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Boundedness of Singular Integral Operators on Herz-Morrey Spaces with Variable Exponent*

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Abstract Let $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > 1)$ be a homogeneous function of degree zero and b be a BMO function or Lipschitz function. In this paper, the authors obtain some boundedness of the Calderón-Zygmund singular integral operator T_Ω and its commutator $[b, T_\Omega]$ on Herz-Morrey spaces with variable exponent.

Keywords Calderón-Zygmund singular integral, Commutator, Herz-Morrey space, Variable exponent
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1 Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [7] appeared in 1991. In [12–17], Tan, Wang et al. studied the boundedness of some integral operators on variable exponent spaces, respectively.

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x \le 1\Big\}.$$

These spaces are referred to as variable L^p spaces, since they generalize the standard L^p spaces: If $p(x) = p$ is constant, $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

Define $\mathcal{P}^0(E)$ to be the set of $p(\cdot) : E \to (0, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 0, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty.$$

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Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot):\Omega\to[1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x\in\Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x\in\Omega\} < \infty.$$

Denote $p'(x)=\frac{p(x)}{p(x)-1}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1 (cf. [2]) *If* $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *satisfying*

$$|p(x)-p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y|\le\frac{1}{2} \tag{1.1}$$

and

$$|p(x)-p(y)| \le \frac{C}{\log(|x|+\mathrm{e})}, \quad |y|\ge|x|, \tag{1.2}$$

then $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$*, that is, the Hardy-Littlewood maximal operator* M *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 (cf. [7]) *Let* $p(\cdot)\in\mathcal{P}(\Omega)$. *If* $f\in L^{p(\cdot)}(\Omega)$ *and* $g\in L^{p'(\cdot)}(\Omega)$*, then* fg *is integrable on* Ω *and*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \le r_p\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1+\frac{1}{p^-}-\frac{1}{p^+}.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 (cf. [4]) *Let* $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S\subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2},$$

where δ_1,δ_2 *are constants with* $0<\delta_1,\delta_2<1$ *and* χ_S *and* χ_B *are the characteristic functions of* S *and* B*, respectively.*

Throughout this paper, δ_2 is the same as in Lemma 1.3.

Lemma 1.4 (cf. [4]) *Suppose* $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. *Then there exists a constant* $C>0$ *such that for all balls* B *in* $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

In a way similar to the method of [5], we will give the definition of the Herz-Morrey spaces with variable exponent. Let $B_k=\{x\in\mathbb{R}^n : |x|\le 2^k\}$ and $A_k=B_k\setminus B_{k-1}$ for $k\in\mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k=\chi_{A_k}$ for $k\in\mathbb{Z}$, $\widetilde{\chi}_k=\chi_k$ if $k\in\mathbb{Z}_+$ and $\widetilde{\chi}_0=\chi_{B_0}$.

Definition 1.1 *Let* $\alpha\in\mathbb{R}$, $0<p\le\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *and* $0\le\lambda<\infty$. *The homogeneous Herz-Morrey space with variable exponent* $M\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *is defined by*

$$M\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f\in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\}) : \|f\|_{M\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{1}{p}}.$$

The non-homogeneous Herz-Morrey space with variable exponent $MK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$MK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{MK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{MK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}_+} 2^{-L\lambda}\Big\{\sum_{k=0}^{L} 2^{k\alpha p}\|f\widetilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{1}{p}}.$$

Remark 1.1 If $\lambda = 0$, then

$$M\dot{K}^{\alpha,0}_{q(\cdot),p}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$$

and

$$MK^{\alpha,0}_{q(\cdot),p}(\mathbb{R}^n) = K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n),$$

where $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ are the Herz spaces with variable exponent.

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s > 1$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1.3}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$. The Calderón-Zygmund singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y.$$

Let b be a locally integrable function on $\mathbb{R}^n$. The commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator T_Ω and b is defined by

$$[b, T_\Omega]f(x) = \mathrm{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}[b(x) - b(y)]f(y)\mathrm{d}y.$$

Motivated by [9, 13, 16], we will study the boundedness of the Calderón-Zygmund singular integral operator T_Ω and its commutator $[b, T_\Omega]$ on Herz-Morrey spaces with variable exponent.

