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Persistence Approximation Property for Maximal Roe Algebras*

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Abstract Persistence approximation property was introduced by Hervé Oyono-Oyono and Guoliang Yu. This property provides a geometric obstruction to Baum-Connes conjecture. In this paper, the authors mainly discuss the persistence approximation property for maximal Roe algebras. They show that persistence approximation property of maximal Roe algebras follows from maximal coarse Baum-Connes conjecture. In particular, let X be a discrete metric space with bounded geometry, assume that X admits a fibred coarse embedding into Hilbert space and X is coarsely uniformly contractible, then $C^*_{\max}(X)$ has persistence approximation property. The authors also give an application of the quantitative K-theory to the maximal coarse Baum-Connes conjecture.

Keywords Quantitative K-theory, Persistence approximation property, Maximal coarse Baum-Connes conjecture, Maximal Roe algebras
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1 Introduction

Initially quantitative K-theory was introduced by Yu in [16]. It was applied to compute higher indices of elliptic operators for noncompact spaces with finite asymptotic dimension. It is very reflexible to compute the K-theory of filtered C^*-algebras. Not only we can use quantitative Mayer-Vietoris sequence to prove coarse Baum-Connes conjecture, but also we can use quantitative K-theory to construct obstructions for Baum-Connes conjecture by propagations.

The quantitative K-theory of filtered C^*-algebras was introduced by Oyono-Oyono and Yu in [10]. The quantitative Bott periodicity and quantitative six-term exact sequence were also proved in the above paper. The quantitative Mayer-Vietoris sequence was introduced in [11], which is quite important in proving the coarse Baum-Connes conjecture.

Persistence approximation property of filtered C^*-algebras was also defined by Oyono-Oyono and Yu (see [12]). It has a close relationship with the Baum-Connes conjecture. It has been proven that: If Γ is a finitely generated group that satisfies the Baum-Connes conjecture with coefficients and admits a cocompact universal example for proper actions, then for any Γ-C^*-algebra A, the reduced crossed product $A \rtimes_{\text{red}} \Gamma$ satisfies the persistence approximation

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property. In this view, it gives a geometric obstruction to the Baum-Connes conjecture. Clément Dell'Aiera also characterized this property especially for the crossed products with groupoids (see [3]).

The notion of fibred coarse embedding into Hilbert spaces, which is a generalized notion of coarse embedding into Hilbert spaces. It turns out that this property is a sufficient condition for the maximal Baum-Connes conjecture (see [2]). Some kinds of expanders which do not embed coarsely into Hilbert spaces are counterexamples to the coarse Baum-Connes conjecture (see [7, 15]). However, they admit a fibred coarse embedding into Hilbert space and satisfy the maximal Baum-Connes conjecture (see [1–2, 9, 15]). Fibred coarse embedding can also be characterized by boundary a-T-menable groupoids (see [4–5]).

This paper is organized as follows. In Section 2, we give a quick review of quantitative K-theory. Most of the results come from the paper of Oyono-Oyono and Yu in [10]. Then we use the method of Aiera (see [3]) to construct the quantitative maximal assembly map. We characterize the relationship between the quantitative maximal coarse Baum-Connes conjecture and the maximal coarse Baum-Connes conjecture. In Section 3, we introduce persistence approximation property from Oyono-Oyono and Yu. We show that the persistence approximation property of maximal Roe algebras comes from the maximal coarse Baum-Connes conjecture, which is our first main theorem.

Theorem 1.1 *Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Assume that*

(1) *X is coarsely uniformly contractible*;

(2) *$\mu_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}$ is onto and $\mu_{X,A,\max,*}$ is one to one.*

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ and every $r > 0$, there exists a $r' > 0$ such that $r \leq r'$ and $\mathcal{PA}_*(C^*_{\max}(X,A), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.

If X admits a fibred coarse embedding into Hilbert space, then the maximal coarse Baum-Connes conjecture holds. So it is natural to consider whether or not the persistence approximation property holds for $C^*_{\max}(X)$ when X admits a fibred coarse embedding into Hilbert space, and we prove our second main result.

Theorem 1.2 *Let X be a discrete metric space with bounded geometry. Assume that X admits a fibred coarse embedding into Hilbert space and X is coarsely uniformly contractible. Then there exists a universal constant $\lambda_{PA} \geq 1$ such that: For any $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(C^*_{\max}(X), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.*

One of the very important example which admits a fibred coarse embedding into Hilbert space comes from the box space of residually finite groups. Let Γ be a finitely generated residually finite group with respect to a family of finite index normal subgroups $\Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n \supseteq \cdots$. The box space $X(\Gamma)$ admits a fibred coarse embedding into Hilbert space if and only if Γ has Haagerup property (see [1]). This type of examples does not require the first assumption of Theorem 1.1, but satisfies the persistence approximation property for maximal Roe algebras. This is our third main theorem.

Theorem 1.3 *Let Γ be a finitely generated residually finite group with haagerup property and admits a cocompact universal example for proper actions. Then there exists a univer-*

sal constant $\lambda_{PA} \geq 1$ *for any* $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ *and any* $r > 0$ *there exists* $r' > r$ *such that* $\mathcal{PA}_*(C^*_{\max}(X(\Gamma)), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ *holds.*

Example 1.1 Both $\mathbb{F}_2$ and $SL_2(\mathbb{Z})$ are finitely generated residually finite group with haagerup property and admits a cocompact universal example for proper actions. Then the maximal Roe algebras of the box spaces have persistence approximation property.

In Section 4, we construct the quantitative maximal Baum-Connes assembly map for a family of metric spaces. We give an application of quantitative K-theory for coarse Baum-Connes conjecture and show the following theorem.

Theorem 1.4 *Let* $\mathcal{X} = (X_i)_{i \in \mathbb{N}}$ *be a family of discrete metric space with bounded geometry. Let* $\Sigma = \bigsqcup_{i \in \mathbb{N}} X_i$. *Assume that*

(1) *for any* $\varepsilon \in \left(0, \frac{1}{4}\right)$ *and positive numbers d,r such that* $\alpha_{\mathcal{X}}(\varepsilon) \cdot d \leq r$, *there exists* d' *with* $d \leq d'$, *such that* $QI_{\mathcal{X},\max,*}(d, d', \varepsilon, r)$ *is holds*;

(2) *For some* $\lambda > 1$ *and any* $\varepsilon \in (0, \frac{1}{4\lambda})$, $r > 0$, *there exists* $d > 0$, $r' > r$ *with* $\alpha_{\mathcal{X}}(\varepsilon) \cdot d \leq r'$ *such that* $QS_{\mathcal{X},\max,*}(d, r, r', \varepsilon, \lambda\varepsilon)$.

Then Σ *satisfies the maximal coarse Baum-Connes conjecture.*

Through out this paper, denote by H the separable Hilbert space and by $K(H)$ the operator algebra consist of compact operators on the Hilbert space H.

2 Quantitative K-theory

In this section, let us give a quick review of quantitative K-theory for filtered C^*-algebras,most of the results come from Oyono-Oyono and Yu in [10]. Firstly, let us introduce filtered C^*-algebras, this is the basic objects for quantitative K-theory.

Definition 2.1 (see [10]) *A filtered* C^**-algebra* A *is a* C^**-algebra equipped with a family* $(A_r)_{r>0}$ *of closed linear subspaces indexed by positive numbers such that*:

(1) $A_r \subseteq A_{r'}$ *if* $r \leq r'$;

(2) A_r *is stable by involution,i.e. for any* $x \in A_r$, *then* $x^* \in A_r$;

(3) $A_r \cdot A_{r'} \subseteq A_{r+r'}$;

(4) *the subalgebra* $\bigcup_{r>0} A_r$ *is dense in* A.

If A is unital, we suppose that $1 \in A_r$, for any $r > 0$. If A is non unital filtered C^*-algebra, then its unitization $\widetilde{A}$ is filtered by $(A_r + \mathbb{C})_{r>0}$. We can define the homomorphism

$$\rho_A : \widetilde{A} \to \mathbb{C}, \quad a + z \to z$$

for $a \in A$ and $z \in \mathbb{C}$.

Let A and B be two C^*-algebras filtered by $(A_r)_{r>0}$ and $(B_r)_{r>0}$. A *-homomorphism $\phi : A \to B$ is said to be filtered if $\phi(A_r) \subseteq B_r$ for all $r > 0$.

There are so many kinds of filtered C^*-algebras, such as Roe algebra, group C^*-algebra, crossed product by étale groupoid G and so on. Through this paper, we mainly study Roe algebras and maximal Roe algebras.

Next we are going to define ε-r-projections and ε-r-unitaries, this is the main elements in quantitative K-theory. Let A be a unital filtered C^*-algebra. for any $r > 0$ and $\varepsilon \in \left(0, \frac{1}{4}\right)$, we call

(1) an element p in A an ε-r-projection if p belongs to A_r, $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε-r-projections will be denoted by $P^{\varepsilon,r}(A)$.

(2) an element u in A is an ε-r-unitary if u belongs to A_r, $\|u^*u - 1\| < \varepsilon$ and $\|uu^* - 1\| < \varepsilon$. The set of ε-r-unitaries in A will be denoted by $U^{\varepsilon,r}(A)$.

Notice that for any ε-r-projection, it has a spectrum gap around $\frac{1}{2}$, by the functional calculus we can construct a projection $k_0(p)$ satisfying $\|k_0(p) - p\| < 2\varepsilon$. For a ε-r-unitary, we can construct a unitary $k_1(u)$, $k_1(u) = u(u^*u)^{-\frac{1}{2}}$ and $\|k_1(u) - u\| < \varepsilon$.

For integer n, we set $U_n^{\varepsilon,r}(A) = U^{\varepsilon,r}(M_n(A))$ and $P_n^{\varepsilon,r}(A) = P^{\varepsilon,r}(M_n(A))$.

Consider the inclusions

$$P_n^{\varepsilon,r} \to P_{n+1}^{\varepsilon,r}, \quad p \mapsto \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$U_n^{\varepsilon,r} \to U_{n+1}^{\varepsilon,r}, \quad u \mapsto \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix},$$

then we can define $P_\infty^{\varepsilon,r}(A) = \bigcup\limits_{n\in\mathbb{N}} P_n^{\varepsilon,r}(A)$ and $U_\infty^{\varepsilon,r}(A) = \bigcup\limits_{n\in\mathbb{N}} U_n^{\varepsilon,r}(A)$.

For a unital filtered C^* algebra A, we can define the following equivalent relation on $P_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $U_\infty^{\varepsilon,r}(A)$.

(1) if p and q are elements of $P_\infty^{\varepsilon,r}(A)$, l and l' are positive integers, $(p, l) \sim (q, l')$ if there exists a positive integer k and an element h of $P_\infty^{\varepsilon,r}(A[0,1])$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.

(2) if u and v are elements of $U_\infty^{\varepsilon,r}(A)$, $u \sim v$ if there exists an element h of $U_\infty^{3\varepsilon,2r}(A[0,1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $P_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p, l]_{\varepsilon,r}$ the equivalent class of (p, l) modulo $\sim$. And if u is an element of $U_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalent class modulo $\sim$.

Definition 2.2 (see [10]) *Let $r > 0$ and $\varepsilon \in \left(0, \frac{1}{4}\right)$. We define*

(1) $K_0^{\varepsilon,r}(A) = P_\infty^{\varepsilon,r}(A) \times \mathbb{N}/\sim$ unital and $K_0^{\varepsilon,r}(A) = P_\infty^{\varepsilon,r}(\widetilde{A}) \times \mathbb{N}/\sim$ such that Rank $k_0(\rho_A(p)) = l$ for A non unital;

(2) $K_1^{\varepsilon,r}(A) = U_\infty^{\varepsilon,r}/\sim$ (with $A = \widetilde{A}$ if A is already unital).

Then $K_0^{\varepsilon,r}(A)$ turns to be a n abelian group where

$$[p, l]_{\varepsilon,r} + [p', l']_{\varepsilon,r} = [\mathrm{diag}(p, p'), l + l']_{\varepsilon,r},$$

$K_1^{\varepsilon,r}(A)$ is also a abelian group with

$$[u]_{\varepsilon,r} + [u']_{\varepsilon,r} = [\mathrm{diag}(u, u')]_{\varepsilon,r}.$$

Next we introduce some basic properties od quantitative K-theory.

Lemma 2.1 (see [10]) *If A is a filtered C^*-algebra, then $K_*^{\varepsilon,r}(A) = K_0^{\varepsilon,r}(A) \oplus K_1^{\varepsilon,r}(A)$ is a $\mathbb{Z}_2$-graded abelian group.*

For any filtered C^*-algebra A and any positive numbers $\varepsilon, \varepsilon'$ and r, r' with $\varepsilon < \varepsilon' < \frac{1}{4}$ and $r \leq r'$, there exists natural group homomorphisms:

(1) $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \to K_0(A), [p,l]_{\varepsilon,r} \mapsto [k_0(p)] - [I_l]$;
(2) $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \to K_1(A), [u]_{\varepsilon,r} \to [k_1(u)]$;
(3) $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$;
(4) $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \to K_0^{\varepsilon',r'}(A), [p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$;
(5) $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \to K_1^{\varepsilon',r'}(A), [u]_{\varepsilon,r} \to [u]_{\varepsilon',r'}$;
(6) $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$.

The next proposition is very useful, it reveals the relationship between K-theory and quantitative K-theory. The second property is quite important to the persistence approximation property that we are going to study in next section.

