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# Character Formulas for a Class of Simple Restricted Modules over the Simple Lie Superalgebras of Witt Type*

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**Abstract** Let $F$ be an algebraically closed field of prime characteristic, and $W(m,n,\mathbf{1})$ be the simple restricted Lie superalgebra of Witt type over $F$, which is the Lie superalgebra of superderivations of the superalgebra $\mathfrak{A}(m;\mathbf{1})\otimes\wedge(n)$, where $\mathfrak{A}(m;\mathbf{1})$ is the truncated polynomial algebra with $m$ indeterminants and $\wedge(n)$ is the Grassmann algebra with $n$ indeterminants. In this paper, the author determines the character formulas for a class of simple restricted modules of $W(m,n,\mathbf{1})$ with atypical weights of type I.

**Keywords** Restricted Lie superalgebra, Witt type Lie superalgebra, Restricted representation, Typical (atypical) weight, Character formula
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## 1 Introduction

Recall that Kac classified finite-dimensional simple Lie superalgebras over the field of complex numbers (cf. [4]). They fall into two types: Classical type and Cartan type. Cartan type Lie superalgebras consist of three families: The Witt type $W(n)$, the special types $S(n)$, $\widetilde{S}(n)$, and the Hamiltonian type $H(n)$. Serganova determined the simplicity of the so-called Kac modules for $W(n), S(n), H(n)$ in [6], and obtained their character formulas. Serganova's results have been extensively extended to modular case (cf. [8]).

Although until now, the classification of finite-dimensional simple Lie superalgebras over a field of prime characteristic is unknown, one naturally expects that there would be some modular version of the classification of iso-classes of finite-dimensional complex simple Lie superalgebras. And Cartan type Lie superalgebras in prime characteristic would be main series of simple Lie superalgebras, apart from classical series. Those Lie superalgebras can be referred to [9].

Let $F$ be an algebraically closed field of characteristic $p>2$. Let $\mathfrak{g}=W(m,n,\mathbf{1})$ be the Lie superalgebra of Witt type over $F$, which belongs to the first class of Lie superalgebras of Cartan type (cf. [9]). Recall that simple restricted $\mathfrak{g}$-modules are homomorphic image of Kac modules and parameterized by the "highest" weights (cf. [7, Corollary 3.16]). Those Kac modules with typical weights are simple modules. And there are two classes of atypical weights, called type I and type II (see Proposition 3.3). The aim of this paper is to present the character formulas for a class of simple restricted $\mathfrak{g}$-modules with atypical weights of type I (see Theorem 3.4), based

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on the assumption that the corresponding ones are known for the general linear Lie superalgebra $\mathfrak{gl}(m \mid n)$.

## 2 Preliminaries

In this paper, we always assume that the ground field $F$ is algebraically closed and of characteristic $p > 2$, and that all modules (vector spaces) are over $F$. For a $\mathbb{Z}_2$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$, we denote the parity of a homogeneous element $x \in V$ by $\overline{x}$.

### 2.1 Restricted Lie superalgebras and their restricted representations

The following definition is a generalization of the notion of restricted Lie algebras to the case of Lie superalgebras.

**Definition 2.1** (cf. [5]) *A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is called a restricted one if $\mathfrak{g}_{\overline{0}}$ is a restricted Lie algebra and $\mathfrak{g}_{\overline{1}}$ is a restricted $\mathfrak{g}_{\overline{0}}$-module under the adjoint action. This is equivalent to saying that there exists a so-called p-mapping $[p]$ on $\mathfrak{g}_{\overline{0}}$ such that the following properties hold*:

(i) $(\operatorname{ad} x)^p = \operatorname{ad}(x^{[p]}), \forall\, x \in \mathfrak{g}_{\overline{0}}$;
(ii) $(ax)^{[p]} = a^p x^{[p]}, \forall\, a \in F, x \in \mathfrak{g}_{\overline{0}}$;
(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y), \forall\, x, y \in \mathfrak{g}_{\overline{0}}$;

*where $s_i(x,y) \in \mathfrak{g}_{\overline{0}}$ are defined via the following formula*:

$$\operatorname{ad}(tx+y)^{p-1}(x) = \sum_{i=1}^{p-1} i s_i(x,y) t^{i-1}, \quad \forall\, x, y \in \mathfrak{g}_{\overline{0}}.$$

*Here $t$ is an indeterminant.*

Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra. As in the case of restricted Lie algebras, one can define the so-called restricted enveloping superalgebra $u(\mathfrak{g})$ to be the quotient of $U(\mathfrak{g})$ by the ideal generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{g}_{\overline{0}}\}$, where $U(\mathfrak{g})$ denotes the universal enveloping superalgebra of $\mathfrak{g}$. A representation $(V, \rho)$ of $\mathfrak{g}$ is said to be restricted if $\rho$ satisfies

$$\rho(x)^p - \rho(x^{[p]}) = 0, \quad \forall\, x \in \mathfrak{g}_{\overline{0}}.$$

All restricted $\mathfrak{g}$-modules constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $u(\mathfrak{g})$-module category denoted by $u(\mathfrak{g})$-mod.

### 2.2 The Witt type Lie superalgebra $W(m, n, \mathbf{1})$

Let $m, n$ be two positive integers. Denote by $A(m; \mathbf{1})$ the index set $\{\alpha = (\alpha_1, \cdots, \alpha_m) \mid 0 \le \alpha_i \le p-1, i = 1, 2, \cdots, m\}$. Let $\epsilon_i\,(1 \le i \le m)$ be the $m$-tuple $(0, 0, \cdots 1, 0, 0, \cdots 0)$ with 1 in the $i$-th position and 0 elsewhere. We have a truncated polynomial algebra $\mathfrak{A}(m; \mathbf{1}) = F[x_1, \cdots, x_m]/(x_1^p, \cdots, x_m^p)$ which has a basis $\{x^\alpha \mid \alpha \in A(m; \mathbf{1})\}$, where we denote the canonical image of $\prod\limits_{i=1}^{m} x_i^{\alpha_i} \in F[x_1, \cdots, x_m]$ in $\mathfrak{A}(m; \mathbf{1})$ by $x^\alpha$ for $\alpha \in A(m; \mathbf{1})$. We make the convention that $x^\alpha = 0$ if $\alpha \notin A(m; \mathbf{1})$.

