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The Initial-Boundary-Value Problems for the Hirota Equation on the Half-Line*

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Abstract An initial boundary-value problem for the Hirota equation on the half-line, $0 < x < \infty$, $t > 0$, is analysed by expressing the solution $q(x,t)$ in terms of the solution of a matrix Riemann-Hilbert (RH) problem in the complex k-plane. This RH problem has explicit (x,t) dependence and it involves certain functions of k referred to as the spectral functions. Some of these functions are defined in terms of the initial condition $q(x,0) = q_0(x)$, while the remaining spectral functions are defined in terms of the boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$ and $q_{xx}(0,t) = g_2(t)$. The spectral functions satisfy an algebraic global relation which characterizes, say, $g_2(t)$ in terms of $\{q_0(x), g_0(t), g_1(t)\}$. The spectral functions are not independent, but related by a compatibility condition, the so-called global relation.

Keywords Hirota equation, Riemann-Hilbert problem, Initial-boundary value problem, Global relation
2000 MR Subject Classification 35Q15, 35Q55

1 Introduction

The Hirota equation reads

$$\mathrm{i}q_t + \alpha(q_{xx} - 2|q|^2 q) + \mathrm{i}\beta(q_{xxx} - 6|q|^2 q_x) = 0, \quad \alpha, \beta \in \mathbb{R}, \tag{1.1}$$

which includes Schrödinger equation and complex MKdV equation as two special cases when $\alpha = 1,\ \beta = 0$ and $\alpha = 0,\ \beta = 1$, respectively. The Hirota equation is a modified nonlinear Schrödinger equation that takes into account higher order dispersion and time-delay corrections to the cubic nonlinearity. In describing wave propagation in the ocean and optical fibers, it can be viewed as an approximation which is more accurate than the nonlinear Schrödinger equation (see [1]). The Hirota equation was studied in a number of papers (see [2–4]) and their references. We shall get the Riemann-Hilbert problem formulation for the initial-boundary value problem of the Hirota equation via the unified method which introduce by Fokas [5].

In 1997, Fokas [5] (also can see [6–7]) announced a new unified approach for the analysis of initial-boundary value problems for linear and nonlinear integrable PDEs. The unified method provides a generalization of the inverse scattering formalism from initial value to initial-

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boundary value problems. This method has been implemented to linear and integrable nonlinear evolutions PDEs on the half-line and the finite interval (see [8–21]).

In this paper, we use Fokas and Lenells method to analyze the initial-boundary value problem for the Hirota equation (1.1) on the half-line domain, that is, in the domain

$$\Omega = \{(x,t) \in \mathbb{R}^2 \mid 0 \le x < \infty, 0 \le t < T\}, \tag{1.2}$$

where $T < \infty$ is a given positive constant. Assuming that a solution exists, we show that $u(x,t)$ can be recovered from the initial and boundary values $u_0(x)$, $g_0(t), g_1(t), g_2(t)$ defined by

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \tag{1.3}$$

$$q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t), \quad q_{xx}(0,t) = g_2(t), \quad 0 < t < T. \tag{1.4}$$

The organization of the paper is as follows. Section 2 contains the spectral analysis of the Lax pair for (1.1). The spectral functions a, b, A, B are further investigated in Section 3 with some proofs postponed to the two appendices in [22]. Finally, the Riemann-Hilbert problem is presented in Section 4.

2 Lax Pair, Eigenfunctions and Spectral Functions

2.1 The closed one-form

The Hirota equation (1.1) admits the following Lax pair representation (see [4]):

$$\psi_x = M\psi, \quad \psi_t = N\psi, \tag{2.1}$$

where

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad M = \begin{pmatrix} -\mathrm{i}k & q \\ r & \mathrm{i}k \end{pmatrix} = -\mathrm{i}k\sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \equiv -\mathrm{i}k\sigma_3 + U, \tag{2.2}$$

$$N = V_3k^3 + (V_2 - 2\mathrm{i}\alpha\sigma_3)k^2 + V_1k + V_0 \equiv -4\mathrm{i}\beta\sigma_3k^3 - 2\mathrm{i}\alpha\sigma_3k^2 + V, \tag{2.3}$$