Proposition 2.1 (see [10]) *let B be a filtered C^*-algebra.*

(i) *For any $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $y \in K_*(B)$, there exist a positive number r and an element $x \in K_*^{\varepsilon,r}(B)$ such that $\iota_*^{\varepsilon,r}(x) = y$;*

(ii) *There exists a positive number $\lambda > 1$ independent on B such that the following is satisfies:*
Let $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $r > 0$, and let x and x' be two elements in $K_*^{\varepsilon,r}(B)$ such that $\iota^{\varepsilon,r}(x) = \iota^{\varepsilon,r}(x')$ in $K_*(B)$. Then there exists r' with $r' > r$ such that $\iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x) = \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x')$ in $K_*^{\lambda\varepsilon,r'}(B)$.

Next we are going to review the controlled morphisms between quantitative operator K-theory. Recall that a control pair is a pair (λ, h), where

(1) $\lambda > 1$;

(2) $h : \left(0, \frac{1}{4\lambda}\right) \to (1, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ is a map such that exists a non-increasing map $g : \left(0, \frac{1}{4\lambda}\right) \to (1, +\infty)$, with $h \leq g$.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\varepsilon \leq h'_\varepsilon$ for all $\varepsilon \in \left(0, \frac{1}{4\lambda'}\right)$.

For any filtered C^*-algebra A, define the families $\mathcal{K}_0(A) = (K_0^{\varepsilon,r}(A))_{0<\varepsilon<\frac{1}{4},r>0}$, $\mathcal{K}_1(A) = (K_1^{\varepsilon,r}(A))_{0<\varepsilon<\frac{1}{4},r>0}$, $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0<\varepsilon<\frac{1}{4},r>0}$.

Definition 2.3 (see [10]) *Let (λ, h) be a control pair, A and B be two filtered C^*-algebras, and i, j be elements of $\{0, 1, *\}$. A (λ, h)-controlled morphism*

$$\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$$

is a family $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ of group homomorphisms

$$F^{\varepsilon,r} : K_i^{\varepsilon,r}(A) \to K_j^{\lambda\varepsilon,h_\varepsilon r}(B)$$

such that for any positive numbers $\varepsilon, \varepsilon'$ and r, r' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{4\lambda}$, $r \leq r'$ and $h_\varepsilon r \leq h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_i^{\varepsilon,\varepsilon',r,r'} = \iota_j^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon r,h_{\varepsilon'}r'} \circ F^{\varepsilon,r}.$$

Let A and B be two filtered C^*-algebras. If $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ is a (λ, h)-controlled morphism, then there is a group homomorphism $F : K_i(A) \to K_j(B)$ uniquely defined by $F \circ \iota_i^{\varepsilon,r} = \iota_j^{\lambda\varepsilon,h_\varepsilon r} \circ F^{\varepsilon,r}$. The homomorphism F is called the (λ, h)-controlled morphism induced by $\mathcal{F}$.

If (λ, h) and (λ', h') are two control pairs,define

$$h * h' : \left(0, \frac{1}{4\lambda\lambda'}\right) \to (0, +\infty), \quad \varepsilon \mapsto h_{\lambda'\varepsilon} h'_\varepsilon.$$

Then $(\lambda\lambda', h * h')$ is a control pair. Let A, B_1 and B_2 be filtered C^*-algebras, i, j and l in $\{0, 1, *\}$. Let $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism, let $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{G}}},r>0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_l(B_2)$ is the $(\alpha_{\mathcal{G}}\alpha_{\mathcal{F}}, k_{\mathcal{G}} * k_{\mathcal{F}})$-controlled morphism defined by the family $(G^{\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\varepsilon}r} \circ F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}},r>0}$.

Let A and B be filtered C^*-algebras, and (λ, h) is a control pair. Let $\mathcal{F} = (F^{\varepsilon,E})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ (resp. $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{G}}},r>0}$) be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism (resp. a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism). Then we write $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ if

(1) $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \le (\lambda, h)$ and $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \le (\lambda, h)$;

(2) for any $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$ and $r > 0$, then

$$\iota_j^{\alpha_{\mathcal{F}}\varepsilon,\lambda\varepsilon,k_{\mathcal{F},\varepsilon}r,h_\varepsilon r} \circ F^{\varepsilon,r} = \iota_j^{\alpha_{\mathcal{G}}\varepsilon,\lambda\varepsilon,k_{\mathcal{G},\varepsilon}r,h_\varepsilon r} \circ G^{\varepsilon,r}.$$

Definition 2.4 (see [10]) *Let (λ, h) be a control pair and $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \le (\lambda, h)$.*

(1) *$\mathcal{F}$ is called left (λ, h)-invertible if there exists a controlled morphism*

$$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$$

such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_i(A)}$. and $\mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_j(B)}$.

(2) *$\mathcal{F}$ is (λ, h)-isomorphism if there exists a controlled morphism*

$$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A),$$

which is a (λ, h)-inverse for $\mathcal{F}$.

Let (λ, h) be a control pair and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism.

(1) $\mathcal{F}$ is called (λ, h)-injective if $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \le (\lambda, h)$ and for any $0 < \varepsilon < \frac{1}{4\lambda}$, any $r > 0$ and any $x \in K_i^{\varepsilon,r}(A)$, then $F^{\varepsilon,r}(x) = 0$ in $K_j^{\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\varepsilon}r}(B)$ implies that $\iota_i^{\varepsilon,\lambda\varepsilon,r,h_\varepsilon r}(x) = 0$ in $K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$;

(2) $\mathcal{F}$ is called (λ, h)-surjective, if for any $0 < \varepsilon < \frac{1}{4\lambda\alpha_{\mathcal{F}}}$, any $r > 0$ and $y \in K_j^{\varepsilon,r}(B)$, there exists an element $x \in K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$ such that

$$F^{\lambda\varepsilon,h_{\lambda\varepsilon}r}(x) = \iota_j^{\varepsilon,\alpha_{\mathcal{F}}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(y)$$

in $K_j^{\alpha_{\mathcal{F}}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(B)$.

The exact sequence is quite important to K-theory, we also have the controlled exact sequence for quantitative K-theory.

Definition 2.5 (see [10]) *Let (λ, h) be a control pair. Let $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism, and let $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism, where i, j and l are in $\{0, 1, *\}$ and A, B_1, B_2 are filtered C^*-algebras. Then the composition*

$$\mathcal{K}_i(A) \overset{\mathcal{F}}{\to} \mathcal{K}_j(B_1) \overset{\mathcal{G}}{\to} \mathcal{K}_l(B_2)$$

is said to be (λ, h)-exact at $\mathcal{K}_j(B_1)$ if $\mathcal{G} \circ \mathcal{F} = 0$ and if for any $0 < \varepsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}},\alpha_{\mathcal{G}}\}}$, any $r > 0$ and $y \in K_j^{\varepsilon,r}(B_1)$ such that $G^{\varepsilon,r}(y) = 0$ in $K_j^{\alpha_{\mathcal{G}}\varepsilon,k_{\mathcal{G},\varepsilon}r}(B_2)$, there exists an element x in $K_i^{\lambda\varepsilon,h_\varepsilon r}(A)$ such that

$$F^{\lambda\varepsilon,h_{\lambda\varepsilon}r}(x) = \iota_j^{\varepsilon,\alpha_{\mathcal{F}}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(y)$$

in $K_j^{\alpha_{\mathcal{F}}\lambda_\varepsilon, k_{\mathcal{F}\lambda_\varepsilon} h_\varepsilon r}(B_1)$.

The six-term exact sequence in K-theory for short exact sequence of C^*-algebras is very important. But for the quantitative K-theory, we need the short exact sequence to be completely filtered. Here we give the definition below.

Definition 2.6 (see [10]) *Let A be a filtered C^*-algebra. let J be an ideal of A and set $J_r = J \cap A_r$. The extension of C^*-algebras*

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of C^*-algebras if the bijection continuous linear map

$$A_r/J_r \to (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry, i.e., for any integer n, any $r > 0$ and $x \in M_n(A_r)$, then

$$\inf_{y \in M_n(J_r)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\|.$$

Lemma 2.2 (see [10]) *Any semi-split extension of filtered C^*-algebra is completely filtered.*

Theorem 2.1 (see [10]) *There exists a control pair (λ, h) such that for any completely filtered extensions of C^*-algebras*

$$0 \to J \overset{j}{\to} A \overset{q}{\to} A/J \to 0,$$

the following six-term sequence is (λ, h)-exact

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{j_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J,A}} \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{j_*} & \mathcal{K}_1(J) \end{array}$$

Quantitative coarse assembly map was constructed in Clément Dell'Aiera's thesis (see [3]). Let us briefly introduced the quantitative coarse assembly map. We call that a discrete metric space has bounded geometry, if for any $r > 0$, there exists an integer N_r such that $\#B(x, r) \leq N_r$ for any $x \in X$, where $\#B(x, r)$ means the number of the elements in $B(x, r)$.

Let X be a discrete metric space with bounded geometry. Let A be a C^*-algebra, the Roe algebra with coefficient A is denoted by $C^*(X, A)$ is defined in the following. Let $\mathbb{C}_r[X, A]$ is the subspace of $\mathcal{L}_A(l^2(X) \otimes H \otimes A)$:

$$\mathbb{C}_r[X, A] = \{T \in \mathcal{L}_A(l^2(X) \otimes H \otimes A) \text{ locally compact s.t. supp} T \subseteq \Delta_r\},$$

where $\Delta_r = \{(x, y); d(x, y) \leq r\}$ and an operator $T \in \mathcal{L}_A(l^2(X) \otimes H \otimes A)$ is called locally compact if $T_{x,y} \in K(H) \otimes A$.

It is easy to see that

$$\mathbb{C}[X, A] = \bigcup_{r > 0} \mathbb{C}_r[X, A]$$

is $*$-subalgebra of $\mathcal{L}_A((l^2(X) \otimes H \otimes A)$. The Roe algebra with coefficient A is the completion of $\mathbb{C}[X, A]$ under the operator norm $\mathcal{L}_A(l^2(X) \otimes H \otimes A)$.

Next we are going to define the maximal Roe algebra with coefficient A. We need a lemma first.

Lemma 2.3 *Let X be a discrete metric space with bounded geometry. For any positive number r, there exists a positive number c_r, such that for any $*$-representation ϕ of $\mathbb{C}[X,A]$ on a Hilbert A module E_A and any $T \in \mathbb{C}_r[X,A]$, then*

$$\|\phi(T)\| \leq c_r \sup_{x,y\in X} \|T(x,y)\|.$$

Proof For any $r > 0$, there exists c_r partial isometries $v_1, \cdots, v_{c_r} \in M(\mathbb{C}[X,A])$, where $M(\mathbb{C}[X,A])$ is the multiplier of $\mathbb{C}[X,A]$, such that any $T \in \mathbb{C}_r[X,A]$ can be written as

$$T = \sum_{i=1}^{c_r} f_i v_i,$$

where each f_i is an element of $l^\infty(X, K(H) \otimes A)$, then we have

$$\|\phi(T)\| \leq c_r \sup_{x,y\in X} \|T(x,y)\|.$$

Definition 2.7 *The maximal Roe algebra with coefficient A is denoted by $C^*_{\max}(X,A)$, is the completion of $\mathbb{C}[X,A]$ in the norm*

$$\|T\| = \sup\{\|\pi(T)\|_{L_A(E_A)}; \pi : \mathbb{C}[X,A] \to L_A(E_A) \ a * \text{-representation}\}.$$

There are basic functor properties for $C^*(X,\cdot)$ and $C^*_{\max}(X,\cdot)$.

Theorem 2.2 *Let X be a discrete space with bounded geometry, A and B be two C^*-algebras.*

(1) *If $\phi : A \to B$ is a $*$-homomorphism, then there exists $*$-homomorphism $\phi_X : C^*(X,A) \to C^*(X,B)$ and $\phi_{X,\max} : C^*_{\max}(X,A) \to C^*_{\max}(X,B)$.*

(2) *If $\phi : A \to B$ is a completely positive map, then there exists completely positive map $\phi_X : C^*(X,A) \to C^*(X,B)$ and $\phi_{X,\max} : C^*_{\max}(X,A) \to C^*_{\max}(X,B)$.*

Next to define the quantitative maximal assembly map. The method is similar in Clément Dell'Aiera's thesis (see [3]). The construction of quantitative maximal assembly map is divided into two steps. The first step: Controlled Roe transformation.

For any $z \in KK_1(A,B)$, then z can be represent by a triple (H_A, π, T), where

(1) $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B;

(2) $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator;

(3) $[T, \pi(a)], \pi(a)[T^2 - Id_{H_B}]$ are compact operators in $K(H_B) \cong K(H) \otimes B$.

Let $P = \left(\frac{1+T}{2}\right) \in \mathcal{L}_B(H_B)$ and

$$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in B \otimes K(H)\}.$$

Then we have a semi-split exact extension:

$$0 \to B \otimes K(H) \to E^{(\pi,T)} \to A \to 0,$$

where the completely positive section is $s : A \to E^{\pi,T}; a \mapsto (a, P\pi(a)P)$. By the functor property of $C^*_{\max}(X,\cdot)$, then we have a semi-split exact extension:

$$0 \to C^*_{\max}(X,B) \to E^{\pi,T}_{X,\max} \to C^*_{\max}(X,A) \to 0,$$

where $E^{\pi,T}_{X,\max} = C^*_{\max}(X, E^{\pi,T})$.