Let $\Lambda(n)$ be the free commutative superalgebra with $n$ odd generators $\xi_1, \cdots, \xi_n$. Then $\Lambda(n)$ is isomorphic to the Grassmann algebra. Let $I$ denote the sequence $i_1, \cdots, i_s$ where $1 \le i_1 < \cdots < i_s \le n$. Let $\mathcal{J}$ be the set of all such sequences including the empty one. For every $I \in \mathcal{J}$, let $|I|$ denote the length of $I$. For $I=\{i_1,\cdots,i_s\}$, denote $\xi_{i_1}\cdots\xi_{i_s}\in\Lambda(n)$ by $\xi_I$ for brevity. Then $\Lambda(n)$ is $\mathbb{Z}_2$-graded with $\Lambda(n)_{\overline{0}}=\mathrm{span}_F\{\xi_I\mid |I| \text{ is even}\}$ and $\Lambda(n)_{\overline{1}}=\mathrm{span}_F\{\xi_I\mid |I| \text{ is odd}\}$.

Let $\Lambda(m,n):=\mathfrak{A}(m;\mathbf{1})\otimes_F\Lambda(n)$. Then $\Lambda(m,n)$ is a superalgebra with the $\mathbb{Z}_2$-gradation given as follows: $\Lambda(m,n)_{\overline{0}}=\mathfrak{A}(m,\mathbf{1})\otimes_F\Lambda(n)_{\overline{0}}$ and $\Lambda(m,n)_{\overline{1}}=\mathfrak{A}(m,\mathbf{1})\otimes_F\Lambda(n)_{\overline{1}}$. For brevity, we denote the element $f\otimes g\in\Lambda(m,n)$ by $fg$, where $f\in\mathfrak{A}(m;\mathbf{1}),g\in\Lambda(n)$.

The restricted Witt type Lie superalgebra $W(m,n,\mathbf{1})$ is by definition the superderivation algebra of the superalgebra $\Lambda(m,n)$. Then by [9], $W(m,n,\mathbf{1})=\mathrm{span}_F\{x^\alpha\xi_ID_i\mid\alpha\in A(m;\mathbf{1}),I\in\mathcal{J},1\le i\le n\}\oplus\mathrm{span}_F\{x^\alpha\xi_Id_j\mid\alpha\in A(m;\mathbf{1}),I\in\mathcal{J},1\le j\le m\}$ where the $D_i\,(1\le i\le n)$ and $d_j\,(1\le j\le m)$ are superderivations on $\Lambda(m,n)$ defined as follows:

$$D_i(x^\alpha\xi_I):=\begin{cases}(-1)^{|i<I|}x^\alpha\xi_{I\setminus i}, & \text{if } i\in I,\\ 0, & \text{otherwise,}\end{cases}$$

$$d_j(x^\alpha\xi_I):=\begin{cases}x^{\alpha-\epsilon_j}\xi_I, & \text{if } \alpha_j>0,\\ 0, & \text{otherwise,}\end{cases}$$

where $|i<I|$ is the number of indices in $I$ that are smaller than $i$. The Lie product in $W(m,n,\mathbf{1})$ is defined as follows:

$$[fD,gE]=fD(g)E-(-1)^{\overline{fD}\,\overline{gE}}gE(f)D,$$

where $f,g\in\Lambda(m,n);D,E\in\{D_1,\cdots,D_n,d_1,\cdots,d_m\}$.

The $\mathbb{Z}_2$-gradation on $\Lambda(m,n)$ induces the $\mathbb{Z}_2$-gradation on $W(m,n,\mathbf{1})$ :

$$W(m,n,\mathbf{1})=W(m,n,\mathbf{1})_{\overline{0}}\oplus W(m,n,\mathbf{1})_{\overline{1}},$$

where

$$W(m,n,\mathbf{1})_{\overline{0}}=\mathrm{span}_F\{x^\alpha\xi_ID_i,x^\beta\xi_Jd_j\mid |I|\,\text{is odd},\ |J|\,\text{is even}\}$$

and

$$W(m,n,\mathbf{1})_{\overline{1}}=\mathrm{span}_F\{x^\alpha\xi_ID_i,x^\beta\xi_Jd_j\mid |I|\,\text{is even},\ |J|\,\text{is odd}\}.$$

The $\mathbb{Z}$-gradation of $\Lambda(m,n)$ induced by $\deg x_i=\deg\xi_j=1$ for all $1\le i\le m,1\le j\le n$ determines the $\mathbb{Z}$-gradation of the Witt type Lie superalgebra $W(m,n,\mathbf{1})$:

$$W(m,n,\mathbf{1})=\bigoplus_{i=-1}^{n+m(p-1)-1}W(m,n,\mathbf{1})_{[i]},$$

where

$$W(m,n,\mathbf{1})_{[i]}=\mathrm{span}_F\{x^\alpha\xi_ID_k,x^\beta\xi_Jd_j\mid 1\le k\le n,1\le j\le m,|\alpha|+|I|=|\beta|+|J|=i+1\}.$$

Associated with this gradation, there is a natural filtration:

$$W(m,n,\mathbf{1})=W(m,n,\mathbf{1})_{-1}\supseteq W(m,n,\mathbf{1})_0\supseteq\cdots,$$

where $W(m,n,\mathbf{1})_i=\bigoplus_{j\ge i}W(m,n,\mathbf{1})_{[j]}$. It is easy to check that $W(m,n,\mathbf{1})$ is a restricted Lie superalgebra in the sense of Definition 2.1. The $p$-mapping on $W(m,n,\mathbf{1})_{\overline{0}}$ is just given as the usual $p$-th power of superderivations. For $x^\alpha\xi_ID_i,x^\beta\xi_Jd_j\in W(m,n,\mathbf{1})_{\overline{0}}$, a straightforward calculation implies that

$$(x^\alpha\xi_ID_i)^{[p]}=\begin{cases}\xi_iD_i, & \text{if } \alpha=0 \text{ and } I=\{i\},\\ 0, & \text{otherwise,}\end{cases}$$

$$(x^\beta \xi_J d_j)^{[p]} = \begin{cases} x_j d_j, & \text{if } J=\emptyset \text{ and } \beta=\epsilon_j, \\ 0, & \text{otherwise.} \end{cases}$$

From now on, we always assume $\mathfrak{g} = W(m,n,\mathbf{1})$, unless otherwise indicated.

