Chin. Ann. Math. Ser. B
43(4), 2022, 509–522
DOI: 10.1007/s11401-022-0343-4

Chinese Annals of Mathematics, Series B

Existence in the Large for Pressure-Gradient System*

Shuxin ZHANG[1] Zejun WANG[2]

Abstract In this paper, the authors use Glimm scheme to study the global existence of BV solutions to Cauchy problem of the pressure-gradient system with large initial data. To this end, some important properties of the shock curves of the pressure-gradient system in the Riemann invariant coordinate system and verify that the shock curves satisfy Diperna's conditions (see [Diperna, R. J., Existence in the large for quasilinear hyperbolic conservation laws, *Arch. Ration. Mech. Anal.,* **52**(3), 1973, 244–257]) are studied. Then they construct the approximate solution sequence through Glimm scheme. By establishing accurate local interaction estimates, they prove the boundedness of the approximate solution sequence and its total variation.

Keywords Pressure-gradient system, Riemann problem, Diperna's conditions, Glimm scheme, BV space
2000 MR Subject Classification 35A01, 35B35, 35B40, 35L65

1 Introduction

In this paper, we study the following pressure-gradient system

$$\begin{cases} u_t + p_x = 0, \\ \Big(p + \dfrac{1}{2}u^2\Big)_t + (pu)_x = 0. \end{cases} \tag{1.1}$$

Here $u = u(x,t)$, $p = p(x,t)$ are velocity, pressure, respectively. For smooth solution, it can be simplified as

$$\begin{cases} u_t + p_x = 0, \\ p_t + pu_x = 0. \end{cases} \tag{1.2}$$

We will study the Cauchy problem of (1.1) and the initial condition is given by

$$u(x,0) = u_0(x), \quad p(x,0) = p_0(x), \quad -\infty < x < +\infty. \tag{1.3}$$

System (1.1) can be obtained from the following 1-dimensional Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \Big(\rho\Big(\dfrac{1}{2}u^2 + e\Big)\Big)_t + \Big(\rho u\Big(\dfrac{1}{2}u^2 + e\Big) + pu\Big)_x = 0 \end{cases} \tag{1.4}$$

Manuscript received April 15, 2021. Revised January 12, 2022.
[1]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Key Laboratory of Mathematical and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 210016, China. E-mail: zhangshuxin@nuaa.edu.cn
[2]Corresponding author. Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Key Laboratory of Mathematical and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 210016, China. E-mail: wangzejun@gmail.com
*This work was supported by the National Natural Science Foundation of China (No. 11671193).

by deleting the nonlinear convective terms, in the case of only considering the effect of differential pressure (see [1, 9]). Here $u = u(x,t)$, $p = p(x,t)$, $\rho = \rho(x,t)$ and $e = e(x,t)$ represent speed, pressure, density and internal energy, respectively. Pressure-gradient system is an important model in the theoretical research of conservation law system. We will study the existence of global BV solution to problem (1.1), (1.3) for large initial data.

In 1965, Glimm [7] used the method of random choice to establish the global existence of weak solutions of the hyperbolic conservation law system for small initial data. For the general system of conservation law

$$U_t + F(U)_x = 0, \quad -\infty < x < \infty,\ t > 0 \tag{1.5}$$

satisfying Diperna's conditions, Diperna [4] considered the existence of global solutions for a class of nonlinear hyperbolic system by studying the shock curve described by the Riemann invariants of (1.5) and proved the existence of weak solution to the Cauchy problem. In the same year, Diperna [5] proved the existence of solutions for a class of quasi-linear hyperbolic conservation laws system with large initial data. Ding et al. [3] proved the global existence of solutions of p-system with $\gamma > 1$ by using Glimm scheme for a special class of large initial data. Li et. al. [8] gave the existence of global entropy solutions to the relativistic Euler equations for a class of large initial data.

As a special and important system of conservation law, the following p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0 \end{cases} \tag{1.6}$$

has been studied by many authors. The initial data is given by

$$v(x,0) = v_0(x), \quad u(x,0) = u_0(x). \tag{1.7}$$

Here v is the specific volume, $v = \frac{1}{\rho}$, ρ is the density and u is the velocity of the gas. p is the pressure satisfying $p(v) = \frac{k^2}{v^\gamma}$, where $\gamma > 1$ is a constant. For the p-system with $\gamma = 1$, Nishida [10] proved the global existence of weak solutions to Cauchy problem via Glimm scheme for large initial data. In 1973, Nishida and Smoller [11] used Glimm scheme to obtain the global existence of solutions to problem (1.6)–(1.7) when

$$(\gamma - 1) \cdot \mathrm{TV}\{v_0(x), u_0(x)\}$$

is sufficiently small. Frid [6] presented a periodic version of Glimm scheme applicable to p-system (for $\gamma = 1$) and proved that the global BV solution always exists in $L^\infty \cap BV_{\mathrm{loc}}(R \times R_+)$. This result was further improved in [12].

For pressure-gradient system (1.1), Zhang and Sheng [15] studied the one-dimensional piston problem of (1.1). Yang and Sheng [9] studied the interaction of a class of waves of the aerodynamic pressure-gradient system. Xu and Huang [13] studied global existence of shock front solution to piston problem of pressure-gradient system. Ding [2] studied stability of rarefaction wave to the 1-dimensional piston problem for the pressure-gradient system. Zhang et al. [16] studied interactions between two rarefaction waves for the pressure-gradient system.

In this paper, we use Glimm scheme and the methods proposed by Diperna in [4] to prove the existence of weak solutions of problem (1.1), (1.3). The main theorem of this paper is as follows.

Theorem 1.1 *Suppose that the initial data $U_0(x) = (u_0(x), p_0(x))^{\mathrm{T}}$ of (1.1) and the total variation of $U_0(x)$ are bounded. In addition, $U_0(x)$ satisfies*

$$z_0 = \sup_{x \in R} z(U_0(x)) \le w_0 = \inf_{x \in R} w(U_0(x)).$$

Then problem (1.1), (1.3) *admits a solution* $U(x,t) \in L^\infty \cap BV_{\mathrm{loc}}(R \times R_+)$.

The rest of this paper is arranged as follows. In Section 2, we study the shock curves of (1.1) and prove that the shock curves satisfy Diperna's conditions in [4]. In Section 3, we use Glimm scheme to construct an approximate solution sequence and prove that the sequence and its total variation are uniformly bounded. Then we define the Glimm functional and prove its monotonicity under Diperna's conditions. In Section 4, we finish the proof of Theorem 1.1 by combining the previous properties.

