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GLOBAL CLASSICAL SOLUTIONS TO QUASILINEAR HYPERBOLIC SYSTEMS WITH WEAK LINEAR DEGENERACY**

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Abstract

Consider the following Cauchy problem for the first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0,$$
$$t = 0: \quad u = f(x).$$

We let

$$M = \sup_{x\in R} |f'(x)| < +\infty.$$

The main result of this paper is that under the assumption that the system is weakly linearly degenerated, there exists a positive constant ε independent of M, such that the above Cauchy problem admits a unique global C^1 solution $u = u(t,x)$ for all $t \in R$, provided that

$$\int_{-\infty}^{+\infty} |f'(x)|dx \le \varepsilon,$$
$$\int_{-\infty}^{+\infty} |f(x)|dx \le \frac{\varepsilon}{M}.$$

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§ 1. Introduction and Main Results

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t,x) and $A(u)$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j = 1,\cdots,n)$.

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By the definition of strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \qquad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \qquad (\text{equivalently} \quad \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

Without loss of generality, we suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \qquad (i, j = 1, \cdots, n), \tag{1.5}$$

$$r_i(u)^T r_i(u) \equiv 1 \qquad (i = 1, \cdots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

It was proved by Li Tatsien, Zhou Yi and Kong Dexing [5–7] that the Cauchy problem for system (1.1) with "small" and decaying initial data admits a unique global classical solution provided that system (1.1) is weakly linearly degenerate. In this paper, we shall reprove the global existence result with less restriction on the initial data. In particular, the supreme norm of the derivatives of the initial data is not assumed to be small. We shall also get global stability results in this case.

To state our result precisely, we shall first recall the concept of weak linear degeneracy (see [5] and [6]) as follows.

Definition 1.1. *The i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate, if , along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by*

$$\frac{du}{ds} = r_i(u), \tag{1.7}$$

$$s = 0 : \quad u = 0, \tag{1.8}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \qquad \forall\, |u| \text{ small}, \tag{1.9}$$

namely

$$\lambda_i(u^i(s)) \equiv \lambda_i(0), \qquad \forall\, |s| \text{ small}. \tag{1.10}$$

If all characteristics $\lambda_i(u)$ $(i = 1, \cdots, n)$ are weakly linearly degenerate, then system (1.1) *is called weakly linearly degenerate.*

We now consider the Cauchy problem for system (1.1) with the following initial data

$$t = 0 : \quad u = f(x). \tag{1.11}$$

Our main results can be summarized as follows:

Theorem 1.1. *Suppose that system* (1.1) *is strictly hyperbolic and weakly linearly degenerated. Suppose furthermore that* $A(u) \in C^2$ *in a neighbourhood of* $u = 0$ *and* $f \in C^1$ *with bounded* C^1 *norm. Let*

$$M = \sup_{x \in R} |f'(x)|. \tag{1.12}$$

Then there exists a positive constant ε *independent of* M*, such that Cauchy problem* (1.1) *and* (1.11) *admits a unique global* C^1 *solution* $u = u(t, x)$ *for all* $t \in R$*, provided that*

$$\int_{-\infty}^{+\infty} |f'(x)| dx \le \varepsilon, \tag{1.13}$$

$$\int_{-\infty}^{+\infty} |f(x)| dx \le \frac{\varepsilon}{M}. \tag{1.14}$$

Remark 1.1. The condition that the system is weakly linearly degenerate is necessary to ensure global existence with small and decaying initial data. If the i-th characteristics is not weakly linearly degenerate for some $i \in \{1, \cdots, n\}$, then we can find a simple wave solution

$$u = u^{(i)}(s(t, x)), \tag{1.15}$$

where $u^{(i)}(s)$ is defined by (1.7) and (1.8), moreover

$$s_t + \lambda_i(u^{(i)}(s)) s_x = 0. \tag{1.16}$$

It is well known that if $\lambda_i(u^{(i)}(s))$ is not a constant in any neighbourhood of origin, then the solution of (1.16) will in general develop singularity in finite time even for small and decaying initial data.

Remark 1.2. Assumptions (1.13) and (1.14) are scaling invariant. If $u(t, x)$ is a solution to system (1.1), then for any $\lambda > 0$, $u_\lambda(t, x) = u(\lambda t, \lambda x)$ is also a solution with initial data $f_\lambda(x) = f(\lambda x)$. We have

$$\int_{-\infty}^{+\infty} |f'_\lambda(x)| dx = \int_{-\infty}^{+\infty} |f'(x)| dx, \tag{1.17}$$

$$M_\lambda = \sup_{x \in R} |f'_\lambda(x)| = \lambda M, \tag{1.18}$$

$$\int_{-\infty}^{+\infty} |f_\lambda(x)| dx = \lambda^{-1} \int_{-\infty}^{+\infty} |f(x)| dx. \tag{1.19}$$

Remark 1.3. The conclusion of Theorem 1.1 was obtained by Li Tatsien, Zhou Yi and Kong Dexing [5–7] under the assumption that

$$\theta = \sup_{x \in R} \{(1 + |x|)^{1+\mu} (|f(x)| + |f'(x)|)\} \tag{1.20}$$

is sufficiently small for some $\mu > 0$. Condition (1.20) implies that

$$\sup_{x \in R} |f'(x)| \le \theta \tag{1.21}$$

as well as

$$\int_{-\infty}^{+\infty} |f'(x)|dx \le C\theta, \tag{1.22}$$

$$\int_{-\infty}^{+\infty} |f(x)|dx \le C\theta, \tag{1.23}$$

where C is a positive constant. Thus the result of [5, 6] can be deduced from Theorem 1.1.

Remark 1.4. Theorem 1.1 was also obtained by Yan Ping [7] under the additional hypothesis that the initial data has compact support

$$\text{supp} f \subset [\alpha_0, \beta_0] \tag{1.24}$$

and she required that (1.13) holds with ε depending on $\beta_0 - \alpha_0$ and M.

We shall also have

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *if* $u^{(1)}$, $u^{(2)}$ *are two solutions given by Theorem* 1.1 *with initial data* $f^{(1)}$ *and* $f^{(2)}$ *respectively, then*

$$\begin{aligned}&\int_{-\infty}^{+\infty} |u^{(1)}(t,x) - u^{(2)}(t,x)|dx \\ &\le C\int_{-\infty}^{+\infty} |f^{(1)}(x) - f^{(2)}(x)|dx, \qquad \forall t \ge 0,\end{aligned} \tag{1.25}$$

where C *is a positive constant independent of* M *and* t.

