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The Coefficient Inequalities for a Class of Holomorphic Mappings in Several Complex Variables *

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Abstract The authors establish the coefficient inequalities for a class of holomorphic mappings on the unit ball in a complex Banach space or on the unit polydisk in $\mathbb{C}^n$, which are natural extensions to higher dimensions of some Fekete and Szegö inequalities for subclasses of the normalized univalent functions in the unit disk.

Keywords Coefficient inequality, Fekete-Szegö problem, Quasi-convex mappings

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1 Introduction

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

The following notions were introduced by Robertson [12].

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ if it satisfies the following inequality:

$$\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad z \in \mathbb{U};\ 0 \le \alpha < 1.$$

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_\alpha$ of convex functions of order α in $\mathbb{U}$ if it satisfies the following inequality:

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha, \quad z \in \mathbb{U};\ 0 \le \alpha < 1.$$

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It is clear that there is an Alexander type result relating $\mathcal{S}^*(\alpha)$ and $\mathcal{K}_\alpha$:

$$f \in \mathcal{K}_\alpha \Longleftrightarrow g \in \mathcal{S}^*(\alpha), \tag{1.2}$$

where $g(z) = zf'(z),\ z \in \mathbb{U}$.

In [1], Fekete and Szegö obtained the following classical result:

Let $f(z)$ be defined by (1.1). If $f \in \mathcal{S}$, then

$$\max_{f \in \mathcal{S}} |a_3 - \lambda a_2^2| = 1 + 2\mathrm{e}^{-\frac{2\lambda}{1-\lambda}}$$

for $\lambda \in [0, 1]$.

The above inequality is known as the Fekete and Szegö inequality. After that, there are many papers to deal with the corresponding problems for various subclasses of the class $\mathcal{S}$, and many interesting results have been obtained.

In contrast, although Fekete and Szegö inequalities for various subclasses of the class $\mathcal{S}$ were established, only a few results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in several complex variables (see for details [2–3, 5–7, 9, 11, 14–18]).

Now, we first recall the Fekete and Szegö inequality for the class $\mathcal{S}^*_\alpha$ which was proved by Keogh and Merkes [8].

Suppose that $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in \mathcal{S}^*_\alpha$. Then

$$|b_3 - \lambda b_2^2| \le (1 - \alpha) \max\{1, |3 - 2\alpha - 4\lambda(1 - \alpha)|\}, \quad \lambda \in \mathbb{C}.$$

The above estimation is sharp.

By combining the above relation with (1.2), we may easily prove the following result.

Theorem A *Let $f(z)$ be defined by* (1.1). *If $f \in \mathcal{K}_\alpha$, then*

$$|a_3 - \lambda a_2^2| \le \frac{1 - \alpha}{3} \max\{1,\ |3 - 2\alpha - 3\lambda(1 - \alpha)|\}, \quad \lambda \in \mathbb{C}.$$

The above estimation is sharp.

In this paper, we will establish inequalities between the second and third coefficients of homogeneous expansions for a class of holomorphic mappings defined on the unit ball in Banach complex spaces and the unit polydisc in $\mathbb{C}^n$, which generalize Theorem A and other known results.

Let X be a complex Banach space with norm $\|\cdot\|$, X^* be the dual space of X, and E be the unit ball in X. Also, let $\partial\mathbb{U}^n$ denote the boundary of $\mathbb{U}^n$, and $\partial_0\mathbb{U}^n$ be the distinguished boundary of $\mathbb{U}^n$.

For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1,\ T_x(x) = \|x\|\}.$$

According to the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(E)$ denote the set of all holomorphic mappings from E into X. It is well known that if $f \in H(E)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y - x)^n)$$

for all y in some neighborhood of $x \in E$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \ge 1$,

$$D^n f(x)((y - x)^n) = D^n f(x)(\underbrace{y - x, \cdots, y - x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

A holomorphic mapping $f : E \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(E)$. A mapping $f \in H(E)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in E$. If $f : E \to X$ is a holomorphic mapping, then f is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X.

Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded circular domain. The first Fréchet derivative and the $m(m \geqslant 2)$-th Fréchet derivative of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ are written by $Df(z)$ and $D^m f(z)(a^{m-1}, \cdot)$, respectively. The matrix representations are

$$Df(z) = \Big(\frac{\partial f_p(z)}{\partial z_k}\Big)_{1 \leqslant p,k \leqslant n},$$

$$D^m f(z)(a^{m-1}, \cdot) = \Big(\sum_{l_1,l_2,\cdots,l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} \cdots \partial z_{l_{m-1}}} a_{l_1} \cdots a_{l_{m-1}}\Big)_{1 \leqslant p,k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))'$, $a = (a_1, a_2, \cdots, a_n)' \in \mathbb{C}^n$.