$$V_3 = \begin{pmatrix} -4\mathrm{i}\beta & 0 \\ 0 & 4\mathrm{i}\beta \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 4q\beta \\ 4\beta r & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -2\mathrm{i}qr\beta & 2\mathrm{i}q_x\beta + 2q\alpha \\ -2\mathrm{i}r_x\beta + 2r\alpha & 2\mathrm{i}qr\beta \end{pmatrix},$$

$$V_0 = \begin{pmatrix} -\mathrm{i}qr\alpha + \beta(-qr_x + q_xr) & \mathrm{i}q_x\alpha - \beta(q_{xx} - 2q^2r) \\ -\mathrm{i}r_x\alpha + \beta(-r_{xx} + 2qr^2) & \mathrm{i}qr\alpha - \beta(-qr_x + q_xr) \end{pmatrix},$$

$$U := \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \tag{2.4}$$

$$V := V_2z^2 + V_1z + V_0, \tag{2.5}$$

where $r = \overline{q}$, $\overline{q}$ denotes complex conjugate of q, the subscripts denote differentiation with respect to the corresponding variables.

Extending the column vector ψ to a 2×2 matrix and letting

$$\psi = \Psi \mathrm{e}^{-\mathrm{i}kx\sigma_3 - (4\mathrm{i}\beta k^3 + 2\mathrm{i}\alpha k^2)t\sigma_3},$$

we obtain an equivalent Lax pair

$$\Psi_x + \mathrm{i}k[\sigma_3, \Psi] = U\Psi, \tag{2.6a}$$

$$\Psi_t + (2\mathrm{i}\alpha k^2 + 4\mathrm{i}\beta k^3)[\sigma_3, \Psi] = V\Psi, \tag{2.6b}$$

where U, V are defined by (2.4) and (2.5), respectively, which can be written in full derivative form

$$\mathrm{d}(\mathrm{e}^{\mathrm{i}(xk+4\beta k^3 t+2\alpha k^2 t)\widehat{\sigma}_3}\Psi(x,t,k)) = \mathrm{e}^{\mathrm{i}(xk+4\beta k^3 t+2\alpha k^2 t)\widehat{\sigma}_3}W(x,t,k)\Psi, \tag{2.7}$$

where

$$\begin{aligned} W(x,t,k) &= U\mathrm{d}x + V\mathrm{d}t \\ &= \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\mathrm{d}x + (V_2k^2 + V_1k + V_0)\mathrm{d}t. \end{aligned} \tag{2.8}$$

2.2 Eigenfunctions and spectral functions

We assume that there exists $q(x,t)$ with sufficient smoothness and decay satisfying (2.1) in $0 < x < \infty,\ 0 < t < T,\ T \neq \infty$.

Define $\mu_n(x,t,k),\ n = 1,2,3$ as 2×2-matrix valued solutions of the integral equations

$$\mu_n(x,t,k) = I + \int_{(x_n,t_n)}^{(x,t)} \mathrm{e}^{-\mathrm{i}(kx+(4\beta k^3+2\alpha k^2)t)\widehat{\sigma}_3}W(y,\tau,k), \tag{2.9}$$

where $(x_1,t_1) = (0,T)$, $(x_2,t_2) = (0,0)$, $(x_3,t_3) = (\infty,t)$ and the paths of integration are chosen to be parallel to the x-and t-axes,

$$\begin{aligned} \mu_1(x,t,k) = I &+ \int_0^x \mathrm{e}^{-\mathrm{i}(k(x-y))\widehat{\sigma}_3}(U\mu_1)(y,t,k)\mathrm{d}y \\ &- \mathrm{e}^{-\mathrm{i}kx\widehat{\sigma}_3}\int_t^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)(t-\tau)\widehat{\sigma}_3}(V\mu_1)(0,\tau,k)\mathrm{d}\tau, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \mu_2(x,t,k) = I &+ \int_0^x \mathrm{e}^{-\mathrm{i}(k(x-y))\widehat{\sigma}_3}(U\mu_2)(y,t,k)\mathrm{d}y \\ &+ \mathrm{e}^{-\mathrm{i}kx\widehat{\sigma}_3}\int_0^t \mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)(t-\tau)\widehat{\sigma}_3}(V\mu_2)(0,\tau,k)\mathrm{d}\tau, \end{aligned} \tag{2.11}$$

$$\mu_3(x,t,k) = I - \int_x^{\infty} \mathrm{e}^{-\mathrm{i}k(x-y)\widehat{\sigma}_3}(U\mu_3)(y,t,k)\mathrm{d}y. \tag{2.12}$$

