] generalized this result. Huang and Lu [5] generalized it to the general Banach algebra setting, and proved that the crossed product Banach algebra, which was defined by Dirkson [2], is isometrically isomorphic to the reduced crossed product Banach algebra associated with a Banach algebra dynamical system with the amenable group.

As a prominent generalization of the idea of C^*-algebra crossed product, Joita introduced the concept of pro-C^*-algebra (also called locally C^*-algebra) crossed products and Hilbert

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pro-C^*-modules, and gave a detailed systematic exposition of the recent development in it, see for example [6–9].

Motivated by the study of crossed product of pro-C^*-algebras (see [6]), in the present paper, we define the reduced crossed product of pro-Banach algebra associated with a Banach algebra dynamical system, and show that a pro-Banach reduced crossed product is an inverse limit of an inverse system of Banach algebra crossed products, which generalize the corresponding result of the pro-C^*-algebra reduced crossed products. More importantly, we establish the equality between the crossed product and the reduced crossed product with the amenable group in the pro-Banach algebra dynamical system setting.

We now recollect some basic definitions, notations and results. The details on pro-Banach algebras are available in [11].

A pro-Banach algebra is a complete Hausdorff topological complex algebra A whose topology is given by a directed family $S(A)$ of continuous submultiplicative seminorms. For $p \in S(A)$, $\ker(p)$ is a closed bilateral ideal of A, and the quotient algebra $A/\ker(p)$ is a normed algebra in the norm induced by p, and its completion is denoted by A_p. The canonical map from A to $A/\ker(p)$ is denoted by φ_p. For $p, q \in S(A)$ with $p \geq q$, there is a unique continuous morphism $\varphi_{pq} : A_p \to A_q$ with dense range such that $\varphi_{pq} \circ \varphi_p = \varphi_q$, and $\{A_p, \varphi_{pq}\}_{p,q \in S(A)}$ is an inverse system of Banach algebras and its inverse limit is a pro-Banach algebra which is topologically isomorphic to A (see [11, Chapter III, Theorem 3.1]).

2 Pro-Banach Algebra Dynamical Systems

Let A be a pro-Banach algebra and G be a locally compact group. An action G on A is a strongly continuous representation $\alpha : G \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of bounded automorphism of A. An action α of G on A is an inverse limit action, if $p(\alpha_r(a)) = p(a)$ for all $p \in S(A)$, $r \in G$, $a \in A$. If α is an inverse limit action, then, for all $p \in S(A)$, there is an action $\alpha^{(p)}$ of G on A_p, such that $\alpha_s^{(p)} \circ \varphi_p = \varphi_p \circ \alpha_s$ for all $s \in G$, and $(A_p, G, \alpha^{(p)})$ is a Banach algebra dynamical system for each $p \in S(A)$.

Definition 2.1 *A triple (A, G, α) is called a pro-Banach algebra dynamical system, if A is a pro-Banach algebra, G is a locally compact group, and $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of G on A. If α is an inverse limit action of G on A, then (A, G, α) is an inversely pro-Banach algebra dynamical system.*

Definition 2.2 *Let (A, G, α) be a pro-Banach algebra dynamical system. A pair (π, U) is called a covariant representation of (A, G, α) on a Banach space X, if π is a representation of A on X, U is a representation of G on X, such that for all $a \in A, s \in G$,*

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1}.$$

If π is non-degenerate, then (π, U) is called non-degenerate; if π is continuous and U is strongly *continuous, then (π, U) is called a continuous covariant representation.*

Suppose that (A, G, α) is an inversely pro-Banach algebra dynamical system. We define convolution on the linear space $C_c(G, A)$ of a continuous function from G to A with compact supports by

$$(f * g)(r) = \int_G f(s)\alpha_s(g(s^{-1}r))\,\mathrm{d}\mu(s)$$

for all $f, g \in C_c(G, A)$. Straightforward computations show that $C_c(G, A)$ becomes an associative algebra with convolution. Also assume that X is a Banach space, for each $f \in C_c(G, X)$, define $\|f\|_p = (\int_G \|f(t)\|^p\,\mathrm{d}\mu(t))^{\frac{1}{p}}$, $1 \le p < \infty$. Then $C_c(G, X)$ is a normed algebra, and we denote by $L^p(G, X)$ its completion. To avoid confusion between symbols, we will be concerned only with $L^1(G, X)$. For a given Banach algebra dynamical system (A, G, α), the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the completion of $C_c(G, A)$ corresponding to the uniformly bounded class $\mathcal{R}$ of continuous covariant representations of (A, G, α). More details on Banach algebra crossed products are available in (see [1–2, 10]).