By [7, Lemma 2.2], $\mathfrak{g}_{[0]} \cong \mathfrak{gl}(m \mid n)$ under the mapping sending any $\Sigma a_{ij}x_i d_j + \Sigma b_{ij}\xi_i D_j + \Sigma c_{ij}x_i D_j + \Sigma d_{ij}\xi_i d_j \in \mathfrak{g}_{[0]}$ to $\Sigma a_{ij}E_{ij} + \Sigma b_{ij}E_{m+i,m+j} + \Sigma c_{ij}E_{i,m+j} + \Sigma d_{ij}E_{m+i,j} \in \mathfrak{gl}(m \mid n)$, where $E_{kl}$ denotes the $(m+n)\times(m+n)$ matrix with 1 in the position of the $k$-th row and $l$-th column, and zero elsewhere. Furthermore, we have a standard triangular decomposition: $\mathfrak{g}_{[0]} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where

$$\mathfrak{n}^- = \sum_{1\le j<i\le m} Fx_i d_j + \sum_{1\le j<i\le n} F\xi_i D_j + \sum_{\substack{1\le i\le n\\ 1\le j\le m}} F\xi_i d_j,$$

$$\mathfrak{h} = \sum_{i=1}^{m+n} Fh_i \quad \text{with } h_i = x_i d_i \text{ for } 1\le i\le m \text{ and } h_j = \xi_{j-m}D_{j-m} \text{ for } m+1\le j\le m+n$$

and

$$\mathfrak{n}^+ = \sum_{1\le i<j\le m} Fx_i d_j + \sum_{1\le i<j\le n} F\xi_i D_j + \sum_{\substack{1\le i\le m\\ 1\le j\le n}} Fx_i D_j.$$

Set $\mathfrak{b}^\pm := \mathfrak{h} + \mathfrak{n}^\pm$ and $\mathfrak{b}^+$ is usually simply denoted by $\mathfrak{b}$. Set $N^- := \mathfrak{n}^- \oplus \mathfrak{g}_{[-1]}$, $N^+ := \mathfrak{n}^+ \oplus \mathfrak{g}_1$, $B^- := \mathfrak{h} \oplus N^-$, $B^+ := \mathfrak{h} \oplus N^+$, $\mathfrak{g}^+ := \mathfrak{g}_{[0]} \oplus \mathfrak{g}_1$ and $\mathfrak{g}^- := \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[-1]}$. Then it is easy to check that $\mathfrak{b}^\pm$, $N^\pm$, $B^\pm$ and $\mathfrak{g}^\pm$ are restricted subalgebras of $\mathfrak{g}$.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{[0]}$ is also a Cartan subalgebra of $\mathfrak{g}$. We then have a root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\Delta = \{a_1\varepsilon_1 + \cdots + a_m\varepsilon_m + \eta_{i_1} + \cdots + \eta_{i_k} - \eta_j \mid 0 \le a_k \le p-1, 1 \le i_1 < \cdots < i_k \le n, 1 \le j \le n\} \cup \{-\varepsilon_i, -\eta_j \mid 1\le i\le m, 1\le j\le m\}$, and $\{\varepsilon_1, \cdots, \varepsilon_m, \eta_1, \cdots, \eta_n\}$ is the standard basis in $\mathfrak{h}^*$ dual to $\{x_1d_1, \cdots, x_md_m, \xi_1D_1, \cdots, \xi_nD_n\}$, i.e.,

$$\varepsilon_i(x_jd_j) = \delta_{ij}, \quad \varepsilon_i(\xi_kD_k) = 0, \quad \forall\, 1\le k\le n,\ 1\le i\le m,\ 1\le j\le m$$

and

$$\eta_s(x_jd_j) = 0, \quad \eta_s(\xi_tD_t) = \delta_{st}, \quad \forall\, 1\le j\le m,\ 1\le s\le n,\ 1\le t\le n.$$

## 3 Character Formulas for Simple Restricted $\mathfrak{g}$-Modules

### 3.1 Simple restricted $\mathfrak{g}$-modules

Recall that the iso-classes of irreducible restricted $\mathfrak{g}_{[0]}$-modules are parameterized by the set of restricted weights $\Lambda := \{\lambda = (\lambda_1, \cdots, \lambda_{m+n}) \mid \lambda_i \in \mathbb{F}_p, i = 1, \cdots, m+n\}$. More precisely, for a given $\lambda \in \Lambda$, there is a one-dimensional restricted $\mathfrak{b}$-module $F_\lambda$ on which every element of $\mathfrak{h}$ acts as a scalar determined by $\lambda$, while $\mathfrak{n}^+$ acts trivially. Then one has the so-called baby Verma module $Z(\lambda) := u(\mathfrak{g}_{[0]}) \otimes_{u(\mathfrak{b})} F_\lambda$ which has a simple head denoted by $L^0(\lambda)$, where $u(\mathfrak{g}_{[0]})$ and $u(\mathfrak{b})$ are restricted enveloping superalgebras of $\mathfrak{g}_{[0]}$ and $\mathfrak{b}$, respectively. Then $\{L^0(\lambda) \mid \lambda \in \Lambda\}$ constitute the set of representatives of restricted simple $\mathfrak{g}_{[0]}$-modules. Since $\mathfrak{g}_1$ is a restricted nilpotent ideal of $\mathfrak{g}_0$, each simple restricted $\mathfrak{g}_{[0]}$-module can be extended to a $\mathfrak{g}_0$-module with trivial action of $\mathfrak{g}_1$. Moreover, each simple restricted $\mathfrak{g}_0$-module is also a simple $\mathfrak{g}_{[0]}$-module with trivial action by $\mathfrak{g}_1$.

**Definition 3.1** *Keep notations as above. For $\lambda \in \Lambda$, set $K(\lambda) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L^0(\lambda)$, where $L^0(\lambda)$ is regarded canonically as a $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_1$-action. The induced module $K(\lambda)$ is called a Kac module.*

**Remark 3.1** For each $\lambda \in \Lambda$, the Kac module $K(\lambda)$ has a unique maximal submodule $J(\lambda)$ which is the sum of all proper submodules of $K(\lambda)$. Therefore $K(\lambda)$ has a unique simple quotient denoted by $L(\lambda)$.