2 Properties of Shock Curves

Denote $U = (u,p)^{\mathrm{T}}$, then problem (1.2)–(1.3) can be rewritten as

$$U_t + AU_x = 0, \tag{2.1}$$

$$U_0(x) = (u_0(x), p_0(x))^{\mathrm{T}}, \tag{2.2}$$

where $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. System (2.1) is strictly hyperbolic for $p \neq 0$ with two eigenvalues $\lambda_1 = -\sqrt{p}$, $\lambda_2 = \sqrt{p}$, and the corresponding eigenvectors are

$$r_1 = (1, -\sqrt{p})^{\mathrm{T}}, \quad r_2 = (1, \sqrt{p})^{\mathrm{T}}. \tag{2.3}$$

It is easy to verify that

$$D\lambda_1 \cdot r_1 = \Big(0, -\frac{1}{2\sqrt{p}}\Big) \cdot (1, -\sqrt{p})^{\mathrm{T}} = \frac{1}{2} > 0, \tag{2.4}$$

$$D\lambda_2 \cdot r_2 = \Big(0, \frac{1}{2\sqrt{p}}\Big) \cdot (1, \sqrt{p})^{\mathrm{T}} = \frac{1}{2} > 0. \tag{2.5}$$

Thus both characteristics of (2.1) are genuinely nonlinear. 1-Riemann invariant w and 2-Riemann invariant z of (2.1), which are defined as two functions satisfying $Dw \cdot r_1 = 0$ and $Dz \cdot r_2 = 0$, respectively, can be given explicitly as

$$w = u + 2\sqrt{p}, \quad z = u - 2\sqrt{p}. \tag{2.6}$$

The function pair (z, w) is also called Riemann invariant coordinates system. For the general definition of Riemann invariant coordinates system, one can refer to [14].

For a given left state $U_0 = (u_0, p_0)^{\mathrm{T}}$, the i-rarefaction wave curve $R_i(U_0)$ $(i = 1,2)$ of (2.1) is defined as all the right states $U = (u,p)^{\mathrm{T}}$ that can connect U_0 by an i-rarefaction wave. These two curves in the (u,p) plane can be given explicitly by $w(u,p) = w(u_0,p_0)$ and $z(u,p) = z(u_0,p_0)$, respectively, that is

$$R_1(U_0): \ u_0 + 2\sqrt{p_0} = u + 2\sqrt{p}, \quad u_0 \le u, \ p_0 \ge p, \tag{2.7}$$

$$R_2(U_0): \ u_0 - 2\sqrt{p_0} = u - 2\sqrt{p}, \quad u_0 \le u, \ p_0 \le p, \tag{2.8}$$

where the range of p and u in (2.7) and (2.8) can be obtained by (2.4) and (2.5).

For a given left state $U_0 = (u_0, p_0)^{\mathrm{T}}$, the i-shock curve $S_i(U_0)$ $(i = 1,2)$ of (2.1) is defined as all the right states that can connect U_0 by an i-shock wave. It can be given in the (u,p) plane by Rankine-Hugoniot conditions, that is

$$s(u - u_0) = (p - p_0), \tag{2.9}$$

$$s\Big(p+\frac{1}{2}u^2-p_0-\frac{1}{2}{u_0}^2\Big)=(pu-p_0u_0), \tag{2.10}$$

where s is the speed of the i-shock.

Eliminating s from (2.9), (2.10), we can obtain

$$u=u_0\pm\sqrt{\frac{2}{p+p_0}}(p-p_0). \tag{2.11}$$

Taking the Lax entropy conditions, that is, $\lambda_1(U_0)>\lambda_1(U)$ for S_1 and $\lambda_2(U_0)>\lambda_2(U)$ for S_2, into account, we can give the equations of the two shock curves as follows

$$S_1(U_0):\ u=u_0-\sqrt{\frac{2}{p+p_0}}(p-p_0),\quad u_0\geq u,\ p_0\leq p, \tag{2.12}$$

$$S_2(U_0):\ u=u_0-\sqrt{\frac{2}{p+p_0}}(p_0-p),\quad u_0\geq u,\ p_0\geq p. \tag{2.13}$$

For a shock or rarefaction wave with right state $U=(u,p)^{\mathrm{T}}$ and left state $U_0=(u_0,p_0)^{\mathrm{T}}$, we denote

$$[p]=p-p_0,\quad [u]=u-u_0, \tag{2.14}$$

$$[z]=u-u_0-2(\sqrt{p}-\sqrt{p_0}),\quad [w]=u-u_0+2(\sqrt{p}-\sqrt{p_0}). \tag{2.15}$$

Combining (2.12)–(2.15) and by simple calculation, we can get the following lemma.

Lemma 2.1 *On the shock curves $S_1(U_0)$, $S_2(U_0)$, the changes of z, w satisfy*

$$|[z]|\geq|[w]|\quad \textit{for } U\in S_1(U_0); \tag{2.16}$$

$$|[z]|\leq|[w]|\quad \textit{for } U\in S_2(U_0). \tag{2.17}$$

Denote $\sigma=w+z$, $\eta=w-z$, by using (2.6), we have

$$\sigma=w+z=2u, \tag{2.18}$$

$$\eta=w-z=4\sqrt{p}, \tag{2.19}$$

$$u=\frac{\sigma}{2},\quad p=\frac{\eta^2}{16}. \tag{2.20}$$

Thus $S_1(U_0)$, $S_2(U_0)$ can be rewritten as

$$S_1(\sigma_0,\eta_0):\quad \sigma{=}\sigma_0-\frac{1}{\sqrt{2}}\frac{\eta^2-\eta_0^2}{\sqrt{\eta^2+\eta_0^2}},\quad \eta\geq\eta_0>0; \tag{2.21}$$

$$S_2(\sigma_0,\eta_0):\quad \sigma{=}\sigma_0-\frac{1}{\sqrt{2}}\frac{\eta_0^2-\eta^2}{\sqrt{\eta^2+\eta_0^2}},\quad 0<\eta\leq\eta_0. \tag{2.22}$$

For any fixed state U_0, all states U that can connect U_0 from the left by a 1-shock or 2-shock also form a curve. We denote them as $S_1^{-1}(U_0)$ and $S_2^{-1}(U_0)$, respectively. Their equations can be easily obtained as

$$S_1^{-1}(\sigma_0,\eta_0):\quad \sigma{=}\sigma_0+\frac{1}{\sqrt{2}}\frac{\eta_0^2-\eta^2}{\sqrt{\eta^2+\eta_0^2}},\quad 0<\eta\leq\eta_0; \tag{2.23}$$

$$S_2^{-1}(\sigma_0,\eta_0):\quad \sigma{=}\sigma_0+\frac{1}{\sqrt{2}}\frac{\eta^2-\eta_0^2}{\sqrt{\eta^2+\eta_0^2}},\quad \eta\geq\eta_0>0. \tag{2.24}$$