Definition 2.3 *Let (A, G, α) be a pro-Banach algebra dynamical system, and $\mathcal{R}$ be a family of continuous covariant representations of (A, G, α). Then $\mathcal{R}$ is called uniformly semi-bounded, if there exists a function $\zeta : G \to [0, \infty)$, which is bounded on compact subsets of G, such that $\|U_r\| \le \zeta(r)$ for all $(\pi, U) \in \mathcal{R}$ and $r \in G$.*

Now let $\mathcal{R}$ be a non-empty uniformly semi-bounded class of continuous covariant representations of inversely pro-Banach algebra dynamical system (A, G, α) on a Banach space X. For $(\pi, U) \in \mathcal{R}$, we define the reduced algebra representation $\widetilde{\pi}$ and the left regular representation Λ on $L^1(G, X)$ by the formula

$$[\widetilde{\pi}(a)h](s) := \pi(\alpha_{s^{-1}}(a))h(s), \quad a \in A,\ s \in G,\ h \in L^1(G, X),$$
$$(\Lambda_r h)(s) := h(r^{-1}s), \quad r, s \in G,\ h \in L^1(G, X).$$

Lemma 2.1 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system, $\mathcal{R}$ be a family of continuous representations of (A, G, α) on a Banach space X, and the representations $\widetilde{\pi}$ and Λ be defined as above.*

(1) *There exists $p \in S(A)$, such that $\|\widetilde{\pi}(a)\| \le p(a)$ for any $a \in A$.*

(2) *For all $r \in G$, Λ_r is an invertible isometry on $L^1(G, X)$.*

(3) *$(\widetilde{\pi}, \Lambda)$ is also a continuous covariant representation of (A, G, α).*

(4) *Suppose $(\pi, U) \in \mathcal{R}$, where U satisfies $M := \sup\limits_{t \in G} \|U_t\| < \infty$. Then $\|\widetilde{\pi}\| \le M^2\|\pi\|$. Specially, if U is an isometric representation of G on a Banach space X, we have $\|\widetilde{\pi}\| \le \|\pi\|$.*

Proof (1) If π is a continuous representation of A, then there is $p \in S(A)$, such that $\|\pi(a)\| \le p(a)$, $a \in A$. Since α is an inverse limit action of G on A, for all $a \in A, h \in L^1(G, X), s \in G$, we have

$$\|[\widetilde{\pi}(a)h](s)\| = \|\pi(\alpha_{s^{-1}}(a))h(s)\|$$

$$\begin{aligned}&\le \|\pi(\alpha_{s^{-1}}(a))\|\|h(s)\|\\&\le \|p(\alpha_{s^{-1}}(a))\|\|h(s)\|\\&= p(a)\|h(s)\|.\end{aligned}$$

(2) and (3) are straightforward verifications.

(4) From the convariance relation $\pi(\alpha_r(a)) = U_r\pi(a)U_r^{-1}$, it follows that $\|\pi(\alpha_r(a))\| \le M^2\|\pi\|\|a\|$ for all $a \in A$ and $r \in G$, hence

$$\|\widetilde{\pi}\| \le M^2\|\pi\|.$$

From now on, we will call $(\widetilde{\pi}, \Lambda)$ the regular covariant representation of pro-Banach algebra dynamical system (A, G, α) associated with π. In what follows, let us denote by $\widetilde{\mathcal{R}}$ the uniformly semi-bounded class of continuous covariant representations (A, G, α). By Lemma 2.1, there is $p \in S(A)$, such that $\|\widetilde{\pi}(a)\| \le p(a)$ for $a \in A$. Note that if we write

$$\widetilde{\mathcal{R}}_p = \{(\widetilde{\pi}, \Lambda) \in \widetilde{\mathcal{R}} : \|\widetilde{\pi}(a)\| \le p(a), a \in A\},$$

then $\widetilde{\mathcal{R}} = \bigcup\limits_{p\in S(A)} \widetilde{\mathcal{R}}_p$. To define reduced crossed product of pro-Banach algebra, it is necessary to consider the representation of $\widetilde{\pi} \times \Lambda$ of $C_c(G, A)$ on $L^1(G, X)$. For all $f \in C_c(G, A)$, $\xi \in C_c(G, X)$, $s \in G$, we have

$$\begin{aligned}[\widetilde{\pi} \times \Lambda(f)\xi](s) &= \int_G [\widetilde{\pi}(f(r))(\Lambda_r\xi)](s)\,\mathrm{d}\mu(r)\\&= \int_G \pi(\alpha_s^{-1}(f(r)))(\Lambda_r\xi)(s)\,\mathrm{d}\mu(r)\\&= \int_G \pi(\alpha_s^{-1}(f(r)))\xi(r^{-1}s)\,\mathrm{d}\mu(r).\end{aligned}$$