Proposition 2.2 *The controlled boundary map $\mathcal{D}^{\pi,T} = \mathcal{D}_{C^*_{\max}(X,B),E^{\pi,T}_{X,\max}}$ of the extension*

$$0 \to C^*_{\max}(X,B) \to E^{\pi,T}_{X,\max} \to C^*_{\max}(X,A) \to 0$$

only depends on the class z.

Then we can define the controlled Roe transformation.

Definition 2.8 *For any $z = [H_B, \pi, T] \in KK_1(A,B)$, the controlled maximal Roe transformation $\widehat{\sigma}_{X,\max}(z) = \mathcal{D}_{C^*_{\max}(X,B),E^{\pi,T}_{X,\max}}$. It is an (α_X, k_X)-controlled morphism $\mathcal{K}_*(C^*_{\max}(X,A)) \to \mathcal{K}_{*+1}(C^*_{\max}(X,B))$.*

Proposition 2.3 *Let A and B be two C^*-algebra. then there exists a control pair (α_X, k_X) such that for any $z \in KK_1(A,B)$, there exists a (α_X, k_X)-controlled morphism*

$$\widehat{\sigma}_{X,\max}(z) : \mathcal{K}_*(C^*_{\max}(X,A)) \to \mathcal{K}_{*+1}(C^*_{\max}(X,B))$$

such that

(i) *$\widehat{\sigma}_{X,\max}(z)$ induces right multiplication by $\sigma_{X,\max}(z)$ in K-theory.*

(ii) *$\widehat{\sigma}_{X,\max}(z)$ is additive, i.e.,*

$$\widehat{\sigma}_{X,\max}(z+z') = \widehat{\sigma}_{X,\max}(z) + \widehat{\sigma}_{X,\max}(z').$$

(iii) *For any $*$-homomorphism $f : A_1 \to A_2$, we have*

$$\widehat{\sigma}_{X,\max}(f^*(z)) = \widehat{\sigma}_{X,\max}(z) \circ f_{X,*}$$

for any $z \in KK_1(A,B)$.

(iv) *For any $*$-homomorphism $g : B_1 \to B_2$, we have*

$$\widehat{\sigma}_{X,\max}(g_*(z)) = g_{X,*} \circ \widehat{\sigma}_{X,\max}(z)$$

for any $z \in KK_1(A,B)$.

(v) *Let $0 \to J \to A \to A/J \to 0$ be a semi-split exact extension and $[\partial_{J,A}] \in KK_1(A/J, J)$ is its boundary element. Then*

$$\widehat{\sigma}_{X,\max}([\partial_{J,A}]) = \mathcal{D}_{C^*_{\max}(X,J),C^*_{\max}(X,A)}.$$

For the even case, we need a lemma first. Let A be a C^*-algebra, we have a short exact sequence

$$0 \to SA \to CA \to A \to 0.$$

Under the functor $C^*_{\max}(X,\cdot)$, we have a semi-split exact extension

$$0 \to C^*_{\max}(X,SA) \to C^*_{\max}(X,CA) \to C^*_{\max}(X,A) \to 0.$$

Let $\mathcal{D}_{X,A,\max} : \mathcal{K}_*(C^*_{\max}(X,A)) \to \mathcal{K}_{*+1}(C^*_{\max}(X,SA))$ be its boundary map.

Lemma 2.4 *Let A be a C^*-algebra, then there exists a control pair (λ, h) independent the choice of X and A such that $\mathcal{D}_{X,A,\max}$ is (λ, h)-invertible.*

Proof The proof is similar in [3] Proposition 4.2.1.

Let A and B be two C^*-algebras, for any $z \in KK_0(A, B)$. For the short exact sequence

$$0 \to SB \to CB \to B \to 0.$$

Let $[\partial_{SB}] \in KK_1(B, SB)$ be its boundary map. Since $\mathcal{D}_{X,A,\max}$ and $\mathcal{D}_{C^*_{\max}(X,A),C^*_{\max}(X,\mathcal{T}_0\otimes A)}$ are controlled inverse to each other. We use $\mathcal{T}_{X,A,\max}$ represents $\mathcal{D}_{C^*_{\max}(X,A),C^*_{\max}(X,\mathcal{T}_0\otimes A)}$. Since $z \otimes_B [\partial_{SB}] \in KK_1(A, SB)$, we define $\widehat{\sigma}_{X,\max}(z) := \mathcal{T}_{X,A,\max} \circ \widehat{\sigma}_{X,\max}(z \otimes_B [\partial_{SB}])$.

Proposition 2.4 *Let A and B be two C^*-algebras, for any $z \in KK_0(A, B)$, there exists a control pair (α_X, k_X) and even degree (α_X, k_X)-controlled morphism*

$$\widehat{\sigma}_{X,\max}(z) : \mathcal{K}_*(C^*_{\max}(X, A)) \to \mathcal{K}_*(C^*_{\max}(X, B))$$

such that

(i) $\widehat{\sigma}_{X,\max}(z)$ *induces right multiplication by $\sigma_{X,\max}(z)$ in K-theory.*

(ii) $\widehat{\sigma}_{X,\max}(z)$ *is additive, i.e.,*

$$\widehat{\sigma}_{X,\max}(z + z') = \widehat{\sigma}_{X,\max}(z) + \widehat{\sigma}_{X,\max}(z').$$

(iii) *For any $*$-homomorphism $f : A_1 \to A_2$, we have*

$$\widehat{\sigma}_{X,\max}(f^*(z)) = \widehat{\sigma}_{X,\max}(z) \circ f_{X,*}$$

for any $z \in KK_1(A, B)$.

(iv) *For any $*$-homomorphism $g : B_1 \to B_2$, we have*

$$\widehat{\sigma}_{X,\max}(g_*(z)) = g_{X,*} \circ \widehat{\sigma}_{X,\max}(z)$$

for any $z \in KK_1(A, B)$.

(v) $\widehat{\sigma}_{X,\max}([\mathrm{id}_A]) \overset{(\alpha_X,k_X)}{\sim} \mathrm{id}_{\mathcal{K}(C^*_{\max}(C,A))}$.

The second step is quite similar to the cut off function in group case. For any positive number d and probability η of the Rips complex $P_d(X)$ can be written as $\eta = \sum_{x\in X} \lambda_x(\eta)\delta_x$, where δ_x is the Dirac probability at x, and $\lambda_x : P_d(X) \to [0, 1]$ is a continuous function. Let

$$h_d : \begin{cases} X \times X \to C_0(P_d(X)), \\ (x, y) \mapsto \lambda_x^{\frac{1}{2}} \lambda_y^{\frac{1}{2}}. \end{cases}$$

Let $(e_x)_{x\in X}$ be the canonical basis of $l^2(X)$, e is a rank one projection in H and P_d be defined as the extension by linearity and continuity of

$$P_d(e_x \otimes \xi \otimes f) = \sum_{y\in X} e_y \otimes (e\xi) \otimes (h(x, y)f)$$

for every $x \in X, \xi \in H$ and $f \in C_0(P_d(X))$. As $\sum_{x\in X} \lambda_x = 1$, P_d is projection of $K(l^2(X)) \otimes C_0(P_d(X))$ of controlled support. Since $\lambda_x^{\frac{1}{2}}\lambda_y^{\frac{1}{2}} = 0$ if $d(x,y) \geq d$. We give the maximal completion of P_d in $C^*_{\max}(X, C_0(P_d(X)))$. Hence P_d define a class

$$[P_d, 0]_{\varepsilon,r'} \in K_0^{\varepsilon,r'}(C^*_{\max}(X, C_0(P_d(X))))$$

for any $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $r' \geq r$.

For every C^*-algebra A and r, r' satisfying $r \leq r'$, the inclusion $P_d(X) \to P_{d'}(X)$ induce a*-homomorphism

$$(q_d^{d'})^* : KK_*(C_0(P_d(X)), A) \to KK_*(C_0(P_{d'}(X)), A)$$

in KK-theory. And a map

$$((q_d^{d'})_X)_* : K_*(C^*_{\max}(X, C_0(P_d(X)))) \to K_*(C^*_{\max}(X, C_0(P_{d'}(X))))$$

in K-theory. The family of projections P_d are compatible with the morphism $q_d^{d'}$, i.e.,

$$((q_d^{d'})_X)_*[P_{d'}, 0]_{\varepsilon,r'} = [p_d, 0]_{\varepsilon,r},$$

for $\varepsilon \in \left(0, \frac{1}{4}\right)$.

Definition 2.9 (see [3]) *Let A be a C^*-algebra, $\varepsilon \in \left(0, \frac{1}{4}\right)$ and positive numbers d, r satisfying that $k_X(\varepsilon)d \leq r$. The quantitative assembly map $\widehat{\mu}_{X,A,\max,*} = (\mu^{\varepsilon,d,r}_{X,A,\max,*})_{\varepsilon,r}$ is defined as the family of maps*

$$\mu^{\varepsilon,d,r}_{X,A,\max,*} : \begin{cases} KK_*(C_0(P_d(X)), A) \to K_*^{\varepsilon,r}(C^*_{\max}(X, A)), \\ z \mapsto \iota_*^{\alpha_X\varepsilon',\varepsilon,k_X(\varepsilon')r',r} \circ \sigma^{\varepsilon',r'}_{X,\max}(z)[P_d, 0]_{\varepsilon',r'}, \end{cases}$$

where ε' and r' satisfy:

(1) *$\varepsilon' \in \left(0, \frac{1}{4}\right)$ such that $\alpha_X\varepsilon' \leq \varepsilon$.*

(2) *$d \leq r'$ such that $k_X(\varepsilon')r' \leq r$.*

Remark 2.1 The controlled coarse assembly map is compatible with the structure morphism $q_d^{d'}$. Indeed, for any d and d' satisfying $d \leq d'$.

$$\begin{aligned} \widehat{\sigma}_{X,\max}((q_d^{d'})^*(z))[P_{d'}, 0]_{\varepsilon,r'} &= \widehat{\sigma}_{X,\max}(z) \circ ((q_d^{d'})_X)_*[P_{d'}, 0]_{\varepsilon,r'} \\ &= \widehat{\sigma}_{X,\max}(z)[P_d, 0]_{\varepsilon,r}. \end{aligned} \tag{2.1}$$

Hence $\widehat{\mu}^{\varepsilon,d,r}_{X,A,\max,*} \circ (q_d^{d'})^* = \widehat{\mu}^{\varepsilon,d',r}_{X,A,\max,*}$.

Remark 2.2 The quantitative maximal coarse assembly map is also compatible with the structure morphism $\iota_*^{\varepsilon,\varepsilon',r,r'}$, i.e., $\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \widehat{\mu}^{\varepsilon,d,r}_{X,A,\max,*} = \widehat{\mu}^{\varepsilon',d,r'}_{X,A,\max,*}$ for every $r \leq r'$ and $\varepsilon \leq \varepsilon'$ such that this equality is defined.

Remark 2.3 The maximal quantitative coarse assembly map $\widehat{\mu}_{X,A,\max,*}$ induces the maximal coarse assembly map $\mu_{X,A,\max,*}$ in K-theory.

Throughout this paper, using $KK_*(P_d(X), A)$ represents $KK_*(C_0(P_d(X)), A)$. Let A be a G-algebra, we say that

(1) (Quantitative injectivity) $\mu_{X,A,\max,*}$ is quantitative injective if for any $d>0$, there exists $\varepsilon\in\left(0,\frac{1}{4}\right)$ such that for any $r>0$ satisfying $k_X(\varepsilon)d\le r$, there exists $d'>d$ such that for any $z\in KK_*(P_d(X),A)$, $\mu^{\varepsilon,d,r}_{X,A,\max,*}(z)=0$ implies that $(q_d^{d'})^*(z)=0$.

(2) (Quantitative surjectivity) $\mu_{X,A,\max,*}$ is quantitative surjective if there exists $\varepsilon\in\left(0,\frac{1}{4}\right)$ such that for any $r>0$ such that, there exists $\varepsilon'\in(\varepsilon,\frac{1}{4})$ and positive numbers d,r' such that $r\le r'$ and $k_X(\varepsilon')d\le r'$, for any $y\in K_*^{\varepsilon,r}(C^*_{\max}(X,A))$ there exists $z\in KK_*(P_d(X),A)$ such that $\mu^{\varepsilon',d,r'}_{X,A,\max,*}(z)=\iota_*^{\varepsilon,\varepsilon',r,r'}(y)$.

Then we show that the maximal coarse Baum-Connes conjecture follows from quantitative maximal coarse Baum-Connes conjecture.