The following definition is due to Liu and Liu [10].

Definition 1.1 (see [10]) *Suppose that $\alpha \in [0, 1)$ and $f : E \to X$ is a normalized locally biholomorphic mapping. If*

$$\mathrm{Re}\{T_x[(Df(x))^{-1}(D^2 f(x)(x^2) + Df(x)x)]\} \geq \alpha\|x\|, \quad x \in E \setminus \{0\},\ T_x \in T(x), \tag{1.3}$$

then f is called a quasi-convex mapping of type B and order α on E. If $X = \mathbb{C}^n, E = \mathbb{U}^n$, then it is obvious that the above condition is equivalent to

$$\mathrm{Re}\frac{g_j(z)}{z_j} > \alpha, \quad \forall z \in \mathbb{U}^n \setminus \{0\},$$

where $g(z) = (g_1(z), \cdots, g_n(z))' = (Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)z)$ is a column vector in $\mathbb{C}^n$, and j satisfies $|z_j| = \|z\| = \max_{1 \leq k \leq n}\{|z_k|\}$.

Especially, when $X = \mathbb{C}$, $E = \mathbb{U}$, the condition (1.3) *reduces to*

$$\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha, \quad z \in \mathbb{U},$$

which is the usual condition for the class $\mathcal{K}_\alpha$ in the unit disc $\mathbb{U}$.

When $\alpha = 0$, Definition 1.1 is the definition of the quasi-convex mapping of type B, which was introduced by Roper and Suffridge [13].

Let $\mathcal{K}_\alpha(E)$ denote the class of quasi-convex mappings of type B and order α on E.

Definition 1.2 *Let $h : \mathbb{U} \to \mathbb{C}$ be a biholomorphic function such that $h(0) = 1$, $\mathrm{Re}\, eh(\xi) > 0$ on $\mathbb{U}$. We define $\mathcal{M}_h$ to be the class of mappings given by*

$$\mathcal{M}_h = \Big\{p \in H(E) : p(0) = 0,\ Dp(0) = \mathrm{I}, \quad \frac{T_x(p(x))}{\|x\|} \in h(\mathbb{U}),\ x \in E\backslash\{0\},\ T_x \in T(x)\Big\}.$$

When $X = \mathbb{C}^n, E = \mathbb{U}^n$, the above relation is equivalent to

$$\mathcal{M}_h = \Big\{p \in H(\mathbb{U}^n) : p(0) = 0,\ Dp(0) = \mathrm{I},\ \frac{p_j(z)}{z_j} \in h(\mathbb{U}),\ z \in \mathbb{U}^n\backslash\{0\}\Big\},$$

where $p(z) = (p_1(z), \cdots, p_n(z))'$ is a column vector in $\mathbb{C}^n$, j satisfies $|z_j| = \|z\| = \max_{1 \leq k \leq n}\{|z_k|\}$.

Remark 1.1 Let $F \in H(E)$ be a normalized locally biholomorphic mapping. If

$$(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)(x)) \in \mathcal{M}_h,$$

then there are many choices of the function h which would provide interesting subclasses of holomorphic mappings. For example, if we let $h(\xi) = \frac{1+(1-2\alpha)\xi}{1-\xi}$ in Definition 1.2, then we easily obtain $F \in \mathcal{K}_\alpha(E)$.

2 Some Lemmas

In order to prove the desired results, we give some lemmas.

Lemma 2.1 (see [4]) *Let* $s(\xi) = 1 + \sum\limits_{k=1}^{\infty} b_k\xi^k \in H(\mathbb{U})$, *and* $\operatorname{Re} s(\xi) > 0$, $\xi \in \mathbb{U}$. *Then*

$$\Big|b_2 - \frac{1}{2}b_1^2\Big| \le 2 - \frac{1}{2}|b_1|^2.$$

Lemma 2.2 *Suppose that* $s \in H(\mathbb{U})$, *h is a biholomorphic function on* $\mathbb{U}$, *and* $s(0) = h(0)$, $s(\xi) \in h(\mathbb{U})$, $\forall \xi \in \mathbb{U}$. *Then*

$$\Big|\frac{s''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(s'(0))^2\Big| \le |h'(0)| - \frac{|s'(0)|^2}{|h'(0)|}. \tag{2.1}$$