We remark that in the case that system (1.1) is linearly degenerated, the global existence has been established by Bressan [1], global L^1 stability also follows from the work of T. P. Liu and T. Yong [8], A. Bressan, T. P. Liu and T. Yang [2] as a special case.

This paper is organized as follows: In Section 2, we recall and generalize John's formula on the decomposition of waves. In Section 3, we introduce two basic lemmas concerning L^1 estimates. In Section 4 to Section 5, we prove Theorem 1.1 and Theorem 1.2 respectively.

§ 2 . Preliminaries

Suppose that $A(u) \in C^2$. By Lemma 2.5 in [5], there exists an invertible C^3 transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in the $\tilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \qquad \forall\, |\tilde{u}_i| \ \text{ small } \quad (i = 1, \cdots, n), \tag{2.1}$$

where

$$e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)^T. \tag{2.2}$$

Such a transformation is called a normalized transformation and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)$ are called normalized variables or normalized coordinates.

Let

$$v_i = l_i(u)u \qquad (i = 1, \cdots, n), \tag{2.3}$$

$$w_i = l_i(u)u_x \qquad (i = 1, \cdots, n). \tag{2.4}$$

By (1.5), we have

$$u = \sum_{k=1}^{n} v_k r_k(u), \tag{2.5}$$

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.6}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.7}$$

be the directional derivative along the i-th characteristic. We have (cf. [5, 6])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k, \tag{2.8}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.9}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \qquad \forall\, j. \tag{2.10}$$

Noting (2.6), we have

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u) v_i)}{\partial x} = \sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k, \tag{2.11}$$

or equivalently,

$$d[v_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k dx dt, \tag{2.12}$$

where

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla\lambda_i(u) r_k(u) \delta_{ij}. \tag{2.13}$$

By (2.9), we have, in normalized coordinates,

$$B_{ijj}(u_j e_j) \equiv 0, \qquad \forall\, i \neq j. \tag{2.14}$$

Furthermore, when the system is weakly linearly degenerate, in normalized coordinates, we have

$$B_{ijj}(u_j e_j) \equiv 0, \qquad \forall\, j \in \{1, \cdots, n\}. \tag{2.15}$$

On the other hand, we have

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k, \tag{2.16}$$

where

$$\gamma_{ijk}(u) = (\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u) \delta_{ik}. \tag{2.17}$$

Thus, we have

$$\gamma_{ijj}(u) \equiv 0, \qquad \forall\, j \neq i. \tag{2.18}$$

When the system is weakly linearly degenerate, we have

$$\gamma_{iii}(u_i e_i) \equiv 0, \qquad \forall\, i \in \{1, \cdots, n\}. \tag{2.19}$$

Similarly to (2.11), we can get

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k, \tag{2.20}$$

or equivalently,

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dx dt, \tag{2.21}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u)[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.22}$$

Thus, we have

$$\Gamma_{ijj}(u) \equiv 0, \qquad \forall\, j \in \{1, \cdots, n\}. \tag{2.23}$$

We next derive a formula on the decomposition of waves for the difference of two solutions $u^{(1)}$, $u^{(2)}$ to system (1.1). We denote

$$w_i^{(1)} = l_i(u^{(1)}) u_x^{(1)}, \quad w_i^{(2)} = l_i(u^{(2)}) u_x^{(2)} \qquad (i = 1, \cdots, n). \tag{2.24}$$

By (1.5), we have

$$u_x^{(1)} = \sum_{i=1}^{n} w_i^{(1)} r_i(u^{(1)}), \quad u_x^{(2)} = \sum_{i=1}^{n} w_i^{(2)} r_i(u^{(2)}). \tag{2.25}$$

Let

$$u^{(0)} = u^{(1)} - u^{(2)}, \tag{2.26}$$

$$\xi_i^{(1)} = l_i(u^{(1)}) u^{(0)}, \quad \xi_i^{(2)} = l_i(u^{(2)}) u^{(0)}. \tag{2.27}$$

Then by (1.5), we have

$$u^{(0)} = \sum_{j=1}^{n} \xi_j^{(1)} r_j(u^{(1)}) = \sum_{j=1}^{n} \xi_j^{(2)} r_j(u^{(2)}). \tag{2.28}$$

Thus

$$\begin{aligned} u_t^{(0)} &= \sum_{j=1}^{n} [\xi_{jt}^{(1)} r_j(u^{(1)}) + \xi_j^{(1)} \nabla r_j(u^{(1)}) u_t^{(1)}] \\ &= \sum_{j=1}^{n} [\xi_{jt}^{(1)} r_j(u^{(1)}) - \xi_j^{(1)} \nabla r_j(u^{(1)}) A(u^{(1)}) u_x^{(1)}] \\ &= \sum_{j=1}^{n} \xi_{jt}^{(1)} r_j(u^{(1)}) - \sum_{j,k=1}^{n} \lambda_k(u^{(1)}) \nabla r_j(u^{(1)}) r_k(u^1) \xi_j^{(1)} w_k^{(1)}. \end{aligned} \tag{2.29}$$

Taking the inner product with $l_i(u^{(1)})$, we get

$$l_i(u^{(1)})u_t^{(0)} = \xi_{it}^{(1)} - \sum_{j,k=1}^{n}[\lambda_k(u^{(1)})l_i(u^{(1)})\nabla r_j(u^{(1)})r_k(u^1)]\xi_j^{(1)}w_k^{(1)}. \tag{2.30}$$

On the other hand, by (2.26), we have

$$\begin{aligned}-u_t^{(0)} &= u_t^{(2)} - u_t^{(1)}\\ &= A(u^{(1)})u_x^{(1)} - A(u^{(2)})u_x^{(2)}\\ &= A(u^{(1)})(u_x^{(1)} - u_x^{(2)}) + (A(u^{(1)}) - A(u^{(2)}))u_x^{(2)}\\ &= A(u^{(1)})u_x^{(0)} + \sum_{j=1}^{n}(A(u^{(1)}) - A(u^{(2)}))r_j(u^{(2)})w_j^{(2)}.\end{aligned} \tag{2.31}$$

