Chin. Ann. Math. Ser. B
44(6), 2023, 803–818
DOI: 10.1007/s11401-023-0045-6

Chinese Annals of Mathematics, Series B

An Overview of the Dynamic Framework in Earth-System Model and Its Well-Posedness*

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Abstract The well-posedness of the dynamic framework in earth-system model (ESM for short) is a common issue in earth sciences and mathematics. In this paper, the authors first introduce the research history and fundamental roles of the well-posedness of the dynamic framework in the ESM, emphasizing the three core components of ESM, i.e., the atmospheric general circulation model (AGCM for short), land-surface model (LSM for short) and oceanic general circulation model (OGCM for short) and their couplings. Then, some research advances made by their own research group are outlined. Finally, future research prospects are discussed.

Keywords Earth-System model, Dynamic framework, Well-posedness
2000 MR Subject Classification 35Q30, 35Q35, 86A10

1 Introduction

The earth-system model (ESM for short) is a mathematical expression describing the coupled evolution of global climate and eco-environmental systems (see [1]). Specifically, it is composed of complex numerical models based on various mathematical equations describing the processes of the atmosphere, hydrosphere, cryosphere, surface layer of the earth, lithosphere and biosphere as well as their coupled evolutions (see [2]). The ESM plays a crucial role in predicting global climate change and planning for sustainable development (see [1]), so its advancement is an important index for measuring the overall progress of a country's earth science research (see [3]).

During the development of the ESM, many major breakthroughs have been closely related to the development of mathematical theories and the introduction of advanced mathematical methods. In the 1970s and 1980s, Zeng [4–5] developed the three-dimensional compressible baroclinic hydrodynamic equations coupled with thermodynamic equations with reasonable initial data and boundary conditions, which provided the precise dynamic frameworks for atmospheric and oceanic general circulation models (i.e., the "climate-system model"). The well-posedness

Manuscript received June 2, 2023. Revised June 26, 2023.
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*This work was supported by the National Natural Science Foundation of China (Nos. 41975129, 41630530), Key Research Program of Frontier Sciences, Chinese Academy of Sciences (No. QYZDY-SSW-DQC002) and the National Key Scientific and Technological Infrastructure project "Earth System Science Numerical Simulator Facility" (EarthLab).

of this dynamic framework was preliminarily analyzed. Similarly, equations describing the physical processes of the earth's surface layer, the ecological processes of vegetation and geochemical processes were later developed, and provided the dynamic frameworks for the "environmental ecosystem model." The coupling between the climate-system model and environmental ecosystem model creates the ESM.

After mathematical modeling, in the 1980s and 1990s, Zeng et al. [6–15] developed several effective numerical algorithms for solving the primitive equations of atmospheric dynamics in a spherical coordinate system, gradually constructing three generations of an atmospheric general circulation model (IAP-AGCM for short). Zhang et al. [16–18] also gradually developed two generations of an oceanic general circulation model (IAP-OGCM for short). After considering the mathematical consistency between the frameworks of IAP-AGCM and IAP-OGCM, Zeng and his research team in [19] developed the first climate-system model IAP-NCSM and numerical climate prediction system IAP-NCP in China, which consisted of atmospheric, ocean and land-surface models; IAP-NCSM was also a component of the IPCC climate simulation assessment. Since 1998, IAP-NCP was applied in practical real time prediction of seasonal climate anomalies in China (see [20–23]) to predict large-scale patterns of summer rainfall anomalies, and major climatic disasters have been successfully predicted such as the catastrophic flood over the Changjiang River basin during the summer of 1998, floods in South China and droughts in North China in 1999, and massive droughts nationwide in 2000 and 2001.

Since 2007, the Institute of Atmospheric Physics (IAP for short) in the Chinese Academy of Sciences (CAS for short), has begun to develop the ESM of China. On the basis of the climate-system model IAP-NCSM, it has gradually incorporated a sea-ice model, dynamic global-vegetation model, aerosol and atmospheric chemistry model, marine biogeochemical model and a land-surface biogeochemical model, among others. In 2015, the first generation of the Chinese Academy of Sciences-Earth System Model (CAS-ESM 1.0 for short) was released, indicating that China had made significant progress in the research and development of the ESM. After continuous improvement, the second generation of CAS-ESM (CAS-ESM 2.0 for short) was released in 2018. It is worth noting that CAS-ESM was independently developed by China, possessing distinct Chinese characteristics and unique advantages (see [1, 24]).

CAS-ESM 2.0 has also participated in the sixth phase of the Coupled Model Intercomparison Project (CMIP6 for short) and produced a large number of public simulation datasets (see [25]). Many foreign research teams have used these simulation datasets and created comparisons between CAS-ESM 2.0 and more than thirty other models worldwide. The results show that CAS-ESM 2.0 is one of the best models for comprehensiveness and simulation performance in the world. Now, CAS-ESM 2.0 has become the core software of "Earth System Science Numerical Simulator Facility" which is a major national infrastructure component for science and technology completed by the IAP and some other institutes of mathematics and computational technology. In addition, the immense number of databases in this facility are publicly available worldwide.