Proposition 2.5 *Let X be a discrete metric space with bounded geometry and A be a C^*-algebra.*

(1) *If $\mu_{X,A,\max,*}$ is quantitative injective then $\mu_{X,A,\max,*}$ is one to one.*

(2) *If $\mu_{X,A,\max,*}$ is quantitative surjective then $\mu_{X,A,\max,*}$ is onto.*

Proof For the first point,we only prove the even case. The odd case is similar. For any positive number d, any $x\in KK_0(P_d(X),A)$ such that $\mu_{X,A,\max,*}(x)=0$. Then for any $\varepsilon\in\left(0,\frac{1}{4}\right)$ and $r>0$ such that $k_X(\varepsilon)d\le r$, $\iota^{\varepsilon,r}\circ\mu^{\varepsilon,d,r}_{X,A,\max,*}(x)=0$, choose $\varepsilon''>0$ and $r''>0$ such that $\alpha_X\varepsilon''\le\varepsilon$ and $k_X(\varepsilon'')r''\le r$. Then there exist a universal $\lambda\ge 1$ and r' with $r'\ge r$ such that

$$
\begin{aligned}
0&=\iota_0^{\varepsilon,\lambda\varepsilon,r,r'}\circ\mu^{\varepsilon,d,r}_{X,A,\max,*}(x)\\
&=\iota_0^{\varepsilon,\lambda\varepsilon,r,r'}\circ\iota_*^{\alpha_X\varepsilon'',\varepsilon,k_X(\varepsilon'')r'',r}(\sigma^{\varepsilon'',r''}_{X,\max,*}(x))[P_d,0]_{\varepsilon'',r''}\\
&=\widehat{\sigma}^{\lambda\varepsilon,r'}_{X,\max,*}(x)[P_d,0]_{\lambda\varepsilon',r'}\\
&=\mu^{\lambda\varepsilon,d,r'}_{X,A,\max,*}(x).
\end{aligned}
\tag{2.2}
$$

Since it is quantitative injective, then we have $(q_d^{d'})^*(x)=0$ in $KK_0(C_0(P_{d'}(X),A))$, then $x=0$ in $\lim\limits_{d>0}KK_0(C_0(P_{d'}(X),A))$.

For the second point, we only prove the even case. The odd case is similar. For any $y\in K_0(C^*_{\max}(X,A))$, and any $\varepsilon\in\left(0,\frac{1}{4}\right)$, there exists $r>0$ and $x\in K_0^{\varepsilon,r}(C^*_{\max}(X,A))$ such that $\iota_0^{\varepsilon,r}(x)=y$. Then there exist $\varepsilon'\in\left(0,\frac{1}{4}\right)$, $d,r'>0$ and $z\in KK_0(P_d(x),A))$ such that $\varepsilon\le\varepsilon', k_X(\varepsilon')\cdot d\le r', r\le r'$ and $\mu^{\varepsilon',d,r'}_{X,A,\max,*}(z)=\iota^{\varepsilon,\varepsilon',r,r'}(x)$, hence $\mu_{X,A,\max,*}(z)=y$.

For any C^*-algebra A. We define the following quantitative statements.

(1) $QI_{X,A,\max,*}(d,d',\varepsilon,r)$: For any $x\in KK_*(P_d(X),A)$, then $\mu^{\varepsilon,d,r}_{X,A,\max,*}(x)=0$ implies $(q_d^{d'})^*(x)=0$ in $KK_*(P_{d'}(X),A)$.

(2) $QS_{X,A,\max,*}(d,\varepsilon,\varepsilon',r,r')$: For any $y\in K_*^{\varepsilon,r}(C^*_{\max}(X,A))$, then there exists a

$$x\in KK_*(P_d(X),A)$$

such that $\mu^{\varepsilon',d,r'}_{X,A,\max,*}(x)=\iota_*^{\varepsilon,\varepsilon',r,r'}(y)$.

Next we are going to study more relationship between maximal quantitative Baum-Connes conjecture and the maximal Baum-Conjecture. Firstly, we introduce some useful lemmas. If $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ is any family of filtered C^*-algebras and H a separable Hilbert space. Set $\mathcal{A}^\infty_{c,r}=\prod\limits_{i\in\mathbb{N}}K(H)\otimes A_{i,r}$ for any $r>0$ and define the C^*-algebra $\mathcal{A}^\infty_c$ as the closure of $\bigcup\limits_{r>0}\mathcal{A}^\infty_{c,r}$ in $\prod\limits_{i\in\mathbb{N}}K(H)\otimes A_{i,r}$.

Lemma 2.5 (see [13]) *Let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is any family of filtered C^*-algebras and let*

$$\mathcal{F}_{\mathcal{A},*} = (F_{\mathcal{A}}, \varepsilon, r)_{0<\varepsilon<\frac{1}{4}, r>0} : \mathcal{K}_*(\mathcal{A}_c^\infty) \to \Pi\mathcal{K}_*(A_i),$$

where

$$F_{\mathcal{A},*}^{\varepsilon,r} : K_*^{\varepsilon,r}(\mathcal{A}_c^\infty) \to \prod_{i\in\mathbb{N}} K_*^{\varepsilon,r}(A_i)$$

is the map induced n the j-th factor and up to the Morita equivalence by the restriction to $\mathcal{A}_c^\infty$ of the evaluation $\prod\limits_{i\in\mathbb{N}} K(H)\otimes A_i \to K(H)\otimes A_j$ at $j\in\mathbb{N}$. Then $\mathcal{F}_{\mathcal{A},*}$ is a (α,h)-controlled isomorphism for a control pair (α,h) independent on the family $\mathcal{A}$.

For a family of C^*-algebra $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$, denote $\mathcal{A}_{X,\max}^\infty$ is the closure of $\bigcup\limits_{r>0}\prod\limits_{i\in\mathbb{N}} \mathbb{C}[X, A_i]_r$ in $\prod\limits_{i\in\mathbb{N}} C_{\max}^*(X, A_i)$.

Lemma 2.6 *Let X is a discrete matric space with bounded geometry, and a family of C^*-algebra $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$, then there is a control pair (λ,h) and a (λ,h)-isomorphism*

$$\mathcal{K}_*(\mathcal{A}_{X,\max}^\infty) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(C_{\max}^*(X, A_i)).$$

Proof We are going to prove it in the even case, the odd case is similar. Firstly, there exists a (λ,h)-controlled morphism

$$\mathcal{K}_*(\mathcal{A}_{X,\max}^\infty) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(C_{\max}^*(X, A_i))$$

induced by the projection $\prod\limits_{i\in\mathbb{N}} C_{\max}^*(X, A_i) \to C_{\max}^*(X, A_i)$ restrict at $\mathcal{A}_{X,\max}^\infty$. For any positive integer n and $i\in\mathbb{N}$, since $M_n(l^\infty\ (X, A_i\otimes K(H))) \subseteq l^\infty(X, A_i)\otimes K(H)$, hence we have $M_n(C_{\max}^*(X, A_i)) \subseteq C_{\max}^*(X, A_i)$. Hence for any positive number r, $\varepsilon\in\left(0,\frac{1}{4}\right)$ and any $x\in\prod\limits_{i\in\mathbb{N}} K_*^{\varepsilon,r}(C_{\max}^*(X, A_i))$. Then $x = [\mathcal{P}, l]$, where l is a integer, $\mathcal{P}\in P_n\Big(\prod\limits_{i\in\mathbb{N}} C_{\max}^*(X, A_i)\Big)$, we write $\mathcal{P} = (p_i)_{i\in\mathbb{N}}$, for any $p_i\in P_n(C_{\max}^*(X, A_i))$. So we can assume $p_i\in P(C_{\max}^*(X, A_i))$, hence the constructed controlled morphism is (λ,h)-surjective.

To prove this controlled morphism is (λ,h)-injective, we need [10, Proposition 1.30], the homotopy of ε-r-projections can be chosen to be Lipschitz homotopy in larger matrix size. Easily this controlled morphism is (λ,h)-injective.

Lemma 2.7 *Let X is a discrete metric space with bounded geometry, and a family of C^*-algebra $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$, then we have a filtered isomorphism*

$$\phi : C_{\max}^*\Big(X, \prod_{i\in\mathbb{N}} A_i\Big) \to \mathcal{A}_{X,\max}^\infty.$$

Proof Obviously, by the universal property of $C_{\max}^*\Big(X, \prod\limits_{i\in\mathbb{N}} A_i\Big)$, there is a filtered homomorphism

$$\phi : C_{\max}^*\Big(X, \prod_{i\in\mathbb{N}} A_i\Big) \to \mathcal{A}_{X,\max}^\infty.$$

Since the range of this filtered homomorphism under the dense subalgebra $\mathbb{C}\Big[X, \prod_{i\in\mathbb{N}} A_i\Big]$ is dense in $\mathcal{A}^{\infty}_{X,\max}$. Hence we only need to prove this filtered homomorphism ϕ is injective.

Since for any $i \in \mathbb{N}$, we have the inclusion $A_i \to \prod_{i\in\mathbb{N}} A_i$. So we have filtered homomorphism

$$C^*_{\max}(X, A_i) \to C^*_{\max}\Big(X, \prod_{i\in\mathbb{N}} A_i\Big).$$

Hence ϕ is a filtered isomorphism.

By the former lemmas we have the following result.

Corollary 2.1 *Let X is a discrete metric space with bounded geometry, and a family of C^*-algebra $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$. Then there exists a control pair (λ, h) and a (λ, h)-isomorphism*:

$$\mathcal{K}_*\Big(C^*_{\max}\Big(X, \prod_{i\in\mathbb{N}} A_i\Big)\Big) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(C^*_{\max}(X, A_i)).$$

The following results give the relationship between quantitative maximal Baum-Connes conjecture and maximal Baum-Connes conjecture.

Theorem 2.3 *Let X be a discrete matric space with bounded geometry and A is a C^*-algebra. The following are equivalent*:

(1) $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ *is one to one.*

(2) *For any $d > 0, \varepsilon \in \big(0, \frac{1}{4}\big)$ and $r > 0$ with $k_X(\varepsilon)d \le r$, there exists d' such that $d \subseteq d'$ and $QI_{X,A,\max,*}(d, d', \varepsilon, r)$ holds.*

Proof Assume that condition (2) holds. Let $x \in KK_*(P_d(X), l^\infty(\mathbb{N}, K(H) \otimes A))$ such that $\mu^d_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = 0$. Since the maximal quantitative assembly map is compatible with the maximal assembly map and by Proposition 2.1. Then there exist $\varepsilon > 0$ and $r > 0$ such that $k_X(\varepsilon)d \le r$ satisfying $\mu^{\varepsilon,d,r}_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = 0$ in $K^{\varepsilon,r}_*(C^*_{\max}(X, l^\infty(\mathbb{N}, K(H) \otimes A)))$. Denote $(x_j)_{j\in\mathbb{N}}$ the element of $\prod_j KK_*(P_d(X), A)$ corresponding to x under the isomorphism of

$$\prod_j KK_*(P_d(X), A) \cong KK_*(P_d(X), l^\infty(\mathbb{N}, K(H) \otimes A)).$$

Let $d' > 0$ with $d \le d'$ such that $QI_{X,A,\max,*}(d, d', \varepsilon, r)$ holds. By the naturality of quantitative assembly map, then we have $\mu^{\varepsilon,d,r}_{X,A,\max,*}(x_i) = 0$ and hence $(q^{d'}_d)^*(x_j) = 0$ in $KK_*(P_{d'}(X), A)$. So $(q^{d'}_d)_*(x) = 0$. Then we have $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ is one to one.

Let us prove the converse. Assume that there exist positive numbers $d > 0, \varepsilon \in \big(0, \frac{1}{4}\big)$ and $r > 0$ such that $QI_{X,A,\max,*}(d, d', \varepsilon, r)$ is not true for all $d \le d'$. Then we can get a increasing exhausting sequence $d_j \ge d$ with $\lim_j d_j = \infty$ and $x_j \in KK_*(P_d(X), A)$ such that $\mu^{\varepsilon,d,r}_{X,A,\max,*}(x_j) = 0$ and $(q^{d_j}_d)^*(x_j) \ne 0$ in $KK_*(P_{d_j}(X), A)$. Let $x \in KK_*(P_d(X), l^\infty(\mathbb{N}, K(H) \otimes A))$ corresponding to each $(x_j) \in KK_*(P_d(X), A)$. By [10, Corollary 2.1, Proposition 1.30] and $\mu^{\varepsilon,d,r}_{X,A,\max,*}(x_j) = 0$, up to rescaling, the image of maximal quantitative assembly map for coefficient $l^\infty(\mathbb{N}, K(H) \otimes A)$ is 0. So we have $\mu^d_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = 0$ and $(q^{d'}_d)^*(x) \ne 0$ in $KK_*(P_{d'}(X), l^\infty(\mathbb{N}, K(H) \otimes A))$ at least with one d' such that $d \le d'$. So $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ is not one to one.

Theorem 2.4 *Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Then there exists $\lambda > 1$ such that the following are equivalent:*

(1) $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ *is onto.*

(2) *For any positive numbers ε with $\varepsilon < \frac{1}{4\lambda}$ and $r > 0$, there exist $d > 0$ and $r' > 0$ with $k_X(\varepsilon)d \leq r$ and $r \leq r'$ for which $QS_{X,A,\max,*}(d,r,r',\varepsilon,\lambda\varepsilon)$ is satisfied.*

Proof Choose λ as in Proposition 2.1. Assume that the condition (2) holds. Let z be an element in $K_*(C^*_{\max}(X,l^\infty(\mathbb{N},K(H)\otimes A)))$ and let y be element in $K^{\varepsilon,r}_*(C^*_{\max}(X,l^\infty(\mathbb{N},K(H)\otimes A)))$ such that $\iota^{\varepsilon,r}_*(y) = z$, with $0 < \varepsilon < \frac{1}{4\lambda}$ and $r > 0$. Up to a rescaling of parameters, by Corollary 2.1, let y_i be the image of y under the controlled isomorphism

$$\mathcal{K}_*(C^*_{\max}(X,l^\infty(\mathbb{N},K(H)\otimes A))) \to \prod_j \mathcal{K}_*(C^*_{\max}(X,A)).$$

Hence $(y_j)_j \in \prod_j K^{\varepsilon,r}_*(C^*_{\max}(X,A))$. Let $d > 0$ and $r > 0$ with $r \leq r'$ and $k_X(\varepsilon)d \leq r$ and such that $QS_{X,A,\max,*}(d,r,r',\varepsilon,\lambda\varepsilon)$ holds. Then for any integer i, there exists a x_i in $KK_*(P_d(X),A)$ such that $\mu^{\lambda\varepsilon,d,r'}_{X,A,\max,*}(x_i) = \iota^{\varepsilon,\lambda\varepsilon,r,r'}_*(y_i)$ in $K^{\lambda\varepsilon,r'}_*(C^*_{\max}(X,A))$. Let x in $KK_*(P_d(X),l^\infty(\mathbb{N},K(H)\otimes A))$ corresponding x_j under the isomorphism

$$KK_*(P_d(X),l^\infty(\mathbb{N},K(H)\otimes A)) \cong \prod_{j\in\mathbb{N}} KK_*(P_d(X),K(H)\otimes A).$$

By naturality of quantitative asembly maps and Corollary 2.1, we get

$$\mu^{\lambda\varepsilon,d,r'}_{G,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = \iota^{\varepsilon,\lambda\varepsilon,r,r'}_*(y)$$

in $K^{\lambda\varepsilon,r'}_*(C^*_{\max}(l^\infty\mathbb{N},K(H)\otimes A))$. Then we have

$$\mu^d_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = \iota^{\lambda\varepsilon,r'}_*(y) = \iota^{\varepsilon,r}_*(y) = z$$

and therefore $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ is onto.