Thus, noting (2.28), we get

$$\begin{aligned}-l_i(u^{(1)})u_t^{(0)} &= \lambda_i(u^{(1)})l_i(u^{(1)})u_x^{(0)} + \sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}\\ &= \lambda_i(u^{(1)})l_i(u^{(1)})\sum_{j=1}^{n}(\xi_j^{(1)}r_j(u^{(1)}))_x\\ &\quad + \sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}\\ &= \lambda_i(u^{(1)})\xi_{ix}^{(1)} + \sum_{j=1}^{n}\lambda_i(u^{(1)})l_i(u^{(1)})\nabla r_j(u^{(1)})\xi_j^{(1)}u_x^{(1)}\\ &\quad + \sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}\\ &= \lambda_i(u^{(1)})\xi_{ix}^{(1)} + \sum_{j,k=1}^{n}\lambda_i(u^{(1)})l_i(u^{(1)})\nabla r_j(u^{(1)})r_k(u^{(1)})\xi_j^{(1)}w_k^{(1)}\\ &\quad + \sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}.\end{aligned} \tag{2.32}$$

Noting (2.9), combining (2.30) and (2.32) gives

$$\begin{aligned}&\xi_{it}^{(1)} + \lambda_i(u^{(1)})\xi_{ix}^{(1)}\\ &= \sum_{j,k=1}^{n}\beta_{ijk}(u^{(1)})\xi_j^{(1)}w_k^{(1)} - \sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}.\end{aligned} \tag{2.33}$$

Noting (1.5), we have

$$\begin{aligned}&-\sum_{j=1}^{n}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})r_j(u^{(2)})w_j^{(2)}\\ &= -(\lambda_i(u^{(1)}) - \lambda_i(u^{(2)}))l_i(u^{(1)})r_i(u^{(2)})w_i^{(2)}\\ &\quad + \sum_{j\neq i}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})(r_j(u^{(1)}) - r_j(u^{(2)}))w_j^{(2)}.\end{aligned} \tag{2.34}$$

Thus

$$\begin{aligned}&\xi^{(1)}_{it}+\lambda_i(u^{(1)})\xi^{(1)}_{ix}\\&=\sum_{j,k=1}^{n}\beta_{ijk}(u^{(1)})\xi^{(1)}_j w^{(1)}_k-(\lambda_i(u^{(1)})-\lambda_i(u^{(2)}))l_i(u^{(1)})r_i(u^{(2)})w^{(2)}_i\\&\quad+\sum_{j\neq i}^{n}(\lambda_i(u^{(1)})-\lambda_j(u^{(2)}))l_i(u^{(1)})(r_j(u^{(1)})-r_j(u^{(2)}))w^{(2)}_j.\end{aligned}\tag{2.35}$$

Similarly to (2.11) , we can also get

$$\begin{aligned}&\xi^{(1)}_{it}+(\lambda_i(u^{(1)})\xi^{(1)}_{i})_x\\&=\sum_{j,k=1}^{n}B_{ijk}(u^{(1)})\xi^{(1)}_j w^{(1)}_k-(\lambda_i(u^{(1)})-\lambda_i(u^{(2)}))l_i(u^{(1)})r_i(u^{(2)})w^{(2)}_i\\&\quad+\sum_{j\neq i}(\lambda_i(u^{(1)})-\lambda_j(u^{(2)}))l_i(u^{(1)})(r_j(u^{(1)})-r_j(u^{(2)}))w^{(2)}_j.\end{aligned}\tag{2.36}$$

In a similar way, we get

$$\begin{aligned}&\xi^{(2)}_{it}+(\lambda_i(u^{(2)})\xi^{(2)}_{i})_x\\&=\sum_{j,k=1}^{n}B_{ijk}(u^{(2)})\xi^{(2)}_j w^{(2)}_k+(\lambda_i(u^{(1)})-\lambda_i(u^{(2)}))l_i(u^{(2)})r_i(u^{(1)})w^{(1)}_i\\&\quad-\sum_{j\neq i}(\lambda_i(u^{(2)})-\lambda_j(u^{(1)}))l_i(u^{(2)})(r_j(u^{(1)})-r_j(u^{(2)}))w^{(1)}_j.\end{aligned}\tag{2.37}$$

§ 3 . L^1 Estimates

In this section, we give some basic L^1 estimates. They are essentially due to Schartzman [9, 10].

Lemma 3.1. *Let $\phi=\phi(t,x)\in C^1$ satisfy*

$$\phi_t+(\lambda(t,x)\phi)_x=F,\qquad 0<t\le T,\ x\in R,\tag{3.1}$$

$$\phi(0,x)=g(x),\tag{3.2}$$

where $\lambda\in C^1$. Then

$$\int_{-\infty}^{+\infty}|\phi(t,x)|dx\le\int_{-\infty}^{+\infty}|g(x)|dx+\int_0^T\int_{-\infty}^{+\infty}|F(t,x)|dxdt,\qquad\forall\, t\le T,\tag{3.3}$$

provided that the right hand side of the inequality is bounded.

Proof. To estimate $\int_{-\infty}^{+\infty}|\phi(t,x)|dx$, we need only to estimate

$$\int_{-l}^{l}|\phi(t,x)|dx\tag{3.4}$$

for any given $l>0$ and then let $l\to+\infty$.