Proof From the condition of Lemma 2.2, we have $s \prec h$. So, there exists $\varphi \in H(\mathbb{U}, \mathbb{U})$, $\varphi(0) = 0$ such that

$$s(\xi) = h(\varphi(\xi)), \quad \xi \in \mathbb{U}.$$

A simple computation shows that

$$s'(\xi) = h'(\varphi(\xi))\varphi'(\xi), \quad s''(\xi) = h''(\varphi(\xi))(\varphi'(\xi))^2 + h'(\varphi(\xi))\varphi''(\xi).$$

Therefore, we have

$$\varphi'(0) = \frac{s'(0)}{h'(0)}, \quad \varphi''(0) = \frac{s''(0)(h'(0))^2 - h''(0)(s'(0))^2}{(h'(0))^3}. \tag{2.2}$$

Define

$$k(\xi) = \frac{1 + \varphi(\xi)}{1 - \varphi(\xi)}, \quad \xi \in \mathbb{U}.$$

We thus find that

$$k(\xi) = 1 + 2\varphi(\xi) + 2\varphi^2(\xi) + \cdots \quad \text{and} \quad \operatorname{Re} k(\xi) > 0, \quad \xi \in \mathbb{U}.$$

Consequently, we have

$$k'(0) = 2\varphi'(0), \quad \frac{k''(0)}{2} = \varphi''(0) + 2(\varphi'(0))^2. \tag{2.3}$$

By Lemma 2.1 and (2.2)–(2.3), we obtain (2.1), as desired. This completes the proof.

3 Main Results

In this section, we state and prove the main results of our present investigation.

Theorem 3.1 *Let $h : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of Definition* 1.2*, $f \in H(E, \mathbb{C})$, $f(x) \neq 0$, $x \in E$, $f(0) = 1$, $F(x) = xf(x)$ and suppose that $(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)(x)) \in \mathcal{M}_h$. Then*

$$\begin{aligned}&\Big|\frac{T_x(D^3F(0)(x^3))}{3!\|x\|^3} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!\|x\|^2}\Big)^2\Big| \\ &\leq \frac{|h'(0)|}{6}\max\Big\{1, \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\Big\}, \quad \lambda \in \mathbb{C},\ x \in E \setminus \{0\},\ T_x \in T(x). \end{aligned} \tag{3.1}$$

The above estimation is sharp.

Proof Fix $x \in X\backslash\{0\}$, and denote $x_0 = \frac{x}{\|x\|}$. Let $g : \mathbb{U} \to \mathbb{C}$ be given by

$$g(\xi) = \begin{cases} \dfrac{T_x((DF(\xi x_0))^{-1}(D^2F(\xi x_0)((\xi x_0)^2) + DF(\xi x_0)\xi x_0))}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Then $g \in H(\mathbb{U})$, $g(0) = h(0) = 1$, and since $(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)x) \in \mathcal{M}_h$, we deduce that

$$\begin{aligned} g(\xi) &= \frac{T_x((DF(\xi x_0))^{-1}(D^2F(\xi x_0)((\xi x_0)^2) + DF(\xi x_0)\xi x_0))}{\xi} \\ &= \frac{T_{x_0}((DF(\xi x_0))^{-1}(D^2F(\xi x_0)((\xi x_0)^2) + DF(\xi x_0)\xi x_0))}{\xi} \\ &= \frac{T_{\xi x_0}((DF(\xi x_0))^{-1}(D^2F(\xi x_0)((\xi x_0)^2) + DF(\xi x_0)\xi x_0))}{\|\xi x_0\|} \in h(\mathbb{U}), \quad \xi \in \mathbb{U}. \end{aligned}$$

By Lemma 2.2, we obtain

$$\Big|\frac{g''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(g'(0))^2\Big| \leq |h'(0)| - \frac{|g'(0)|^2}{|h'(0)|}. \tag{3.2}$$

Using a similar method as in [4, Theorem 7.1.14], we have

$$(DF(x))^{-1} = \frac{1}{f(x)}\Big(I - \frac{\frac{xDf(x)}{f(x)}}{1 + \frac{Df(x)x}{f(x)}}\Big).$$

We easily compute that

$$D^2F(x)(x^2) + DF(x)(x) = (D^2f(x)(x^2) + 3Df(x)(x) + f(x))x.$$

From this it follows that

$$(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)(x)) = \frac{D^2f(x)(x^2) + 3Df(x)(x) + f(x)}{f(x) + Df(x)(x)}x. \tag{3.3}$$