Lemma 2.2 *On the shock wave curves* $S_1(\sigma_0,\eta_0)$, $S_2(\sigma_0,\eta_0)$, $S_1^{-1}(\sigma_0,\eta_0)$, $S_2^{-1}(\sigma_0,\eta_0)$, *there hold*:

$$\text{on } S_1(\sigma_0,\eta_0): \quad -1 \le \frac{\partial\sigma}{\partial\eta} \le -\frac{1}{\sqrt{2}}; \tag{2.25}$$

$$\text{on } S_2(\sigma_0,\eta_0): \quad \frac{1}{\sqrt{2}} \le \frac{\partial\sigma}{\partial\eta} \le 1; \tag{2.26}$$

$$\text{on } S_1^{-1}(\sigma_0,\eta_0): \quad -1 \le \frac{\partial\sigma}{\partial\eta} \le -\frac{1}{\sqrt{2}}; \tag{2.27}$$

$$\text{on } S_2^{-1}(\sigma_0,\eta_0): \quad \frac{1}{\sqrt{2}} \le \frac{\partial\sigma}{\partial\eta} \le 1. \tag{2.28}$$

Proof For a fixed state (σ_0,η_0) (see Figure 1), denote $\varepsilon=\frac{\eta}{\eta_0}$, then $\varepsilon \ge 1$ on S_1. Along S_1, differentiating (2.21) with respect to η, we have

$$\frac{\partial\sigma}{\partial\eta} = -\frac{1}{\sqrt{2}}\frac{(\eta^3+3\eta\eta_0^2)}{(\eta^2+\eta_0^2)\sqrt{\eta^2+\eta_0^2}} = -\frac{1}{\sqrt{2}}\frac{\varepsilon^3+3\varepsilon}{(\varepsilon^2+1)\sqrt{\varepsilon^2+1}} < 0.$$

Denote $c(\varepsilon) = -\frac{1}{\sqrt{2}}\frac{\varepsilon^3+3\varepsilon}{(\varepsilon^2+1)\sqrt{\varepsilon^2+1}}$, then along S_1, we can easily verify that

$$c'(\varepsilon) = \frac{3}{\sqrt{2}}\frac{\varepsilon^2-1}{(\varepsilon^2+1)\sqrt{\varepsilon^2+1}} \ge 0.$$

That is, along S_1, $c(\varepsilon)$ is monotonically increasing and satisfies $-1 = c(1) < c(\varepsilon)$. In addition, direct calculation gives

$$\lim_{\varepsilon\to\infty} c(\varepsilon) = \lim_{\varepsilon\to\infty} -\frac{1}{\sqrt{2}}\frac{\varepsilon^3+3\varepsilon}{(\varepsilon^2+1)\sqrt{\varepsilon^2+1}} = -\frac{1}{\sqrt{2}}.$$

Thus we can get (2.25), and (2.26)–(2.28) can be similarly proved.

From the properties obtained above, in the (z,w) plane and on every curve of S_1, S_2, S_1^{-1} and S_2^{-1}, z can be looked as a function of w. For convenience, we denote them as $z = S_1(w,z_0,w_0)$, $z = S_2(w,z_0,w_0)$, $z = S_1^{-1}(w,z_0,w_0)$ and $z = S_2^{-1}(w,z_0,w_0)$, respectively. Among them, $z = S_1(w,z_0,w_0)$ and $z = S_2(w,z_0,w_0)$ are convex functions, $z = S_1^{-1}(w,z_0,w_0)$ and $z = S_2^{-1}(w,z_0,w_0)$ are concave functions. Figure 1 shows the shapes and positions of the four curves in the (σ,η) plane.

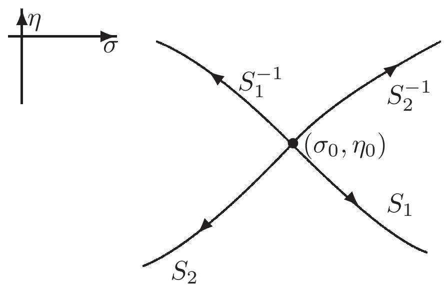

Figure 1

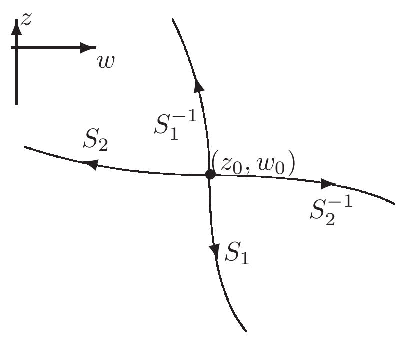

Figure 2

Lemma 2.3 *On (z,w) plane, the curves S_1, S_2, S_1^{-1}, S_2^{-1} satisfy*:

$$\text{on } S_1:\ -\infty \le \frac{\partial z}{\partial w} \le -(\sqrt{2}+1)^2; \tag{2.29}$$

$$\text{on } S_2:\ -(\sqrt{2}-1)^2 \le \frac{\partial z}{\partial w} \le 0; \tag{2.30}$$

$$\text{on } S_1^{-1}:\ -\infty \le \frac{\partial z}{\partial w} \le -(\sqrt{2}+1)^2; \tag{2.31}$$

$$\text{on } S_2^{-1}:\ -(\sqrt{2}-1)^2 \le \frac{\partial z}{\partial w} \le 0. \tag{2.32}$$

Proof From (2.18) and (2.19), we can get $\frac{\mathrm{d}\sigma}{\mathrm{d}w} = 1 + \frac{\partial z}{\partial w}$, $\frac{\mathrm{d}\eta}{\mathrm{d}w} = 1 - \frac{\partial z}{\partial w}$. Then we have $\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = \frac{1+\frac{\partial z}{\partial w}}{1-\frac{\partial z}{\partial w}}$, $\frac{\partial z}{\partial w} = \frac{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}-1}{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}+1}$. On the curve S_1, from $\frac{\mathrm{d}\sigma}{\mathrm{d}\eta} \in \big[-1, -\frac{1}{\sqrt{2}}\big]$, we can get $-1 \le \frac{1+\frac{\partial z}{\partial w}}{1-\frac{\partial z}{\partial w}} \le -\frac{1}{\sqrt{2}}$. Thus we have $\frac{\partial z}{\partial w} \le -(\sqrt{2}+1)^2$. In addition, since $\lim\limits_{\eta\to\infty} \frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = -\frac{1}{\sqrt{2}}$, we can get

$$\lim_{\eta\to\infty} \frac{\partial z}{\partial w} = \lim_{\eta\to\infty} \frac{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}-1}{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}+1} = -(\sqrt{2}+1)^2.$$

Similarly, we can use the property of $\lim\limits_{\eta\to\eta_0} \frac{\mathrm{d}\sigma}{\mathrm{d}\eta} = -1$ to get

$$\lim_{\eta\to\eta_0} \frac{\partial z}{\partial w} = \lim_{\eta\to\eta_0} \frac{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}-1}{\frac{\mathrm{d}\sigma}{\mathrm{d}\eta}+1} = -\infty.$$

Therefore, we have $-\infty \le \frac{\partial z}{\partial w} \le -(\sqrt{2}+1)^2$. So we can get (2.29), and (2.30)–(2.32) can be similarly proved. On (z,w) plane, the shock wave curves S_1, S_2, S_1^{-1}, S_2^{-1} are shown in Figure 2.