From point (T, L), we draw a C^1 characteristic curve $x = x_r(t)$ such that

$$\frac{dx_r(t)}{dt} = \lambda(t, x_r(t)), \qquad t \le T, \tag{3.5}$$

$$t = T: \ x_r = L. \tag{3.6}$$

From point $(T, -L)$, we draw a C^1 characteristic curve $x = x_l(t)$ such that

$$\frac{dx_l(t)}{dt} = \lambda(t, x_l(t)), \qquad t \le T, \tag{3.7}$$

$$t = T: \ x_l = -L. \tag{3.8}$$

Let L be sufficiently large so that

$$x_l(t) \le -l < l \le x_r(t), \qquad 0 \le t \le T. \tag{3.9}$$

From (3.1), we have

$$|\phi|_t + (\lambda|\phi|)_x = \mathrm{sgn}(\phi)F. \tag{3.10}$$

Thus

$$\begin{aligned} \frac{d}{dt}\int_{x_l(t)}^{x_r(t)} |\phi(t,x)|dx &= \int_{x_l(t)}^{x_r(t)} \frac{\partial}{\partial t}|\phi(t,x)|dx + x_r'(t)|\phi(t,x_r(t))| - x_l'(t)|\phi(t,x_l(t))| \\ &= \int_{x_l(t)}^{x_r(t)} \mathrm{sgn}(\phi)F dx - \int_{x_l(t)}^{x_r(t)} (\lambda(t,x)|\phi(t,x)|)_x dx \\ &\quad + x_r'(t)|\phi(t,x_r(t))| - x_l'(t)|\phi(t,x_l(t))| \\ &= \int_{x_l(t)}^{x_r(t)} \mathrm{sgn}(\phi)F dx - (\lambda(t,x_r(t)) - x_r'(t))|\phi(t,x_r(t))| \\ &\quad - (x_l'(t) - \lambda(t,x_l(t)))|\phi(t,x_l(t))| \\ &= \int_{x_l(t)}^{x_r(t)} \mathrm{sgn}(\phi)F dx \le \int_{x_l(t)}^{x_r(t)} |F(t,x)|dx. \end{aligned} \tag{3.11}$$

Therefore it follows that

$$\begin{aligned} \int_{x_l(t)}^{x_r(t)} |\phi(t,x)|dx &\le \int_{x_l(0)}^{x_r(0)} |g(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |F(t,x)|dxdt \\ &\le \int_{-\infty}^{+\infty} |g(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |F(t,x)|dxdt. \end{aligned} \tag{3.12}$$

Thus

$$\int_{-l}^{l} |\phi(t,x)|dx \le \int_{-\infty}^{+\infty} |g(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |F(t,x)|dxdt. \tag{3.13}$$

Letting $l \to +\infty$, the desired conclusion follows.

Lemma 3.2. *Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying*

$$\phi_t + (\lambda(t,x)\phi)_x = F, \qquad 0 \le t \le T, \ x \in R, \tag{3.14}$$

$$\phi(0,x) = g_1(x) \tag{3.15}$$

and

$$\psi_t + (\mu(t,x)\psi)_x = G, \qquad 0 \le t \le T,\ x \in R, \tag{3.16}$$
$$\psi(0,x) = g_2(x), \tag{3.17}$$

respectively, where $\lambda, \mu \in C^1$ *such that there exists a positive constant* δ_0 *independent of* T *verifying*

$$\mu(t,x) - \lambda(t,x) \ge \delta_0, \qquad 0 \le t \le T,\ x \in R. \tag{3.18}$$

Then

$$\begin{aligned}&\int_0^T \int_{-\infty}^{+\infty} |\phi(t,x)||\psi(t,x)|dxdt \\ &\le C\Big(\int_{-\infty}^{+\infty} |g_1(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |F(t,x)|dxdt\Big) \\ &\quad \cdot \Big(\int_{-\infty}^{+\infty} |g_2(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |G(t,x)|dxdt\Big),\end{aligned} \tag{3.19}$$

provided that the two factors on the right hand side of the inequality are bounded.

Proof. To estimate

$$\int_0^T \int_{-\infty}^{+\infty} |\phi(t,x)||\psi(t,x)|dxdt, \tag{3.20}$$

it is enough to estimate

$$\int_0^T \int_{-l}^{l} |\phi(t,x)||\psi(t,x)|dxdt \tag{3.21}$$

for any given $l > 0$ and then let $l \to +\infty$.

From point (T, L), we draw a C^1 characteristic curve $x = x_r(t)$ such that

$$\frac{dx_r(t)}{dt} = \lambda(t, x_r(t)), \qquad t \le T, \tag{3.22}$$
$$t = T:\ x_r = L. \tag{3.23}$$

From point $(T, -L)$, we draw a C^1 characteristic curve $x = x_l(t)$ such that

$$\frac{dx_l(t)}{dt} = \mu(t, x_l(t)), \qquad t \le T, \tag{3.24}$$
$$t = T:\ x_l = -L. \tag{3.25}$$

Let L be sufficiently large so that

$$x_l(t) \le -l < l \le x_r(t), \qquad 0 \le t \le T. \tag{3.26}$$

We introduce the "continuous Glimm's functional"

$$Q(t) = \iint_{x_l(t)<x<y<x_r(t)} |\psi(t,x)||\phi(t,y)|dxdy. \tag{3.27}$$

Then, it is easy to see that

$$
\begin{aligned}
\frac{dQ(t)}{dt} &= x_r'(t)|\phi(t,x_r(t))|\int_{x_l(t)}^{x_r(t)}|\psi(t,x)|dx - x_l'(t)|\psi(t,x_l(t))|\int_{x_l(t)}^{x_r(t)}|\phi(t,x)|dx \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}\frac{\partial}{\partial t}|\psi(t,x)|\cdot|\phi(t,y)|dxdy \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}|\psi(t,x)|\cdot\frac{\partial}{\partial t}|\phi(t,y)|dxdy \\
&= x_r'(t)|\phi(t,x_r(t))|\int_{x_l(t)}^{x_r(t)}|\psi(t,x)|dx - x_l'(t)|\psi(t,x_l(t))|\int_{x_l(t)}^{x_r(t)}|\phi(t,x)|dx \\
&\quad - \iint_{x_l(t)<x<y<x_r(t)}\frac{\partial}{\partial x}(\mu|\psi(t,x)|)\cdot|\phi(t,y)|dxdy \\
&\quad - \iint_{x_l(t)<x<y<x_r(t)}|\psi(t,x)|\cdot\frac{\partial}{\partial y}(\lambda|\phi(t,y)|)dxdy \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}\mathrm{sgn}(\psi)G(t,x)|\phi(t,y)|dxdy \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}\mathrm{sgn}(\phi)|\psi(t,x)|F(t,y)dxdy \\
&= (x_r'(t)-\lambda(t,x_r(t)))|\phi(t,x_r(t))|\int_{x_l(t)}^{x_r(t)}|\psi(t,x)|dx \\
&\quad + (\mu(t,x_l(t))-x_l'(t))|\psi(t,x_l(t))|\int_{x_l(t)}^{x_r(t)}|\phi(t,x)|dx \\
&\quad - \int_{x_l(t)}^{x_r(t)}(\mu(t,x)-\lambda(t,x))|\psi(t,x)||\phi(t,x)|dx \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}\mathrm{sgn}(\psi)G(t,x)|\phi(t,y)|dxdy \\
&\quad + \iint_{x_l(t)<x<y<x_r(t)}\mathrm{sgn}(\phi)|\psi(t,x)|F(t,y)dxdy \\
&\le -\delta_0\int_{x_l(t)}^{x_r(t)}|\psi(t,x)||\phi(t,x)|dx + \int_{x_l(t)}^{x_r(t)}|G(t,x)|dx\int_{x_l(t)}^{x_r(t)}|\phi(t,x)|dx \\
&\quad + \int_{x_l(t)}^{x_r(t)}|F(t,x)|dx\int_{x_l(t)}^{x_r(t)}|\psi(t,x)|dx. \qquad (3.28)
\end{aligned}
$$