Therefore

$$\frac{T_x((DF(x))^{-1}(D^2F(x)(x^2) + DF(x)(x)))}{\|x\|} = \frac{D^2f(x)(x^2) + 3Df(x)(x) + f(x)}{f(x) + Df(x)(x)}. \tag{3.4}$$

In view of (3.4), we obtain

$$\begin{aligned} g(\xi) &= \frac{T_{\xi x_0}((DF(\xi x_0))^{-1}(D^2F(\xi x_0)((\xi x_0)^2) + DF(\xi x_0)\xi x_0))}{\|\xi x_0\|} \\ &= \frac{D^2 f(\xi x_0)((\xi x_0)^2) + 3Df(\xi x_0)(\xi x_0) + f(\xi x_0)}{f(\xi x_0) + Df(\xi x_0)(\xi x_0)}, \end{aligned}$$

or, equivalently,

$$g(\xi)(f(\xi x_0) + Df(\xi x_0)(\xi x_0)) = D^2 f(\xi x_0)((\xi x_0)^2) + 3Df(\xi x_0)(\xi x_0) + f(\xi x_0).$$

Using Taylor series expansions in ξ, we obtain

$$\begin{aligned} &\Big(1 + g'(0)\xi + \frac{g''(0)}{2}\xi^2 + \cdots\Big)\Big(1 + 2Df(0)(x_0)\xi + \frac{3}{2}D^2 f(0)(x_0^2)\xi^2 + \cdots\Big) \\ &= 1 + 4Df(0)(x_0)\xi + \frac{9}{2}D^2 f(0)(x_0^2)\xi^2 + \cdots . \end{aligned}$$

Comparing the homogeneous expansions of two sides of the above equality, we deduce that

$$g'(0) = 2Df(0)(x_0), \qquad \frac{g''(0)}{2} = 3D^2 f(0)(x_0^2) - 4(Df(0)(x_0))^2.$$

That is

$$g'(0)\|x\| = 2Df(0)(x), \qquad \frac{g''(0)}{2}\|x\|^2 = 3D^2 f(0)(x^2) - 4(Df(0)(x))^2. \tag{3.5}$$

Moreover, from $F(x) = xf(x)$, we have

$$\frac{D^3F(0)(x^3)}{3!} = \frac{D^2 f(0)(x^2)}{2!}x, \qquad \frac{D^2F(0)(x^2)}{2!} = Df(0)(x)x. \tag{3.6}$$

From (3.6), we conclude that

$$\frac{T_x(D^3F(0)(x^3))}{3!} = \frac{D^2 f(0)(x^2)\|x\|}{2!} \tag{3.7}$$

and

$$\frac{T_x(D^2F(0)(x^2))}{2!} = Df(0)(x)\|x\|. \tag{3.8}$$

Thus, from (3.2), (3.5), (3.7) and (3.8), we obtain

$$\begin{aligned} &\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big| \\ &= \Big|\|x\|^2\frac{D^2 f(0)(x^2)}{2!} - \lambda\|x\|^2(Df(0)(x))^2\Big| \\ &= \frac{1}{6}\Big|3\|x\|^2 D^2 f(0)(x^2) - 6\lambda\|x\|^2(Df(0)(x))^2\Big| \\ &= \frac{1}{6}|3\|x\|^2 D^2 f(0)(x^2) - 4\|x\|^2(Df(0)(x))^2 + (4 - 6\lambda)\|x\|^2(Df(0)(x))^2| \\ &= \frac{1}{6}\|x\|^4\Big|\frac{g''(0)}{2} + \Big(1 - \frac{3}{2}\lambda\Big)(g'(0))^2\Big| \\ &= \frac{1}{6}\|x\|^4\Big|\frac{g''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(g'(0))^2 + \Big(\frac{1}{2}\frac{h''(0)}{(h'(0))^2} + 1 - \frac{3}{2}\lambda\Big)(g'(0))^2\Big| \\ &\le \frac{1}{6}\|x\|^4\Big(|h'(0)| - \frac{|g'(0)|^2}{|h'(0)|} + \Big|\frac{1}{2}\frac{h''(0)}{(h'(0))^2} + 1 - \frac{3}{2}\lambda\Big||g'(0)|^2\Big) \\ &= \frac{1}{6}\|x\|^4\Big(|h'(0)| - \frac{|g'(0)|^2}{|h'(0)|} + \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\frac{|g'(0)|^2}{|h'(0)|}\Big). \end{aligned}$$

Now, we consider the following two cases.