Proposition 2.1 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system, and let $\mathcal{R}$ be the uniformly semi-bounded class of continuous covariant representations of (A, G, α) on a Banach space X. Also assume that $\widetilde{\mathcal{R}}$ are the associated regular covariant representations of (A, G, α) on $L^1(G, X)$. For all $f \in C_c(G, A)$, define $\rho^{\widetilde{\mathcal{R}}_p}(f) := \sup\limits_{(\widetilde{\pi},\Lambda)\in\widetilde{\mathcal{R}}_p} \|\widetilde{\pi} \times \Lambda(f)\|$. Then $\rho^{\widetilde{\mathcal{R}}_p}$ is a finite sub-multiplicative semi-norm on $C_c(G, A)$.*

Proof Let $f, g \in C_c(G, A)$, $\xi \in C_c(G, X)$ and $s \in G$, we have

$$\begin{aligned}&[\widetilde{\pi} \times \Lambda(f * g)\xi](s)\\&= \int_G \pi(\alpha_s^{-1}(f * g(r)))\xi(r^{-1}s)\,\mathrm{d}\mu(r)\\&= \int_G \pi\Big(\alpha_s^{-1}\Big(\int_G f(t)\alpha_t(g(t^{-1}r))\,\mathrm{d}\mu(t)\Big)\Big)\xi(r^{-1}s)\,\mathrm{d}\mu(r)\\&= \int_G\int_G \pi(\alpha_s^{-1}(f(t)))\pi(\alpha_{s^{-1}t}(g(t^{-1}r)))\xi(r^{-1}s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(r)\\&= \int_G\int_G \pi(\alpha_s^{-1}(f(t)))\pi(\alpha_{s^{-1}t}(g(r)))\xi(r^{-1}t^{-1}s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(r)\end{aligned}$$

$$
\begin{aligned}
&= \int_G \int_G \pi(\alpha_s^{-1}(f(t)))\widetilde{\pi}(\alpha_t(g(r)))\xi(r^{-1}t^{-1}s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(r) \\
&= \int_G \int_G \pi(\alpha_s^{-1}(f(t)))\Lambda_t\widetilde{\pi}(\alpha_t(g(r)))\Lambda_{t^{-1}}\xi(r^{-1}t^{-1}s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(r) \\
&= \int_G \pi(\alpha_s^{-1}(f(t)))\Lambda_t\,\mathrm{d}\mu(t)\int_G \widetilde{\pi}(g(r))\Lambda_r\xi(s)\,\mathrm{d}\mu(r) \\
&= [\widetilde{\pi}\times\Lambda(f)\cdot\widetilde{\pi}\times\Lambda(g)\xi](s).
\end{aligned}
$$

And then,

$$
\begin{aligned}
\rho^{\widetilde{\mathcal{R}}_p}(f*g) &= \sup_{(\widetilde{\pi},\Lambda)\in\widetilde{\mathcal{R}}_p}\|\widetilde{\pi}\rtimes\Lambda(f*g)\| \\
&\leq \sup_{(\widetilde{\pi},\Lambda)\in\widetilde{\mathcal{R}}_p}\|\widetilde{\pi}\rtimes\Lambda(f)\|\cdot\sup_{(\widetilde{\pi},\Lambda)\in\widetilde{\mathcal{R}}_p}\|\widetilde{\pi}\rtimes\Lambda(g)\| \\
&= \rho^{\widetilde{\mathcal{R}}_p}(f)\cdot\rho^{\widetilde{\mathcal{R}}_p}(g).
\end{aligned}
$$

From this, it follows that $\rho^{\widetilde{\mathcal{R}}_p}$ is sub-multiplicative. For $f\in C_c(G,A)$, $\xi\in C_c(G,X)$, we have

$$
\begin{aligned}
\|\widetilde{\pi}\times\Lambda(f)(\xi)\| &= \int_G \|[\widetilde{\pi}\times\Lambda(f)(\xi)](s)\|\,\mathrm{d}\mu(s) \\
&= \int_G \left\|\int_G \pi(\alpha_s^{-1}(f(r)))\xi(r^{-1}s)\,\mathrm{d}\mu(r)\right\|\mathrm{d}\mu(s) \\
&\leq \int_G\int_G \|\pi(\alpha_s^{-1}(f(r)))\xi(r^{-1}s)\|\,\mathrm{d}\mu(r)\,\mathrm{d}\mu(s) \\
&\leq \int_G\int_G p(\alpha_s^{-1}(f(r))))\|\xi(r^{-1}s)\|\,\mathrm{d}\mu(r)\,\mathrm{d}\mu(s) \\
&= \int_G\int_G p(f(r))\|\xi(r^{-1}s)\|\,\mathrm{d}\mu(r)\,\mathrm{d}\mu(s) \\
&= \int_G p(f(r))\,\mathrm{d}\mu(r)\int_G \|\xi(r^{-1}s)\|\,\mathrm{d}\mu(s) \\
&= N_p(f)\cdot\|\xi\|_1<\infty,
\end{aligned}
$$

where

$$
N_p(f)=\int_G p(f(r))\,\mathrm{d}\mu(r).
$$

This shows that $\rho^{\widetilde{\mathcal{R}}_p}$ is a finite sub-multiplicative semi-norm on $C_c(G,A)$.