2 Boundedness of the Calderón-Zygmund Singular Integral Operator

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to A_p $(1 < p < \infty)$, if there is a constant $C > 0$ such that

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\mathrm{d}x\Big)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$.

The weighted (L^p, L^p) boundedness of T_Ω was proved by Lu, Ding and Yan [8].

Lemma 2.1 (cf. [8]) *Suppose that $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s>1)$ is a homogeneous function of degree zero and satisfies* (1.3). *If $\omega \in A_{\frac{p}{s'}}$, $s' \le p < \infty$, then there is a constant C independent of f, such that*

$$\int_{\mathbb{R}^n} |T_\Omega(f)(x)|^p \omega(x)\mathrm{d}x \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x.$$

Lemma 2.2 (cf. [1]) *Given a family $\mathcal{F}$ and an open set $E \subset \mathbb{R}^n$, assume that for some p_0, $0 < p_0 < \infty$ and for every $\omega \in A_\infty$,*

$$\int_E f(x)^{p_0}\omega(x)\mathrm{d}x \le C_0 \int_E g(x)^{p_0}\omega(x)\mathrm{d}x, \quad (f,g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0(E)$ such that $p(\cdot)$ satisfies (1.1)–(1.2) *in Lemma* 1.1. *Then for all $(f,g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(E)$,*

$$\|f\|_{L^{p(\cdot)}(E)} \le C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{\frac{p}{s'}} \subset A_\infty$, by Lemmas 2.1–2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the operator T_Ω.

Next, we will give the corresponding result about the operator T_Ω on Herz-Morrey spaces with variable exponent.

Theorem 2.1 *Suppose that $0 < \nu \le 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions* (1.1)–(1.2) *in Lemma* 1.1, *$\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > q'^-)$. Let $0 < p_1 \le p_2 < \infty$ and $0 < \lambda < \alpha < n\delta_2 - \nu - \frac{n}{s}$ $\big($or $0 < \lambda < \alpha_2 \le \alpha_1 < n\delta_2 - \nu - \frac{n}{s}\big)$. Then T_Ω is bounded from $M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ (or $MK^{\alpha_1,p_1}_{q(\cdot)}(\mathbb{R}^n)$) to $M\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ (or $MK^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$).*

In the proof of Theorem 2.1, we also need the following lemmas.

Lemma 2.3 (cf. [11]) *Define a variable exponent $\widetilde{q}(\cdot)$ by $\frac{1}{p(x)} = \frac{1}{\widetilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{\widetilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

for all measurable functions f and g.

Lemma 2.4 (cf. [3]) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions* (1.1)–(1.2) *in Lemma* 1.1. *Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}}, & \text{if } |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}}, & \text{if } |Q| \ge 1 \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim\limits_{x\to\infty} p(x)$.

Lemma 2.5 (cf. [10]) *If $a > 0$, $1 \le s \le \infty$, $0 < d \le s$ and $-n + \frac{(n-1)d}{s} < \nu < \infty$, then*

$$\Big(\int_{|y|\le a|x|} |y|^\nu |\Omega(x-y)|^d \mathrm{d}y\Big)^{\frac{1}{d}} \le C|x|^{\frac{\nu+n}{d}}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}.$$

Proof of Theorem 2.1 We only prove the homogeneous case. In a way similar to the method of [18], it is easy to prove that $MK^{\alpha_1,p_2}_{q(\cdot)}(\mathbb{R}^n) \subset MK^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$ for $0 < \alpha_2 \le \alpha_1$. So the non-homogeneous case can be proved in the same way. Let $f \in M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$. Then we have $f(x) = \sum\limits_{j=-\infty}^{\infty} f_j(x)$. Noting that $p_1 \le p_2$, we have

$$\|T_\Omega(f)\|^{p_1}_{M\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha p_2}\|T_\Omega(f)\chi_k\|^{p_2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{p_1}{p_2}}$$

$$\begin{aligned}
&\le \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{k\alpha p_1} \|T_\Omega(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{k\alpha p_1} \Big(\sum_{j=-\infty}^{k-2} \|T_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{k\alpha p_1} \Big(\sum_{j=k-1}^{\infty} \|T_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned} \tag{2.1}$$