When constructing and developing a numerical model (e.g., introducing new sub-system models or developing/improving parameterizations), we also pay substantial attention to the research on the well-posedness of the dynamic framework of models. If the physical models are not well-posed in a mathematical sense, the simulation results are "unreasonable", and uncertainty may substantially increase with simulation time. Therefore, in order to correctly simulate or predict the progressive evolution of the Earth system, it is necessary to further investigate the well-posedness of the model's framework.

In this paper, we will firstly focus on the major core components of CAS-ESM 2.0: The atmospheric general circulation model (AGCM for short), land-surface model (LSM for short) and oceanic general circulation model (OGCM for short), and then make a review on research advances on their well-posedness made by our research team in recent years. In the conclusions, future research prospects will be outlined as well.

2 Climate-System Models

The well-posedness is the cornerstone of developing climate-system/earth-system models. As early as 1979, Zeng [4] dedicated a chapter in book "Mathematical and Physical Foundations of Numerical Weather Prediction" to discuss the well-posedness theory of several types of atmospheric models using applied mathematical methods, and clearly showed that the corresponding partial differential equations with boundary conditions and initial data should be internally consistent, and the existence, uniqueness and stability of the classical solution (with continuous partial derivatives of any order) to a two-dimensional barotropic model and three-dimensional baroclinic model can be proved. By using mathematical tools and analytical methods (such as the functional space and generalized functions), Zeng also has proved the existence of a generalized solution to the non-divergent barotropic model, and showed that the solution is not unique; the differences among the solutions are any time-dependent functions (this problem was encountered in early numerical weather prediction). However, because the viscosity terms were not introduced in the model as they are in the Navier-Stokes equations, the relevant problems of the generalized solution have not been studied for the three-dimensional compressible atmosphere model in that time.

In the 1980s, the rapid development of computers facilitated the AGCM for numerical weather forecasting and climate prediction; these were further supported by studies on the well-posedness of mathematical models and numerical algorithms. Under this background, the IAP of CAS developed the IAP-AGCM. Meanwhile, Zeng [5] also preliminarily put forward an ocean-atmosphere coupled model in 1983 and outlined the boundary conditions of the ocean-atmosphere interface. Later, these research results received attention from mathematicians both domestically and internationally.

On the basis of Zeng's research results, in the early 1990s, Lions, together with Teman and Wang [26–28] proved the existence of global weak solutions to the atmospheric equations and coupled ocean-atmosphere equations with viscosities, diffusions in the form as in the Navier-Stokes equation, etc. However, in their papers, the atmospheric dynamics equations adopted many approximations that are inconsistent with the actual physical processes, such as: (1)

The upper atmospheric pressure was prescribed as a positive constant; (2) Dirichlet boundary conditions or Neumann boundary conditions were used but not the actural ones; (3) the non-divergence of the vertically integrated wind was required; and (4) the external forcing terms were known functions.

To address these shortcomings, Zeng [29] made some improvements as follows: (1) The upper atmospheric pressure was set to zero; (2) tangential flow at the interface was allowed to ensure the boundary conditions more reasonable; (3) the non-divergence approximation was eliminated; and (4) the external forcing terms were related to temperature and pressure. Later, Mu and Zeng et al. [30] proved the existence of a global weak solution to the initial boundary value problem of such improved atmospheric dynamics model in 2001 and also proved the uniqueness of a global strong solution. Zeng and Mu [31] presented a new dynamic framework of the ocean-land-atmosphere coupled model in 2001. In this section, we firstly introduce some referential dynamic frameworks (see [31]).

2.1 Atmospheric general circulation model (AGCM for short)

Let the atmosphere cover a rotating sphere with radius a and angular velocity ω, subject to Earth's gravity and the Coriolis force. Denote the standard reference atmospheric pressure $\widetilde{p}(z)$, the standard reference temperature $\widetilde{T}(z)$ and the standard reference geopotential $\widetilde{\Phi}(z)$, which are only dependent on the height variable z. The standard reference earth-surface pressure $\widetilde{p}_s(\theta,\lambda)$ is a function related to the colatitude θ and the longitude λ. Since $\widetilde{p}(z)$ is a monotonic function of z, we can also take p instead of z as the vertical coordinate. Then, the hydrostatic conditions and Earth's standard surface pressure can be given as follows:

$$\begin{cases} R\widetilde{T} = -p\dfrac{\mathrm{d}\widetilde{\Phi}(p)}{\mathrm{d}p}, \\ \widetilde{p}|_{z=z_s(\theta,\lambda)} = \widetilde{p}_s(\theta,\lambda), \end{cases} \tag{2.1}$$

where $z_s(\theta,\lambda)$ is the elevation of Earth's surface. For convenience, we introduce the terrain coordinate $\zeta = \frac{p}{p_s} \in [0,1]$, where $p_s = p(\theta,\lambda,z_s)$ is the pressure at Earth's surface. Thus, we have

$$\begin{cases} \vec{V}(\theta,\lambda,\zeta,t) = \vec{\theta_0} v_\theta(\theta,\lambda,\zeta,t) + \vec{\lambda_0} v_\lambda(\theta,\lambda,\zeta,t), \\ T'(\theta,\lambda,\zeta,t) = T(\theta,\lambda,\zeta,t) - \widetilde{T}(p), \\ \Phi'(\theta,\lambda,\zeta,t) = \Phi(\theta,\lambda,\zeta,t) - \widetilde{\Phi}(p), \\ p'_s(\theta,\lambda,t) = p_s(\theta,\lambda,t) - \widetilde{p}_s(\theta,\lambda) \end{cases} \tag{2.2}$$

and the following dynamical equations:

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}t} = A + B + C + D + Q, \\ \kappa_a \dfrac{\partial p'_s}{\partial t} + \nabla\cdot(p_s\vec{V}) + \dfrac{\partial p_s\dot{\zeta}}{\partial\zeta} = \kappa_a D_{sa}, \\ RT' = -\zeta\dfrac{\partial\Phi'}{\partial\zeta}, \end{cases} \tag{2.3}$$

where

$$F = (v_\theta, v_\lambda, c_p T'), \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v_\theta \frac{\partial}{a\partial\theta} + v_\lambda \frac{\partial}{a\sin\theta\partial\lambda} + \dot{\zeta}\frac{\partial}{\partial\zeta} \tag{2.5}$$

and $\kappa_a = 1$ ($\kappa_a = 0$ means the approximation of non-divergence of the vertically integrated wind). A represents the pure dynamical terms (Coriolis force term, advection term and pure kinetic term), B and C are the parameterized process operators (they are very complex and also not fully understood in the climate-system model), D is the turbulent dissipation term (positive-definite operator), $\nabla\cdot$ is the two-dimensional divergence operator on the sphere, $\dot{\zeta} \equiv \frac{\mathrm{d}\zeta}{\mathrm{d}t}$ and Q represents the source terms (such as radiation heating). The concrete formulas for A and D are as follows:

$$\begin{cases} A_{v_\theta} = -\Big(2\omega\cos\theta + \dfrac{v_\lambda}{a}\cot\theta\Big)v_\lambda - \dfrac{\partial\Phi'}{a\partial\theta} - \dfrac{RT'}{\widetilde{p}_s}\dfrac{\partial\widetilde{p}_s}{a\partial\theta}, \\ A_{v_\lambda} = -\Big(2\omega\cos\theta + \dfrac{v_\lambda}{a}\cot\theta\Big)v_\theta - \dfrac{\partial\Phi'}{a\sin\theta\partial\lambda} - \dfrac{RT'}{\widetilde{p}_s}\dfrac{\partial\widetilde{p}_s}{a\sin\theta\partial\lambda}, \\ A_{c_pT'} = \dfrac{c_pC_0^2}{R\widetilde{p}_s\zeta}\Big(\widetilde{p}_s\dot{\zeta} + \zeta\kappa\Big(\dfrac{\partial p'_{sa}}{\partial t} + \vec{V}\cdot\nabla\widetilde{p}_s - \kappa D_{sa}\Big)\Big), \\ D_F = \nabla\cdot\dfrac{k_{hF}}{\widetilde{p}_s}\nabla F + \dfrac{\partial}{\partial\zeta}\Big(\dfrac{k_{zF}}{\widetilde{p}_s}\Big(\dfrac{g\zeta}{R\widetilde{T}}\Big)^2\dfrac{\partial\widetilde{p}_sF}{\partial\zeta}\Big), \\ D_{sa} \equiv \nabla\cdot k_{sa}\widetilde{\rho}_{sa}\nabla\dfrac{p'_s}{\widetilde{\rho}_{sa}}. \end{cases} \tag{2.6}$$

But the concrete formulas for B, C and Q are omitted in this paper.

The upper bound of the atmosphere is infinity, i.e., $z \to \infty$ or $p \to 0$, i.e., $\zeta \to 0$. Note that, from z_s to infinity, the whole air mass is finite, so it is natural to assume that the energy of the whole column is also finite (see [4]). Therefore, the upper boundary conditions of the atmospheric dynamics equations should be

$$\begin{cases} \dot{\zeta} = 0 \quad \text{as } \zeta \to 0, \\ \vec{V} \in L^2_\zeta, \quad \text{i.e.,} \quad \displaystyle\int_0^1 |\vec{V}|^2\mathrm{d}\zeta < \infty, \\ c_pT' \in L^2_\zeta, \quad \text{i.e.,} \quad \displaystyle\int_0^1 c_p^2R^2\Big(\zeta\frac{\partial\Phi'}{\partial\zeta}\Big)^2\mathrm{d}\zeta < \infty. \end{cases} \tag{2.7}$$

The bottom boundary of the atmosphere is the Earth's surface. If it is a land surface, its altitude does not change with time. In the case of the ocean surface, the height is variable with time (see Subsection 2.3). The bottom boundary of the atmospheric model often interacts with the upper interface of the underlying material (OGCM or LSM). Thus, we can formulate the corresponding bottom boundary conditions.

Note, if $\kappa_a = 0$ is taken, so that we have

$$\int_0^1 \Big(\nabla \cdot (p_s \vec{V}) + \frac{\partial p_s \dot{\zeta}}{\partial \zeta}\Big) \mathrm{d}\zeta = 0. \tag{2.8}$$

This means that there is no prediction equation for p'_s, if the non-divergence appoximation of the vertically integrated wind is taken. Further analysis shows that if $\kappa_a = 0$, the solution p_s is not unique, and the difference among the different solutions is an arbitrary time-dependent function (see [4]). That is why, in the early years of numerical weather prediction, the computed mean value of p_s for the whole region fluctuated even monotonically increases or decreases with time.