3 Reduced Crossed Products

In this section, we mainly study the reduced crossed products associated with pro-Banach algebra dynamical systems. As we know if $\mathcal{R}$ is a non-empty uniformly semi-bounded class of continuous covariant representations of inversely pro-Banach algebra dynamical system (A,G,α), then for all

$$
f\in C_c(G,A),\quad \rho^{\mathcal{R}_p}(f):=\sup_{(\pi,U)\in\mathcal{R}_p}\|\pi\times U(f)\|
$$

is a sub-multiplicative semi-norm on $C_c(G, A)$. Crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is thus the completion of $C_c(G, A)$ in $\{\rho^{\mathcal{R}_p} : p \in S(A)\}$. We define the reduced crossed products of pro-Banach algebras analogously.

Definition 3.1 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system, $\mathcal{R}$ be a non-empty uniformly semi-bounded class of continuous covariant representations of (A, G, α), and $\widetilde{\mathcal{R}}$ be the associated class of regular covariant representations of (A, G, α). By Proposition 2.1, $\{\rho^{\widetilde{\mathcal{R}}_p} : p \in S(A)\}$ is a set of sub-multiplicative semi-norms on $C_c(G, A)$. Then the completion of $C_c(G, A)$ in $\{\rho^{\widetilde{\mathcal{R}}_p} : p \in S(A)\}$ is called the reduced crossed product of (A, G, α) associated to $\widetilde{\mathcal{R}}$, denoted by $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$.*

3.1 Inverse limit of Banach algebra crossed products

Let π be a continuous representation of pro-Banach algebra A on a Banach space X, and $\widetilde{\pi}$ be the associated regular representation of A on $L^1(G, X)$. From the continuity of $\widetilde{\pi}$, there exists $p \in S(A)$, such that $\|\widetilde{\pi}\| \le p(a)$. Define a map $\widetilde{\pi}_p$ by $a + \ker(p) \mapsto \widetilde{\pi}(a)$, then $\widetilde{\pi}_p$ is a bounded homomorphism from $A/\ker(p)$ to $B(L^1(G, X))$, and thus it can be extended to a bounded homomorphism from A_p to $B(L^1(G, X))$, still denoted by $\widetilde{\pi}_p$. It is easy to verify that the following diagram

$$\begin{tikzcd} A \arrow[r, "\widetilde{\pi}"] \arrow[d, "\varphi_p"'] & B(L^1(G, X)) \\ A_p \arrow[ur, "\widetilde{\pi}_p"'] & \end{tikzcd}$$

is commutative. That is, $\widetilde{\pi}_p \circ \varphi_p = \widetilde{\pi}$ for all $a \in A$. Suppose that (π, U) is a continuous covariant representation of pro-Banach algebra dynamical system (A, G, α), and $(\widetilde{\pi}, \Lambda)$ is the regular covariant representation of (A, G, α). Then it follows from Lemma 2.1 that $(\widetilde{\pi}, \Lambda)$ is a continuous covariant representation of (A, G, α) on $L^1(G, X)$, so $(\widetilde{\pi}_p, \Lambda)$ is a continuous covariant representation of Banach algebra dynamical system $(A_p, G, \alpha^{(p)})$ on $L^1(G, X)$. In fact, for $a \in A$, $s \in G$, we have

$$\begin{aligned} \widetilde{\pi}_p(\alpha_s^{(p)}(a + \ker(p))) &= \widetilde{\pi}_p(\alpha_s^{(p)} \circ \varphi_p)(a) = \widetilde{\pi}_p(\varphi_p \circ \alpha_s)(a) \\ &= (\widetilde{\pi}_p \circ \varphi_p)(\alpha_s(a)) = \widetilde{\pi}(\alpha_s(a)) \\ &= \Lambda_s \widetilde{\pi}(a) \Lambda_{s^{-1}} = \Lambda_s(\widetilde{\pi}_p \circ \varphi_p)(a)\Lambda_{s^{-1}} \\ &= \Lambda_s \widetilde{\pi}_p(a + \ker(p))\Lambda_{s^{-1}}. \end{aligned}$$

Note that $A/\ker(p)$ is dense in A_p, we get

$$\widetilde{\pi}_p(\alpha_s^{(p)}(a)) = \Lambda_s \widetilde{\pi}_p(a)\Lambda_{s^{-1}}, \quad a \in A_p.$$

To differentiate, we denote by $\widetilde{\mathcal{R}}(p)$ the uniformly bounded set of continuous covariant representations of Banach algebra dynamical system $(A_p, G, \alpha^{(p)})$.