**Proposition 3.1** (cf. [7, Corollary 3.16]) *The family $\{L(\lambda) \mid \lambda \in \Lambda\}$ constitute the set of iso-classes of restricted irreducible $\mathfrak{g}$-modules.*

**Definition 3.2** (1) *$L(\lambda)$ is called a typical simple $\mathfrak{g}$-module if $L(\lambda) = K(\lambda)$. In this case, $\lambda$ is called a typical weight.*

(2) *$L(\lambda)$ is called an atypical simple $\mathfrak{g}$-module if $L(\lambda) \neq K(\lambda)$. In this case, $\lambda$ is called an atypical weight.*

Each simple restricted $\mathfrak{g}_{[0]}$-module $L^0(\lambda)$ ($\lambda \in \Lambda$) can also be extended to a $\mathfrak{g}^-$-module with trivial action by $\mathfrak{g}_{[-1]}$. Moreover, each simple restricted $\mathfrak{g}^-$-module is also a simple $\mathfrak{g}_{[0]}$-module with trivial action by $\mathfrak{g}_{[-1]}$. For each $\lambda \in \Lambda$, set $K^-(\lambda) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^-)} L^0(\lambda)$, where $L^0(\lambda)$ is regarded canonically as a $\mathfrak{g}^-$-module with trivial $\mathfrak{g}_{[-1]}$-action. Then $K^-(\lambda)$ has a unique maximal submodule $J^-(\lambda)$ which is the sum of all proper submodules of $K^-(\lambda)$. Therefore, $K^-(\lambda)$ has a unique simple quotient denoted by $L^-(\lambda)$. Similar to Proposition 3.1, we have the following analogous result.

**Proposition 3.2** *The family $\{L^-(\lambda) \mid \lambda \in \Lambda\}$ constitute the set of iso-classes of restricted irreducible $\mathfrak{g}$-modules.*

Combining Proposition 3.1 with Proposition 3.2, we can define a bijection " $'$ " on $\Lambda$ via: $L(\lambda) \cong L^-(\lambda')$. Then by [7, Corollary 3.9, Theorem 3.13], we have the following result.

**Proposition 3.3** (cf. [7, Corollary 3.9, Theorem 3.13, Proposition 3.6, Proposition 3.8]) *Let $\lambda \in \Lambda$. Then the following statements hold.*

(1) *$\lambda$ is typical if and only if $\lambda' = \lambda - \sum\limits_{i=1}^{n} \eta_i - \sum\limits_{j=1}^{m} (p-1)\epsilon_j$.*

(2) *$\lambda$ is atypical if and only if $\lambda$ is one of the following forms*:

(i) *Type* I: $\{a\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m \mid 0 < a \leq p-1,\ 1 \leq i \leq n\}$.

(ii) *Type* II: $\{(p-1)\epsilon_j + \cdots + (p-1)\varepsilon_m \mid 1 \leq j \leq m\} \cup \{0\}$.

(3) *Let $\lambda = a\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m$ be an atypical weight of type* I *with $1 \leq i \leq n$ and $a \neq 0$, then $\lambda' = -\eta_1 - \cdots - \eta_{i-1} + a\eta_i$.*

### 3.2 $u(\mathfrak{g})$–$T$-module category

Let $T$ be the canonical maximal torus of $\mathrm{GL}(m, F) \times \mathrm{GL}(n, F)$ which consists of diagonal matrices $\mathrm{Diag}(t_1, \cdots, t_{m+n})$, $t_i \in F^*, i = 1, \cdots, m+n$. Clearly, $\mathrm{Lie}(T) = \mathfrak{h}$. Let $X(T)$ be the character group of $T$. Then $X(T)$ is a free Abelian group of rank $m+n$, identified with $\mathbb{Z}^{m+n}$. By a rational $T$-module $V$, we mean that $V = \bigoplus\limits_{\lambda \in X(T)} V_\lambda$, where $V_\lambda = \{v \in V \mid \boldsymbol{t}v = t_1^{\lambda_1} \cdots t_{m+n}^{\lambda_{m+n}} v\}$ for $\lambda = (\lambda_1, \cdots, \lambda_{m+n}) \in \mathbb{Z}^{m+n}$ and $\boldsymbol{t} =\mathrm{Diag}(t_1, \cdots, t_{m+n}) \in T$. For $\lambda \in X(T)$, its differential $\mathrm{d}\lambda : \mathfrak{h} \to F$ is a homomorphism of restricted Lie algebras, i.e., $\mathrm{d}\lambda(h^{[p]}) = (\mathrm{d}\lambda(h))^p$. This implies that $\mathrm{d}\lambda \in \Lambda$. The map $\varphi : X(T) \to \Lambda$ sending any $\lambda \in X(T)$ to $\mathrm{d}\lambda \in \Lambda$ has kernel $pX(T)$. So it induces a bijection $X(T)/pX(T) \cong \Lambda$. Denote $\mathrm{d}\lambda \in \Lambda \cong X(T)/pX(T)$ by $\overline{\lambda}$ for brevity. It follows from an easy calculation that $u(\mathfrak{g})$ is a rational $T$-module. The action of $\boldsymbol{t} \in T$ on $a \in u(\mathfrak{g})$ is denoted by $\mathrm{Ad}(t)(a)$, defined as the usual way. A character $\lambda \in X(T)$ is called typical (atypical), if $\overline{\lambda} \in \Lambda$ is typical (atypical).