Let us prove the converse. Assume that there exist $\varepsilon \in \left(0,\frac{1}{4\lambda}\right)$ and $r > 0$ such that for any $d > 0$ and $r' > 0$ with $r \leq r'$ and $k_X(\varepsilon)d \leq r$ and $QS_{X,A,\max,*}(d,r,r',\varepsilon,\lambda\varepsilon)$ is not hold. Let us prove that $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}$ is not onto. Then we can find increasing and unbounded sequences $(d_j)_{j\in\mathbb{N}}$ and $(r_j)_{j\in\mathbb{N}}$ such that $k_X(\varepsilon)d_j \leq r_j$ and $r \leq r_j$. Let y_j be an element in $K^{\varepsilon,r}_*(C^*_{\max}(X,A))$ such that $\iota^{\varepsilon,\lambda\varepsilon,r,r_j}_*(y_i)$ is not in the image of $\mu^{\lambda\varepsilon,d_i,r_i}_{X,A,\max,*}$. There exists an element y in $K^{\varepsilon,r}_*(C^*_{\max}(X,l^\infty(\mathbb{N},K(H)\otimes A)))$ corresponding to each y_i in $K^{\varepsilon,r}_*(C^*_{\max}(X,A))$ by Corollary 2.1. Assume that for some $d' > 0$, there exists an x in $KK_*(P_{d'}(X),l^\infty(\mathbb{N},K(H)\otimes A))$ such that $\iota^{\varepsilon,r}_*(y) = \mu^{d'}_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x)$. Using Proposition 2.4, there exists a positive number r' with $r \leq r'$ and $k_X(\lambda\varepsilon)d' \leq r'$ such that

$$\iota^{\varepsilon,\lambda\varepsilon,r,r'}_* \circ \mu^{\varepsilon,d',r}_{X,l^\infty(\mathbb{N},K(H)\otimes A),\max,*}(x) = \iota^{\varepsilon,\lambda\varepsilon,r,r'}_*(y).$$

But if we choose j such that $r' \leq d_j$ and $r' \leq r_j$ we get by using the naturality of assembly map that $\iota^{\varepsilon,\lambda\varepsilon,r,r_j}_*(y_j)$ belongs to the image of $\mu^{\lambda\varepsilon,d_i,r_i}_{X,A,\max,*}$ which contradicts our assumption.

Replacing the algebra $l^\infty(\mathbb{N},K(H)\otimes A)$ by $\prod_{i\in\mathbb{N}} K(H)\otimes A_i$, using the similar method in the proof of the former theorems, we can get the following result.

Corollary 2.2 *Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Then we have the following results.*

(1) $\mu_{X,A,\max,*}$ *is one to one. Then for any* $\varepsilon \in \left(0, \frac{1}{4}\right)$ *and every* $d > 0, r > 0$ *such that* $k_X(\varepsilon)d \leq r$*, there exists* d' *with* $d \leq d'$ *such that* $QI_{X,A,\max,*}(d, d', \varepsilon, r)$ *holds.*

(2) $\mu_{X,A,\max,*}$ *is onto. Then for some* $\lambda \geq 1$ *and any* $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$ *and every* $r > 0$*, there exists* $d > 0$ *and* $r' > 0$ *such that* $k_X(\varepsilon)d \leq r$ *and* $r \leq r'$ *such that* $QS_{X,A,\max,*}(d, r, r', \varepsilon, \lambda\varepsilon)$ *holds.*

3 Persistence Approximation Property

Persistence approximation property was introduced by Oyono-Oyono and Yu in [12]. It has a strong relationship with Baum-Connes conjecture with coefficient and provide geometric obstruction for Baum-Connes conjecture. The persistence approximation property was defined as follows.

Let B be a filtered C^*-algebra and positive numbers $\varepsilon, \varepsilon'$, and r' such that $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $0 < r \leq r'$.

Definition 3.1 (see [12]) *We call that* $K_*(B)$ *has persistence approximation property if: For any* $\varepsilon \in \left(0, \frac{1}{4}\right)$ *and* $r > 0$*, there exists* $\varepsilon' \in \left(\varepsilon, \frac{1}{4}\right)$ *and* $r' \geq r$ *such that for any* $x \in K_*^{\varepsilon,r}(B)$*, then* $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) \neq 0$ *in* $K_*^{\varepsilon',r'}(B)$ *implies that* $\iota_*^{\varepsilon,r}(x) \neq 0$ *in* $K_*(B)$*.*

We give the following quantitative statements. $\mathcal{PA}_*(B, \varepsilon, \varepsilon', r, r')$: *For any* $x \in K_*^{\varepsilon,r}(B)$*, then* $\iota_*^{\varepsilon,r}(x) = 0$ *in* $K_*(B)$ *implies that* $\iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0$ *in* $K_*^{\varepsilon',r'}(B)$*.*

For the crossed product with groups, we have the following theorem.

Theorem 3.1 (see [12]) *Let* Γ *be a finite generated group and* A *be a* C^**-algebra. Assume that*

(1) $\mu_{\Gamma,l^\infty(\mathbb{N},A\otimes K(H))}$ *is onto and* $\mu_{\Gamma,A}$ *is one to one.*

(2) Γ *admits a cocompact universal example for proper actions.*

Then for some universal constant $\lambda_{PA} \geq 1$*, any* $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$*, any* $r > 0$*, and any* Γ-C^* *algebra* A *there exists* $r' \geq r$ *such that* $\mathcal{PA}_*(A \rtimes_{\text{red}} \Gamma, \varepsilon, \lambda_{PA}\varepsilon, r, r')$ *holds.*

Remark 3.1 A finite generated group with Haagerup property will satisfy Baum-Connes conjecture for any coefficients (see [6]). If a group is hyperbolic or the fundamental group of a compact oriented 3-manifolds will admit a cocompact universal example for proper actions (see [8]). Both of $\mathbb{F}_2$ and $SL_2(\mathbb{Z})$ satisfy the hypothesis of the former theorem.

For the groupoid C^*-algebras, we have the following theorem.

Theorem 3.2 (see [3]) *Let* G *be an étale groupoid such that*

(1) $G^{(0)}$ *is compact;*

(2) G *admits a cocompact example for universal space for proper actions.*

Then there exists a universal constant $\lambda_{PA} \geq 1$ *such that for any* G*-algebra* A*, if* $\mu_{G,l^\infty(\mathbb{N},A\otimes K(H))}$ *is onto and* $\mu_{G,A}$ *is one to one, then for any* $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ *and every* $F \in \mathcal{E}$*, there exists a* F *such that* $F \subseteq F'$ *and* $\mathcal{PA}_*(A \rtimes_{\text{red}} G, \varepsilon, \lambda_{PA}\varepsilon, F, F')$ *holds.*

Remark 3.2 Let X is a discrete metric space with bounded geometry. If X embeds coarsely into a Hilbert space, then the groupoid $G(X)$ is *a*-*T*-menable (see [14]). Hence $G(X)$ will satisfy

Baum-Connes conjecture with coefficients. If this groupoid $G(X)$ also admits a cocompact universal example for proper actions, then the reduced crossed product with this groupoid will have persistence approximation property.

For the metric space, we have the following theorem. We also need a hypothesis to replace that the group (groupoid) admits a cocompact universal example for proper actions.

Definition 3.2 (see [12]) *A discrete metric space is coarsely uniformly contractible*: *If for every $d > 0$, there exists $d' > d$ such that any compact subset of $P_d(X)$ lies in a contractible invariant compact subset of $P_{d'}(X)$.*

Example 3.1 (see [8]) Any discrete hyperbolic metric space is coarsely uniformly contractible.

The following theorem describe the persistence approximation property for maximal Roe algebras.

Theorem 3.3 *Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Assume that*

(1) *X is coarsely uniformly contractible.*

(2) *$\mu_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}$ is onto and $\mu_{X,A,\max,*}$ is one to one.*

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ and every $r > 0$, there exists a $r' > 0$ such that $r \leq r'$ and $\mathcal{PA}_*(C^*_{\max}(X,A), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.

Proof Let (α, h) be the control pair in Corollary 2.1 and λ be the constant in Proposition 2.1. Set $\lambda_{PA} = \lambda\alpha$. Assume that the conclusion is not hold. Then there exists $\varepsilon > 0$ and $r > 0$ such that $\mathcal{PA}_*(C^*_{\max}(X,A), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ is not true for any $r' > 0$ with $r \leq r'$. Hence we can find an nondecreasing and unbounded sequence of r_j and elements $x_j \in K_*^{\varepsilon,r}(C^*_{\max}(X,A))$ such that $\iota^{\varepsilon,r}(x_j) = 0$ but $\iota^{\varepsilon,\lambda_{PA}\varepsilon,r,r_j}(x_j) \neq 0$.

Let x be the element of $K_*^{\alpha\varepsilon,h_\varepsilon r}(C^*_{\max}(X, l^\infty(\mathbb{N}, A \otimes K(H))))$ corresponding to $(x_j) \in \prod_j K_*^{\varepsilon,r}(C^*_{\max}(X,A))$ under the controlled isomorphism of Corollary 2.1. If $\iota_*^{\lambda_{PA}\varepsilon,h_\varepsilon r}(x)$ is in the range of $\mu_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}$, then there exists $d > 0$ and $z \in KK_*(P_d(X), l^\infty(\mathbb{N}, A\otimes K(H)))$, such that $\mu^d_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}(z) = \iota_*^{\lambda_{PA}\varepsilon,h_\varepsilon r}(x)$. Let (z_j) be the element of $\prod_j KK_*(P_d(X), A)$ corresponding the isomorphism

$$KK_*(P_d(X), l^\infty(\mathbb{N}, A \otimes K(H))) \cong \prod_j KK_*(P_d(X), A).$$

By Proposition 2.1, there exists r'' with $h_\varepsilon r \leq r''$ such that

$$\mu^{\lambda_{PA},d,r''}_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}(z) = \iota_*^{\alpha\varepsilon,\lambda_{PA},h_\varepsilon r,r''}(x).$$

Since the quantitative maximal assembly map is compatible with the maximal assembly map, we have $\mu^d_{X,A,\max,*}(z_j) = 0$. Since X is coarsely uniformly contractible and $\mu_{X,A,\max,*}$ is injective. There exists d' with $d \leq d'$ and $(q_d^{d'})^*(z) = 0$. Since the following is compatible:

$$\mu^{\lambda_{PA}\varepsilon,d,r''}_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*}(z) = \mu^{\lambda_{PA},\varepsilon,d',r''}_{X,l^\infty(\mathbb{N},A\otimes K(H)),\max,*} \circ (q_d^{d'})^*(z).$$

Then we have $\iota_*^{\alpha\varepsilon,\lambda_{PA},h_\varepsilon\cdot r,r''}(x) = 0$ in $K_*^{\lambda_{PA}\varepsilon,r''}(C^*_{\max}(X, l^\infty(\mathbb{N}, A \otimes K(H))))$. Choosing $i \in \mathbb{N}$ such that $r'' \leq r_i$, then we have $\iota_*^{\varepsilon,\lambda_{PA}\varepsilon,r,r_i}(x_i) = 0$, which contradicts our assumption.

From this theorem, the persistence approximation property provides the geometric obstruction to the maximal coarse Baum-Connes conjecture.

Let X be a discrete metric space with bounded geometry. If X admits a fibred coarse embedding into Hilbert space, then X satisfies the maximal coarse Baum-Connes conjecture (see [2]). So whether or not the persistence approximation property holds for $C^*_{\max}(X)$ when X admits a fibred coarse embedding Hilbert space?

Theorem 3.4 *Let X be a discrete metric space with bounded geometry. Assume that X admits a fibred coarse embedding into Hilbert space and X is coarsely uniformly contractible. Then there exists a universal constant $\lambda_{PA} \geq 1$ such that: For any $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(C^*_{\max}(X), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.*

Proof Since X can be fibred coarse embedding into Hilbert space, then X satisfies the quantitative maximal coarse Baum-Connes conjecture with coefficient $K(H)$. By Theorem 2.4, $\mu_{X,l^\infty(\mathbb{N},K(H)),\max,*}$ is surjective. Since $\mu_{X,K(H),\max,*}$ is injective and X is coarsely uniformly contractible, then by Theorem 3.3, there exists a universal constant $\lambda_{PA} \geq 1$ such that: For any $\varepsilon \in \left(0, \frac{1}{4\lambda_{PA}}\right)$ and any $r > 0$, there exists $r' > r$ such that $\mathcal{PA}(C^*_{\max}(X), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.