We first estimate I_1. For each $k \in \mathbb{Z}$, $j \le k-2$ and a.e. $x \in A_k$, using the generalized Hölder inequality we have

$$\begin{aligned}
|T_\Omega(f_j)(x)| &\le C \int_{B_j} \frac{|\Omega(x-y)|}{|x-y|^n} |f_j(y)|\mathrm{d}y \\
&\le C 2^{-kn} \int_{B_j} |\Omega(x-y)||f_j(y)|\mathrm{d}y \\
&\le C 2^{-kn} \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Noting $s > q'^{-}$, we denote $\widetilde{q}'(\cdot) > 1$ and $\frac{1}{q'(x)} = \frac{1}{\widetilde{q}'(x)} + \frac{1}{s}$. By Lemmas 2.3 and 2.5, we have

$$\begin{aligned}
&\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\le \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)} \|\chi_j(\cdot)\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \\
&\le \|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{-j\nu} \Big(\int_{A_j} |\Omega(x-y)|^s |y|^{s\nu} \mathrm{d}y \Big)^{\frac{1}{s}} \|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{-j\nu} 2^{k(\nu+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_j| \le 2^n$ and $x_j \in B_j$, by Lemma 2.4 we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{\widetilde{q}'(x_j)}} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}}.$$

When $|B_j| \ge 1$ we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{\widetilde{q}'(\infty)}} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}}$.

By Lemmas 1.3–1.4 we have

$$\begin{aligned}
&\|T_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{-kn} 2^{-j\nu} 2^{k(\nu+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{-kn} 2^{-j\nu} 2^{k(\nu+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= C 2^{-kn+(k-j)(\nu+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{(k-j)(\nu+\frac{n}{s})} \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}
\end{aligned}$$

$$\leq C2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{2.2}$$

So we have

$$\begin{aligned}\mathrm{I}_1 &= C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\|T_\Omega(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}.\end{aligned}$$

When $1<p_1<\infty$, take $\frac{1}{p_1}+\frac{1}{p_1'}=1$. Since $n\delta_2-\nu-\frac{n}{s}-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}\mathrm{I}_1 &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\ &\quad\times\Big(\sum_{j=-\infty}^{k-2}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1'}\Big)^{\frac{p_1}{p_1'}}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{L}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.3}$$

When $0<p_1\leq 1$, we have

$$\begin{aligned}\mathrm{I}_1 &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &= C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{L}2^{(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|\Omega\|^{p_1}_{L^s(\mathrm{S}^{n-1})}\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.4}$$

Next we estimate I_2. By the $(L^{q(\cdot)}(\mathbb{R}^n),L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator T_Ω we have

$$\begin{aligned}\mathrm{I}_2 &= C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|T_{\Omega,\sigma}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\ &\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\end{aligned}$$

$$= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \Big(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha} 2^{j\alpha} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{p_1}.$$

If $0 < p_1 \le 1$, then we have

$$\begin{aligned}
\mathrm{I}_2 &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&=: \mathrm{I}_{21} + \mathrm{I}_{22}.
\end{aligned} \tag{2.5}$$

For I_{21}, we have

$$\begin{aligned}
\mathrm{I}_{21} &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{2.6}$$

For I_{22}, by $0 < \lambda < \alpha$ we have

$$\begin{aligned}
\mathrm{I}_{22} &= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} 2^{-j\lambda p_1} \Big(\sum_{m=-\infty}^{j} 2^{m\alpha p_1} \|f_m\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big) \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \\
&= C \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{k\alpha p_1} \sum_{j=L}^{\infty} 2^{j(\lambda-\alpha)p_1} \\
&\le C \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1} 2^{L(\lambda-\alpha)p_1} \\
&= C \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{2.7}$$

If $1 < p_1 < \infty$, noting $\lambda < \alpha$, we can take a constant $\eta > 1$ so that $\lambda - \frac{\alpha}{\eta} < 0$. By the Hölder inequality we have

$$\begin{aligned}
\mathrm{I}_2 &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \Big(\sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \Big) \Big(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1' \frac{\eta-1}{\eta}} \Big)^{\frac{p_1}{p_1'}} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=k-1}^{L-1}2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}$$
$$+C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty}2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}$$
$$=: \mathrm{I}_{23}+\mathrm{I}_{24}. \tag{2.8}$$