If $\mathcal{R}$ is a non-empty uniformly semi-bounded class of continuous covariant representations of inversely pro-Banach algebra dynamical system (A, G, α), by Arens-Michael decomposition theorem (see [11]), we may show that pro-Banach algebra crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is an inverse limit of a family of Banach algebra crossed products $(A_p \rtimes_{\alpha^{(p)}} G)^{\mathcal{R}(p)}$, that is,

$$(A \rtimes_\alpha G)^{\mathcal{R}} \cong \varprojlim_{\rho^{\mathcal{R}_p}} (A_p \rtimes_{\alpha^{(p)}} G)^{\mathcal{R}(p)}.$$

The proof for reduced crossed product $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ is similar. To make our exposition self-contained, we will give a detailed proof.

Theorem 3.1 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system, and $\mathcal{R}$ be a non-empty uniformly semi-bounded class of continuous covariant representations of (A, G, α) on a Banach space X. Suppose that $\widetilde{\mathcal{R}}$ is the corresponding semi-bounded class of regular covariant representations of (A, G, α) on $L^1(G, X)$. Then for all $p \in S(A)$, we have*

$$(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}} \cong \varprojlim_{\rho^{\widetilde{\mathcal{R}}_p}} (A_p \rtimes_{\alpha^{(p)}} G)^{\widetilde{\mathcal{R}}(p)},$$

up to a topological algebraic isomorphism.

Proof Note that $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ is a complete pro-Banach algebra, using Arens-Michael deposition theorem (see [11]), we have

$$(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}} \cong \varprojlim_{\rho^{\widetilde{\mathcal{R}}_p}} ((A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}})_{\rho^{\widetilde{\mathcal{R}}(p)}}.$$

So it is enough to prove that $((A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}})_{\rho^{\widetilde{\mathcal{R}}_p}}$ is topologically isomorphic to $(A_p \rtimes_{\alpha^{(p)}} G)^{\widetilde{\mathcal{R}}(p)}$. Define a map $T : C_c(G, A, \alpha)/\ker(\rho^{\widetilde{\mathcal{R}}_p}) \to C_c(G, A_p, \alpha^{(p)})/\ker(\sigma^{\widetilde{\mathcal{R}}(p)})$ by

$$f + \ker(\rho^{\widetilde{\mathcal{R}}_p}) \mapsto \varphi_p \circ f + \ker(\sigma^{\widetilde{\mathcal{R}}(p)})$$

for all $f \in C_c(G, A, \alpha)$, where φ_p is the canonical map from A to A_p.

If $f \in \ker(\rho^{\widetilde{\mathcal{R}}_p})$, then for all $(\widetilde{\pi}, \Lambda) \in \widetilde{\mathcal{R}}_p$, $\int_G \widetilde{\pi}(f(s))\Lambda_s \,\mathrm{d}\mu(s) = 0$. And thus

$$\int_G \widetilde{\pi}_p((\varphi_p \circ f)(s))\Lambda_s \,\mathrm{d}\mu(s) = \int_G \widetilde{\pi}(f(s))\Lambda_s \,\mathrm{d}\mu(s) = 0.$$

So the definition of T is unambiguous. To show that T is a topological isomorphism, we divide our proof in three steps.

First, we need to verify that T is a linear homomorphism from $C_c(G, A, \alpha)/\ker(\rho^{\widetilde{\mathcal{R}}_p})$ to $C_c(G, A_p, \alpha^{(p)})/\ker(\sigma^{\widetilde{\mathcal{R}}(p)})$. Clearly, T is linear, so it suffices to show that T is a homomorphism. Note that for all $f, g \in C_c(G, A)$,

$$T[(f + \ker(\rho^{\widetilde{\mathcal{R}}_p}))(g + \ker(\rho^{\widetilde{\mathcal{R}}_p}))]$$

$$
\begin{aligned}
&= T[f * g + \ker(\rho^{\widetilde{\mathcal{R}}_p})] \\
&= \varphi_p \circ (f * g) + \ker(\sigma^{\widetilde{\mathcal{R}}(p)}) \\
&= (\varphi_p \circ f)(\varphi_p \circ g) + \ker(\sigma^{\widetilde{\mathcal{R}}(p)}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(\varphi_p \circ (f * g))(s) \\
&= \varphi_p \int_G f(r)\alpha_r(g(r^{-1}s))\,\mathrm{d}\mu(r) \\
&= \int_G (\varphi_p \circ f)(r)(\varphi_p \circ \alpha_r)(g(r^{-1}s))\,\mathrm{d}\mu(r) \\
&= \int_G (\varphi_p \circ f)(r)(\alpha_r^{(p)} \circ \varphi_p)(g(r^{-1}s))\,\mathrm{d}\mu(r) \\
&= \int_G (\varphi_p \circ f)(r)\alpha_r^{(p)}((\varphi_p \circ g)(r^{-1}s))\,\mathrm{d}\mu(r) \\
&= ((\varphi_p \circ f) * (\varphi_p \circ g))(s).
\end{aligned}
$$