**Definition 3.3** (cf. [1, 3]) *The category $u(\mathfrak{g})$–$T$-mod is defined as such a category whose objects are finite-dimensional $F$-superspaces endowed with both $u(\mathfrak{g})$-module and rational $T$-module structure satisfying the following compatible conditions for $V \in u(\mathfrak{g})$–$T$-mod:*

(1) *The action of $\mathfrak{h}$ coincides with the action of $\mathrm{Lie}(T)$ induced from $T$.*

(2) $\boldsymbol{t}(av) = (\mathrm{Ad}\,\boldsymbol{t}(a))\boldsymbol{t}v$ *for any $a \in u(\mathfrak{g}), \boldsymbol{t} \in T$ and $v \in V$.*

*The morphisms in $u(\mathfrak{g})$–$T$-mod are defined to be linear maps of $F$-superspaces as both $u(\mathfrak{g})$-module homomorphisms and rational $T$-module homomorphisms. Each object in $u(\mathfrak{g})$–$T$-mod is called a $\widehat{u}(\mathfrak{g})$-module.*

**Example 3.1** For $\lambda \in X(T)$, one can define a rational $T$-module $\widehat{K}(\lambda) := u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} \widehat{V}(\lambda)$ on which $T$ acts diagonally, where $\widehat{V}(\lambda)$ is a simple $\widehat{u}(\mathfrak{g}_{[0]})$-module. One can also similarly define $\widehat{L}(\lambda)$, $\widehat{K}^-(\lambda)$ and $\widehat{L}^-(\lambda)$ for $\lambda \in X(T)$. Moreover, all these modules are $\widehat{u}(\mathfrak{g})$-modules.

We have the following obvious facts.

**Proposition 3.4** *Keep notations as above. Then the following statements hold.*

(1) *$\widehat{K}(\lambda)$ is irreducible if and only if $K(\overline{\lambda})$ is irreducible.*

(2) *The iso-classes of irreducible modules in $u(\mathfrak{g})$–$T$-mod are in one-to-one correspondence with $X(T)$. Precisely speaking, each simple object in $u(\mathfrak{g})$–$T$-mod is isomorphic to $\widehat{L}(\lambda)$ for $\lambda \in X(T)$.*

(3) *$\widehat{K}(\lambda)\big|_{u(\mathfrak{g})} \cong K(\overline{\lambda})$ and $\widehat{L}(\lambda)\big|_{u(\mathfrak{g})} \cong L(\overline{\lambda})$. Furthermore, sending $\lambda \in X(T)$ to $\overline{\lambda} \in \Lambda = X(T)/pX(T)$ gives rise to the map $\widehat{L}(\lambda) \mapsto L(\overline{\lambda})$ from the set of iso-classes of simple objects in $u(\mathfrak{g})$–$T$-mod to the set of iso-classes of simple objects in $u(\mathfrak{g})$-mod.*

**Remark 3.2** $\lambda \in X(T)$ is typical (atypical) if and only if $\widehat{K}(\lambda) \cong \widehat{L}(\lambda)$ (resp. $\widehat{K}(\lambda) \ncong \widehat{L}(\lambda)$).

For a rational $T$-module $M$, define the length of $M$, denoted by $l(M)$, as the number of $\mathrm{Ad}\boldsymbol{t}$ weights of $M$ minus one, where $\boldsymbol{t} = \mathrm{Diag}(t, \cdots, t) \in T$ for $t \in F^*$. It is a routine to check that the set of weights of $\widehat{L}(\lambda)$ is $\{\lambda(\boldsymbol{t}), \lambda(\boldsymbol{t}) - 1, \cdots, \lambda'(\boldsymbol{t})\}$. It follows that $l(\widehat{L}(\lambda)) = |\lambda| - |\lambda'| \leq n + m(p-1)$ where $|\lambda| = \sum\limits_{i=1}^{m+n} \lambda_i$ for $\lambda = \sum\limits_{i=1}^{m} \lambda_i \varepsilon_i + \sum\limits_{j=1}^{n} \lambda_{m+j}\eta_j$. The equality holds if and only if $\lambda$ is typical.

**Lemma 3.1** *Let $\lambda = a\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_i + \cdots + (p-1)\varepsilon_m$ with $0 \neq a \in \mathbb{Z}, 1 \leq i \leq n$. Then $l(\widehat{L}(\lambda)) = n + m(p-1) - 1$.*

**Proof** Since $\lambda' = -\eta_1 - \cdots - \eta_{i-1} + a\eta_i$ by Proposition 3.3(3), it follows that $l(\widehat{L}(\lambda)) = |\lambda| - |\lambda'| = n + m(p-1) - 1$.

For $\lambda, \mu \in X(T)$, let $m(\lambda, \mu) = [\widehat{K}(\lambda) : \widehat{L}(\mu)]$ be the number of the simple module $\widehat{L}(\mu)$ appearing as a decomposition factor in the decomposition series of the Kac module $\widehat{K}(\lambda)$.

**Proposition 3.5** *Let $\lambda = a\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m$ be atypical of type* I*, where $a \in \mathbb{Z}, 1 \leq i \leq n$. Then the following statements hold.*

(1) *Suppose that $\mu \neq 0$, then $m(\lambda, \mu) \neq 0$ if and only if $\mu = \lambda$ or $\mu' = \lambda - \sum\limits_{i=1}^{n} \eta_i - \sum\limits_{j=1}^{m} (p-1)\epsilon_j$. In such case, $m(\lambda, \mu) = 1$.*

(2) *$m(\lambda, 0) = 0$, if $a \neq 1$.*

(3) *$m(\lambda, 0) = 1$, if $a = 1$.*

**Proof** (1) As a vector space $\widehat{K}(\lambda)\cong u(\mathfrak{g}_{[-1]})\otimes_F L^0(\lambda)$. Let $Y=D_1\cdots D_n d_1^{p-1}\cdots d_m^{p-1}\in u(\mathfrak{g}_{[-1]})$. For any $\mathbb{Z}_2$-graded submodule $M$ of $\widehat{K}(\lambda)$, with some action by $D_1,\cdots,D_n,d_1,\cdots,d_m$ on any nonzero element in $M$, one can easily obtain that $M$ contains $Y\otimes v$ for some nonzero homogeneous element $v\in L^0(\lambda)$. Note that $[Y,\mathfrak{n}^{\pm}]=0$. The simplicity of $L^0(\lambda)$ as a $\mathfrak{g}_{[0]}$-module implies that $Y\otimes L^0(\lambda)\subseteq M$. It is obvious that $Y\otimes v_\lambda$ is a maximal vector of weight $\lambda-\sum\limits_{i=1}^{n}\eta_i-\sum\limits_{j=1}^{m}(p-1)\epsilon_j$ with respect to the Borel subalgebra $B^-$. Hence $\mathrm{Soc}\,(\widehat{K}(\lambda))=\widehat{L}^-(\lambda-\sum\limits_{i=1}^{n}\eta_i-\sum\limits_{j=1}^{m}(p-1)\epsilon_j)$. Therefore, it follows from Lemma 3.1 that $\widehat{L}(\mu)$ occurs as a composition factor of $\widehat{K}(\lambda)$ if and only if $\widehat{L}(\mu)$ is a head or a socle of $\widehat{K}(\lambda)$. Then $m(\lambda,\mu)\neq 0$ if and only if $\mu=\lambda$ or $\mu'=\lambda-\sum\limits_{i=1}^{n}\eta_i-\sum\limits_{j=1}^{m}(p-1)\epsilon_j$. In these cases $m(\lambda,\mu)=1$.