Moreover, we can also consider the well-posedness of the framework in AGCM with the water vapor phase transformation. Using methods similar to those in Zeng [29], we know that the external forcing term in the temperature equation is taken as a nonlinear function related to pressure and temperature, and the accurate method should directly represent the process of the water-vapor phase transition and heat release according to the laws of thermodynamics. Additionally, the equations describing specific humidity and the liquid water content variable should be also introduced.

2.2 AGCM coupled a simplified land-surface model

It is the surface soil (with a small amount of rock) that interacts with the atmosphere, where vegetation above the ground is considered to be the physical component of the soil (without detained canopy and ice-snow cover), and no volcanic eruption is assumed. The state functions in the LSM are the temperature deviation T'_l and the wetness deviation w'_g. Then, we have the governing equation for T'_l as follows:

$$\rho_l c_l \frac{\partial T'_l}{\partial t} = h_l^{-2} \frac{\partial}{\partial \eta}\Big(\rho_l k_l \frac{\partial c_l T'_l}{\partial \eta}\Big), \quad -1 \le \eta = \frac{z}{h_l} \le 0, \tag{2.9}$$

where ρ_l, c_l, h_l are the soil density, relative heat capacity and thickness of the active layer of the soil, respectively. The governing equation for w'_g is similar and is omitted here.

Usually, on the land surface, there are upward heat conduction fluxes and heat radiation from below the soil layers and downward heat conduction flux and radiation flux from the atmosphere, allowing the upper boundary thermal conditions of the soil layer and the bottom boundary thermal conditions of the atmosphere can be obtained. The bottom boundary of the soil layer is set as having no heat flow or known function. In addition, the surface of the soil is fixed, but there is frictional resistance to the atmosphere. Then, we have the bottom conditions of AGCM and the upper boundary conditions of the soils ($\zeta = 1$, $\eta = 0$) as follows:

$$\begin{cases} \dot{\zeta} = 0, \\ \dfrac{\partial \widetilde{p}_s \vec{V}}{\partial \zeta} = -\alpha_s \vec{V}, \quad \zeta \to 1,\ \eta \to 0, \\ \dfrac{\partial \widetilde{p}_s T'}{\partial \zeta} = \beta_s \dfrac{\partial c_l T_l'}{\partial \eta} + Q_{st}, \end{cases} \tag{2.10}$$

furthermore, the bottom boundary conditions of the soils ($\eta = -1$) are as follows:

$$\frac{\partial c_l T_l'}{\partial \eta} = Q_{sb}, \quad \eta \to -1, \tag{2.11}$$

where α_s and β_s denote two functions dependent on (θ, λ, t). However, the conductive heat flux Q_{st} and Q_{sb} may be two functions related to state functions.

Therefore, for the AGCM with a simplified LSM, the energy equation is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{al} = -(\mathcal{D}_a + \mathcal{D}_l + \mathcal{D}_{as}) + (\mathcal{F}_b + \mathcal{F}_e + \mathcal{F}_q), \tag{2.12}$$

where

$$\mathcal{E}_{al} = \mathcal{E}_{ak} + \mathcal{E}_{ae} + \mathcal{E}_l + \kappa_a \mathcal{E}_{als}, \tag{2.13}$$

and $\mathcal{E}_{ak}$, $\mathcal{E}_{ae}$, $\mathcal{E}_l$ are the atmospheric kinetic energy, available relative potential energy and heat energy of the soil, respectively. $\mathcal{E}_{als}$ is the available atmospheric surface potential energy. As $\kappa_a = 1$, $\mathcal{E}_{ak}$, $\mathcal{E}_{ae}$, $\mathcal{E}_l$ and $\mathcal{E}_{als}$ satisfy

$$(\mathcal{E}_{ak}, \mathcal{E}_{ae}) = \int_{S^2} \int_0^1 \frac{\widetilde{p}_s}{g} \Big[\frac{|\vec{V}|^2}{2}, \frac{(RT')^2}{2C_0^2} \Big] \mathrm{d}\zeta \mathrm{d}S, \tag{2.14}$$

$$\mathcal{E}_l = \int_{S^2} \int_{-1}^0 h_g \rho_g \frac{c_g}{c_p} \frac{(T_g')^2}{2C_0^2} R^2 \mathrm{d}\zeta \mathrm{d}S, \tag{2.15}$$

$$\mathcal{E}_{als} = \int_{S^2} \widetilde{\rho}_{sa} g \frac{1}{2} \Big(\frac{p_s'}{\widetilde{\rho}_{sa} g} \Big)^2 \mathrm{d}S, \tag{2.16}$$

where $\mathcal{D}_a$, $\mathcal{D}_l$ and $\mathcal{D}_{as}$ are positive definite functions, and $\mathcal{F}_b$, $\mathcal{F}_e$ and $\mathcal{F}_q$ are external forcings in the atmosphere and terrestrial surface system. In particular, there is no surface potential energy if $\kappa_a = 0$. It is easy to provide the initial data on the atmosphere and the soil layer. Equations (2.12)–(2.13) and the positive definite functions (2.14)–(2.16) are very useful for proving the well-posedness of the initial boundary value problems relevant to this coupled system.