For I_{23}, we have

$$\begin{aligned}\mathrm{I}_{23}&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=-\infty}^{j+1}2^{\frac{(k-j)\alpha p_1}{\eta}}\\&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\&\leq C\|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}.\end{aligned}\tag{2.9}$$

For I_{24}, by $0<\lambda<\frac{\alpha}{\eta}$ we have

$$\begin{aligned}\mathrm{I}_{24}&=C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty}2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty}2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\lambda p_1}\|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\\&=C\|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{\frac{k\alpha p_1}{\eta}}\sum_{j=L}^{\infty}2^{j(\lambda-\frac{\alpha}{\eta})p_1}\\&\leq C\|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}2^{\frac{L\alpha p_1}{\eta}}2^{L(\lambda-\frac{\alpha}{\eta})p_1}\\&=C\|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}.\end{aligned}\tag{2.10}$$

Thus, by (2.1) and (2.3)–(2.10) we complete the proof of Theorem 2.1.

3 BMO Boundedness for the Commutator of Calderón-Zygmund Singular Integral Operator

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_*=\sup_Q\frac{1}{|Q|}\int_Q|f(x)-f_Q|\mathrm{d}x<\infty,$$

where $f_Q=|Q|^{-1}\int_Q f(y)\mathrm{d}y$, the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denoting the Lebesgue measure of Q.

Lemma 3.1 (cf. [6]) *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, k be a positive integer and B be a ball in $\mathbb{R}^n$. Then we have that for all $b\in\mathrm{BMO}(\mathbb{R}^n)$ and all $j,i\in\mathbb{Z}$ with $j>i$,*

$$\frac{1}{C}\|b\|_*^k\leq\sup_B\frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^k\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C\|b\|_*^k,$$

$$\|(b-b_{B_i})^k\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^k\|b\|_*^k\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \le 2^i\}$ *and* $B_j = \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p, L^p) boundedness of $[b, T_\Omega]$ was proved by Lu, Ding and Yan [8].

Lemma 3.2 (cf. [8]) *Suppose that* $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > 1)$ *is a homogeneous function of degree zero and satisfies* (1.3). *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $\omega \in A_{\frac{p}{s'}}$, $s' \le p < \infty$, *then there is a constant* C *independent of* f, *such that*

$$\int_{\mathbb{R}^n} |[b, T_\Omega](f)(x)|^p\omega(x)\mathrm{d}x \le C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)\mathrm{d}x.$$

Since $A_{\frac{p}{s'}} \subset A_\infty$, by Lemmas 3.2 and 2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_\Omega]$.

Next, we will give the corresponding result about the commutator $[b, T_\Omega]$ on Herz-Morrey spaces with variable exponent.

Theorem 3.1 *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \nu \le 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1)–(1.2) *in Lemma* 1.1 *and* $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > q'^-)$. *Let* $0 < p_1 \le p_2 < \infty$ *and* $0 < \lambda < \alpha < n\delta_2 - \nu - \frac{n}{s}$ $\big($*or* $0 < \lambda < \alpha_2 \le \alpha_1 < n\delta_2 - \nu - \frac{n}{s}\big)$. *Then* $[b, T_\Omega]$ *is bounded from* $M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ (*or* $MK^{\alpha_1,p_1}_{q(\cdot)}(\mathbb{R}^n)$) *to* $M\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ (*or* $MK^{\alpha_2,p_2}_{q(\cdot)}(\mathbb{R}^n)$).

Proof In a way similar to Theorem 2.1, we only prove the homogeneous case. Let $f \in M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Denote $f_j = f\chi_j$ for each $j \in \mathbb{Z}$. Then we have $f(x) = \sum\limits_{j=-\infty}^{\infty} f_j(x)$. Noting that $p_1 \le p_2$, we have

$$\begin{aligned}
\|[b, T_\Omega](f)\|^{p_1}_{M\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\Big\{\sum_{k=-\infty}^{L} 2^{k\alpha p_2}\|[b, T_\Omega](f)\chi_k\|^{p_2}_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{p_1}{p_2}} \\
&\le \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\|[b, T_\Omega](f)\chi_k\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\|[b, T_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|[b, T_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \\
&=: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned} \tag{3.1}$$