Let (z_0,w_0) and $(\overline{z},\overline{w})$ be two points in the (z,w) plane. Denote $\Delta z = |[z]| = |z-z_0|$, $\Delta w = |[w]| = |w-w_0|$, $\Delta z' = |[z']| = |z'-\overline{z}|$, $\Delta w' = |[w']| = |w'-\overline{w}|$.

Lemma 2.4 *Along the four curves S_1, S_2, S_1^{-1}, S_2^{-1}, there hold*
(i) *if $z = S_1(w,z_0,w_0)$, $z' = S_1(w',\overline{z},\overline{w})$, and $\Delta z = \Delta z'$, then $\Delta w' \le \Delta w$*;
(ii) *if $z = S_2(w,z_0,w_0)$, $z' = S_2(w',\overline{z},\overline{w})$, and $\Delta w = \Delta w'$, then $\Delta z' \le \Delta z$*;
(iii) *if $z = S_1^{-1}(w,z_0,w_0)$, $z' = S_1^{-1}(w',\overline{z},\overline{w})$, and $\Delta z = \Delta z'$, then $\Delta w' \le \Delta w$*;
(iv) *if $z = S_2^{-1}(w,z_0,w_0)$, $z' = S_2^{-1}(w',\overline{z},\overline{w})$, and $\Delta w = \Delta w'$, then $\Delta z' \le \Delta z$.*

Proof We only prove (i) since the proofs of the others are similar. The property (i) is shown in Figure 3.

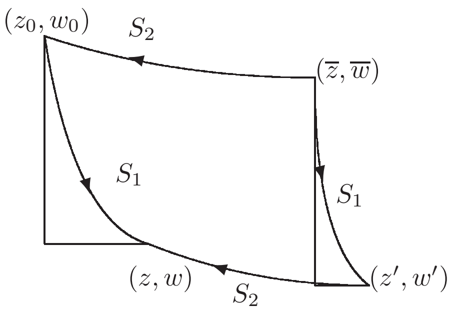

Figure 3

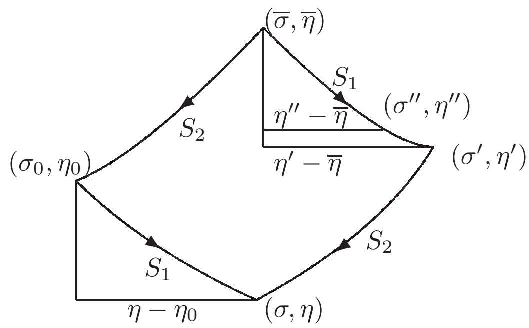

Figure 4

To prove (i), we only need to prove $w'-\overline{w} \le w-w_0$. Since $\Delta z = \Delta z'$, we have $z_0 - z = \overline{z} - z'$, and by (2.18)–(2.19), we only need to prove $\eta' - \overline{\eta} \le \eta - \eta_0$. If it is not true, we can suppose $\overline{\eta} > \eta_0$, that is $\eta' - \overline{\eta} > \eta - \eta_0$. From the convexity properties of shock curves, we see that there exist a point (σ'', η'') on $\sigma' = \sigma(\eta', \overline{\eta}, \overline{\sigma})$ such that $\eta'' - \overline{\eta} \le \eta - \eta_0$, and $k(\sigma_0, \eta_0; \sigma, \eta) = k(\overline{\sigma}, \overline{\eta}; \sigma'', \eta'')$, i.e., $\frac{\overline{\sigma}-\sigma''}{\eta''-\overline{\eta}} = \frac{\sigma_0-\sigma}{\eta-\eta_0}$. Therefore,

$$\frac{\eta''}{\overline{\eta}} = \frac{\eta}{\eta_0} = \varepsilon > 1,$$

and we have $\overline{\eta}(\varepsilon - 1) = \eta'' - \overline{\eta} \le \eta - \eta_0 = \eta_0(\varepsilon - 1)$ or $\overline{\eta} \le \eta_0$, which contradicts with the hypothesis. The proof is complete.

Denote

$$Q(z_0, w_0) = \{(z, w) \mid z \le z_0, w \ge w_0\}, \tag{2.33}$$

where $z_0 = \sup\limits_{x\in R} z(U(x, 0))$, $w_0 = \inf\limits_{x\in R} w(U(x, 0))$. Then $Q(z_0, w_0)$ has the following property.

Lemma 2.5 (see [4]) *If* $U(x,t) = (u,p)^{\mathrm{T}}$ *is a solution to the Riemann problem* (2.1)*, then* $(z, w) \triangleq (z(u, p), w(u, p)) \in Q(z_0, w_0)$.

By (2.6), Lemmas 2.2–2.4, the pressure-gradient system satisfies the following Diperna's conditions.

(A_1) $\nabla w \cdot r_1 = \nabla z \cdot r_2 = 0$, $\nabla z \cdot r_1 > 0$, $\nabla w \cdot r_2 > 0$.

(A_2) In the (z, w) plane, shock curves satisfy:

$$\text{on } S_1: \ -\infty \le \frac{\partial z}{\partial w} \le -1; \tag{2.34}$$

$$\text{on } S_2: \ -1 \le \frac{\partial z}{\partial w} \le 0; \tag{2.35}$$

$$\text{on } S_1^{-1}: \ -\infty \le \frac{\partial z}{\partial w} \le -1; \tag{2.36}$$

$$\text{on } S_2^{-1}: \ -1 \le \frac{\partial z}{\partial w} \le 0. \tag{2.37}$$

(A_3) If $z_r = S_i(w_r; z_l, w_l)$, then $S_i(w; z_l, w_l) \le S_i^{-1}(w; z_r, w_r)$, $i = 1, 2$, $w \in [w_r, w_l]$. That is, S_i^{-1} is at the top of S_i.

(A_4) Let $(\overline{z}, \overline{w}) \in Q(z_0, w_0)$, along the shock curves S_1, S_2 respectively, then there hold

(1) if $z = S_1(w, z_0, w_0)$, $z' = S_1(w', \overline{z}, \overline{w})$ and $\Delta z = \Delta z'$, then $\Delta w' \le \Delta w$;

(2) if $z = S_2(w, z_0, w_0)$, $z' = S_2(w', \overline{z}, \overline{w})$ and $\Delta w = \Delta w'$, then $\Delta z' \le \Delta z$.

Lemma 2.6 *Let* $z = S_i(w, z_0, w_0)$, $z' = S_i(w', \overline{z}, \overline{w})$, $i = 1, 2$ (*or equivalently,* $z = S_i^{-1}(w, z_0, w_0)$, $z' = S_i^{-1}(w', \overline{z}, \overline{w})$, $i = 1, 2$). *Suppose that* $(\overline{z}, \overline{w}) \in Q(z_0, w_0)$, *then we have*

(i) *if* $i = 1$, *and* $\Delta z \le \Delta z'$, *then* $\Delta\sigma \le \Delta\sigma'$;

(ii) *if* $i = 2$, *and* $\Delta w \le \Delta w'$, *then* $\Delta\sigma \le \Delta\sigma'$.