The next is to show that T is isometric. To do it, let $f \in C_c(G, A, \alpha)$. Then

$$
\begin{aligned}
&\|T(f + \ker(\rho^{\widetilde{\mathcal{R}}_p}))\|_{\rho^{\sigma^{\mathcal{R}(p)}}} \\
&= \|\varphi_p \circ f + \ker(\sigma^{\widetilde{\mathcal{R}}(p)})\|_{\sigma^{\widetilde{\mathcal{R}}(p)}} = \sigma^{\widetilde{\mathcal{R}}(p)}(\varphi_p \circ f) \\
&= \sup_{(\widetilde{\pi}_p, \Lambda) \in \widetilde{\mathcal{R}}(p)} \left\| \int_G \widetilde{\pi}_p((\varphi_p \circ f)(s))\Lambda_s\,\mathrm{d}\mu(s) \right\| \\
&= \sup_{(\widetilde{\pi}, \Lambda) \in \widetilde{\mathcal{R}}_p} \left\| \int_G \widetilde{\pi}(f(s))\Lambda_s\,\mathrm{d}\mu(s) \right\| \\
&= \rho^{\widetilde{\mathcal{R}}_p}(f) = \|f + \ker(\rho^{\widetilde{\mathcal{R}}_p})\|_{\rho^{\widetilde{\mathcal{R}}_p}}.
\end{aligned}
$$

Now, we have already proved that T is an isometric homomorphism. Therefore, it can be extended to an isometric homomorphism from $((A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}})_{\rho^{\widetilde{\mathcal{R}}_p}}$ to $(A_p \rtimes_{\alpha^{(p)}} G)^{\widetilde{\mathcal{R}}(p)}$, denoted by $\widetilde{T}$.

Finally, we have to show that $\widetilde{T}$ is surjective. Let $\xi \in C_c(G)$, $a \in A$; then

$$
\begin{aligned}
\widetilde{T}(\xi \otimes a + \ker(\rho^{\widetilde{\mathcal{R}}_p})) &= \varphi_p \circ (\xi \otimes a) + \ker(\sigma^{\widetilde{\mathcal{R}}(p)}) \\
&= \xi \otimes \varphi_p(a) + \ker(\sigma^{\widetilde{\mathcal{R}}(p)}).
\end{aligned}
$$

It follows from the surjection of $\varphi_p : A \to A/\ker(p)$ that the range of $\widetilde{T}$ contains the set

$$
Y := \mathrm{span}\{\xi \otimes b + \ker(\sigma^{\widetilde{\mathcal{R}}(p)}) : \xi \in C_c(G), b \in A/\ker(p)\}.
$$

Since $A/\ker(p)$ is dense in A_p, Y is also dense in $(A_p \rtimes_{\alpha^{(p)}} G)^{\widetilde{\mathcal{R}}(p)}$(see [2, Corollary 3.6]). This implies that $\widetilde{T}$ is surjective. The proof is complete.

3.2 Reduced crossed products with amenable groups

Takai [13] proved that the crossed products and the associated reduced crossed products of C^*-algebra dynamical system are equal in the amenable group condition. The same holds for a Banach algebra dynamical system with the amenable group (see [5]). To establish this theorem in the pro-Banach algebra dynamical system setting, we introduce the following lemma.

Lemma 3.1 (see [4]) *Let G be a locally compact group with Haar measure μ. If G is amenable, then, for every $\varepsilon > 0$ and compact set $K \subset G$ containing the identity element of G, there exists a compact set U with*

$$\mu(U) > 0 \quad \textit{and} \quad \frac{\mu(KU \triangle U)}{\mu(U)} < \varepsilon.$$

Here $KU \triangle U = (KU \setminus U) \cup (U \setminus KU)$ is the symmetric difference of the sets KU and U.