It then follows from Lemma 3.1 that

$$
\begin{aligned}
&\frac{dQ(t)}{dt} + \delta_0\int_{x_l(t)}^{x_r(t)}|\psi(t,x)||\phi(t,x)|dx \\
\le &\int_{x_l(t)}^{x_r(t)}|G(t,x)|dx\Big(\int_{-\infty}^{+\infty}|g_1(x)|dx + \int_0^T\int_{x_l(t)}^{x_r(t)}|F(t,x)|dxdt\Big) \\
&+ \int_{x_l(t)}^{x_r(t)}|F(t,x)|dx\Big(\int_{-\infty}^{+\infty}|g_2(x)|dx + \int_0^T\int_{x_l(t)}^{x_r(t)}|G(t,x)|dxdt\Big). \qquad (3.29)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\delta_0 \int_0^T \int_{x_l(t)}^{x_r(t)} |\psi(t,x)||\phi(t,x)|dxdt \\
\le\ & Q(0) + \int_0^T \int_{x_l(t)}^{x_r(t)} |G(t,x)|dxdt \\
&\cdot \Big(\int_{-\infty}^{+\infty} |g_1(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |F(t,x)|dxdt \Big) \\
&+ \int_0^T \int_{x_l(t)}^{x_r(t)} |F(t,x)|dxdt \\
&\cdot \Big(\int_{-\infty}^{+\infty} |g_2(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |G(t,x)|dxdt \Big).
\end{aligned} \tag{3.30}
$$

Noting

$$
Q(0) \le \int_{-\infty}^{+\infty} |g_1(x)|dx \int_{-\infty}^{+\infty} |g_2(x)|dx, \tag{3.31}
$$

we get

$$
\begin{aligned}
&\delta_0 \int_0^T \int_{x_l(t)}^{x_r(t)} |\psi(t,x)||\phi(t,x)|dxdt \\
\le\ & 2\Big(\int_{-\infty}^{+\infty} |g_1(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |F(t,x)|dxdt \Big) \\
&\cdot \Big(\int_{-\infty}^{+\infty} |g_2(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |G(t,x)|dxdt \Big).
\end{aligned} \tag{3.32}
$$

It thus follows

$$
\begin{aligned}
&\int_0^T \int_{x_l(t)}^{x_r(t)} |\phi(t,x)||\psi(t,x)|dxdt \\
\le\ & C\Big(\int_{-\infty}^{+\infty} |g_1(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |F(t,x)|dxdt \Big) \\
&\cdot \Big(\int_{-\infty}^{+\infty} |g_2(x)|dx + \int_0^T \int_{x_l(t)}^{x_r(t)} |G(t,x)|dxdt \Big).
\end{aligned} \tag{3.33}
$$

Therefore

$$
\begin{aligned}
&\int_0^T \int_{-l}^{l} |\phi(t,x)||\psi(t,x)|dxdt \\
\le\ & C\Big(\int_{-\infty}^{+\infty} |g_1(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |F(t,x)|dxdt \Big) \\
&\cdot \Big(\int_{-\infty}^{+\infty} |g_2(x)|dx + \int_0^T \int_{-\infty}^{+\infty} |G(t,x)|dxdt \Big)
\end{aligned} \tag{3.34}
$$

and the desired conclusion follows by taking $l \to +\infty$.

§ 4 . Proof of Theorem 1.1

By Lemma 2.5 in [5], there exists a normalized transformation. Without loss of generality, we assume that $u = (u_1, \cdots, u_n)^T$ are already normalized variables.

By the existence and uniqueness of local C^1 solution to the Cauchy problem, in order to prove Theorem 1.1, it suffices to establish a uniform a priori estimate on the C^0 norm of u and $\frac{\partial u}{\partial x}$ on the existence domain of C^1 solution $u = u(t, x)$.

By (1.2), there exist positive constants δ and δ_0 so small that

$$\lambda_{j+1}(u) - \lambda_j(v) \geq \delta_0, \qquad \forall\, |u|,\ |v| \leq \delta \quad (j = 1, \cdots, n-1). \tag{4.1}$$

For the time being it is supposed that on the existence domain of C^1 solution $u = u(t, x)$, we have

$$|u(t, x)| \leq \delta. \tag{4.2}$$

In what follows, we shall explain that this hypothesis is reasonable. Thus, to prove Theorem 1.1, we only need to establish a uniform a priori estimate on the supreme norm of v and w defined by (2.3) and (2.4) on any given time interval $[0, T]$.

Let

$$V_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in R} |v(t, x)|, \tag{4.3}$$

$$U_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in R} |u(t, x)|, \tag{4.4}$$

$$W_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in R} |w(t, x)|, \tag{4.5}$$

$$V_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |v(t, x)| dx, \tag{4.6}$$

$$\widetilde{V}_1(T) = \max_{i=1,\cdots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |v_i| dt, \tag{4.7}$$

where C_j stands for any given j-th characteristic on the domain $0 \leq t \leq T$.

$$W_1(T) = \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |w(t, x)| dx, \tag{4.8}$$

$$\widetilde{W}_1(T) = \max_{i=1,\cdots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |w_i| dt, \tag{4.9}$$

where $v = (v_1, \cdots, v_n)^T$, $w = (w_1, \cdots, w_n)^T$ and $|v| = \sqrt{v_1^2 + \cdots + v_n^2}$, etc. Noting (2.3)–(2.6) and (4.2), $V_\infty(T)$ is obviously equivalent to $U_\infty(T)$.