We first estimate J_1. For each $k \in \mathbb{Z}$, $j \le k-2$ and *a.e.* $x \in A_k$, using the generalized Hölder inequality we have

$$\begin{aligned}
|[b, T_\Omega](f_j)(x)| &\le C\int_{B_j} \frac{|\Omega(x-y)|}{|x-y|^n}|b(x)-b(y)||f_j(y)|\mathrm{d}y \\
&\le C2^{-kn}\int_{B_j} |\Omega(x-y)||b(x)-b(y)||f_j(y)|\mathrm{d}y \\
&\le C2^{-kn}\Big(|b(x)-b_{B_j}|\int_{B_j} |\Omega(x-y)||f_j(y)|\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_j}|\Omega(x-y)||b_{B_j}-b(y)||f_j(y)|\mathrm{d}y\Big)\\
&\le C2^{-kn}(|b(x)-b_{B_j}|\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}).
\end{aligned}
$$

Noting $s>q'^{-}$, we denote $\widetilde{q}'(\cdot)>1$ and $\frac{1}{q'(x)}=\frac{1}{\widetilde{q}'(x)}+\frac{1}{s}$. By Lemmas 2.3 and 2.5 we have

$$
\begin{aligned}
\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}&\le\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)}\|\chi_j(\cdot)\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-j\nu}\Big(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}\mathrm{d}y\Big)^{\frac{1}{s}}\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-j\nu}2^{k(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

When $|B_j|\le 2^n$ and $x_j\in B_j$, by Lemma 2.4 we have

$$
\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_j|^{\frac{1}{\widetilde{q}'(x_j)}}\approx\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}.
$$

When $|B_j|\ge 1$ we have

$$
\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_j|^{\frac{1}{\widetilde{q}'(\infty)}}\approx\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}.
$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\approx\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}$.

So we have

$$
\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\le C2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{3.2}
$$

Similarly, by Lemma 3.1 we have

$$
\begin{aligned}
&\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\le\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)}\|(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\|\chi_{B_j}\|_{L^{\widetilde{q}'(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{s}(\mathbb{R}^n)}\\
&\le C\|b\|_*2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.3}
$$

By (3.2)–(3.3), Lemmas 1.3–1.4 and 3.1, we have

$$
\begin{aligned}
&\|[b,T_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}(2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-b_{B_j})\chi_k(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|b\|_*2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)})\\
&\le C2^{-kn}((k-j)\|b\|_*2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|b\|_*2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)})\\
&\le C(k-j)\|b\|_*2^{-kn}2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C(k-j)\|b\|_*2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C\|b\|_*(k-j)2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.4}
$$

So we have

$$\begin{aligned}
\mathrm{J}_1 &= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}(k-j)2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}.
\end{aligned}$$

When $1<p_1<\infty$, take $\frac{1}{p_1}+\frac{1}{p_1'}=1$. Since $n\delta_2-\nu-\frac{n}{s}-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}
\mathrm{J}_1 &\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac12(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\Big)\\
&\quad\times\Big(\sum_{j=-\infty}^{k-2}2^{\frac12(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1'}(k-j)^{p_1'}\Big)^{\frac{p_1}{p_1'}}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac12(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=j+2}^{L}2^{\frac12(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\|f\|_{M\dot K_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (3.5)
\end{aligned}$$

When $0<p_1\le 1$, we have

$$\begin{aligned}
\mathrm{J}_1 &\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})p_1}(k-j)^{p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&= C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=j+2}^{L}2^{(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}(k-j)^{p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\|b\|_*^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\|f\|_{M\dot K_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (3.6)
\end{aligned}$$

Next we estimate J_2. By the $(L^{q(\cdot)}(\mathbb{R}^n),L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b,T_\Omega]$, we have

$$\begin{aligned}
\mathrm{J}_2 &= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=k-1}^{\infty}2^{(k-j)\alpha}2^{j\alpha}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}.
\end{aligned}$$

If $0 < p_1 \le 1$, then we have

$$\begin{aligned}
\mathrm{J}_2 &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&=: \mathrm{J}_{21} + \mathrm{J}_{22}. \qquad (3.7)
\end{aligned}$$