Theorem 3.2 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system, $\mathcal{R}$ be a non-empty uniformly semi-bounded class of continuous covariant representations of (A, G, α), and suppose that $\sup\{\|U_r\| : (\pi, U) \in \mathcal{R}_p, r \in G\} < \infty$ for some $p \in S(A)$. Let $\widetilde{\mathcal{R}}$ be the class of regular covariant representations of (A, G, α) associated with $\mathcal{R}$. If G is amenable, then $(A \rtimes_\alpha G)^{\mathcal{R} \cup \widetilde{\mathcal{R}}}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ are topologically isomorphic.*

Proof We will prove the theorem directly without using the lemma. Let (π, U) be in $\mathcal{R}$, then there exists $p \in S(A)$ such that $\|\pi(a)\| \le p(a)$ for all $a \in A$. Suppose that $(\widetilde{\pi}, \Lambda)$ is the regular representation associated with (π, U). By Lemma 2.1, $(\widetilde{\pi}, \Lambda)$ is a continuous covariant representation of (A, G, α) satisfying $\|\widetilde{\pi}(a)\| \le p(a)$. For convenience, we write $M = \sup\{\|U_r\| : (\pi, U) \in \mathcal{R}_p, r \in G\}$. First, we will show that

$$\|\widetilde{\pi} \times \Lambda(f)\| \ge \frac{1}{M^2} \|(\pi \times U(f)\|.$$

Define a bounded map $h : L^1(G, X) \to L^1(G, X)$ as follows:

$$[h(\xi)](s) := U_s^{-1}(\xi(s)), \quad \xi \in L^1(G, X),\ s \in G.$$

It is easy to check that h is a linear bijection bounded by M. In fact, for all $\xi \in L^1(G, X)$, we have

$$\begin{aligned} \|h(\xi)\| &= \int_G \|[h(\xi)](s)\| \, \mathrm{d}\mu(s) \\ &= \int_G \|U_s^{-1}(\xi(s))\| \, \mathrm{d}\mu(s) \\ &\le \int_G \|U_s^{-1}\| \|\xi(s)\| \, \mathrm{d}\mu(s) \\ &\le M \|\xi\|. \end{aligned}$$

Similarly, it follows from the equality $[h^{-1}(\xi)](s) := U_s(\xi(s))$ that h^{-1} is also M-bounded.

Let $f \in C_c(G, A)$. Notice the fact that

$$\|h^{-1}(\widetilde{\pi} \times \Lambda(f))h\| \le \|h^{-1}\|\|h\|\|\widetilde{\pi} \times \Lambda(f)\| \le M^2 \cdot \|\widetilde{\pi} \times \Lambda(f)\|.$$

So the proof will be finished if one shows, given $\varepsilon > 0$, that

$$\|h^{-1}(\widetilde{\pi} \times \Lambda(f))h\| > \|(\pi \times U)(f)\| - \varepsilon.$$

Without loss of generality, we assume that $(\pi \times U)(f) \neq 0$ and $\varepsilon < \|(\pi \times U)(f)\|$. Choose $x_0 \in X_0$ such that $\|x_0\| = 1$ and $\|(\pi \times U)(f)x_0\| > \|(\pi \times U)(f)\| - \frac{\varepsilon}{2}$. Let

$$\delta = \frac{\|(\pi \times U)(f)\| - \frac{\varepsilon}{2}}{\|(\pi \times U)(f)\| - \varepsilon} - 1,$$

then $\delta > 0$. Write $S = \mathrm{supp}(f) \cup \{e\}$. Since $f \in C_c(G, A)$, S and S^{-1} are both compact. Therefore, there is a compact subset $K \subset G$ such that $0 < \mu(K) < \infty$ and $\mu(S^{-1}K \bigtriangleup K) < \delta\mu(K)$ by Lemma 3.1, and thus $\mu(S^{-1}K) < (1 + \delta)\mu(K)$ which is due to the latter inequality. From Uryshon's lemma, we may define $\eta \in C_c(G, X)$ by

$$\eta(s) = \begin{cases} x_0, & s \in S^{-1}K; \\ 0, & s \notin S^{-1}K. \end{cases}$$

Then

$$\|\eta\| = \int_G \|\eta(r)\| \,\mathrm{d}\mu(r) = \int_G \|x_0\| \,\mathrm{d}\mu(r) < \mu(S^{-1}K) < (1 + \delta)\mu(K).$$

For all $s \in G$, we have $(h\eta)(s) = U_{s^{-1}}(\eta(s)) = U_{s^{-1}}(x_0)$, it follows from the strong continuity of U that $h\eta \in_c (G, X)$. For any $r \in K$, noting $\eta(s^{-1}r) = x_0$ if $s \in S$ with $f(s) \neq 0$, we have