2.3 Ocean general circulation model (OGCM for short)

Seawater contains salt, denoted by salinity S. The density ρ of the oceanic water can be described by the following state equation:

$$\rho = \rho_0 (1 - \alpha_T (T - T_0) + \alpha_S (S - S_0)), \tag{2.17}$$

where ρ_0, T_0 and S_0 are standard values, and α_T and α_S are two positive constants. By using the hydrostatic conditions, we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\rho g. \tag{2.18}$$

The height coordinate of the sea surface is z (Let the average of sea surface height with respect to time be zero, i.e., $\widetilde{z_{so}} = 0$), the upper elevation of the ocean is z_{so}, and the height of the sea floor is $z_b(\theta,\lambda) = -\widetilde{h}(\theta,\lambda)$. Denote the standard reference profiles of $\widetilde{T}(z)$, $\widetilde{S}(z)$, $\widetilde{p}(z)$ and $\widetilde{\rho}(z)$. Let $\xi = \frac{z-z_{so}}{z_{so}+\widetilde{h}} \in [-1,0]$, and let the vertical coordinates of the surface and lower interfaces of the ocean be $\xi = 0$ and $\xi = -1$, respectively. As in AGCM, we have equations for the OGCM as follows:

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}t} = A + B + C + D + Q, \\ \kappa_0 \dfrac{\partial z'_{so}}{\partial t} + \nabla\cdot(h^*\vec{V}) + \dfrac{\partial h^*\dot{\xi}}{\partial \xi} = \kappa_0 D_{so}, \\ \dfrac{\partial p'}{\partial \xi} = -h^* g \rho', \end{cases} \tag{2.19}$$

where h^* is some smoothed value of $\widetilde{h}$. Note that p' and ρ' here are the deviations of the pressure and density in the sea water from the standard values $\widetilde{p}$ and $\widetilde{\rho}$ at their given heights, $T' = T - \widetilde{T}(z)$, $S' = S - \widetilde{S}(z)$, $z'_{so} = z_{so} - \widetilde{z_{so}}$ and

$$F = (v_\theta, v_\lambda, c_0 T', S'). \tag{2.20}$$

The terms A, B, C, D and Q are similar as in AGCM, where κ_0 is taken as 1 or 0. The upper boundary conditions are

$$\begin{cases} \dot{\xi} = 0, \\ p' = \kappa'_0 p'_s + \rho_0 g z'_{so}, \\ \widetilde{h}^{-2} \rho_0 k_{z0V} \dfrac{\partial h^*\vec{V}}{\partial \xi} = -\vec{\tau}_a, \quad \xi \to 0, \\ \widetilde{h}^{-2} \rho_0 c_{0T} k_{z0T'} \dfrac{\partial h^* T'}{\partial \xi} = H'_{s0}, \\ \widetilde{h}^{-2} \rho_0 k_{z0S'} \dfrac{\partial h^* S'}{\partial \xi} = F'_{s0}, \end{cases} \tag{2.21}$$

and the bottom boundary conditions are

$$\begin{cases} \dot{\xi} = 0, \\ \widetilde{h}^{-2} \rho_0 k_{z0V} \dfrac{\partial h^*\vec{V}}{\partial \xi} = -\vec{\tau}_b, \quad \xi \to -1, \\ \dfrac{\partial h^* T'}{\partial \xi} = \dfrac{\partial h^* S'}{\partial \xi} = 0, \end{cases} \tag{2.22}$$

where $\vec{\tau}_a$, H'_{s0}, F'_{s0} and $\vec{\tau}_b$ are all flux functions. Since the oceanic surface is time-varying, when $\xi = 0$, z_{so} or z'_{so} (the sea surface height or sea surface height deviation) and p_s or p'_s (sea surface pressure or sea surface pressure deviation) are two important predicted variables. Using $(2.21)_2$, we know that the pressure deviation at the upper ocean interface is calculated by the atmospheric pressure deviation at the mean sea level plus the sea water mass deviation of the water column due to the deviation of the sea surface height.

2.4 Land-atmosphere-ocean coupled model

The earth-surface area (S) is divided into the marine area (O) and land area ($L = S - O$). Because lateral interaction between the sea and land are often neglected, the total energy of the coupled land-atmosphere-ocean model is described as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_{ak} + \mathcal{E}_{ap} + \mathcal{E}_l + \mathcal{E}_{ok} + \mathcal{E}_{op} + \mathcal{E}_{laos})\\ &= -(\mathcal{D}_a + \mathcal{D}_l + \mathcal{D}_o + \mathcal{D}_{laos}) + (\mathcal{F}_a + \mathcal{F}_o + \mathcal{F}_{laos}),\end{aligned} \tag{2.23}$$