Proof Here we only prove (i) (the proof of (ii) is similar). At this place there hold $z = S_1(w, z_0, w_0)$, $z' = S_1(w', \overline{z}, \overline{w})$. We take $\overline{z} = z_0$ (the case of $\overline{z} < z_0$ can be similarly proved), then $(\overline{z}, \overline{w}) \in Q(z_0, w_0)$. On the curve $z = S_1(w, z_0, w_0)$, we choose a point $(\widetilde{z}, \widetilde{w})$ satisfying $\widetilde{z} - z_0 = z' - \overline{z}$. It follows from Lemma 2.4 that $0 < w' - \overline{w} \le \widetilde{w} - w_0$. Now we have $0 < z_0 - z = \Delta z \le \Delta z' = z' - \overline{z}$, $z \ge \widetilde{z}$, it follows from (2.16) that $\widetilde{\sigma} - \sigma = \widetilde{w} - w + \widetilde{z} - z \le 0$, namely, $\widetilde{\sigma} \le \sigma$. Therefore

$$\Delta\sigma = \sigma_0 - \sigma \le \sigma - \widetilde{\sigma} = z_0 - \widetilde{z} + w_0 - \widetilde{w}$$

$$= \overline{z} - z' + w_0 - \widetilde{w} < \overline{z} - z' + \overline{w} - w'$$
$$= \overline{\sigma} - \sigma' = \Delta\sigma'.$$

Thus we can get $\Delta\sigma \leq \Delta\sigma'$.

3 Construction of Approximate Solutions

We use Glimm scheme to construct an approximate solution sequence, which is denoted as $\{U^h(x,t)\}$ for $t \geq 0$. Fix a spatial mesh-length $l = \Delta x > 0$ and a temporal mesh-length $h = \Delta t > 0$ satisfying the Courant-Friedrichs-Lewy condition

$$\lambda h \leq l, \tag{3.1}$$

where $\lambda = \sup\limits_{i=1,2} |\lambda_i(U)|$. Denote $x_m = ml$, $t_n = nh$ and denote the time strip $s_n = \{(x,t) : nh \leq t < (n+1)h\}$, $n = 0,\ 1,\ 2, \cdots$, and $\sum\limits_n = \bigcup\limits_{k=1}^{n} s_k$.

Let $\alpha_n (n = 1, 2, \cdots)$ be a random point in $(-1, 1)$ and denote $y_{m,n} = (x_m + \alpha_n)l$, where m is an integer, and $m + n$ is an even number.

For $t = 0$, $\{U^h(x,0)\}$ is defined as

$$U^h(x,0) = \frac{1}{2l}\int_{(m-1)l}^{(m+1)l} U_0(x)\mathrm{d}x, \quad (m-1)l < x < (m+1)l,\ m = \pm 1, \pm 3, \cdots. \tag{3.2}$$

Obviously $\{U^h(x,0)\}$ satisfies

$$\lim_{h\to 0} \frac{1}{2l}\int_0^{2l} (U^h(x,0) - \overline{U}_0)\mathrm{d}x = 0, \tag{3.3}$$

where $\overline{U}_0 = \frac{1}{2l}\int_0^{2l} U_0(x)\mathrm{d}x$. Obviously, $\{U^h(x,0)\}$ satisfy

$$\lim_{h\to 0} U^h(\cdot\,,0) = U_0(\cdot), \tag{3.4}$$
$$\mathrm{TV}U^h(\cdot\,,0) \leq \mathrm{TV}U_0(\cdot), \tag{3.5}$$
$$\sup z(U^h(\cdot\,,0)) \leq \sup z(U_0(\cdot)), \tag{3.6}$$
$$\inf w(U^h(\cdot\,,0)) \geq \inf w(U(\cdot)). \tag{3.7}$$

Assuming that $\{U^h(x,t)\}$ has been constructed in $\bigcup\limits_{k=1}^{n} s_k$, we continue to construct approximate solution $\{U^h(x,t)\}$ in s_{n+1}. Define

$$U^h(x,nh) = U^h(y_{m,n},\ nh - 0), \quad (m-1)l < x < (m+1)l,\ m+n \text{ is odd}. \tag{3.8}$$

Then we solve Riemann problem (2.1), (3.8) in $t > t_n$, and the solution in between $t_n < t < t_{n+1}$ is defined as $\{U^h(x,t)\}$ in s_{n+1}.

In order to prove the convergence of the sequence $\{U^h(x,t)\}$, we need to prove that there exists some positive constant C, such that

$$\sup_{x\in(-\infty,+\infty)} |U^h(x,t)| \leq C \sup_{x\in(-\infty,+\infty)} |U_0(x)|, \tag{3.9}$$

$$\mathrm{TV}(U^h(\cdot,t))|_{(-\infty,+\infty)} \leq \mathrm{TV}(U_0^h(\cdot))|_{(-\infty,+\infty)}, \quad 0 \leq t < +\infty, \tag{3.10}$$

$$\int_{-\infty}^{+\infty} |U^h(x,t_1) - U^h(x,t_2)|\mathrm{d}x$$
$$\leq C(|t_1 - t_2| + h)\mathrm{TV}(U_0(\cdot))|_{(-\infty,+\infty)}, \quad \forall t_1, t_2 \in [0, +\infty). \tag{3.11}$$

Thus due to the Helly's theorem, as $h \to 0$, there exists a convergent subsequence of $\{U^h(x,t)\}$. Denote the limit function as $U(x,t)$, then by standard process we can verify that $U(x,t)$ is a weak solution to problem (2.1)–(2.2).