A very important example of metric space which admits a fibred coarse embedding into Hilbert space comes from the box space residually finite group. The box space of residually finite group admits a fibred coarse embedding into Hilbert space if and only if the group has Haagerup property (see [1]).

Here we give some examples of maximal Roe algebra satisfying the persistence approximation property. For a finitely generated residually finite group Γ with Haagerup property and $\{\Gamma_i\}_{i\in\mathbb{N}}$ be a family of finite index normal group with trivial intersection. We endow Γ/Γ_i with the metric $d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2); \gamma_1, \gamma_2 \in \Gamma_i\}$. We set $X(\Gamma) = \bigsqcup_{i\in\mathbb{N}} \Gamma/\Gamma_i$ and equip $X(\Gamma)$ with the following metric d :

(1) On Γ/Γ_i, then d is the metric defined above;

(2) $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) \geq i + j$ if $i \neq j$;

(3) The group Γ acts on $X(\Gamma)$ by isometries.

As in [9, Proposition 2.8], let $B_\Gamma = l^\infty(X(\Gamma), K(H))$, $B_{\Gamma,0} = C_0(X(\Gamma), K(H))$ and $A_\Gamma = B_\Gamma/B_{\Gamma,0}$. There is a short exact sequence:

$$0 \to K(l^2(X(\Gamma)) \otimes H) \to C^*_{\max}(X(\Gamma)) \to A_\Gamma \rtimes_{\max} \Gamma \to 0.$$

$C^*_{\max}(X(\Gamma))$ is filtered by $(\mathbb{C}[X(\Gamma)]_r)_{r>0}$. Let $J_r = \mathbb{C}[X(\Gamma)]_r \cap K(l^2(X(\Gamma) \otimes H)$ and $J = K(l^2(X(\Gamma) \otimes H)$. Let $q : C^*_{\max}(X(\Gamma)) \to A_\Gamma \rtimes_{\max} \Gamma$ be quotient map, from the prove of the Proposition 2.8 in [9], the propagation of $q(\mathbb{C}[X(\Gamma)]_r)$ is no more than r.

To use the six-term exact sequence of quantitative K-theory, we need the extension is completely filtered. Firstly, we will show the above extension is completely filtered.

Lemma 3.1 *The extension*

$$0 \to K(l^2(X(\Gamma)) \otimes H) \to C^*_{\max}(X(\Gamma)) \to A_\Gamma \rtimes_{\max} \Gamma \to 0$$

is completely filtered, i.e., for any integer n, $r > 0$ and $x \in M_n(\mathbb{C}[X(\Gamma)]_r)$, we have

$$\inf_{y \in M_n(J_r)} \|x - y\| = \inf_{y \in M_n(J)} \|x - y\|.$$

Proof Since for any positive integer n, we have

$$l^\infty(X(\Gamma), K(H)) \otimes M_n(\mathbb{C}) \subseteq l^\infty(X(\Gamma), K(H)).$$

Hence $C^*_{\max}(X(\Gamma)) \otimes M_n(\mathbb{C}) \subseteq C^*_{\max}(X(\Gamma))$. So we only need to prove for $n = 1$ case. Since for any $x \in \mathbb{C}[X(\Gamma)]_r$, easily we have $\inf\limits_{y \in J_r} \|x - y\| \geq \inf\limits_{y \in J} \|x - y\|$.

Let e_λ be an approximate unit for $K(l^2(X(\Gamma) \otimes H))$, obviously we can choose e_λ to be the 0 propagation operators on $l^2(X(\Gamma) \otimes H)$. We also have

$$\inf_{y \in J} \|x - y\| = \lim_\lambda \|x - xe_\lambda\|$$

and $xe_\lambda \in \mathbb{C}[X(\Gamma)]_r$. Hence $\inf\limits_{y \in J_r} \|x - y\| \leq \inf\limits_{y \in J} \|x - y\|$, then we get our result.

Secondly, we show $K(l^2(X(\Gamma) \otimes H))$ has persistence approximation property, then we use the controlled exact sequence to prove $C^*_{\max}(X(\Gamma))$ also has this property.

Proposition 3.1 *Let Γ be a finitely generated residually finite group with Haagerup property, if Γ admits a cocompact universal example for proper actions. Then there exists a universal constant $\lambda \geq 1$ for any $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(K(l^2(X(\Gamma) \otimes H)), \varepsilon, \lambda\varepsilon, r, r')$ holds.*

Proof We only prove the even case. For the odd case is similar. By Theorem 3.2, there exists λ_{PA} making the persistence approximation property holds. Let $\lambda = \max\{5, \lambda_{PA}\}$ for any $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$, $r > 0$ and $x \in P^{\varepsilon,r}(K(l^2(X(\Gamma) \otimes H)))$, then $x = x' + x''$ with $x' \in K(l^2(\coprod\limits_{i=0}^{n-1} \Gamma/\Gamma_i) \otimes H)$ and $x'' = (x''_i)_{i \geq n} \in \prod\limits_{i \geq n} K(l^2(X(\Gamma_i) \otimes H))$. This decomposition is independent of the choice of x only depend on r. Since $x' \cdot x'' = 0$,then $\iota_0^{\varepsilon,r}(x') = 0$ and $\iota_0^{\varepsilon,r}(x'') = 0$. If $\iota_0^{\varepsilon,r}(x) = 0$, then $\iota_0^{\varepsilon,r}(x') = 0$ and $\iota_0^{\varepsilon,r}(x'') = 0$. Since for any operators in $K\big(l^2(\coprod\limits_{i=0}^{n-1} \Gamma/\Gamma_i) \otimes H\big)$, their propagation is finite. Hence there exists r'_1 with $r'_1 \geq r$ only depend on r, such that x' and $\iota_0^{\varepsilon,r}(x')$ is homotopic in 5ε-r'_1-projections in $K\big(l^2(\coprod\limits_{i=0}^{n-1} \Gamma/\Gamma_i) \otimes H\big)$.

Since there exists a inclusion $B_{\Gamma,0} \rtimes_{\mathrm{red}} \Gamma \to K(l^2(X(\Gamma) \otimes H))$ and

$$x'' = (x''_i)_{i \geq n} \in B_{\Gamma,0} \rtimes_{\mathrm{red}} \Gamma.$$

Since $\iota_0^{\varepsilon,r}(x'') = \iota_0^{\varepsilon,r}(x''_i)_{i \geq n} = 0$ and $\iota_0^{\varepsilon,r}(x''_i)_{i \geq n} \in B_{\Gamma,0} \rtimes_{\mathrm{red}} \Gamma$. If Γ has Haagerup property and admits cocompact universal example for proper actions. Then by Theorem 3.1, for some universal constant $\lambda_2 \geq 1$, any $\varepsilon \in \left(0, \frac{1}{4\lambda_2}\right)$, any $r > 0$, there exists $r'_2 \geq r$ such that $\mathcal{PA}_*(B_{\Gamma,0} \rtimes_{\mathrm{red}} \Gamma, \varepsilon, \lambda_2\varepsilon, r, r'_2)$ holds. Then $\iota_0^{\varepsilon,\lambda_2,r,r_2}(x'') = 0$ in $K_0^{\lambda_2\varepsilon,r_2}(B_{\Gamma,0} \rtimes_{\mathrm{red}} \Gamma)$. Hence $\iota_0^{\varepsilon,\lambda_2,r,r_2}(x'') = 0$ in $K_0^{\lambda_2\varepsilon,r_2}\Big(\prod\limits_{i \geq n} K(l^2(\Gamma/\Gamma_i) \otimes H)\Big)$.

Let $r' = \max\{r_1, r_2\}$, since the choice of r_1, r_2 do not depend on x. Hence there exists a universal constant $\lambda \geq 1$ for any $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(K(l^2(X(\Gamma) \otimes H)), \varepsilon, \lambda\varepsilon, r, r')$ holds.

Theorem 3.5 *Let Γ be a finitely generated residually finite group with haagerup property and admits a cocompact universal example for proper actions. Then there exists a universal constant $\lambda_{PA} \geq 1$ for any $\varepsilon \in \big(0, \frac{1}{4\lambda_{PA}}\big)$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(C^*_{\max}(X(\Gamma)), \varepsilon, \lambda_{PA}\varepsilon, r, r')$ holds.*

Proof Since Γ has Haagerup property, then Γ is K-amenable and satisfies Baum-Connes conjecture with any coefficients (see [6]). By Theorem 5.10 in [10] there is a control pair (λ, h) such that

$$\lambda_{\Gamma,A,*} : \mathcal{K}_*(A_\Gamma \rtimes_{\max} \Gamma) \to \mathcal{K}_*(A_\Gamma \rtimes_{\mathrm{red}} \Gamma)$$

is a (λ, h)-isomorphism. If Γ admits a cocompact universal example for proper actions. Then by Theorem 3.1, for some universal constant $\lambda_{PA_\Gamma} \geq 1$, any $\varepsilon \in \big(0, \frac{1}{4\lambda_{PA_\Gamma}}\big)$,any $r > 0$, and any Γ-C* algebra A there exists $r'_0 \geq r$ such that $\mathcal{PA}_*(A \rtimes_{\mathrm{red}} \Gamma, \varepsilon, \lambda_{PA_\Gamma}\varepsilon, r, r'_0)$ holds. Up to rescaling of parameters, we can assume there exists some universal constant $\lambda_{PA_\Gamma} \geq 1$, any $\varepsilon \in \big(0, \frac{1}{4\lambda_{PA_\Gamma}}\big)$, any $r > 0$, there exists $r'_0 \geq r$ such that $\mathcal{PA}_*(A_\Gamma \rtimes_{\max} \Gamma, \varepsilon, \lambda_{PA_\Gamma}\varepsilon, r, r'_0)$ holds.

Since the extension

$$0 \to K(l^2(X(\Gamma)) \otimes H) \to C^*_{\max}(X(\Gamma)) \to A_\Gamma \rtimes_{\max} \Gamma \to 0$$

is completely filtered. So we have a six-term exact sequence of quantitative K-theory: Denote $\mathcal{D}^0 = \mathcal{D}^0_{K(l^2(X(\Gamma) \otimes H), C^*_{\max}(X(\Gamma))}$, $\mathcal{D}^1 = \mathcal{D}^1_{K(l^2(X(\Gamma) \otimes H), C^*_{\max}(X(\Gamma))}$ by the controlled boundary map. There exists a control pair (λ', h') such that the following six-term sequence is (λ', h')-exact

$$\begin{array}{ccccc}
\mathcal{K}_0(K(l^2(X(\Gamma) \otimes H))) & \xrightarrow{j_*} & \mathcal{K}_0(C^*_{\max}(X(\Gamma))) & \xrightarrow{q_*} & \mathcal{K}_0(A_\Gamma \rtimes_{\max} \Gamma) \\
\uparrow \mathcal{D}^1 & & & & \downarrow \mathcal{D}^0 \\
\mathcal{K}_1(A_\Gamma \rtimes_{\max} \Gamma) & \xleftarrow{q_*} & \mathcal{K}_1(C^*_{\max}(X(\Gamma)) & \xleftarrow{j_*} & \mathcal{K}_1(K(l^2(X(\Gamma) \otimes H)))
\end{array}$$

Since $K(l^2(X(\Gamma) \otimes H))$ has persistence approximation property for some universal constant λ_{PA}, we may assume $\lambda_{PA} \geq \lambda' > 1$. Then for any $\varepsilon \in \big(0, \frac{1}{4\lambda_{PA}}\big)$, $r > 0$ and $x \in K_0^{\varepsilon,r}(C^*_{\max}(X(\Gamma)))$ with $\iota_0^{\varepsilon,r}(x) = 0$ in $K_0(C^*_{\max}(X(\Gamma)))$. Since $\iota_0^{\varepsilon,r}(q(x)) = q(\iota_0^{\varepsilon,r}(x)) = 0$ in $K_0(A_\Gamma \rtimes_{\max} \Gamma)$ and $A_\Gamma \rtimes_{\max} \Gamma$ has persistence approximation property. Then $q(x) = 0$ in $K_0^{\lambda_{A_\Gamma}\varepsilon, r'_0}(A_\Gamma \rtimes_{\max} \Gamma)$. Since the six term sequence is (λ', h') exact. Then there exists an element

$$y \in K_0^{\lambda'\varepsilon, h'_\varepsilon r}(K(l^2(X(\Gamma))))$$

such that

$$j^{\lambda'\varepsilon, h'_\varepsilon r}(y) = \iota_0^{\varepsilon, \lambda'\varepsilon, r, h'_\varepsilon r}(x)$$

in $K_0^{\lambda'\varepsilon, h'_\varepsilon r}(C^*_{\max}(X(\Gamma)))$. By [9, Proposition 2.10], the inclusion

$$K(l^2(X(\Gamma) \otimes H)) \hookrightarrow C^*_{\max}(X(\Gamma))$$

induces an injection $\mathbb{Z} \hookrightarrow K_0(C^*_{\max}(X(\Gamma)))$. Then

$$\iota_0^{\lambda'\varepsilon, h'_\varepsilon r}(j^{\lambda'\varepsilon, h'_\varepsilon r}(y)) = j(\iota_0^{\lambda'\varepsilon, h'_\varepsilon r}(y)) = 0$$

induces $\iota_0^{\lambda'\varepsilon, h'_\varepsilon r}(y) = 0$. Since $K(l^2(X(\Gamma) \otimes H))$ has persistence approximation property, we may choose a bigger λ_{PA} with $\lambda_{PA} \geq \lambda'$ and there exists $r' \geq h'_\varepsilon r$ such that $\iota_0^{\lambda'\varepsilon, \lambda_{PA}\varepsilon, h'_\varepsilon, r'}(y) = 0$ in

$K_0^{\lambda_{PA},r_1'}(K(l^2(X(\Gamma))))$. Hence $j^{\lambda_{PA}\varepsilon,r'}(y)=\iota_0^{\varepsilon,\lambda_{PA}\varepsilon,r,r'}(x)=0$ in $K_0^{\lambda_{PA}\varepsilon,r'}(C^*_{\max}(X(\Gamma)))$ and the choice of r' does not depend on x.