For J_{21}, we have

$$\begin{aligned}
\mathrm{J}_{21} &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\le C \|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (3.8)
\end{aligned}$$

For J_{22}, by $0 < \lambda < \alpha$ we have

$$\begin{aligned}
\mathrm{J}_{22} &= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} 2^{-j\lambda p_1} \Big(\sum_{m=-\infty}^{j} 2^{m\alpha p_1} \|f_m\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \Big) \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} \|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1} \\
&= C \|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{k\alpha p_1} \sum_{j=L}^{\infty} 2^{j(\lambda-\alpha)p_1} \\
&\le C \|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1} 2^{L(\lambda-\alpha)p_1} \\
&= C \|f\|_{M\dot{K}_{q(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (3.9)
\end{aligned}$$

If $1 < p_1 < \infty$, noting $\lambda < \alpha$, we can take a constant $\eta > 1$ so that $\lambda - \frac{\alpha}{\eta} < 0$. By the Hölder inequality we have

$$\begin{aligned}
\mathrm{J}_2 &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \Big(\sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \Big) \Big(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1' \frac{\eta-1}{\eta}} \Big)^{\frac{p_1}{p_1'}} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_1}
\end{aligned}$$

$$+ C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$=: \mathrm{J}_{23} + \mathrm{J}_{24}. \tag{3.10}$$

For J_{23}, we have

$$\begin{aligned}
\mathrm{J}_{23} &\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+1} 2^{\frac{(k-j)\alpha p_1}{\eta}} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.11}$$

For J_{24}, by $0 < \lambda < \frac{\alpha}{\eta}$ we have

$$\begin{aligned}
\mathrm{J}_{24} &= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\alpha p_1} \|f_j\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}} 2^{j\lambda p_1} \|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \\
&= C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{\frac{k\alpha p_1}{\eta}} \sum_{j=L}^{\infty} 2^{j(\lambda-\frac{\alpha}{\eta})p_1} \\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} 2^{\frac{L\alpha p_1}{\eta}} 2^{L(\lambda-\frac{\alpha}{\eta})p_1} \\
&= C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.12}$$

Thus, by (3.1) and (3.5)–(3.12) we complete the proof of Theorem 3.1.

4 Lipschitz Boundedness for the Commutator of Calderón-Zygmund Singular Integral Operator

For $0 < \gamma \le 1$, the Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\gamma(\mathbb{R}^n) = \Big\{ f : \|f\|_{\mathrm{Lip}_\gamma} = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty \Big\}.$$

Let $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. It is easy to know that $|[b, T_\Omega]| \le C\|b\|_{\mathrm{Lip}_\gamma} |T_{\Omega,\gamma}|$, where

$$T_{\Omega,\gamma} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\gamma}} f(y)\mathrm{d}y.$$

Denote $T_\gamma = T_{\Omega,\gamma}$ when $\Omega \equiv 1$. In [12], the authors proved that $T_{\Omega,\gamma}$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\gamma}{n}$ and $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (1.1)–(1.2) in Lemma 1.1 with $q_1^+ < \frac{n}{\gamma}$. So we can get the following theorem.

Theorem 4.1 *Suppose that* $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ *with* $0 < \gamma \le 1$. *If* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1)–(1.2) *in Lemma* 1.1 *with* $q_1^+ < \frac{n}{\gamma}$, $\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\gamma}{n}$, $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > {q_1'}^-)$, *then* $[b, T_\Omega]$ *is bounded from* $L^{q_1(\cdot)}(\mathbb{R}^n)$ *to* $L^{q_2(\cdot)}(\mathbb{R}^n)$.

Next, we will give the Lipschitz estimate about the commutator $[b,T_\Omega]$ on Herz-Morrey spaces with variable exponent.