Noting $\int_{-\infty}^{+\infty} |f'(x)| dx < +\infty$, we conclude that

$$\lim_{x \to -\infty} f(x) = f_-, \qquad \lim_{x \to +\infty} f(x) = f_+ \tag{4.10}$$

exist. By $\int_{-\infty}^{+\infty} |f(x)| dx < +\infty$, we see that $f_- = f_+ = 0$.

By finite propagation speed of waves, we have

$$\lim_{x \to -\infty} u(t, x) = \lim_{x \to +\infty} u(t, x) = 0, \qquad \forall\, t \in [0, T]. \tag{4.11}$$

Thus, it follows

$$u(t,x)=\int_{-\infty}^{x}u_y(t,y)dy, \tag{4.12}$$

$$|u(t,x)|\le\int_{-\infty}^{+\infty}|u_x(t,x)|dx\le CW_1(T). \tag{4.13}$$

Here and hereafter, C will denote a generic constant independent of ε and T, the meaning of C may change from line to line.

Lemma 4.1. *There exists a positive constant C independent of ε, T and M, such that*

$$W_1(T),\widetilde{W}_1(T)\le C\varepsilon, \tag{4.14}$$

$$V_1(T),\widetilde{V}_1(T)\le C\cdot\frac{\varepsilon}{M}, \tag{4.15}$$

$$W_\infty(T)\le CM. \tag{4.16}$$

Proof. We introduce

$$Q_W(T)=\sum_{j=1}^{n}\sum_{i\ne j}\int_0^T\int_R|w_i(t,x)||w_j(t,x)|dxdt \tag{4.17}$$

and let

$$Q_V(T)=\sum_{j=1}^{n}\sum_{i\ne j}\int_0^T\int_R|v_i(t,x)||w_j(t,x)|dxdt. \tag{4.18}$$

By (2.20), it follows from Lemma 3.2 that

$$Q_W(T)\le C\Big(W_1(0)+\int_0^T\int_R|G|dxdt\Big)^2, \tag{4.19}$$

where $G=(G_1,\cdots,G_n)$ with

$$G_i=\sum_{j,k=1}^{n}\Gamma_{ijk}(u)w_jw_k. \tag{4.20}$$

Noting (2.23), we have

$$\int_0^T\int_R|G|dxdt\le CQ_W(T). \tag{4.21}$$

It then follows that

$$Q_W(T)\le C(W_1(0)+Q_W(T))^2. \tag{4.22}$$

We now estimate Q_V. By (2.11) and (2.20), it follows from Lemma 3.2 that

$$Q_V(T)\le C\Big(V_1(0)+\int_0^T\int_R|F|dxdt\Big)\Big(W_1(0)+\int_0^T\int_R|G|dxdt\Big), \tag{4.23}$$

where $F=(F_1,\cdots,F_n)$ with

$$F_i=\sum_{j,k=1}^{n}B_{ijk}(u)v_jw_k. \tag{4.24}$$

Noting (2.15), we can use Hadamard's formula to get

$$\begin{aligned} B_{ijj}(u) &= B_{ijj}(u) - B_{ijj}(u_j e_j) \\ &= \sum_{h \neq j} u_h \int_0^1 \frac{\partial B_{ijj}(su_1, \cdots, su_{j-1}, u_j, su_{j+1}, \cdots, su_n)}{\partial u_h} ds. \end{aligned} \tag{4.25}$$

Thus

$$|B_{ijj}(u)| \leq C \sum_{h \neq j} |u_h|, \qquad \forall i,\, j \in \{1, \cdots, n\}. \tag{4.26}$$

We shall prove that

$$\sum_{h \neq j} |u_h| \leq C \sum_{h \neq j} |v_h|. \tag{4.27}$$

As a matter of fact

$$u_h = u \cdot e_h = \sum_{k=1}^{n} v_k r_k(u) \cdot e_h, \tag{4.28}$$

noting (2.1), we have

$$r_j(u_j e_j) \cdot e_h \equiv 0, \qquad \forall\, j \neq h. \tag{4.29}$$

By Hadamard's formula, we get

$$|r_j(u) \cdot e_h| \leq C \sum_{k \neq j} |u_k|. \tag{4.30}$$

It thus follows from (4.28) that

$$\sum_{h \neq j} |u_h| \leq C \sum_{h \neq j} |v_h| + C|v| \sum_{h \neq j}^{n} |u_h| \leq C \sum_{h \neq j} |v_h| + C\delta \sum_{h \neq j} |u_h|. \tag{4.31}$$

Thus, (4.27) is valid provided that δ is sufficiently small. Thus, it follows

$$|B_{ijj}(u)| \leq C \sum_{h \neq j} |v_h|. \tag{4.32}$$

Thus, we get

$$\int_0^T \int_R |F_i| dx dt \leq C Q_V(T). \tag{4.33}$$

It follows that

$$Q_V(T) \leq C(V_1(0) + C Q_V(T))(\varepsilon + Q_W(T)). \tag{4.34}$$

We now estimate $W_1(T)$. By Lemma 3.1,

$$\int_{-\infty}^{+\infty} |w_i(t,x)| dx \leq W_1(0) + \int_0^T \int_R |G| dx dt \leq W_1(0) + C Q_W(T). \tag{4.35}$$

Therefore

$$W_1(T) \leq C W_1(0) + C Q_W(T). \tag{4.36}$$

To estimate $\widetilde{W}_1(T)$, we need to estimate

$$\int_{C_j} |w_i| dt.$$

We assume that C_j intersects $t=0$ with point A, intersects $t=T$ with point B. We draw an i-th characteristic C_i from B downward and intersects $t=0$ with point C.