$$\begin{aligned}
\|h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\eta\| &= \int_G \|[h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\eta](r)\| \,\mathrm{d}\mu(r) \\
&= \int_G \|U_r[(\widetilde{\pi} \times \Lambda)(f)h\eta(r)]\| \,\mathrm{d}\mu(r) \\
&= \int_G \left\| \int_G U_r\pi(\alpha_r^{-1}f(s))[(h\eta)(s^{-1}r)] \,\mathrm{d}\mu(s) \right\| \mathrm{d}\mu(r) \\
&= \int_G \left\| \int_G \pi(f(s))U_s x_0 \,\mathrm{d}\mu(s) \right\| \mathrm{d}\mu(r) \\
&= \int_G \|(\pi \times U)(f)x_0\| \,\mathrm{d}\mu(r) \\
&\ge \int_K \|(\pi \times U)(f)x_0\| \,\mathrm{d}\mu(r) \\
&= \|(\pi \times U)(f)x_0\| \cdot \mu(K) \\
&> \left(\|(\pi \times U)(f)\| - \frac{\varepsilon}{2}\right) \cdot \mu(K).
\end{aligned}$$

Thus,

$$\|h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\| > \frac{\|h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\eta\|}{\|\eta\|}$$

$$> \frac{\|(\pi \times U)(f)\| - \frac{\varepsilon}{2}}{1+\delta}$$
$$= \|(\pi \times U)(f)\| - \varepsilon.$$

We conclude from the inequality $\|h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\| \leq M^2\|\widetilde{\pi} \times \Lambda(f)\|$ that $\|\widetilde{\pi} \times \Lambda(f)\| \geq \frac{1}{M^2}\|h^{-1}(\widetilde{\pi} \times \Lambda)(f)h\|$, hence that $\|\widetilde{\pi} \times \Lambda(f)\| > \frac{1}{M^2}\|(\pi \times U)(f)\| - \varepsilon$, and this yields that

$$\|\widetilde{\pi} \times \Lambda(f)\| \geq \frac{1}{M^2}\|(\pi \times U)(f)\|.$$

Using the fact that

$$\frac{1}{M^2}\sigma^{\mathcal{R}_p \cup \widetilde{\mathcal{R}}_p}(f) \leq \rho^{\widetilde{R}_p}(f) \leq \sigma^{\mathcal{R}_p \cup \widetilde{\mathcal{R}}_p}(f), \quad f \in C_c(G, A),$$

we conclude that the semi-norms $\sigma^{\mathcal{R}_p \cup \widetilde{\mathcal{R}}_p}$ and $\rho^{\widetilde{R}_p}$ are equivalent on $C_c(G, A)$. This implies that the locally convex topologies determined by $\{\sigma^{\mathcal{R}_p \cup \widetilde{\mathcal{R}}_p} : p \in S(A)\}$ and $\{\rho^{\widetilde{R}_p} : p \in S(A)\}$ are equal. Thus it follows from the definitions of the crossed product and the reduced crossed product that $(A \rtimes_\alpha G)^{\mathcal{R} \cup \widetilde{\mathcal{R}}}$ is equal to $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ up to a topological isomorphism. The proof is completed.

As we know that the crossed product and the associated reduced crossed product are equal for a Banach algebra dynamical system with amenable group (see [5]). Inspired by it, we will give a similar result in case of pro-Banach algebra dynamical system setting.

Theorem 3.3 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system with the amenable group G, $\mathcal{R}$ be a uniformly semi-bounded class of continuous covariant representations on a Banach space such that U_s is an isometry for all $s \in G$, and $\mathcal{R}$ be the class of regular representations of (A, G, α) associated with $\mathcal{R}$. Then the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ and the reduced crossed product $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ are isometrically isomorphic.*

Proof Since U_s is isometrical for each $s \in G$, we obtain $\widetilde{\mathcal{R}}_p \subseteq \mathcal{R}_p$ for some $p \in S(A)$. It follows from the first part of the proof of Theorem 3.2 that

$$\sigma^{\mathcal{R}_p}(f) \leq \rho^{\widetilde{\mathcal{R}}_p}(f) \leq \sigma^{\mathcal{R}_p}(f)$$

for all $f \in C_c(G, A)$. This shows that $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ are isometrically isomorphic.

Since all abelian topological groups are amenable (see [15]), and as a consequence of Theorem 3.3, we naturally get the following result.

Corollary 3.1 *Let (A, G, α) be an inversely pro-Banach algebra dynamical system where G is abelian, and $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be given as in Theorem 3.3. Then $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(A \rtimes_\alpha G)^{\widetilde{\mathcal{R}}}$ are equal. In particular, $(A \rtimes_\alpha \mathbb{Z})^{\mathcal{R}} = (A \rtimes_\alpha \mathbb{Z})^{\widetilde{\mathcal{R}}}$, $(A \rtimes_\alpha \mathbb{R})^{\mathcal{R}} = (A \rtimes_\alpha \mathbb{R})^{\widetilde{\mathcal{R}}}$, where $\mathbb{Z}$ and $\mathbb{R}$ respectively stand for the group of the integer numbers and the group of the real numbers.*

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