Case I If $\left|\frac{1}{2}\frac{h''(0)}{h'(0)}+\left(1-\frac{3}{2}\lambda\right)h'(0)\right|\le 1$, then

$$\begin{aligned}&\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!}-\lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big|\\ &\le \frac{1}{6}\|x\|^4\Big(|h'(0)|-\frac{|g'(0)|^2}{|h'(0)|}+\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\frac{|g'(0)|^2}{|h'(0)|}\Big)\\ &\le \frac{1}{6}|h'(0)|\|x\|^4. \end{aligned}\tag{3.9}$$

Case II If $\left|\frac{1}{2}\frac{h''(0)}{h'(0)}+\left(1-\frac{3}{2}\lambda\right)h'(0)\right|\ge 1$, then

$$\begin{aligned}&\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!}-\lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big|\\ &\le \frac{1}{6}\|x\|^4\Big(|h'(0)|-\frac{|g'(0)|^2}{|h'(0)|}+\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\frac{|g'(0)|^2}{|h'(0)|}\Big)\\ &= \frac{1}{6}|h'(0)|\|x\|^4+\frac{1}{6}\|x\|^4\Big(\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|-1\Big)\frac{|g'(0)|^2}{|h'(0)|}. \end{aligned}$$

Since $|g'(0)|\le|h'(0)|$, we obtain

$$\begin{aligned}&\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!}-\lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big|\\ &\le \frac{1}{6}|h'(0)|\|x\|^4+\frac{1}{6}\|x\|^4\Big(\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|-1\Big)\frac{|g'(0)|^2}{|h'(0)|}\\ &\le \frac{1}{6}|h'(0)|\|x\|^4+\frac{1}{6}\|x\|^4\Big(\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|-1\Big)\frac{|h'(0)|^2}{|h'(0)|}\\ &= \frac{1}{6}|h'(0)|\|x\|^4\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|. \end{aligned}\tag{3.10}$$

From (3.9)–(3.10), we deduce (3.1), as desired.

To see that the estimation of Theorem 3.1 is sharp, it suffices to consider the following examples.

Example 3.1 If $\left|\frac{1}{2}\frac{h''(0)}{h'(0)}+\left(1-\frac{3}{2}\lambda\right)h'(0)\right|\ge 1$, we consider the following example:

$$DF(x)=I\exp\int_0^{T_u(x)}(h(t)-1)\frac{\mathrm{d}t}{t},\ x\in E,\quad \|u\|=1.$$

We deduce that $(DF(x))^{-1}(D^2F(x)(x^2)+DF(x)(x))\in\mathcal{M}_h$, and a short computation yields the relation

$$\frac{D^3F(0)(x^3)}{3!}=\Big(\frac{h''(0)}{12}+\frac{(h'(0))^2}{6}\Big)(T_u(x))^2x,\qquad \frac{D^2F(0)(x^2)}{2!}=\frac{h'(0)}{2}T_u(x)x.$$

From this it follows that

$$\begin{aligned}&\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!}-\lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big|\\ &=\Big|\Big(\frac{h''(0)}{12}+\frac{(h'(0))^2}{6}\Big)(T_u(x))^2\|x\|^2-\lambda\frac{(h'(0))^2}{4}(T_u(x))^2\|x\|^2\Big|\\ &=\frac{(T_u(x))^2\|x\|^2|h'(0)|}{6}\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|. \end{aligned}\tag{3.11}$$

Setting $x = ru$ $(0 < r < 1)$ in (3.11), we have

$$\Big|\frac{T_x(D^3F(0)(x^3))}{3!\|x\|^3} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!\|x\|^2}\Big)^2\Big| = \frac{|h'(0)|}{6}\Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|.$$

If $\big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \big(1-\frac{3}{2}\lambda\big)h'(0)\big| \le 1$, we consider the following example:

$$DF(x) = I\exp\int_0^{T_u(x)}(h(t^2)-1)\frac{\mathrm{d}t}{t}, \quad x\in E,\ \|u\|=1. \tag{3.12}$$

It is elementary to verify that the mapping $F(x)$ defined in (3.12) satisfies $(DF(x))^{-1}(D^2F(x)(x^2)+DF(x)(x)) \in \mathcal{M}_h$, and a simple computation shows that

$$\frac{D^3F(0)(x^3)}{3!} = \frac{h'(0)(T_u(x))^2x}{6}, \quad \frac{D^2F(0)(x^2)}{2!} = 0. \tag{3.13}$$

From (3.13), we have

$$\Big|\frac{T_x(D^3F(0)(x^3))\|x\|}{3!} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!}\Big)^2\Big| = \frac{|h'(0)||T_u(x)|^2\|x\|^2}{6}. \tag{3.14}$$

Taking $x = ru$ $(0 < r < 1)$ in (3.14), we obtain

$$\Big|\frac{T_x(D^3f(0)(x^3))}{3!\|x\|^3} - \lambda\Big(\frac{T_x(D^2f(0)(x^2))}{2!\|x\|^2}\Big)^2\Big| = \frac{|h'(0)|}{6}.$$

This completes the proof of Theorem 3.1.