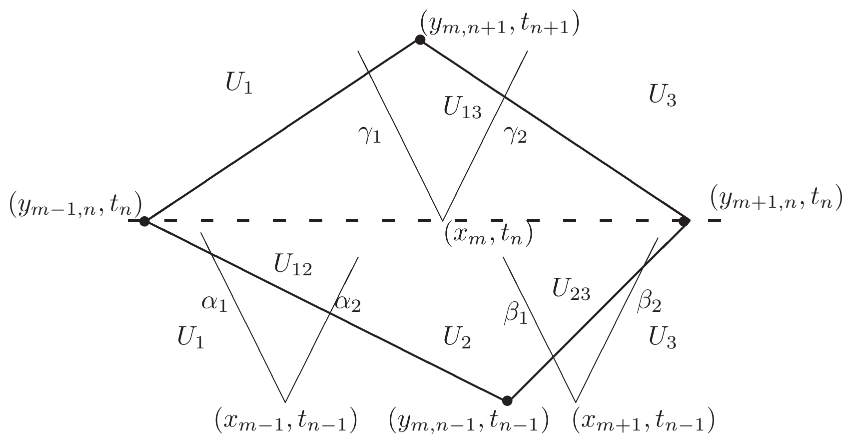

Figure 5

As shown in Figure 5, let $D_{m,n}$ be a "diamond" with $(y_{m-1,n}, t_n)$, $(y_{m,n-1}, t_{n-1})$, $(y_{m+1,n}, t_n)$, $(y_{m,n+1}, t_{n+1})$ as its vertices. All of these "diamonds" cover the upper half of (x,t) plane. The elementary waves issuing from (x_{m-1}, t_{n-1}) and entering $D_{m,n}$ are denoted as $\alpha = (\alpha_1, \alpha_2)$. The left, middle and right states of waves (α_1, α_2) are denoted by U_1, U_{12} and U_2, respectively. The waves issuing from (x_{m+1}, t_{n-1}) and entering $D_{m,n}$ are denoted as $\beta = (\beta_1, \beta_2)$. The left, middle and right states of waves (β_1, β_2) are denoted by U_2, U_{23} and U_3, respectively. The elementary waves issuing from (x_m, t_n) are denoted as $\gamma = (\gamma_1, \gamma_2)$. The left, middle and right states of waves (γ_1, γ_2) are denoted by U_1, U_{13} and U_3, respectively. Every elementary wave may be a shock wave or a rarefaction wave. Denote $|\alpha| = |\alpha_1| + |\alpha_2|$, $|\beta| = |\beta_1| + |\beta_2|$ and $|\gamma| = |\gamma_1| + |\gamma_2|$. Due to (2.18), we define $\sigma_h = w(U_h) + z(U_h)$, and define the strengths of α_1, α_2 as

$$[\sigma_h(\alpha_1)]^+ = (\sigma_h(U_{12}) - \sigma_h(U_1))^+, \quad [\sigma_h(\alpha_2)]^+ = (\sigma_h(U_2) - \sigma_h(U_{12}))^+.$$

The strengths of β_1, β_2, γ_1, γ_2 can be similarly defined. The symbol "+" means the positive part of a number, that is, $a^+ = \max\{a, 0\}$.

A mesh curve, associated with U^h, is a polygonal graph with vertices that from a finite sequence of sample points $(y_{m_1,n_1}, t_{n_1}), \cdots, (y_{m_l,n_l}, t_{n_l})$. A mesh curve J is called an immediate successor of the mesh curve I when $J\backslash I$ is the upper boundary of some diamond, and $I\backslash J$ is the lower boundary of diamond. Thus J has the same vertices as I, save for one, $(y_{m,n-1}, t_{n-1})$, which is replaced by $(y_{m,n+1}, t_{n+1})$. This induces a natural partial ordering in the family of mesh curves: J is a successor of I, denoted $J > I$, whenever there is a finite sequence, namely, $I = I_0, I_1, \cdots, I_n = J$ of mesh curves such that I_l is an immediate successor of I_{l-1}, for $l = 1, \cdots, n$.

On the mesh curve J, define the Glimm functional as follows

$$F(J) = \Big\{ \sum [\sigma_h(\alpha)]^+ : \alpha \text{ crossing } J \Big\}. \tag{3.12}$$

Since $\sigma_h = w(U_h) + z(U_h)$, it is known from (2.25)–(2.26) that crossing a shock curves S_1, S_2 there holds $[\sigma_h]^+ = (\sigma_l - \sigma_r)^+ \geq 0$. From (2.18) that crossing a rarefaction wave curves R_1, R_2, we have $[\sigma_h]^+ = (\sigma_l - \sigma_r)^+ = 0$. Thus, $F(J)$ represents the total variation of the shocks crossing J.

The following proposition gives the monotonicity of the Glimm functional defined by (3.12).

Proposition 3.1 *If $J > I$, we have*

$$F(J) \leq F(I). \tag{3.13}$$

Proof When a mesh curve J is an immediate successor of I, which is shown in Figure 5, we need only to prove $[\sigma_h(\gamma)]^+ \leq [\sigma_h(\alpha)]^+ + [\sigma_h(\beta)]^+$. As shown in Figure 5, α_1, α_2, β_1, β_2, γ_1, γ_2 are shocks or rarefaction waves. When γ_1 and γ_2 are both shocks, whatever the incoming waves are, (3.13) always hold. When γ_1 and γ_2 are both rarefaction waves, we can easily get $0 = F(J) \leq F(I)$. We need only consider the case when γ_1 is a 1-rarefaction wave and γ_2 is a 2-shock wave. According to [4], the waves α_1, α_2, β_1, β_2 can be divided into 16 cases. In the following S or R is used to represent that α_1, α_2, β_1, β_2 is a shock or a rarefaction wave, respectively. For example, when α_1 is a shock, denote α_1 as S. In addition, denote $\sigma_1 = \sigma(U_1)$, $\sigma_{12} = \sigma(U_{12})$, and so on.

(1) Cases $RRRR$, $RRRS$, $RSRR$ and $RSRS$. These four cases are obviously impossible.

(2) Case $SRRR$. On the curve S_1, by Lemma 2.1, we have $|w_{12} - w_{13}| < |z_{12} - z_{13}|$. Since $z_{12} = z_2 < z_{23} = z_3 = z_{13}$, we have $w_{13} < w_{12}$. Thus $w_{12} - w_{13} + z_{12} - z_{13} < 0$, and then

$$F(J) - F(I) = (\sigma_1 - \sigma_{13}) - (\sigma_1 - \sigma_{12}) = w_{12} + z_{12} - z_{13} - w_{13} < 0,$$

which is exactly $[\sigma_h(\gamma)]^+ \leq [\sigma_h(\alpha)]^+ + [\sigma_h(\beta)]^+$. The proofs of cases $SSRR$, $SRRS$, $SSRS$, $SRSR$ and $SRSS$ are similar.

(3) Case $RRSR$. In this case $z_3 = z_{13} = z_{23} < z_1 < z_{12} = z_2$. We can divide it into two subcases: $w_{13} \leq w_{23}$ (see Figure 6) and $w_{13} > w_{23}$ (see Figure 7). When $w_{13} \leq w_{23}$, $z_1 - z_{13} < z_2 - z_{23}$, from Lemma 2.6, we can get $\sigma_1 - \sigma_{13} < \sigma_2 - \sigma_{23}$, which implies that $F(J) - F(I) = (\sigma_1 - \sigma_{13}) + (\sigma_{23} - \sigma_2) < 0$. Thus $[\sigma_h(\gamma)]^+ \leq [\sigma_h(\alpha)]^+ + [\sigma_h(\beta)]^+$ holds.