(2)–(3) Suppose $m(\lambda,0)\neq 0$. Then $\widehat{K}(\lambda)$ contains a nonzero $\widehat{u}(\mathfrak{b})$-module primitive vector $v$ of weight 0. Since $\widehat{K}(\lambda)\cong u(\mathfrak{g}_{[-1]})\otimes L^0(\lambda)$ as $\widehat{u}(\mathfrak{g}_{[0]})$-modules. The weights of all possible $\widehat{u}(\mathfrak{b})$-module primitive vectors in $u(\mathfrak{g}_{[-1]})$ are $-(\eta_{j_1}+\cdots+\eta_{j_s})-(a_1\varepsilon_1+\cdots+a_m\varepsilon_m)$ with $1\le j_1<j_2<\cdots<j_s\le n, 0\le a_k\le p-1, 1\le k\le m$. We denote this set by $P(\mathfrak{g}_{[-1]})$.

Note that the weights of all possible $\widehat{u}(\mathfrak{b})$-module primitive vector in $\widehat{K}(\lambda)$ are of the form $\delta+\lambda$ for $\delta\in P(\mathfrak{g}_{[-1]})$ by the modular superversion of Littlewood-Richardson rule (cf. [2, D10]). Since $m(\lambda,0)\neq 0$, it follows that $\lambda=\eta_{j_1}+\cdots+\eta_{j_s}+a_1\varepsilon_1+\cdots+a_m\varepsilon_m$ with $1\le j_1<j_2<\cdots<j_s\le n, 0\le a_k\le p-1, 1\le k\le m$. Hence $\lambda=\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$. On the other hand, for $\lambda=\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$, note that $D_i\cdots D_n d_1^{p-1}\cdots d_m^{p-1}\otimes v_\lambda$ is a $\widehat{u}(\mathfrak{b})$-module primitive vector with weight 0, then $m(\lambda,0)\ge 1$. While it follows from the Littlewood-Richardson rule that $m(\lambda,0)\le 1$. Therefore $m(\lambda,0)=1$.

By Proposition 3.5, we have the following result.

**Proposition 3.6** *The following statements hold.*

(1) *For $\lambda=a\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$ with $a\neq 0,1$, there is an exact sequence as follows*:

$$0\to\widehat{L}(\lambda-\eta_i)\to\widehat{K}(\lambda)\to\widehat{L}(\lambda)\to 0.$$

(2) *For $\lambda=\eta_i+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$ with $i>1$, there are two exact sequences as follows*:

$$0\to\widehat{J}(\lambda)\to\widehat{K}(\lambda)\to\widehat{L}(\lambda)\to 0;$$
$$0\to\widehat{L}(-\eta_{i-1}+\lambda)\to\widehat{J}(\lambda)\to\widehat{L}(0)\to 0.$$

(3) *For $\lambda=\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$, there is an exact sequence as follows*:

$$0\to\widehat{L}(0)\to\widehat{K}(\lambda)\to\widehat{L}(\lambda)\to 0.$$

**Proof** (1) In this case, by Proposition 3.5, there are only two composition factors $\widehat{L}(\lambda)$ and $\widehat{L}(\mu)$ with $\mu'=\lambda-\sum\limits_{i=1}^{n}\eta_i-\sum\limits_{j=1}^{m}(p-1)\epsilon_j$, which appears once for each one in the decomposition series of $\widehat{K}(\lambda)$. Hence $\mu=\lambda-\eta_i$ by Proposition 3.3(3), so that the exact sequence is obtained.

(2) In this case, by Proposition 3.5, there are three composition factors $\widehat{L}(\lambda),\widehat{L}(\mu)$ and $\widehat{L}(0)$ with $\mu'=\lambda-\sum\limits_{i=1}^{n}\eta_i-\sum\limits_{j=1}^{m}(p-1)\epsilon_j$, which appears once for each one in the decomposition series of $\widehat{K}(\lambda)$. Hence $\mu=\lambda-\eta_{i-1}$ by Proposition 3.3(3), so that the exact sequence is obtained.

(3) In this case, by Proposition 3.5, there are two composition factors $\widehat{L}(\lambda), \widehat{L}(0)$ which appears once for each one in the decomposition series of $\widehat{K}(\lambda)$, so that the exact sequence is obtained.

### 3.3 Character formulas in $u(\mathfrak{g})$–$T$-mod

Let $V$ be an object in $u(\mathfrak{g})$–$T$-mod (resp. $u(\mathfrak{g})$-mod) with $V = \sum\limits_{\lambda\in X(T)} V_\lambda$ (resp. $V = \sum\limits_{\lambda\in\Lambda} V_\lambda$). Denote the character of $V$ by $\mathrm{ch}(V)$ which is by definition equal to $\sum\limits_{\lambda\in X(T)} (\dim V_\lambda)\mathrm{e}^\lambda$ (resp. $\sum\limits_{\lambda\in\Lambda} (\dim V_\lambda)\mathrm{e}^\lambda$). Let $\Pi := \prod\limits_{i=1}^{n}\prod\limits_{j=1}^{m}(1+\mathrm{e}^{-\eta_i})(1+\mathrm{e}^{-\epsilon_j})^{p-1}$ for brevity. The following result is obvious.

**Theorem 3.1** *Let $\lambda$ be typical. Then* $\mathrm{ch}(\widehat{L}(\lambda)) = \mathrm{ch}(\widehat{K}(\lambda)) = \Pi\,\mathrm{ch}(\widehat{L}^0(\lambda))$.

**Proof** Since $\lambda$ is typical, we have $\widehat{L}(\lambda) \cong \widehat{K}(\lambda)$. Moreover, as a vector space, $\widehat{K}(\lambda) \cong u(\mathfrak{g}_{[-1]}) \otimes_F L^0(\lambda)$, from which the statement follows immediately.