where $\mathcal{E}_{ok}$ and $\mathcal{E}_{op}$ are the oceanic kinetic and available relative potential energy, respectively, and $\mathcal{E}_{laos}$ is the available surface potential energy. $\mathcal{D}_a+\mathcal{D}_l+\mathcal{D}_o+\mathcal{D}_{laos}$ are due to the "turbulent dissipation term" in the coupled system and are positive definite functions. $\mathcal{F}_a+\mathcal{F}_o+\mathcal{F}_{laos}$ are due to the external forcing field in the coupled system. Note that the surface potential energy of the land area is different from that of the ocean area, and

$$\begin{aligned}\mathcal{E}_{laos} &= \frac{1}{2}\int_{S-O}\int_{-1}^{0}\widetilde{\rho}_{sa}g\Big(\frac{p'_s}{\widetilde{\rho}_{sa}g}\Big)^2\mathrm{d}\zeta\mathrm{d}S\\ &\quad+\frac{1}{2}\int_O\Big(\frac{p'_s}{\sqrt{\widetilde{\rho}_{sa}g}}+\kappa'_0\sqrt{\widetilde{\rho}_{sa}g}z'_{so}\Big)^2\mathrm{d}S+\frac{1}{2}\int_O\rho_0 g\Big(1-\kappa_0'^2\frac{\widetilde{\rho}_{sa}}{\rho_0}\Big)z_{so}'^2\mathrm{d}S\end{aligned} \tag{2.24}$$

as $\rho_0 >> \widetilde{\rho}_{sa}$, $\mathcal{E}_{laos}$ is positively definite.

3 Well-Posedness of the Atmosphere-Ocean Coupled Model

Since Zeng [29] proposed an improved dynamical framework for IAP-AGCM in 1998, there have been some important theoretical results on the well-posedness of the atmospheric primitive equations. Firstly, Mu and Zeng et al. [30] proved the existence of the weak solution and the uniqueness of the strong solution to the initial boundary value problem of the atmospheric dynamic equations with applicable physical conditions. Since then, many validating research results have been published both domestically and internationally. For example, Huang and Guo [32] proved the existence of attractors to the dynamical framework of IAP-AGCM in 2007. Moreover, Guillén-Gonzaléz et al. [33], Temam and Ziane [34], Cao and Titi [35] and others proved the local and global existence and uniqueness of a strong solution to the initial boundary value problem of the primitive equations. Guo and Huang [36–38] proved the existence of a unique global strong solution and the existence of global attractors to the large-scale dry and moist atmospheric dynamics equations.

Based on multiscale analysis, Cao and Titi [39–41] studied the initial boundary value problem of the primitive equations with only horizontal viscosity or only vertical dissipation, reaffirming the existence and uniqueness of the strong solution. In addition, there are some conclusions related to the existence and uniqueness of the strong solution to the moist atmospheric equations (see references [42–45]). It should be noted that the references [33–45] all take positive constant pressure as the upper boundary of atmosphere and some approximations adopted are inconsistent with the actual physical processes. However, the theoretical achievements in these papers provide some new useful research methods for studying the well-posedness of the models used in real-world forecasting.

Due to the increased complexity of the dynamic framework of the climate-system/earth-system models compared to the primitive equations, there are still some difficulties should be overcomed in order to prove their well-posedness. For example, in the dynamic framework of the AGCM, the difficulties are from: (1) Taking the upper atmospheric pressure as zero, leading to the singularity of some terms; (2) more complex boundary conditions being given in the model, which creates some additional boundary terms to resolve; (3) the non-divergence approximation of the vertically integrated wind being eliminated, making it necessary to prove the higher-order regularity estimations of the atmospheric pressure deviation at the lower atmospheric interface; and (4) the external forcing terms being assumed to be nonlinear functions related to the atmospheric temperature and pressure rather than to known functions, so a lot of complex normal estimations are introduced. When studying the dynamic framework of the ocean-atmosphere coupled model, the difficulties caused by the complex coupled boundary conditions have to be overcome, and the higher-order regularity estimations of the ocean salinity deviation and the high deviation of the ocean interface must be proven.

In recent years, our research group has made great efforts to solve these difficulties. Now there has been some progress in the well-posedness of the dynamic framework of the AGCM and OGCM.

3.1 Well-posedness of the atmospheric general circulation model

a) In 2018, Lian, Zeng and Jin [46] proved the L^1 stability of the global weak solution of IAP-AGCM by using the energy estimation method. Firstly, the dynamic framework of IAP-AGCM was expressed as an operator equation; next, a sequence of weak solutions of the operator equation was given, and the initial data sequence of the global weak solutions sequence was assumed to converge in L^1. Then, the energy estimates were established, and using compactness arguments, the L^1 stability of the global weak solution was shown. By Egorov Theorem, we also proved that the global weak solution was stable almost everywhere. Now let us demonstrate the main results as follows.