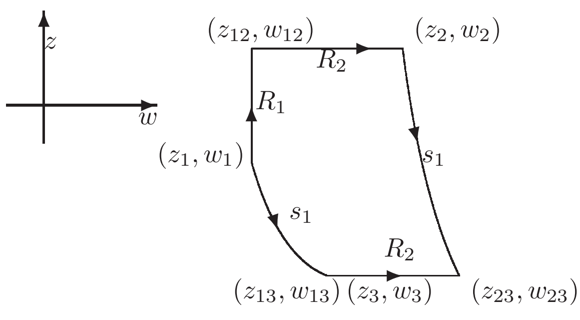

Figure 6

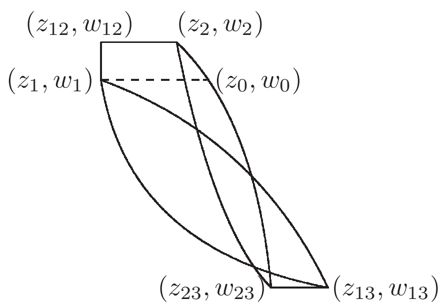

Figure 7

Next we show that $w_{13} > w_{23}$ is impossible. If $w_{13} > w_{23}$ and $z_{13} = z_{23}$, we have $\sigma_{23} = w_{23} + z_{23} < w_{13} + z_{13} = \sigma_{13}$. Choose a point (z_0, w_0), satisfying $z_0 = z_1$ and $z_{23} = S_1^{-1}(w_{23}, z_2, w_2)$. Since $(z_{13}, w_{13}) \in Q(z_{23}, w_{23})$ and $z_0 - z_{23} = z_1 - z_{13}$, we see from Lemma 2.1 that $w_{13} - w_1 < w_{23} - w_0$. Now, $w_{13} > w_{23}$ implies $w_1 - w_0 > w_{13} - w_{23} > 0$, which is a contradiction since we have $w_1 < w_0$. Cases $RRSS$, $RSSS$ and $SSSR$ can be treated analogously.

(4) Case $RSSR$. In this case, we have $z_3 = z_{13} = z_{23} < z_1 < z_{12} < z_2$, then $z_1 - z_{13} < z_2 - z_{23}$. Suppose $w_{13} \leq w_{23}$, then from Lemma 2.6 we know that $\sigma_1 - \sigma_{13} < \sigma_2 - \sigma_{23}$. By using of Lemma 2.1, we have $\sigma_{12} - \sigma_2 = w_{12} - w_2 + z_{12} - z_2 > 0$, that is, $\sigma_{12} > \sigma_2$, thus $\sigma_1 - \sigma_{13} < \sigma_2 - \sigma_{23} < \sigma_{12} - \sigma_{23}$. Therefore, we can get

$$\begin{aligned}[\sigma_h(\gamma)]^+ - [\sigma_h(\alpha)]^+ - [\sigma_h(\beta)]^+ &= (\sigma_1 - \sigma_{13}) - (\sigma_{12} - \sigma_2 + \sigma_2 - \sigma_{23}) \\ &= \sigma_1 - \sigma_{13} + \sigma_{23} - \sigma_{12} \leq 0.\end{aligned}$$

If $w_{13} > w_{23}$, it is obvious that $[\sigma_h(\gamma)]^+ \leq [\sigma_h(\alpha)]^+ + [\sigma_h(\beta)]^+$ holds.

(5) Case $SSSS$. This case implies $w_{23} > w_{13}$, which is impossible for the similar reason to case (3).

Therefore, from Proposition 3.1, we have $F(J) \leq F(0)$, where "0" is the unique I-curve between the two lines $t = 0$ and $t = s$.

In the following we study the strength of the waves for Riemann problem (2.1)–(2.2), where U_l, $U_r \in Q$, $Q = \{U \in R^2; z(U) \leq z_0, w(U) \geq w_0\}$. For simplicity, we use (U_l, U_r) to denote the solution of Riemann problem (2.1)–(2.2). According to Lemma 2.5, there is an intermediate state U_m with U_l as the left state and U_r as the right state.

Define the strength of elementary waves for the solution (U_l, U_r) to Riemann problem (2.1)–(2.2) in two different ways. The first one is

$$d(U_l, U_r) = (\sigma_l - \sigma_m)^+ + (\sigma_m - \sigma_r)^+. \tag{3.14}$$

From the discussion in Section 3, we know that for rarefaction waves, $(\sigma_k - \sigma_{k+1})^+ = 0$, and (3.14) only records the strengths of shocks. The second definition of the strength is

$$\overline{d}(U_l, U_r) = (u_l - u_m)^+ + (p_l - p_m)^+ + (u_m - u_r)^+ + (p_m - p_r)^+. \tag{3.15}$$

It also records only the strengths of the shocks.

Lemma 3.1 *Denote $U = (u,\ p)^{\mathrm{T}}$ as the solution to the Riemann problem* (2.1) *and* (1.3), *then there exists some positive constant c, such that*

$$d(U_l, U_r) \leq c\overline{d}(U_l, U_r). \tag{3.16}$$

Proof From (2.18), (3.14) and (3.15), we have

$$\begin{aligned}d(U_l, U_r) &= (\sigma_l - \sigma_m)^+ + (\sigma_m - \sigma_r)^+ \\ &= 2(u_l - u_m)^+ + 2(u_m - u_r)^+\end{aligned}$$

$$\begin{aligned}&\le 2[(u_l-u_m)^+ + (u_m-u_r)^+ + (p_l-p_m)^+ + (p_m-p_r)^+]\\&= 2\overline{d}(U_l,U_r).\end{aligned}$$

Take $c=2$, then the lemma is proved.

Let $\mathrm{DV}(z(U^h(t)))$ and $\mathrm{DV}(w(U^h(t)))$ be the decreasing total variation of $z(U^h(t))$ and $w(U^h(t))$, respectively.

Lemma 3.2 *For any fixed $t\in[nh,(n+1)h)$, we have*

$$\mathrm{DV}(w(U^h(t)))+\mathrm{DV}(z(U^h(t)))\le\sum_{\alpha,\beta}\{|[\sigma(U^h(t)]|+|[\eta(U^h(t)]|\},\tag{3.17}$$

where $\sum\limits_{\alpha,\beta}$ is taken over all shocks at time t.

Proof From the analysis of Section 3, we know that $w(U^h(t))$ and $z(U^h(t))$ can only increase or remain unchanged when they cross a rarefaction wave, and they can only decrease when they cross a shock. When two states U_l and U_m are connected by a 1-shock, there hold $w(U_l)<w(U_m)$, $z(U_m)<z(U_l)$. By (2.19) we have

$$\eta_m-\eta_l=w_m-w_l+z_l-z_m.$$

Therefore,

$$\begin{aligned}\mathrm{DV}(w(U^h(t)))+\mathrm{DV}(z(U^h(t)))&=z(U_l)-z(U_m)\\&\le z(U_l)-z(U_m)+w(U_m)-w(U_l)\\&=\eta_m-\eta_l=|[\eta(U)]|\le|[\sigma(U)]|+|[\eta(U)]|.\end{aligned}$$

Similarly, it can be shown that when U_m and U_r are connected by a 2-shock, (3.17) still holds.

4 Proof of Main Theorem

To prove Theorem 1.1, we need only to prove that the approximate solution sequence $\{U^h(x,t)\}$ constructed in Section 3 satisfies (3.9)–(3.11). We complete the proof by several lemmas and a main theorem.