Theorem 4.2 *Suppose that* $b\in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ *with* $0<\gamma\le 1$, $0<\nu\le 1$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1)–(1.2) *in Lemma* 1.1 *with* $q_1^+<\frac{n}{\gamma}$, $\frac{1}{q_1(x)}-\frac{1}{q_2(x)}=\frac{\gamma}{n}$, $\Omega\in L^s(\mathrm{S}^{n-1})$ $(s>{q_1'}^-)$. *Let* $0<p_1\le p_2<\infty$ *and* $0<\lambda<\alpha<n\delta_2-\nu-\frac{n}{s}$ $\big($*or* $0<\lambda<\alpha_2\le\alpha_1<n\delta_2-\nu-\frac{n}{s}\big)$. *Then* $[b,T_\Omega]$ *is bounded from* $M\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (*or* $MK^{\alpha_1,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) *to* $M\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ (*or* $MK^{\alpha_2,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).

Proof In a way similar to Theorem 2.1, we only prove the homogeneous case. Let $f\in M\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ and $b\in\mathrm{Lip}_\gamma(\mathbb{R}^n)$. Denote $f_j=f\chi_j$ for each $j\in\mathbb{Z}$. Then we have $f(x)=\sum\limits_{j=-\infty}^{\infty}f_j(x)$. Noting that $p_1\le p_2$, we have

$$\begin{aligned}
\|[b,T_\Omega](f)\|^{p_1}_{M\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)}&=\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\Big\{\sum_{k=-\infty}^{L}2^{k\alpha p_2}\|[b,T_\Omega](f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big\}^{\frac{p_1}{p_2}}\\
&\le\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\|[b,T_\Omega](f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&\quad+C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&=:\mathrm{U}_1+\mathrm{U}_2. \qquad (4.1)
\end{aligned}$$

We first estimate U_1. For each $k\in\mathbb{Z}$, $j\le k-2$ and a.e. $x\in A_k$, we have $|x-y|\sim|x|$. Using the generalized Hölder inequality, we have

$$\begin{aligned}
|[b,T_\Omega](f_j)(x)|&\le C\int_{B_j}\frac{|\Omega(x-y)|}{|x-y|^n}|b(x)-b(y)||f_j(y)|\mathrm{d}y\\
&\le C\|b\|_{\mathrm{Lip}_\gamma}\int_{B_j}\frac{|\Omega(x-y)|}{|x-y|^{n-\gamma}}|f_j(y)|\mathrm{d}y\\
&\le C\|b\|_{\mathrm{Lip}_\gamma}2^{-kn+k\gamma}\int_{B_j}|\Omega(x-y)||f_j(y)|\mathrm{d}y\\
&\le C\|b\|_{\mathrm{Lip}_\gamma}2^{-kn+k\gamma}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Noting $s>{q_1'}^-$, we denote $\widetilde{q'}_1(\cdot)>1$ and $\frac{1}{q_1'(x)}=\frac{1}{\widetilde{q'}_1(x)}+\frac{1}{s}$. By Lemmas 2.3 and 2.5 we have

$$\begin{aligned}
\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}&\le\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_j(\cdot)\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)}\\
&\le\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-j\nu}\Big(\int_{A_j}|\Omega(x-y)|^s|y|^{s\nu}\mathrm{d}y\Big)^{\frac{1}{s}}\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-j\nu}2^{k(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_j|\le 2^n$ and $x_j\in B_j$, by Lemma 2.4 we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)}\approx|B_j|^{\overline{\frac{1}{\widetilde{q'}_1(x_j)}}}\approx\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}.$$

When $|B_j| \geq 1$ we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{\widetilde{q'}_1(\infty)}} \approx \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}.$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q'}_1(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}$.

So we have

$$\|\Omega(x-\cdot)\chi_j(\cdot)\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \leq C2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}. \tag{4.2}$$

Since

$$T_\gamma(\chi_{B_k})(x) \geq \int_{B_k} \frac{\mathrm{d}y}{|x-y|^{n-\gamma}}\chi_{B_k}(x) \geq C2^{k\gamma}\chi_{B_k}(x), \tag{4.3}$$

by (4.2)–(4.3) and Lemmas 1.3–1.4, we have

$$\begin{aligned}
&\|[b,T_\Omega](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}2^{-kn+k\gamma}2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}2^{-kn+(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}\|T_\gamma(\chi_{B_k})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}2^{-kn+(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)(\nu+\frac{n}{s})}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\frac{\|\chi_{B_j}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.4}$$

So we have

$$\begin{aligned}
\mathrm{U}_1 &= C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}.
\end{aligned}$$