Let $\vec{U}^m = (\vec{V}^m, T'^m)$ be a sequence of weak solutions to the dynamic framework of the AGCM (2.3) with the sequence of initial data $\vec{U}^m|_{t=0} = (\vec{V}_0^m, T_0'^m) = \vec{U}_0^m$ satisfying $\|\vec{U}_0^m -$

$\vec{U}_0\|_{L^1(\Omega)} \to 0$ as $m \to +\infty$ and $\|\vec{U}_0^m\|_{L^2(\Omega)} < +\infty$; then, we obtain

$$\|\vec{U}^m - \vec{U}\|_{L^1([0,M];L^1(\Omega))} \to 0 \quad \text{as } m \to +\infty. \tag{3.1}$$

Moreover, assuming $\vec{U}_0^m \to \vec{U}_0$ as $m \to +\infty$ and

$$\|\vec{U}_0^m\|_{L^2(\Omega)} + \|\vec{U}_0\|_{L^2(\Omega)} < +\infty, \tag{3.2}$$

then we have $\vec{U}^m \to \vec{U}$ as $m \to +\infty$ a.e..

b) In the same year, using energy estimation methods and decomposing the velocity field into the barotropic velocity field and baroclinic velocity field, Lian and Zeng [47] proved the existence and uniqueness of a first-order regular global strong solution to the dynamic framework of IAP-AGCM. Firstly, we established the basic energy estimation of state variables $\vec{V}, T'$, and then we decomposed the velocity field $\vec{V}$ into the barotropic velocity field $\overline{\vec{V}}$ and baroclinic velocity field $\tilde{\vec{V}}$. Next, using energy estimation methods, we obtained the a priori L^3 and L^4 estimates of $\tilde{\vec{V}}, T'$, and L^2 estimates of $\nabla\overline{\vec{V}}$ and further obtained the first-order regularity estimations of state variables $\vec{V}, T'$. In conducting the above proof, by Hardy's inequality, the singularity caused by taking the upper atmospheric pressure as zero was overcome, and then we used the method of contradiction to prove the existence of the unique global strong solution. The main results are as follows.

For any given $M > 0$, let $\vec{U}_0 = (v_{\theta 0}, v_{\lambda 0}, T_0') \in H^1(\Omega)$; then, there exists a unique global strong solution $\vec{U} = (\vec{V}, T')$ to the dynamic framework of the AGCM (2.3) satisfying:

$$\vec{V}, T' \in C([0,M]; H^1(\Omega)) \cap L^2([0,M]; H^2(\Omega)). \tag{3.3}$$

This result confirms the hypothesis of the existence of a global strong solution in Mu et al. [30] (2001).

3.2 Well-posedness of the AGCM with the water-vapor phase transformation

a) In 2020, Lian and Ma [48] proved the existence and uniqueness of the first order regular global strong solution to the dynamic framework of IAP-AGCM with the phase transformation of water vapor.

The phase transformation of water vapor is addressed in the following aspects. The radiant heating term and the latent heating term are added to the thermodynamic equation in the following form:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\kappa_a T' - LF_q, \tag{3.4}$$

where $-\kappa_a T'$ denotes the radiant heating, $-LF_q$ denotes the latent heating, and F_q represents the mass of water that is added by condensation or removed by evaporation. Moreover, we also add the specific humidity equation and the liquid water content equation into the dynamic framework of IAP-AGCM as follows:

$$\begin{cases} \dfrac{\partial q}{\partial t} + (\vec{V}^* \cdot \nabla) q + \dot{\zeta}^* \dfrac{\partial q}{\partial \zeta} = \dfrac{k_{hq}}{\widetilde{p}_s} \Delta q + k_{zq} \dfrac{\partial}{\partial \zeta}\Big(\Big(\dfrac{g\zeta}{R\widetilde{T}}\Big)^2 \dfrac{\partial q}{\partial \zeta}\Big) + F_q, \\ \dfrac{\partial m_w}{\partial t} + (\vec{V}^* \cdot \nabla) m_w + \dot{\zeta}^* \dfrac{\partial m_w}{\partial \zeta} = \dfrac{k_{hm_w}}{\widetilde{p}_s} \Delta m_w + k_{zm_w} \dfrac{\partial}{\partial \zeta}\Big(\Big(\dfrac{g\zeta}{R\widetilde{T}}\Big)^2 \dfrac{\partial m_w}{\partial \zeta}\Big) - F_q + P_r, \end{cases} \tag{3.5}$$

where P_r is the precipitation rate, and F_q and P_r are nonlinear functions depending on temperature and pressure; for their specific forms, refer to [29, 48]. Let $\vec{U}_0 = (v_{\theta 0}, v_{\lambda 0}, T'_0, q_0, m_{w0}) \in H^1(\Omega)$; then, there exists a unique global strong solution $\vec{U}$ to the dynamic framework of IAP-AGCM with the phase transformation of water vapor satisfying

$$\vec{V}, T', q, m_w \in C([0, M]; H^1(\Omega)) \cap L^2([0, M]; H^2(\Omega)). \tag{3.6}$$

b)In 2022, Ma, Lian and Zeng [49] also proved the existence and uniqueness of the second-order regular local strong solution to the dynamic framework of IAP-AGCM with the phase transformation of water vapor.

First, the linear equations corresponding to the dynamic framework of IAP-AGCM with the phase transformation of water vapor, the initial data and the boundary conditions were given. Then, under the assumption that the initial data satisfies the second-order regularity, the higher-order regularity estimations of the state functions in local time were established by adopting the energy estimation method. In other words, the second-order regularity estimations of the velocity field, temperature deviation, specific humidity and liquid water content in local time were also proven, and then the existence and uniqueness of the local strong solution with second-order regularity was proved by using the contractive mapping principle. We show the main results as follows.