Theorem 3.2 *Let $h:\mathbb{U}\to\mathbb{C}$ satisfy the conditions of Definition* 1.2, $f\in H(\mathbb{U}^n,\mathbb{C})$, $f(z)\neq 0$, $z\in\mathbb{U}^n$, $f(0)=1$, $F(z)=zf(z)$ *and suppose that* $(DF(z))^{-1}(D^2F(z)(z^2)+DF(z)(z))\in\mathcal{M}_h$. *Then*

$$\begin{aligned}&\Big\|\frac{D^3F(0)(z^3)}{3!} - \lambda\frac{1}{2}D^2F(0)\Big(z,\frac{D^2F(0)(z^2)}{2!}\Big)\Big\|\\ &\le \frac{|h'(0)|\|z\|^3}{6}\max\Big\{1, \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\Big\}, \quad z\in\mathbb{U}^n.\end{aligned} \tag{3.15}$$

Proof For $z\in\mathbb{U}^n\setminus\{0\}$, denote $z_0=\frac{z}{\|z\|}$. Let $q_j:\mathbb{U}\to\mathbb{C}$ be given by

$$q_j(\xi) = \begin{cases}\dfrac{p_j(\xi z_0)\|z\|}{\xi z_j}, & \xi\neq 0,\\ 1, & \xi = 0,\end{cases}$$

where $p(z) = (DF(z))^{-1}(D^2F(z)(z^2)+DF(z)z)$ and j satisfies $|z_j| = \|z\| = \max\limits_{1\le k\le n}\{|z_k|\}$.

Since $(DF(z))^{-1}(D^2F(z)(z^2)+DF(z)z)\in\mathcal{M}_h$, we have $q_j(\xi)\in h(\mathbb{U})$, $\xi\in\mathbb{U}$. Therefore, according to Lemma 2.2, we obtain

$$\Big|\frac{q_j''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(q_j'(0))^2\Big| \le |h'(0)| - \frac{|q_j'(0)|^2}{|h'(0)|}. \tag{3.16}$$

According to (3.3), we have

$$q_j(\xi) = \frac{D^2f(\xi z_0)((\xi z_0)^2) + 3Df(\xi z_0)(\xi z_0) + f(\xi z_0)}{f(\xi z_0) + Df(\xi z_0)(\xi z_0)},$$

or, equivalently,

$$q_j(\xi)(f(\xi z_0) + Df(\xi z_0)(\xi z_0)) = D^2 f(\xi z_0)((\xi z_0)^2) + 3Df(\xi z_0)(\xi z_0) + f(\xi z_0).$$

Using Taylor series expansions in ξ, we obtain

$$\begin{aligned}&\Big(1 + q_j'(0)\xi + \frac{q_j''(0)}{2}\xi^2 + \cdots\Big)\Big(1 + 2Df(0)(z_0)\xi + \frac{3}{2}D^2 f(0)(z_0^2)\xi^2 + \cdots\Big)\\ &= 1 + 4Df(0)(z_0)\xi + \frac{9}{2}D^2 f(0)(z_0^2)\xi^2 + \cdots .\end{aligned}$$

Comparing the homogeneous expansions of two sides of the above equality, we deduce that

$$q_j'(0) = 2Df(0)(z_0), \quad \frac{q_j''(0)}{2} = 3D^2 f(0)(z_0^2) - 4(Df(0)(z_0))^2. \tag{3.17}$$

Moreover, from $F(z_0) = z_0 f(z_0)$, we have

$$\frac{D^3 F_j(0)(z_0^3)}{3!} = \frac{D^2 f(0)(z_0^2)}{2!}\frac{z_j}{\|z\|}, \quad \frac{D^2 F_j(0)(z_0^2)}{2!} = Df(0)(z_0)\frac{z_j}{\|z\|}. \tag{3.18}$$