For the odd case is similar. Hence there exists a universal constant $\lambda_{PA}\geq 1$ for any $\varepsilon\in\left(0,\frac{1}{4\lambda_{PA}}\right)$ and any $r>0$ there exists $r'>r$ such that $\mathcal{PA}_*(C^*_{\max}(X(\Gamma)),\varepsilon,\lambda_{PA}\varepsilon,r,r')$ holds.

Remark 3.3 Both $\mathbb{F}_2$ and $SL_2(\mathbb{Z})$ are finite generated group with Haagerup property. Since their classifying space is a tree and this tree is cocompact. So they admit a cocompact universal example for proper actions. Hence the maximal Roe algebras of their box spaces will have persistence approximation property.

4 Quantitative Assembly Map for a Family of Metric Spaces

Initially the quantitative K-theory was used to prove the coarse Baum-Connes conjecture for the space with finite asymptotic dimension. In this section, we will study an application of quantitative K-theory.

Let $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ be a family of discrete metric space with bounded geometry and $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras. Denote $C^*_{\max}(\mathcal{X},\mathcal{A})$ by the closure of $\bigcup\limits_{r>0}\Big(\prod\limits_{i\in\mathbb{N}}\mathbb{C}[X_i,A_i]\Big)_r$ in $\prod\limits_{i\in\mathbb{N}}C^*_{\max}(X_i,A_i)$. Then $C^*_{\max}(\mathcal{X},\mathcal{A})$ is filtered C^*-algebra.

Lemma 4.1 *There exists a control pair (λ,h) and a (λ,h)-controlled isomorphism*

$$\mathcal{K}_*(C^*_{\max}(\mathcal{X},\mathcal{A}))\to\prod_i\mathcal{K}_*(C^*_{\max}(X_i,A_i)).$$

Proof We will prove it in the even case, for the odd case is similar. Obviously, there is a controlled homomorphism induced by the projection to the i-th factor. For any $x\in\prod\limits_{i\in\mathbb{N}}K_0^{\varepsilon,r}(C^*_{\max}(X_i,A_i))$, then $x=(x_i)$ and for each $x_i\in K_0^{\varepsilon,r}(C^*_{\max}(X_i,A_i))$. Then $x_i=[p_i,l_i]$, $p_i\in P_{n_i}^{\varepsilon,r}(C^*_{\max}(X_i,A_i))$ and $n_i,l_i\in\mathbb{N}$. Since $A_i\otimes K(H)\otimes M_n(\mathbb{C})\cong A_i\otimes K(H)$ for any positive integer n and i. Then we can assume $p_i\in P^{\varepsilon,r}(C^*_{\max}(X_i,A_i))$. Hence there exists a control pair (λ,h) such that this map is (λ,h)-surjective. By [10, Proposition 1.30], up to enlarge matrix size, the homotopy of ε-r projections can be chosen to be Lipschitz homotopy. Hence we can choose a bigger control pair (λ,h) such that this map is (λ,h) isomorphism.

As [3, Proposition 4.2.6], there exists a universal control pair (α,h) such that:

(1) For any family $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ of metric space;

(2) For any family of C^*-algebras $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ and $\mathcal{B}=(B_i)_{i\in\mathbb{N}}$;

(3) For any $z=(z_i)_{i\in\mathbb{N}}$ in $\prod\limits_{i\in\mathbb{N}}KK_*(A_i,B_i)$,

There exists an (α,h)-controlled morphism

$$\widehat{\sigma}^{\infty}_{\mathcal{X},\max}(z)=(\widehat{\sigma}^{\infty,\varepsilon,r}_{\mathcal{X},\max})_{0<\varepsilon<\frac{1}{4\alpha},r>0}:\mathcal{K}_*(C^*_{\max}(\mathcal{X},\mathcal{A}))\to\mathcal{K}_*(C^*_{\max}(\mathcal{X},\mathcal{B}))$$

that satisfies the analogous properties listed in [3, Proposition 4.2.6].

For each metric space X_i and positive number d, there is a projection $P_{d,X_i,\max}$ with propagation less than d in $C^*_{\max}(X_i)$. For a family of metric space $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$, denote $P^{\infty}_{d,\mathcal{X},\max}=(P_{d,X_i,\max})_{i\in\mathbb{N}}$ is projection of propagation less than d in $C^*_{\max}(\mathcal{X},\mathcal{A})$, where $\mathcal{A}$ is a family of C^*-algebras $C_0(P_d(X_i))$.

Then we can defined the quantitative assembly map valued in $C^*_{\max}(\mathcal{X})$. For any $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $d, r > 0$ with $k_{\mathcal{X}}(\varepsilon) \cdot d \leq r$. Define

$$\mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*} : \begin{cases} \prod\limits_{i\in\mathbb{N}} KK_*(C_0(P_d(X_i)), \mathbb{C}) \to K^{\varepsilon,r}_*(C^*_{\max}(\mathcal{X})), \\ z \mapsto \iota_*^{\alpha_{\mathcal{X}}\varepsilon',\varepsilon,k_{\mathcal{X}}(\varepsilon')r',r} \circ \widehat{\sigma}^{\infty}_{\mathcal{X}}(z)[P^{\infty}_{d,\mathcal{X},\max}, 0]_{\varepsilon',r'}, \end{cases}$$

where ε' and r' satisfy

(1) $\varepsilon' \in \left(0, \frac{1}{4}\right)$ such that $\alpha_{\mathcal{X}} \cdot \varepsilon' \leq \varepsilon$;

(2) $d \leq r'$ and $k_{\mathcal{X}}(\varepsilon') \cdot r' \leq r$.

Remark 4.1 The quantitative assembly map is compatible with the inclusion of the inclusion of Rips complex, i.e., For any positive number d and d' with $d \leq d'$, we have $\mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*} \circ (q_d^{\infty,d'})^* = \mu^{\infty,\varepsilon,d',r}_{\mathcal{X},\max,*}$.

Remark 4.2 The quantitative assembly map is also compatible with structure morphism $\iota_*^{\varepsilon,\varepsilon',r,r'}$, i.e., $\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \mu^{\infty,\varepsilon',d,r'}_{\mathcal{X},\max,*} = \mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*}$ for any $\varepsilon \leq \varepsilon'$ and $r \leq r'$.

Similarly we give two quantitative statements.

(1) $QI_{\mathcal{X},\max,*}(d, d', r, \varepsilon)$: for any $x \in \prod\limits_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C})$, then $\mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*}(x) = 0$ implies $(q_d^{d'})^*(x) = 0$ in $\prod\limits_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C})$.

(2) $QS_{\mathcal{X},\max,*}(d, r, r', \varepsilon, \varepsilon')$: for any $y \in K^{\varepsilon,r}_*(C^*_{\max}(\mathcal{X}))$, there exists $x \in \prod\limits_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C})$ such that $\mu^{\infty,\varepsilon',d,r'}_{\mathcal{X},\max,*}(x) = \iota_*^{\varepsilon,\varepsilon',r,r'}(y)$.

Let $\Sigma = \bigsqcup\limits_{i\in\mathbb{N}} X_i$, where $(X_i)_{i\in\mathbb{N}}$ is a family of metric space satisfying: For any $r > 0$, there exists an integer N_r such that for any integer i, any ball of radius r in X_i is no more than N_r element.

The metric d on Σ is defined to be:

(1) On each X_i, the metric is just the usual metric on X_i;

(2) $d(X_i, X_j) \geq i + j$ if $i \neq j$.

Obviously, there is a inclusion of filtered C^*-algebras $j_{\Sigma,\max} : C^*_{\max}(\mathcal{X}) \hookrightarrow C^*_{\max}(\Sigma)$. Let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras and denote $\mathcal{A}^{\oplus} = \bigoplus\limits_{i\in\mathbb{N}} A_i$, for each $i \in \mathbb{N}$, there is an inclusion $A_i \otimes K(l^2(X_i) \otimes H) \hookrightarrow \mathcal{A}^{\oplus} \otimes K(l^2(\Sigma) \otimes H)$ induce the inclusion $j_{\mathcal{A},\Sigma,\max} : C^*_{\max}(\mathcal{X}, \mathcal{A}) \hookrightarrow C^*_{\max}(\Sigma, \mathcal{A}^{\oplus})$. Recall there is an isomorphism

$$\prod_{i\in\mathbb{N}} KK_*(A_i, \mathbb{C}) \cong KK_*(\mathcal{A}^{\oplus}, \mathbb{C}).$$

Lemma 4.2 *For a family of C^*-algebras $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$, $\Sigma = \bigsqcup\limits_{i\in\mathbb{N}} X_i$ and $z \in \prod\limits_{i\in\mathbb{N}} KK_*(A_i, \mathbb{C})$, then we have a commutative diagram*:

$$\begin{array}{ccc} K_*(C^*_{\max}(\mathcal{X}, \mathcal{A})) & \xrightarrow{\sigma^{\infty}_{\mathcal{X},\max}(z)} & K_*(C^*_{\max}(\mathcal{X})) \\ \Big\downarrow{\scriptstyle (j_{\mathcal{A},\Sigma},\max)_*} & & \Big\downarrow{\scriptstyle (j_{\Sigma},\max)_*} \\ K_*(C^*_{\max}(\Sigma, \mathcal{A}^{\oplus})) & \xrightarrow{\sigma_{\Sigma,\max}(z)} & K_*(C^*_{\max}(\Sigma)) \end{array}$$

Proof Firstly we prove the odd case, assume $z \in \prod_{i\in\mathbb{N}} KK_1(A_i,\mathbb{C}) \cong KK_1(\mathcal{A}^{\oplus},\mathbb{C})$ (We write $z=(z_i)_{i\in\mathbb{N}}$ for each $z_i \in KK_1(A_i,\mathbb{C})$). Let us fix a separable Hilbert space H. For each $i\in\mathbb{N}$, let (H,π_i,T_i) be the K-cycle for $KK_1(A_i,\mathbb{C})$ representing z_i,where $\pi_i : A_i \to \mathcal{L}(H)$ a representation and T_i in $\mathcal{L}(H)$ satisfying the K-cycle conditions. Let $P_i = \frac{T_i + Id_H}{2}$ and $E_i = \{(x,T) \in A_i \oplus \mathcal{L}(H)$ such that $P_i\pi_i(x)P_i - T \in K(H)\}$. Then we have a semi-split extension

$$0 \to K(H) \to E_i \to A_i \to 0.$$

The cross-section $s:\ A_i \to E_i;\ x \mapsto (x, P_i x P_i)$. Then for the $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ family of metric spaces, we have a family of semi split filtered C^*-algebras extension

$$0 \to C^*_{\max}(X_i,K(H)) \to C^*_{\max}(X_i,E_i) \to C^*_{\max}(X_i,A_i) \to 0.$$

Let $\mathcal{E}=(E_i)_{i\in\mathbb{N}}$ be the family of C^*-algebras, since $C^*_{\max}(X_i,K(H)) = C^*_{\max}(X_i)$, we can get a semi-split filtered C^*-algebras

$$0 \to C^*_{\max}(\mathcal{X}) \to C^*_{\max}(\mathcal{X},\mathcal{E}) \to C^*_{\max}(\mathcal{X},\mathcal{A}) \to 0.$$

The boundary map associated this extension is $\sigma^{\infty}_{\mathcal{X},\max}(z) : K_*(C^*_{\max}(\mathcal{X},\mathcal{A})) \to K_{*+1}(C^*_{\max}(\mathcal{X}))$.