Let $\vec{U}_0 = (\vec{V}_0, T'_0, q_0, m_{w0}) \in H^2(\Omega)$; there exists a unique local strong solution $\vec{U} = (\vec{V}, T', q, m_w)$ to the dynamic framework of IAP-AGCM with the phase transformation of water vapor on $\Omega \times [0, t_\delta]$ for some small enough $t_\delta > 0$ satisfying

$$\begin{cases} \vec{V}, T', q, m_w \in L^\infty([0, t_\delta]; H^2(\Omega)) \cap L^2([0, t_\delta]; H^3(\Omega)), \\ \dfrac{\partial \vec{V}}{\partial t}, \dfrac{\partial T'}{\partial t}, \dfrac{\partial q}{\partial t}, \dfrac{\partial m_w}{\partial t} \in L^2([0, t_\delta]; H^1(\Omega)). \end{cases} \tag{3.7}$$

On this basis, the authors' group also studied the existence and uniqueness of the second-order regular global strong solution to the dynamic framework of IAP-AGCM with the phase transformation of water vapor. The relevant results have been submitted in a paper.

3.3 Well-posedness of the ocean-atmosphere coupled model

Based on the above dynamic framework of the AGCM and OGCM, using actual physical frictional stress and flux, we give the following physical boundary conditions at the ocean-atmosphere interface:

$$
\begin{cases}
k_{zF}\dfrac{\partial \vec{V}_a}{\partial \zeta}\Big|_{\zeta=1} + k_{s1} f(|\vec{V}_{10}|)(\vec{V}_a|_{\zeta=1} - \vec{V}_o|_{\xi=0}) = 0,\\
k_{zF}\dfrac{\partial T'_a}{\partial \zeta}\Big|_{\zeta=1} + k_{s2} f(|\vec{V}_{10}|)(T'_a|_{\zeta=1} - T'_o|_{\xi=0}) = 0,\\
k_{zoF}\dfrac{\partial \vec{V}_o}{\partial \xi}\Big|_{\xi=0} + k_{s3} f(|\vec{V}_{10}|)(\vec{V}_o|_{\xi=0} - \vec{V}_a|_{\zeta=1}) = 0,\\
k_{zoF}\dfrac{\partial T'_o}{\partial \xi}\Big|_{\xi=0} + k_{s4} f(|\vec{V}_{10}|)(T'_o|_{\xi=0} - T'_a|_{\zeta=1}) = 0,\\
k_{zoF}\dfrac{\partial S'}{\partial \xi}\Big|_{\xi=0} + k_{s5}(P+R-E)S'|_{\xi=0} + \alpha|\vec{V}_{10}|^3 S'|_{\xi=0} = 0,
\end{cases}
\tag{3.8}
$$

where $k_{si}(i = 1, 2, \cdots, 5)$ and α are all positive constants, and $f(|\vec{V}_{10}|) > 0$ is a coefficient depending on the wind velocity at 10m height. $\vec{V}_a|_{\zeta=1}$ and $\vec{V}_o|_{\xi=0}$ stand for the atmospheric velocity and the oceanic velocity at the ocean-atmosphere interface, and $T'_a|_{\zeta=1}$ and $T'_o|_{\xi=0}$ stand for the atmospheric temperature deviation and oceanic temperature deviation at the ocean-atmosphere interface, respectively. The positive constants P, R and E respectively s-tand for the influences of precipitation, river runoff and evaporation, and this ocean salinity boundary condition was proposed by Jin et al. [50]. Then, the ocean-atmosphere coupled model is obtained. Based on the above research on the well-posedness of the dynamic framework of the atmospheric and oceanic general circulation models, we further studied the existence and stability of the global weak solutions as well as the existence and uniqueness of the first-order regular global strong solution to the ocean-atmosphere coupled model, and the relevant results have been submitted.

In addition, Lian et al. [51–53] studied the well-posedness of IAP-AGCM with stochastic external forces, and the well-posedness of IAP-OGCM. In the future, we will study the well-posedness of the fully coupled dynamic framework of CAS-ESM. These studies can theoretically guarantee coupling coordination among different subsystems in the earth-system model and the stability of the long-time integration of the model, providing a theoretical basis and technical guidance for improvements to and developments of the earth-system model.

4 Summary: The Unity of Theory and Practice

Recalling the research history of the well-posedness of the earth-system model, we have specific and profound experience: Theory can guide practical activities, and new theories can also be discovered in practical activities. Therefore, in the future, we will try to apply new methods, new theories and new technologies of applied mathematics to investigate the earth-system model. We will also actively extract and summarize new mathematical problems in the process of model development, introducing the scientific issues to the mathematicians. In addition, through the China Society for Industrial and Applied Mathematics and other high quality platforms, we hope to promote academic communications and project cooperation between the fields of mathematics and earth sciences, carrying out science popularization, information dissemination and talent scouting. Taking the mathematical research of the earth-system model as an example, we hope that a breakthrough in the earth sciences will be made, the theory of the earth-system model will be promoted into a complete knowledge system and multidisciplinary fusion will be more close.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

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