**Theorem 3.2** *Let $\lambda = a\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m$ be atypical of type* I *with $1 \le i \le n$ and $0 \ne a \in \mathbb{Z}$.*

(1) *If $a \in \mathbb{Z}_-$, then*

$$\mathrm{ch}(\widehat{L}(\lambda)) = \sum_{j=0}^{\infty}(-1)^j\Pi\,\mathrm{ch}(\widehat{L}^0(\lambda - j\eta_i)).$$

(2) *If $a \in \mathbb{Z}_+$ and $i > 1$, then*

$$\begin{aligned}\mathrm{ch}(\widehat{L}(\lambda)) = &\sum_{j=0}^{a-1}(-1)^j\Pi\,\mathrm{ch}(\widehat{L}^0(\lambda - j\eta_i)) + \sum_{b=1}^{\infty}(-1)^{a+b-1}\Pi\,\mathrm{ch}(\widehat{L}^0(\lambda - b\eta_{i-1} - (a-1))\eta_i) \\ &+ (-1)^a\mathrm{ch}(\widehat{L}(0)).\end{aligned}$$

(3) *If $a \in \mathbb{Z}_+$ and $i = 1$, then*

$$\mathrm{ch}(\widehat{L}(\lambda)) = \sum_{j=0}^{a-1}(-1)^j\Pi\,\mathrm{ch}(\widehat{L}^0(\lambda - j\eta_1)) + (-1)^a\mathrm{ch}(\widehat{L}(0)).$$

**Proof** (1) By Proposition 3.6(1), we obtain the following exact sequence:

$$\cdots \to \widehat{K}(\lambda - j\eta_i) \to \cdots \to \widehat{K}(\lambda - \eta_i) \to \widehat{K}(\lambda) \to \widehat{L}(\lambda) \to 0. \tag{3.1}$$

The desired character formula follows from (3.1) and Theorem 3.1.

(2) By Proposition 3.6(1)–(2), we obtain the following exact sequences:

$$\begin{aligned}\cdots \to \widehat{K}(\lambda - (a-1)\eta_i - b\eta_{i-1})) \to \cdots \to \widehat{K}(\lambda - (a-1)\eta_i - \eta_{i-1}) \\ \to \widehat{L}(\lambda - (a-1)\eta_i - \eta_{i-1}) \to 0,\end{aligned} \tag{3.2}$$

$$0 \to \widehat{L}(\lambda - (a-1)\eta_i - \eta_{i-1}) \to \widehat{J}(\lambda - (a-1)\eta_i) \to \widehat{L}(0) \to 0, \tag{3.3}$$

$$\begin{aligned}0 \to \widehat{J}(\lambda - (a-1)\eta_i) \to \widehat{K}(\lambda - (a-1)\eta_i) \to \cdots \to \widehat{K}(\lambda - \eta_i) \\ \to \widehat{K}(\lambda) \to \widehat{L}(\lambda) \to 0.\end{aligned} \tag{3.4}$$

The desired character formula follows from (3.2)–(3.4) and Theorem 3.1.

(3) By Proposition 3.6(1) and 3.6(3), we obtain the following exact sequence:

$$\begin{aligned}&0\to \widehat{L}(0)\to \widehat{K}(\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m)\to\cdots\\&\cdots\to\widehat{K}(\lambda-\eta_1)\to\widehat{K}(\lambda)\to\widehat{L}(\lambda)\to 0.\end{aligned}\tag{3.5}$$

The desired character formula follows from (3.5) and Theorem 3.1.

The proof is completed.

### 3.4 Character formulas in $u(\mathfrak{g})$-mod

In this subsection, we compute the character formulas for the simple restricted $\mathfrak{g}$-modules with atypical weights of type I. We first have the following easy observation.

**Theorem 3.3** *Let $\lambda$ be typical. Then* $\mathrm{ch}(L(\lambda))=\Pi\,\mathrm{ch}(L^0(\lambda))$.

**Proof** Since $\lambda$ is typical, we have $L(\lambda)\cong K(\lambda)$. Consequently, $L(\lambda)\cong u(\mathfrak{g}_{[-1]})\otimes_F L^0(\lambda)$ as a vector space, from which we obtain the character formula stated in the theorem.

The following result is a direct consequence of Proposition 3.6.

**Corollary 3.1** *The following statements hold.*

(1) *For $\lambda=a\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$ with $a\neq 0,1$, there is an exact sequence as follows*:

$$0\to L(\lambda-\eta_i)\to K(\lambda)\to L(\lambda)\to 0.$$

(2) *For $\lambda=\eta_i+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$ with $i>1$, there are two exact sequences as follows*:

$$\begin{aligned}&0\to J(\lambda)\to K(\lambda)\to L(\lambda)\to 0;\\&0\to L((p-1)\eta_{i-1}+\lambda)\to J(\lambda)\to L(0)\to 0.\end{aligned}$$

(3) *For $\lambda=\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m$, there is an exact sequence as follows*:

$$0\to L(0)\to K(\lambda)\to L(\lambda)\to 0.$$

**Proof** Since each exact sequence in $u(\mathfrak{g})$–$T$-mod is also an exact sequence in $u(\mathfrak{g})$-mod, the desired assertion follows immediately from Proposition 3.6.

By Corollary 3.1, we then obtain the following main theorem on character formulas of the restricted simple $\mathfrak{g}$-modules with atypical weights of type I.