Lemma 4.1 *Suppose that there exist some positive constants c, p_0 such that $p<p_0$, then $\mathrm{TV}(U^h(x,t))|_{[0,\infty)}$ is uniformly bounded for $t>0$.*

Proof Combining Lemmas 3.1–3.2, Proposition 3.1 and Lemma 2.2, we can get

$$\begin{aligned}\mathrm{TV}(U^h(x,t))|_{[0,\infty)}&\le\sup(1+\sqrt{p})(\mathrm{TV}(w(U^h(x,t)))|_{[0,\infty)}+\mathrm{TV}(z(U^h(x,t)))|_{[0,\infty)})\\&\le 2(1+\sqrt{p_0})(\mathrm{DV}(w(U^h(x,t)))|_{[0,\infty)}+\mathrm{DV}(z(U^h(x,t)))|_{[0,\infty)})\\&\le 2(1+\sqrt{p_0})\sum_{\alpha,\beta}\{|[\sigma(U^h(t))]|+|[\eta(U^h(t))]|\}\\&\le 2(1+\sqrt{p_0})(1+\sqrt{2})\mathrm{DV}(\sigma(U^h(t)))|_{[0,\infty)}\end{aligned}$$

$$
\begin{aligned}
&= 2(1+\sqrt{p_0})(1+\sqrt{2})F(U^h(t)) \le 2(1+\sqrt{p_0})(1+\sqrt{2})F(U^h(0)) \\
&\le 2c(1+\sqrt{p_0})(1+\sqrt{2})\mathrm{TV}(U^h(0))|_{[0,\infty)} \\
&\le 2c(1+\sqrt{p_0})(1+\sqrt{2})\mathrm{TV}(U_0)|_{[0,\infty)},
\end{aligned}
$$

where c is the same as in Lemma 3.1. The proof is complete.

Lemma 4.2 *For any $t > 0$, we have*

$$
\|U^h(x,t)\|_\infty \le \|U_0\|_\infty + \mathrm{TV}_{(-\infty,+\infty)}U_0(x). \tag{4.1}
$$

Proof For any $t > 0$, by Lemmas 3.1 and 4.1, we have

$$
\|U^h(x,t)\|_{L^\infty} - \|U_0(x)\|_{L^\infty} \le \|U^h(x,t) - U_0(x)\|_{L^\infty} \le F(J) \le F(0) \le \mathrm{TV}\|U_0\|.
$$

Thus (4.1) follows.

Theorem 4.1 *For any $t_1, t_2 > 0$, the approximate solution $U^h(x,t)$ satisfies*

$$
\int_{-\infty}^{+\infty} |U^h(x,t_1) - U^h(x,t_2)|\,\mathrm{d}x \le C(|t_1 - t_2| + h), \tag{4.2}
$$

where C depends only on c, p_0.

Proof By Lemmas 4.1–4.2, we can get

$$
\begin{aligned}
|U^h(x,t_1) - U^h(x,t_2)| &\le |U^h(x,t_1) - U^h(x,\tau)| + |U^h(x,t_2) - U^h(x,\tau)| \\
&\le C\mathrm{TV}(U(\cdot,t_1),[x-Nl,x+Nl]) + C\mathrm{TV}(U(\cdot,t_2),[x-Nl,x+Nl]),
\end{aligned}
$$

then in combination with (4.1), we can get (4.2), where $N = \max\left[\frac{t_i-\tau}{h}\right] + 1, i = 1, 2$.

From Lemmas 4.1–4.2 and Theorem 4.1, according to Helly's theorem, there exists a convergent subsequence $\{U^{h_m}(x,t)\}$ of $\{U^h(x,t)\}$, such that

$$
U^{h_m}(x,t) \to U(x,t) \quad \text{as } h_m \to 0, \text{ a.e.}
$$

By standard procedure, we can verify that $U(x,t)$ is a global weak solution of (2.1)–(2.2). (3.9) can be obtained from (4.1). (3.10) can be obtained from Proposition 3.2 and Lemma 4.1. Finally, (3.11) can be obtained from Lemma 4.1 and Theorem 4.1. Thus, we have proved that $U(x,t)$ is a weak solution to the problem (2.1)–(2.2).

References

[1] Dafermos, C. M., Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag, Berlin, Heidelberg, 1999.

[2] Ding, M., Stability of rarefaction wave to the 1-D piston problem for the pressure-gradient equations, *Chin. Ann. Math. Ser. B,* **40**(2), 2019, 161–186.

[3] Ding, X. Q., Zhang, T., Wang, J. H., et al., On the global solution of nonlinear hyperbolic system of conservation laws (in Chinese), *Chin. Sci.,* **16**(3), 1973, 239–254.

[4] Diperna, R. J., Global solutions to a class of nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.,* **26**(1), 1973, 1–28.

[5] Diperna, R. J., Existence in the large for quasilinear hyperbolic conservation laws, *Arch. Ration. Mech. Anal.,* **52**(3), 1973, 244–257.

[6] Frid, H., Periodic solutions of conservation laws constructed through glimm scheme, *Trans. Am. Math. Soc.,* **353**(11), 2001, 4529–4544.

[7] Glimm, J., Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.,* **18**(4), 1965, 697–715.

[8] Li, Y. C., Feng, D. M. and Wang, Z. J., Global entropy solutions to the relativistic euler equations for a class of large initial data, *Z. Angew. Math. Phy.,* **56**(2), 2005, 239–253.

[9] Liu, Y. and Sheng, W. C., The interaction of shock waves of pressure difference equations in gas dynamics, *Acta Math. Sci.,* **25A**(2), 2005, 277–280 (in Chinese).

[10] Nishida, T., Global solution for an initial boundary value problem of a quasilinear hyperbolic system, *Proc. Japan Acad.,* **44**(7), 1968, 642–646.

[11] Nishida, T. and Smoller, J., Solutions in the large for some nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.,* **26**(2), 1973, 183–200.

[12] Wang, Z. J. and Zhang, Q., Periodic solutions to p-system constructed through glimm scheme, *J. Math. Anal. Appl.,* **435**(2), 2016, 1088–1098.

[13] Xu, Y. L. and Huang, H., Global existence of shock front solution to piston problem of pressure-gradient system, *Comm. Pure Appl. Math. Com.,* **30**(3), 2016, 386–393(in Chinese).

[14] Ying, L. A. and Teng, Z. H., Hyperbolic Conservation Laws Equation and Finite Difference Method, Sci. Press., Beijing, 1991, 35–45 (in Chinese).

[15] Zhang, L. Y. and Sheng, W. C., One-dimensional piston problem of pressure-gradient equations, *Jour. Shanghai University.,* **15**(4), 2009, 388–393.

[16] Zhang, T., Yang, H. C. and He, Y. N., Interactions between two rarefaction waves for the pressure-gradient equations in the gas dynamics, *Appl. Math. Com.,* **199**(1), 2008, 231–241.