Using the similar way, let $E=\{((x_i)_{i\in\mathbb{N}},T) \in \mathcal{A}^{\infty} \oplus \mathcal{L}(l^2(\mathbb{N})\otimes H)$ such that $\big(\bigoplus_{i\in\mathbb{N}} P_i\pi_i(x_i)P_i\big) - T \in K(l^2(\mathbb{N})\otimes H)\}$. Then we have a semi-split extension

$$0 \to K(l^2(\mathbb{N})\otimes H) \to E \to \mathcal{A}^{\oplus} \to 0.$$

Hence we get a semi-split filtered C^*-algebras extension

$$0 \to C^*_{\max}(\Sigma, K(l^2(\mathbb{N})\otimes H)) \to C^*_{\max}(\Sigma,E) \to C^*_{\max}(\Sigma,\mathcal{A}^{\infty}) \to 0.$$

The boundary map associated with this extension is

$$\sigma_{\Sigma,\max}(z) : K_*(C^*_{\max}(\Sigma,\mathcal{A}^{\infty})) \to K_{*+1}(C^*_{\max}(\Sigma)).$$

For each integer i, there is obvious a representation of $K(l^2(X_i)\otimes H)\otimes E_i$ on the right E-Hilbert module $H\otimes l^2(\Sigma)\otimes E$ as a corner which gives a C^*-homomorphism $j_{\mathcal{E},\Sigma,\max} : C^*_{\max}(\mathcal{X},\mathcal{E}) \to C^*_{\max}(\Sigma,E)$ such that $j_{\mathcal{E},\Sigma}(C^*_{\max}(\mathcal{X})) \subseteq C^*_{\max}(\Sigma, K(l^2(\mathbb{N})\otimes H))$. Then we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*_{\max}(\mathcal{X}) & \longrightarrow & C^*_{\max}(\mathcal{X},\mathcal{E}) & \longrightarrow & C^*_{\max}(\mathcal{X},\mathcal{A}) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle j_{\mathcal{E},\Sigma,\max}} & & \Big\downarrow {\scriptstyle j_{\mathcal{E},\Sigma,\max}} & & \Big\downarrow {\scriptstyle j_{\mathcal{A},\Sigma,\max}} & & \\ 0 & \longrightarrow & C^*_{\max}(\Sigma, K(l^2(\mathbb{N})\otimes H)) & \longrightarrow & C^*_{\max}(\Sigma,E) & \longrightarrow & C^*_{\max}(\Sigma,\mathcal{A}^{\oplus}) & \longrightarrow & 0 \end{array}$$

By the naturality of the index map and exponential map, we have the commutative diagram:

$$\begin{array}{ccc} K_*(C^*_{\max}(\mathcal{X},\mathcal{A})) & \xrightarrow{\sigma^{\infty}_{\mathcal{X},\max}(z)} & K_{*+1}(C^*_{\max}(\mathcal{X})) \\ \Big\downarrow {\scriptstyle (j_{\mathcal{A},\Sigma,\max})_*} & & \Big\downarrow {\scriptstyle (j_{\Sigma},\max)_*} \\ K_*(C^*_{\max}(\Sigma,\mathcal{A}^{\oplus})) & \xrightarrow{\sigma_{\Sigma,\max}(z)} & K_{*+1}(C^*_{\max}(\Sigma)) \end{array}$$

Next we will prove it in the even case. Let $z \in \prod_{i\in\mathbb{N}} KK_0(A_1, \mathbb{C})$. For a family of extensions

$$0 \to SA_i \to CA_i \to A_i \to 0.$$

Since $\partial_{A_i} \in KK_1(A_i, SA_i)$ then let $z' = ([\partial_{A_i}]^{-1} \otimes_{A_i} z_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_1(SA_i, \mathbb{C})$. We denote $\sigma^{\infty}_{\mathcal{X},\max}(z) = \sigma^{\infty}_{\mathcal{X},\max}(z') \circ \sigma^{\infty}_{\mathcal{X},\max}((\partial_{A_i})_{i\in\mathbb{N}})$:

$$K_*(C^*_{\max}(\mathcal{X}, \mathcal{A})) \to K_{*+1}(C^*_{\max}(\mathcal{X}, S\mathcal{A})) \to K_*(C^*_{\max}(\mathcal{X})).$$

Since $(S\mathcal{A})^{\oplus} \cong S\mathcal{A}^{\oplus}$ and $(C\mathcal{A})^{\oplus} \cong C\mathcal{A}^{\oplus}$ we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*_{\max}(\mathcal{X}, S\mathcal{A}) & \longrightarrow & C^*_{\max}(\mathcal{X}, C\mathcal{A}) & \longrightarrow & C^*_{\max}(\mathcal{X}, \mathcal{A}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_{S\mathcal{A},\Sigma,\max}} & & \downarrow{\scriptstyle j_{C\mathcal{A},\Sigma,\max}} & & \downarrow{\scriptstyle j_{\mathcal{A},\Sigma,\max}} & & \\
0 & \longrightarrow & C^*_{\max}(\Sigma, S\mathcal{A}^{\oplus}) & \longrightarrow & C^*_{\max}(\Sigma, C\mathcal{A}^{\oplus}) & \longrightarrow & C^*_{\max}(\Sigma, \mathcal{A}^{\oplus}) & \longrightarrow & 0
\end{array}$$

Let $\partial_{\mathcal{A}^{\oplus}} \in KK_1(\mathcal{A}^{\oplus}, S\mathcal{A}^{\oplus})$,$z'' = ([\partial_{\mathcal{A}^{\oplus}}]^{-1} \otimes_{\mathcal{A}^{\oplus}} z \in KK_1(S\mathcal{A}^{\oplus}, \mathbb{C})$, then $\sigma_{\Sigma,\max}(z) = \sigma_{\Sigma,\max}(z'') \circ \sigma_{\Sigma,\max}(\partial_{\mathcal{A}^{\oplus}}) : K_*(C^*_{\max}(\Sigma, \mathcal{A}^{\oplus})) \to K_*(C^*_{\max}(\Sigma))$. By the naturality of the boundary map we have the commutative diagram:

$$\begin{array}{ccccc}
K_*(C^*_{\max}(\mathcal{X}, \mathcal{A})) & \longrightarrow & K_{*+1}(C^*_{\max}(\mathcal{X}, S\mathcal{A})) & \longrightarrow & K_*(C^*_{\max}(\mathcal{X})) \\
\downarrow{\scriptstyle (j_{\mathcal{E},\Sigma,\max})_*} & & \downarrow{\scriptstyle (j_{S,\mathcal{E},\Sigma,\max})_*} & & \downarrow{\scriptstyle (j_{\Sigma,\max})_*} \\
K_*(C^*_{\max}(\Sigma, \mathcal{A}^{\oplus})) & \longrightarrow & K_{*+1}(C^*_{\max}(\Sigma, S\mathcal{A}^{\oplus})) & \longrightarrow & K_*(C^*_{\max}(\Sigma))
\end{array}$$

Proposition 4.1 *Let $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$ as above, let s be a positive number, such that $d(X_i, X_j) > s$ if $i \neq j$. Then we have commutative diagram*:

$$\begin{array}{ccc}
\prod_{i\in\mathbb{N}} KK_*(P_s(X_i), \mathbb{C}) & \xrightarrow{\mu^{\infty,s}_{\mathcal{X},\max,*}} & K_*(C^*_{\max}(\mathcal{X})) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle (j_{\Sigma,\max})_*} \\
KK_*(P_s(\Sigma), \mathbb{C}) & \xrightarrow{\mu^{s}_{\Sigma,\max,*}} & K_*(C^*_{\max}(\Sigma))
\end{array}$$

Proof Since $C_0(P_s(\Sigma)) = \bigoplus_{i\in\mathbb{N}} C_0(P_s(X_i))$ then we have the isomorphism

$$\prod_{i\in\mathbb{N}} KK_*(P_s(X_i), \mathbb{C}) \cong KK_*(P_s(\Sigma), \mathbb{C}).$$

Let $z = (z_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_*(P_s(X_i), \mathbb{C})$. Let us consider the family $\mathcal{A} = (C_0(P_s(X_i)))_{i\in\mathbb{N}}$ of C^*-algebras. Since $d(X_i, X_j) > s$ if $i \neq j$. Hence $j_{\mathcal{A},\Sigma,\max}(P^{\infty}_{s,\mathcal{X},\max}) = P_{s,\Sigma,\max}$. Following the previous lemma, we have

$$j_{\Sigma,\max} \circ \sigma^{\infty}_{\mathcal{X},\max}(z)([P^{\infty}_{s,\mathcal{X},\max}, 0]) = \sigma_{\Sigma,\max}(z) \circ j_{\mathcal{A},\Sigma,\max}([P_{s,\Sigma,\max}, 0]).$$

Then we get our commutative diagram.

Theorem 4.1 *Let $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ be a family of discrete metric space with bounded geometry.Let $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$ defined as before. Assume that*

(1) for any $\varepsilon \in \left(0, \frac{1}{4}\right)$ and positive numbers d,r such that $\alpha_{\mathcal{X}}(\varepsilon) \cdot d \leq r$, there exists d' with $d \leq d'$, such that $QI_{\mathcal{X},\max,*}(d, d', \varepsilon, r)$ is holds;

(2) for some $\lambda > 1$ and any $\varepsilon \in \left(0, \frac{1}{4\lambda}\right)$, $r > 0$, there exists $d > 0$ and $r' > r$ with $\alpha_{\mathcal{X}}(\varepsilon)\cdot d \leq r'$ such that $QS_{\mathcal{X},\max,*}(d, r, r', \varepsilon, \lambda\varepsilon)$.

Then Σ satisfies the maximal coarse Baum-Connes conjecture.

Proof Firstly, let us to prove $\mu_{\Sigma,*}$ is one to one. Let d be a positive number and $x \in KK_*(P_d(\Sigma), \mathbb{C})$ such that $\mu^d_{\Sigma,\max,*}(x) = 0$ in $K_*(C^*_{\max}(\Sigma))$. We may assume without loss of generality that $d(X_i, X_j) > d$ if $i \neq j$.Then $P_d(\Sigma) = \bigsqcup_{i\in\mathbb{N}} P_d(X_i)$. Hence $KK_*(P_d(\Sigma), \mathbb{C}) \cong \prod_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C})$. We write $x = (x_i)_{i\in\mathbb{N}}$ corresponding to this identification for each $x_i \in KK_*(P_d(X_i), \mathbb{C})$. By Proposition 4.1, we have

$$(j_{\Sigma,\max})_* \circ \mu^{\infty,d}_{\mathcal{X},\max,*}(x) = 0.$$

Fix a $\varepsilon > 0$ small enough and choose $\lambda > 1$ in Proposition 2.1. Let us fix a positive number r with $\alpha_{\mathcal{X}}(\varepsilon) \cdot d \leq r$, then we have

$$(j_{\Sigma,\max})_* \circ \mu^{\infty,d}_{\mathcal{X},\max,*} = (j_{\Sigma,\max})_* \circ \iota^{\varepsilon,r}_* \circ \mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*} = \iota^{\varepsilon,r}_* \circ (j^{\varepsilon,r}_{\Sigma,\max})_* \circ \mu^{\infty,\varepsilon,r}_{\mathcal{X},\max,*}.$$

Then by Proposition 2.1, there exists r' with $r \leq r'$ such that $(j^{\lambda\varepsilon,r'}_{\Sigma,\max})_* \circ \mu^{\infty,\lambda\varepsilon,d,r'}_{\mathcal{X},\max,*}(x) = 0$ in $K^{\lambda\varepsilon,r'}_*(C^*_{\max}(\Sigma))$. Therefore up to replacing $\lambda\varepsilon$ by ε and r by r', we may assume there exists $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $r > 0$ such that $(j^{\varepsilon,r}_{\Sigma,\max})_* \circ \mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*}(x) = 0$ in $K^{\varepsilon,r}_*(C^*_{\max}(\Sigma))$. We may assume without loss of generality that $d(X_i, X_j) \geq r$ if $i \neq j$. Then we have $\mu^{\infty,\varepsilon,d,r}_{\mathcal{X},\max,*}(x) = 0$ in $K^{\varepsilon,r}_*(C^*_{\max}(\mathcal{X}))$.By assumption then there exists d' with $d \leq d'$ such that $(q^{d'}_d)^*(x) = 0$ in

$$\prod_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C}) \cong KK_*(P_d(\Sigma), \mathbb{C}).$$

Hence $\mu_{\Sigma,\max,*}$ is one to one.

Next to prove $\mu_{\Sigma,\max,*}$ is onto. Let $y \in K_*(C^*_{\max}(\Sigma))$ and positive number ε small enough. Then there exists $r > 0$ and $y' \in K^{\varepsilon,r}_*(C^*_{\max}(\Sigma))$ such that $\iota^{\varepsilon,r}_*(y') = y$. Let $\varepsilon' \in (\varepsilon, \frac{1}{4})$ and positive numbers d, r' with $r \leq r'$ and $\alpha_{\mathcal{X}}(\varepsilon) \cdot d \leq r'$ such that $QS_{\mathcal{X},\max,*}(d, r, r', \varepsilon, \lambda\varepsilon)$ holds. We may assume without loss of generality that $d(X_i, X_j) > \max\{d, r\}$ if $i \neq j$. Then there exists $z \in K^{\varepsilon,r}_*(C^*_{\max}(\mathcal{X}))$ such that $(j^{\varepsilon,r}_{\Sigma,\max})_*(z) = y'$. Hence there exists $x \in \prod_{i\in\mathbb{N}} KK_*(P_d(X_i), \mathbb{C})$ such that $\mu^{\infty,\lambda\varepsilon,d,r'}_{\mathcal{X},\max,*}(x) = \iota^{\varepsilon,\lambda\varepsilon,r,r'}_*(z)$. By Proposition 4.1, we have $\mu^d_{\Sigma,\max,*}(x) = (j_{\Sigma,\max})_* \circ \mu^{\infty,d}_{\mathcal{X},\max,*}(x) = (j_{\Sigma,\max})_* \circ \iota^{\lambda\varepsilon,r'}_* \circ \mu^{\infty,\lambda\varepsilon,d,r'}_{\mathcal{X},\max,*}(x) = \iota^{\lambda\varepsilon,r'}_*(y') = y$. Hence $\mu_{\Sigma,\max,*}$ is onto.

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