**Theorem 3.4** *The following statements on character formulas of simple modules hold.*

(1) *For $1\leq t\leq p-1$ and $1\leq i\leq n$ with $i$ being odd, we have*

$$\begin{aligned}&\mathrm{ch}(L(t\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m))\\=&\sum_{j=1}^{t}(-1)^{j+t}\Pi\mathrm{ch}(L^0(j\eta_i+\eta_{i+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m))\\&+\sum_{s=1}^{i-1}\sum_{k=1}^{p-1}(-1)^{s+k+i+t}\Pi\mathrm{ch}(L^0(k\eta_s+\eta_{s+1}+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots\\&+(p-1)\varepsilon_m))+(-1)^t i.\end{aligned}$$

(2) *For $1 \le t \le p-1$ and $1 \le i \le n$ with $i$ being even, we have*

$$\begin{aligned}
&\mathrm{ch}(L(t\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&= \sum_{j=1}^{t} (-1)^{j+t} \Pi\mathrm{ch}(L^0(j\eta_i + \eta_{i+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad + \sum_{s=1}^{i-1}\sum_{k=1}^{p-1} (-1)^{s+k+i+t-1} \Pi\mathrm{ch}(L^0(k\eta_s + \eta_{s+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots \\
&\quad + (p-1)\varepsilon_m)) + (-1)^t i.
\end{aligned}$$

**Proof** We use induction on $i$ and $t$ to prove the assertion.

By Corollary 3.1(3), $\mathrm{ch}(L(\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m)) = \mathrm{ch}(K(\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m)) - \mathrm{ch}(L(0)) = \Pi\mathrm{ch}(L^0(\eta_1+\cdots+\eta_n+(p-1)\varepsilon_1+\cdots+(p-1)\varepsilon_m)) - 1$. Hence the statement holds for $i=1$ and $t=1$. For any $1 < l \le p-1$, if the assertion holds for $i=1$ and $t<l$, we then show that it also holds for $i=1$ and $t=l$. For that, we have the following exact sequence by Corollary 3.1(1):

$$\begin{aligned}
0 &\to L((l-1)\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \\
&\to K(l\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \\
&\to L(l\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to 0.
\end{aligned}$$

So

$$\begin{aligned}
&\mathrm{ch}(L(l\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&= \mathrm{ch}(K(l\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - \mathrm{ch}(L((l-1)\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&= \Pi\mathrm{ch}(L^0(l\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - \Big(\sum_{j=1}^{l-1} (-1)^{j+l-1} \Pi\mathrm{ch}(L^0(j\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) + (-1)^{l-1}\Big) \\
&= \sum_{j=1}^{l} (-1)^{j+l} \Pi\mathrm{ch}(L^0(j\eta_1 + \eta_2 + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) + (-1)^l,
\end{aligned}$$

which implies that the assertion also holds for $i=1$ and $t=l$. Therefore, we have proved that the assertion holds for $i=1$ and $1 \le t \le p-1$.

Furthermore, for any $1 < i' \le n$ with $i'$ being even, if the assertion holds for $i < i'$ and $1 \le t \le p-1$, next we will verify that it also holds for $i = i'$ and $1 \le t \le p-1$. Indeed, we have the following two exact sequences by Corollary 3.1(2):

$$\begin{aligned}
&0 \to J(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to K(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \\
&\quad \to L(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to 0,
\end{aligned}$$

and

$$\begin{aligned}
0 &\to L((p-1)\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \\
&\to J(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to L(0) \to 0.
\end{aligned}$$

Hence

$$\mathrm{ch}(K(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))$$

$$\begin{aligned}&= \mathrm{ch}(J(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad + \mathrm{ch}(L(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{ch}(J(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &= \mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) + \mathrm{ch}(L(0))\\ &= \mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) + 1.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathrm{ch}(L(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &= \mathrm{ch}(K(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad - \mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) - 1\\ &= \Pi\mathrm{ch}(L^0(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad - \mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_j + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) - 1.\end{aligned}$$

By the inductive hypotheses,

$$\begin{aligned}&\mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &= \sum_{j=1}^{p-1}(-1)^j\Pi\mathrm{ch}(L^0(j\eta_{i'-1} + \eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad + \sum_{s=1}^{i'-2}\sum_{k=1}^{p-1}(-1)^{s+k+i'-1}\Pi\mathrm{ch}(L^0(k\eta_s + \eta_{s+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) + (i'-1),\end{aligned}$$

so that

$$\begin{aligned}&\mathrm{ch}(L(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &= \Pi\mathrm{ch}(L^0(\eta_{i'} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad - \mathrm{ch}(L((p-1)\eta_{i'-1} + \eta_j + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) - 1\\ &= \Pi\mathrm{ch}(L^0(\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))\\ &\quad + \sum_{s=1}^{i'-1}\sum_{k=1}^{p-1}(-1)^{s+k+i'}\Pi\mathrm{ch}(L^0(k\eta_s + \eta_{s+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) - i',\end{aligned}$$

which implies that the assertion holds for $i = i'$ and $t = 1$.

For any $1 < l \leq p-1$, if the assertion holds for $i = i'$ and $t < l$, we then show that it also holds for $i = i'$ and $t = l$. Indeed, we have the following exact sequence by Corollary 3.1(1):

$$\begin{aligned}&0 \to L((l-1)\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to K(l\eta_{i'} + \eta_{i'+1}\\ &\quad + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m) \to L(l\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1\\ &\quad + \cdots + (p-1)\varepsilon_m) \to 0.\end{aligned}$$

So

$$\mathrm{ch}(L(l\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m))$$

$$
\begin{aligned}
&= \mathrm{ch}(K(l\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - \mathrm{ch}(L((l-1)\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&= \Pi\mathrm{ch}(L^0(l\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - \sum_{j=1}^{l-1}(-1)^{j+l-1}\Pi\mathrm{ch}(L^0(j\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - \sum_{s=1}^{i'-1}\sum_{k=1}^{p-1}(-1)^{s+k+i'+l}\Pi\mathrm{ch}(L^0(k\eta_s + \eta_{s+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad - (-1)^{l-1}i' \\
&= \sum_{j=1}^{l}(-1)^{j+l}\Pi\mathrm{ch}(L^0(j\eta_{i'} + \eta_{i'+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad + \sum_{s=1}^{i'-1}\sum_{k=1}^{p-1}(-1)^{s+k+i'+l-1}\Pi\mathrm{ch}(L^0(k\eta_s + \eta_{s+1} + \cdots + \eta_n + (p-1)\varepsilon_1 + \cdots + (p-1)\varepsilon_m)) \\
&\quad + (-1)^l i',
\end{aligned}
$$

which implies that the assertion also holds for $i = i'$ and $t = l$.

For any $1 < i' \leq n$ with $i'$ being odd, if the assertion holds for $i < i'$ and $1 \leq t \leq p-1$, one can verify that it also holds for $i = i'$ and $1 \leq t \leq p-1$ by using similar arguments as above.

Summing up, according to the induction principal, we complete the proof.

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