We rewrite (2.21) as

$$d(|w_i(t,x)|(dx-\lambda_i(u)dt)) = \mathrm{sgn}(w_i)G_i dxdt, \tag{4.37}$$

and we integrate it in the region ABC to get

$$\begin{aligned}\Big|\int_{C_j}|w_i(t,x)|(\lambda_j(u)-\lambda_i(u))dt\Big| &\le \int_A^C |w_i(0,x)|dx + \iint_{ABC}|G_i|dxdt\\ &\le W_1(0)+CQ_W(T).\end{aligned} \tag{4.38}$$

In the definition of $\widetilde{W}_1$, $j\neq i$, thus

$$|\lambda_j(u)-\lambda_i(u)|\ge \delta_0, \tag{4.39}$$

therefore, it follows that

$$\int_{C_j}|w_i(t,x)|dt \le CW_1(0)+CQ_W(T), \tag{4.40}$$

hence

$$\widetilde{W}_1(T)\le CW_1(0)+CQ_W(T). \tag{4.41}$$

We now estimate $V_1(T)$. By Lemma 3.1, we have

$$\int_{-\infty}^{+\infty}|v_i(t,x)|dx \le C\frac{\varepsilon}{M}+\int_0^T\int_R |F|dxdt \le C\frac{\varepsilon}{M}+CQ_V(T). \tag{4.42}$$

Thus we get

$$V_1(T)\le C\frac{\varepsilon}{M}+CQ_V(T). \tag{4.43}$$

In a similar way, we can get

$$\widetilde{V}_1(T)\le C\frac{\varepsilon}{M}+CQ_V(T). \tag{4.44}$$

Finally, we estimate $W_\infty(T)$. By (2.16), we have

$$|w_i|_{C^0}\le W_\infty(0)+\Big|\int_{C_i}\sum_{j,k=1}^n \gamma_{ijk}(u)w_jw_k\Big|. \tag{4.45}$$

Noting (2.18), we have

$$\begin{aligned}\Big|\int_{C_i}\sum_{j,k=1}^n \gamma_{ijk}(u)w_jw_k\Big| &\le \int_{C_i}|\gamma_{iii}(u)w_i^2| + CW_\infty(T)\widetilde{W}_1(T)\\ &\le CW_\infty^2(T)\int_{C_i}|\gamma_{iii}(u)| + CW_\infty(T)\widetilde{W}_1(T).\end{aligned} \tag{4.46}$$

By Hadamard's formula, it holds that

$$|\gamma_{iii}(u)| = |\gamma_{iii}(u)-\gamma_{iii}(u_ie_i)| \le C\sum_{j\neq i}|u_j| \le C\sum_{j\neq i}|v_j|. \tag{4.47}$$

Thus

$$\int_{C_i} |\gamma_{iii}(u)| \le C\widetilde{V}_1(T). \tag{4.48}$$

Therefore

$$W_\infty(T) \le nM + CW_\infty^2(T)\widetilde{V}_1(T) + CW_\infty(T)\widetilde{W}_1(T). \tag{4.49}$$

Combining (4.22), (4.34), (4.36), (4.41), (4.43), (4.44) and (4.49) all together, we can prove that

$$Q_W(T) \le C\varepsilon^2, \tag{4.50}$$

$$Q_V(T) \le C\frac{\varepsilon^2}{M} \tag{4.51}$$

as well as the conclusion of the lemma. This proves Lemma 4.1.

It follows from Lemma 4.1 that

$$U_\infty(T) \le CW_1(T) \le C\varepsilon. \tag{4.52}$$

Taking ε sufficiently small, we get

$$U_\infty(T) \le \frac{1}{2}\delta, \tag{4.53}$$

so the hypothesis (4.2) is reasonable.

Theorem 1.1 is a direct consequences of Lemma 4.1.

§ 5 . Proof of Theorem 1.2

To estimate

$$\int_{-\infty}^{+\infty} |u^{(0)}(t,x)|dx,$$

it suffices to estimate

$$\int_{-\infty}^{+\infty} |\xi_i^{(1)}(t,x)|dx, \qquad \forall\, i \in \{1,\cdots,n\}$$

on a fixed time interval $[0,T]$.

By Lemma 3.1, we get

$$\int_{-\infty}^{+\infty} |\xi_i^{(1)}(t,x)|dx \le \int_{-\infty}^{+\infty} |\xi_i^{(1)}(0,x)|dx + \int_0^T \int_R |F_i^{(1)}(t,x)|dxdt, \tag{5.1}$$

where

$$\begin{aligned} F_i^{(1)} = &\sum_{j,k=1}^n B_{ijk}(u^{(1)})\xi_j^{(1)}w_k^{(1)} - (\lambda_i(u^{(1)}) - \lambda_i(u^{(2)}))l_i(u^{(1)})r_i(u^{(2)})w_i^{(2)} \\ &+ \sum_{j\neq i}(\lambda_i(u^{(1)}) - \lambda_j(u^{(2)}))l_i(u^{(1)})(r_j(u^{(1)}) - r_j(u^{(2)}))w_j^{(2)}. \end{aligned} \tag{5.2}$$

We first estimate the term $\lambda_i(u^{(1)}) - \lambda_i(u^{(2)})$. We have

$$\begin{aligned} \lambda_i(u^{(1)}) &= \lambda_i(u^{(1)}) - \lambda_i(u_i^{(1)}e_i) \\ &= \sum_{j\neq i} u_j^{(1)} \int_0^1 \frac{\partial\lambda_i(su_1^{(1)},\cdots,su_{i-1}^{(1)},u_i^{(1)},su_{i+1}^{(1)}\cdots,su_n^{(1)})}{\partial u_j^{(1)}}ds. \end{aligned} \tag{5.3}$$

In a similar way, we have

$$\lambda_i(u^{(2)}) = \sum_{j\neq i} u_j^{(2)} \int_0^1 \frac{\partial\lambda_i(su_1^{(2)}, \cdots, su_{i-1}^{(2)}, u_i^{(2)}, su_{i+1}^{(2)} \cdots, su_n^{(2)})}{\partial u_j^{(2)}} ds. \tag{5.4}$$

Thus

$$\begin{aligned}
&\lambda_i(u^{(1)}) - \lambda_i(u^{(2)}) \\
&= \sum_{j\neq i} u_j^{(0)} \int_0^1 \frac{\partial\lambda_i(su_1^{(1)}, \cdots, su_{i-1}^{(1)}, u_i^{(1)}, su_{i+1}^{(1)} \cdots, su_n^{(1)})}{\partial u_j^{(1)}} ds \\
&\quad + \sum_{j\neq i} u_j^{(2)} \int_0^1 \Big(\frac{\partial\lambda_i(su_1^{(1)}, \cdots, su_{i-1}^{(1)}, u_i^{(1)} su_{i+1}^{(1)}, \cdots, su_n^{(1)})}{\partial u_j^{(1)}} \\
&\quad - \frac{\partial\lambda_i(su_1^{(2)}, \cdots, su_{i-1}^{(2)}, u_i^{(2)} su_{i+1}^{(2)}, \cdots, su_n^{(2)})}{\partial u_j^{(2)}} \Big) ds.
\end{aligned} \tag{5.5}$$