Thus, from (3.16)–(3.18), we have

$$\begin{aligned}&\Big|\frac{D^3 F_j(0)(z_0^3)\|z\|}{3!z_j} - \lambda\frac{1}{2}D^2 F_j(0)\Big(z_0, \frac{D^2 F(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2 f(0)(z_0^2)}{2} - \lambda\frac{1}{2}D^2 F_j(0)(z_0, Df(z_0)z_0)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2 f(0)(z_0^2)}{2} - \lambda Df(z_0)\frac{1}{2}D^2 F_j(0)(z_0, z_0)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2 f(0)(z_0^2)}{2} - \lambda(Df(z_0)(z_0))^2\Big|\\
&= \frac{1}{6}|3D^2 f(0)(z_0^2) - 6\lambda(Df(0)(z_0))^2|\\
&= \frac{1}{6}|3D^2 f(0)(z_0^2) - 4(Df(0)(z_0))^2 + (4 - 6\lambda)(Df(0)(z_0))^2|\\
&= \frac{1}{6}\Big|\frac{q_j''(0)}{2} + \Big(1 - \frac{3}{2}\lambda\Big)(q_j'(0))^2\Big|\\
&= \frac{1}{6}\Big|\frac{q_j''(0)}{2} - \frac{1}{2}\frac{h''(0)}{(h'(0))^2}(q_j'(0))^2 + \Big(\frac{1}{2}\frac{h''(0)}{(h'(0))^2} + 1 - \frac{3}{2}\lambda\Big)(q_j'(0))^2\Big|\\
&\le \frac{1}{6}\Big(|h'(0)| - \frac{|q_j'(0)|^2}{|h'(0)|} + \Big|\frac{1}{2}\frac{h''(0)}{(h'(0))^2} + 1 - \frac{3}{2}\lambda\Big||q_j'(0)|^2\Big)\\
&= \frac{1}{6}\Big(|h'(0)| - \frac{|q_j'(0)|^2}{|h'(0)|} + \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\frac{|q_j'(0)|^2}{|h'(0)|}\Big).\end{aligned}$$

Using similar arguments as in the proof of Theorem 3.1, we obtain

$$\begin{aligned}&\Big|\frac{D^3 F_j(0)(z_0^3)\|z\|}{3!z_j} - \lambda\frac{1}{2}D^2 F_j(0)\Big(z_0, \frac{D^2 F(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|\\
&\le \frac{|h'(0)|}{6}\max\Big\{1, \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\Big\}.\end{aligned}$$

If $z_0 \in \partial_0\mathbb{U}^n$, then we get

$$\begin{aligned}&\Big|\frac{D^3F_j(0)(z_0^3)}{3!}-\lambda\frac{1}{2}D^2F_j(0)\Big(z_0,\frac{D^2F(0)(z_0^2)}{2!}\Big)\Big|\\ \le&\frac{|h'(0)|}{6}\max\Big\{1,\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\Big\},\quad j=1,2,\cdots,n.\end{aligned}$$

Also since

$$\frac{D^3F_j(0)(z^3)}{3!}-\lambda\frac{1}{2}D^2F_j(0)\Big(z,\frac{D^2F(0)(z^2)}{2!}\Big),\quad j=1,2,\cdots,n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, in view of the maximum modulus theorem of holomorphic functions on the unit polydisc, we obtain

$$\begin{aligned}&\Big|\frac{D^3F_j(0)(z_0^3)}{3!}-\lambda\frac{1}{2}D^2F_j(0)\Big(z_0,\frac{D^2F(0)(z_0^2)}{2!}\Big)\Big|\\ \le&\frac{|h'(0)|}{6}\max\Big\{1,\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\Big\},\quad z_0\in\partial\mathbb{U}^n,\ j=1,2,\cdots,n.\end{aligned}$$

That is

$$\begin{aligned}&\Big|\frac{D^3F_j(0)(z^3)}{3!}-\lambda\frac{1}{2}D^2F_j(0)\Big(z,\frac{D^2F(0)(z^2)}{2!}\Big)\Big|\\ \le&\frac{|h'(0)|\|z\|^3}{6}\max\Big\{1,\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\Big\},\quad z\in\mathbb{U}^n,\quad j=1,2,\cdots,n.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big\|\frac{D^3F(0)(z^3)}{3!}-\lambda\frac{1}{2}D^2F(0)\Big(z,\frac{D^2F(0)(z^2)}{2!}\Big)\Big\|\\ \le&\frac{|h'(0)|\|z\|^3}{6}\max\Big\{1,\Big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\Big(1-\frac{3}{2}\lambda\Big)h'(0)\Big|\Big\},\quad z\in\mathbb{U}^n,\end{aligned}$$

as desired.