When $1 < p_1 < \infty$, take $\frac{1}{p_1}+\frac{1}{p'_1}=1$. Since $n\delta_2-\nu-\frac{n}{s}-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}
\mathrm{U}_1 &\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\Big) \\
&\quad\times\Big(\sum_{j=-\infty}^{k-2}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p'_1}\Big)^{\frac{p_1}{p'_1}} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{j\alpha p_1}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=j+2}^{L}2^{\frac{1}{2}(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}
\end{aligned}$$

$$\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \tag{4.5}$$

When $0 < p_1 \leq 1$, we have

$$\begin{aligned}
\mathrm{U}_1 &\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\nu-\frac{n}{s})p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=j+2}^{L}2^{(j-k)(n\delta_2-\nu-\frac{n}{s}-\alpha)p_1}\\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-2}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}^{p_1}\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}\tag{4.6}$$

Next we estimate U_2. By the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_\Omega]$ we have

$$\begin{aligned}
\mathrm{U}_2 &= C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|[b,T_\Omega](f_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&= C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=k-1}^{\infty}2^{(k-j)\alpha}2^{j\alpha}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}.
\end{aligned}$$

If $0 < p_1 \leq 1$, then we have

$$\begin{aligned}
\mathrm{U}_2 &\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=k-1}^{\infty}2^{(k-j)\alpha p_1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=k-1}^{L-1}2^{(k-j)\alpha p_1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\quad + C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty}2^{(k-j)\alpha p_1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&=: \mathrm{U}_{21}+\mathrm{U}_{22}.
\end{aligned}\tag{4.7}$$

For U_{21}, we have

$$\begin{aligned}
\mathrm{U}_{21} &\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p_1}\\
&\leq C\sup_{L\in\mathbb{Z}}2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\leq C\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}\tag{4.8}$$

For U_{22}, by $0<\lambda<\alpha$ we have

$$\begin{aligned}
U_{22} &= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1}2^{j\lambda p_1}2^{-j\lambda p_1}\Big(\sum_{m=-\infty}^{j} 2^{m\alpha p_1}\|f_m\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\Big)\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1}2^{j\lambda p_1}\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\\
&= C\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}2^{k\alpha p_1}\sum_{j=L}^{\infty}2^{j(\lambda-\alpha)p_1}\\
&\le C\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}2^{L\alpha p_1}2^{L(\lambda-\alpha)p_1}\\
&= C\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (4.9)
\end{aligned}$$

If $1<p_1<\infty$, noting $\lambda<\alpha$, we can take a constant $\eta>1$ so that $\lambda-\frac{\alpha}{\eta}<0$. By the Hölder inequality we have

$$\begin{aligned}
U_2 &\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\Big(\sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\Big)\Big(\sum_{j=k-1}^{\infty}2^{(k-j)\alpha p_1'\frac{\eta-1}{\eta}}\Big)^{\frac{p_1}{p_1'}}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=k-1}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=k-1}^{L-1} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&=: U_{23}+U_{24}. \qquad (4.10)
\end{aligned}$$

For U_{23}, we have

$$\begin{aligned}
U_{23} &\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\sum_{k=-\infty}^{j+1}2^{\frac{(k-j)\alpha p_1}{\eta}}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{j=-\infty}^{L-1}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}. \qquad (4.11)
\end{aligned}$$

For U_{24}, by $0<\lambda<\frac{\alpha}{\eta}$ we have

$$\begin{aligned}
U_{24} &= C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\alpha p_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}\\
&\le C\sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1}\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{\frac{(k-j)\alpha p_1}{\eta}}2^{j\lambda p_1}\|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)}^{p_1}
\end{aligned}$$

$$
\begin{aligned}
&= C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^{L} 2^{\frac{k\alpha p_1}{\eta}} \sum_{j=L}^{\infty} 2^{j(\lambda-\frac{\alpha}{\eta})p_1} \\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda p_1} 2^{\frac{L\alpha p_1}{\eta}} 2^{L(\lambda-\frac{\alpha}{\eta})p_1} \\
&= C\|f\|^{p_1}_{M\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)}. \qquad (4.12)
\end{aligned}
$$

Thus, by (4.1) and (4.5)–(4.12) we complete the proof of Theorem 4.2.

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