Therefore

$$\begin{aligned}
|\lambda_i(u^{(1)}) - \lambda_i(u^{(2)})| &\leq C \sum_{j\neq i}^{n} |u_j^{(0)}| + C \sum_{j\neq i}^{n} |u_j^{(2)}||u^{(0)}| \\
&\leq C \sum_{j\neq i}^{n} |u_j^{(0)}| + C \sum_{j\neq i}^{n} |v_j^{(2)}||u^{(0)}|.
\end{aligned} \tag{5.6}$$

We have

$$u_j^{(0)} = \sum_{k=1}^{n} \xi_k^{(2)} r_k(u^{(2)}) \cdot e_j. \tag{5.7}$$

Noting (2.1), by Hadamard's formula, we get, for $j \neq i$,

$$\begin{aligned}
|r_i(u^{(2)}) \cdot e_j| &= |r_i(u^{(2)}) \cdot e_j - r_i(u_i^{(2)} e_i) \cdot e_j| \\
&\leq C \sum_{h\neq i} |u_h^{(2)}| \leq C \sum_{h\neq i} |v_h^{(2)}|.
\end{aligned} \tag{5.8}$$

Thus

$$\sum_{j\neq i} |u_j^{(0)}| \leq C \sum_{j\neq i} |\xi_j^{(2)}| + C \sum_{j\neq i} |v_j^{(2)}||\xi_i^{(2)}|. \tag{5.9}$$

Finally, we get

$$|\lambda_i(u^{(1)}) - \lambda_i(u^{(2)})| \leq C \sum_{j\neq i} |\xi_j^{(2)}| + C \sum_{j\neq i} |v_j^{(2)}||\xi_i^{(2)}|. \tag{5.10}$$

By a similar argument, we get

$$|r_j(u^{(1)}) - r_j(u^{(2)})| \leq C \sum_{k\neq j} |\xi_k^{(2)}| + C \sum_{k\neq j} |v_k^{(2)}||\xi_j^{(2)}|. \tag{5.11}$$

The estimate of $B_{ijk}(u^{(1)})$ is the same as in the proof of Theorem 1.1. We have

$$|B_{ijj}(u^{(1)})| \leq C \sum_{k\neq j} |v_k^{(1)}|. \tag{5.12}$$

Combining all the above facts, we get

$$\int_0^T \int_R |F_i^{(1)}|dxdt \le C(Q_{\xi^{(1)}}(T) + MD_{\xi^{(1)}}(T) + Q_{\xi^{(2)}}(T) + MD_{\xi^{(2)}}(T)), \tag{5.13}$$

where

$$Q_{\xi^{(1)}}(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_R |\xi_i^{(1)}(t,x)||w_j^{(1)}(t,x)|dxdt, \tag{5.14}$$

$$Q_{\xi^{(2)}}(T) = \sum_{j=1}^n \sum_{i\neq j} \int_0^T \int_R |\xi_i^{(2)}(t,x)||w_j^{(2)}(t,x)|dxdt, \tag{5.15}$$

$$D_{\xi^{(1)}}(T) = \sum_{j=1}^n \sum_{h\neq j} \int_0^T \int_R |v_h^{(1)}||\xi_j^{(1)}|dxdt, \tag{5.16}$$

$$D_{\xi^{(2)}}(T) = \sum_{j=1}^n \sum_{h\neq j} \int_0^T \int_R |v_h^{(2)}||\xi_j^{(2)}|dxdt. \tag{5.17}$$

Therefore, it remains to estimate $Q_{\xi^{(1)}}(T)$, $Q_{\xi^{(2)}}(T)$, $D_{\xi^{(1)}}(T)$ and $D_{\xi^{(2)}}(T)$.

Noting (2.20), (2.37), by Lemma 3.2, we get

$$\begin{aligned} Q_{\xi^{(1)}}(T) &\le C\Big(|f^{(0)}|_{L^1} + \sum_{i=1}^n \int_0^T \int_R |F_i^{(1)}|dxdt\Big) \\ &\quad \cdot \Big(W_1(0) + \sum_{i,j,k=1}^n \int_0^T \int_R |\Gamma_{ijk}(u^{(1)})w_j^{(1)}w_k^{(1)}|dxdt\Big) \\ &\le C\varepsilon\Big(|f^{(0)}|_{L^1} + \sum_{i=1}^n \int_0^T \int_R |F_i^{(1)}|dxdt\Big), \end{aligned} \tag{5.18}$$

where $f^{(0)} = f^{(1)} - f^{(2)}$.

Noting (2.11), (2.37), by Lemma 3.2, we have

$$\begin{aligned} D_{\xi^1}(T) &\le C\Big(|f^{(0)}|_{L^1} + \sum_{i=1}^n \int_0^T \int_R |F_i^{(1)}|dxdt\Big) \\ &\quad \cdot \Big(V_1(0) + \sum_{i,j,k=1}^n \int_0^T \int_R |B_{ijk}(u^{(1)})v_j^{(1)}w_k^{(1)}|dxdt\Big) \\ &\le C\frac{\varepsilon}{M}\Big(|f^{(0)}|_{L^1} + \sum_{i=1}^n \int_0^T \int_R |F_i^{(1)}|dxdt\Big). \end{aligned} \tag{5.19}$$

Similar estimates hold true for $Q_{\xi^{(2)}}(T)$ and $D_{\xi^{(2)}}(T)$. Therefore, we obtain

$$Q_{\xi^{(1)}}(T), Q_{\xi^{(2)}}(T) \le C\varepsilon|f^{(0)}|_{L^1}, \tag{5.20}$$

$$D_{\xi^{(2)}}(T), D_{\xi^{(2)}}(T) \le \frac{C\varepsilon}{M}|f^{(0)}|_{L^1}. \tag{5.21}$$

By (5.13), this implies

$$\sum_{i=1}^n \int_0^T \int_R |F_i^{(1)}|dxdt \le C\varepsilon|f^{(0)}|_{L^1}. \tag{5.22}$$

It then follows from (5.1) that

$$\int_{-\infty}^{+\infty} |\xi_i^{(1)}(t,x)|dx \le C|f^{(0)}|_{L^1}. \tag{5.23}$$

This is exactly what we want.

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