In order to prove the sharpness, it suffices to consider the following examples.

If $\big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\big(1-\frac{3}{2}\lambda\big)h'(0)\big|\ge 1$, we consider the following example:

$$DF(z)=I\exp\int_0^{z_1}(h(t)-1)\frac{\mathrm{d}t}{t},\quad z\in\mathbb{U}^n. \tag{3.19}$$

If $\big|\frac{1}{2}\frac{h''(0)}{h'(0)}+\big(1-\frac{3}{2}\lambda\big)h'(0)\big|\le 1$, we consider the following example:

$$DF(z)=I\exp\int_0^{z_1}(h(t^2)-1)\frac{\mathrm{d}t}{t},\quad z\in\mathbb{U}^n. \tag{3.20}$$

It is not difficult to verify that the mappings $F(z)$ defined in (3.19) and (3.20) satisfy

$$(DF(z))^{-1}(D^2F(z)(z^2)+DF(z)(z))\in\mathcal{M}_h.$$

Taking $z=(r,0,\cdots,0)'$ $(0<r<1)$ in (3.19) and (3.20), respectively, we deduce that the equality in (3.15) holds. This completes the proof of Theorem 3.2.

In view of Remark 1.1, if we set $h(\xi)=\frac{1+(1-2\alpha)\xi}{1-\xi}$ in Theorems 3.1 and 3.2, we can deduce Corollary 3.1, which we merely state here without proof.

Corollary 3.1 *Let $f : E \to \mathbb{C}$, $F(x) = xf(x) \in \mathcal{K}_\alpha(E)$. Then*

$$\Big|\frac{T_x(D^3F(0)(x^3))}{3!\|x\|^3} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!\|x\|^2}\Big)^2\Big|$$
$$\le \frac{1-\alpha}{3}\max\{1, |3-2\alpha-3\lambda(1-\alpha)|\}, \quad \lambda \in \mathbb{C},\ x \in E\setminus\{0\},\ T_x \in T(x).$$

If $X = \mathbb{C}^n$, $E = \mathbb{U}^n$, then

$$\Big\|\frac{D^3F(0)(z^3)}{3!} - \lambda\frac{1}{2}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big)\Big\|$$
$$\le \frac{1-\alpha}{3}\max\{1, |3-2\alpha-3\lambda(1-\alpha)|\}, \quad \lambda \in \mathbb{C},\ z \in \mathbb{U}^n. \tag{3.21}$$

These estimates are sharp.

Especially, when $n = 1$, $E = \mathbb{U}$, (3.21) *reduces to the following*

$$\Big|\frac{F^{(3)}(0)}{3!} - \lambda\Big(\frac{F''(0)}{2!}\Big)^2\Big| \le \frac{1-\alpha}{3}\max\{1, |3-2\alpha-3\lambda(1-\alpha)|\}, \quad \lambda \in \mathbb{C},\ z \in \mathbb{U},$$

which is equivalent to Theorem A.

At present, we do not know whether the assertions of Theorems 3.1 and 3.2 hold true for a normalized locally biholomorphic mapping F satisfying $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathcal{M}_h$. Consequently, we pose the following open problem.

Open Problem Let $F \in H(E)$ be a normalized locally biholomorphic mapping. If

$$(DF(x))^{-1}(D^2F(x)(x^2) + DF(x)(x)) \in \mathcal{M}_h,$$

then

$$\Big|\frac{T_x(D^3F(0)(x^3))}{3!\|x\|^3} - \lambda\Big(\frac{T_x(D^2F(0)(x^2))}{2!\|x\|^2}\Big)^2\Big|$$
$$\le \frac{|h'(0)|}{6}\max\Big\{1, \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\Big\}, \quad \lambda \in \mathbb{C},\ x \in E\setminus\{0\},\ T_x \in T(x).$$

If $X = \mathbb{C}^n$, $E = \mathbb{U}^n$, then

$$\Big\|\frac{D^3F(0)(z^3)}{3!} - \lambda\frac{1}{2}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big)\Big\|$$
$$\le \frac{|h'(0)|\|z\|^3}{6}\max\Big\{1, \Big|\frac{1}{2}\frac{h''(0)}{h'(0)} + \Big(1 - \frac{3}{2}\lambda\Big)h'(0)\Big|\Big\}, \quad \lambda \in \mathbb{C},\ z \in \mathbb{U}^n.$$

These estimates are sharp.

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