We note that this choice implies the following inequalities on the contours

$$\begin{aligned} &(x_1,t_1) \to (x,t) : y - x \le 0, \quad \tau - t \ge 0, \\ &(x_2,t_2) \to (x,t) : y - x \le 0, \quad \tau - t \le 0, \\ &(x_3,t_3) \to (x,t) : y - x \ge 0. \end{aligned}$$

Let the columns of a 2×2 matrix μ be denoted as $(\mu^{(1)}, \mu^{(2)})$. Then the columns of the μ_n are analytic and bounded in the following domains in the complex k-plane. We find that the second column of the matrix equation (2.9) involves $\mathrm{e}^{2\mathrm{i}(k(y-x)+(4\beta k^3+2\alpha k^2)(\tau-t))}$, and using the above inequalities it implies that the exponential term of μ_j is bounded in the following regions of the complex k-plane.

Table 1 Domain of analyticity and bounded for eigenfunctions.

Eigenfunction	Domain of analyticity and bounded
$\mu_1^{(1)}(x,t,k)$	$\{k \mid \Im(k) \le 0,\ \Im(4\beta k^3 + 2\alpha k^2) \le 0\} = \mathrm{IV} \cup \mathrm{VI}$
$\mu_1^{(2)}(x,t,k)$	$\{k \mid \Im(k) \ge 0,\ \Im(4\beta k^3 + 2\alpha k^2) \ge 0\}$
$\mu_2^{(1)}(x,t,k)$	$\{k \mid \Im(k) \le 0,\ \Im(4\beta k^3 + 2\alpha k^2) \ge 0\}$
$\mu_2^{(2)}(x,t,k)$	$\{k \mid \Im(k) \ge 0,\ \Im(4\beta k^3 + 2\alpha k^2) \le 0\}$
$\mu_3^{(1)}(x,t,k)$	$\{k \mid \Im(k) \ge 0\}$
$\mu_3^{(2)}(x,t,k)$	$\{k \mid \Im(k) \le 0\}$

Setting $k = k_1 + \mathrm{i}k_2$, $k_1, k_2 \in \mathbb{R}$, then we have

$$\Im(k) = k_2, \tag{2.13}$$

$$\Im(4\beta k^3 + 2\alpha k^2) = k_2(4\alpha k_1 + 12\beta k_1^2 - 4\beta k_2^2). \tag{2.14}$$

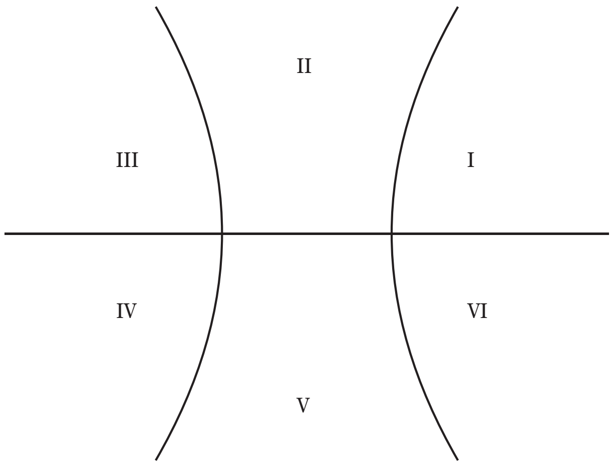

Figure 1 The k-plane.

μ_1 (for $T < \infty$) and μ_2 are the entire functions of k. Thus, in each domain $\mathrm{I}, \cdots, \mathrm{VI}$ (see Figure 1), one has a 2×2-matrix-valued eigenfunction, analytic and bounded, consisting of the appropriate vectors $\mu_n^{(j)}, n = 1,2,3, j = 1,2$ (see Table 1).

For particular values of x or t, the domains of boundedness of the eigenfunctions are larger than above. Particularly, for $t = 0$, the domain of boundedness of μ_2 are as Table 2.

Table 2 Domain of analyticity and bounded for eigenfunctions as $t = 0$.

Eigenfunction	Domain of analyticity and bounded
$\mu_1^{(1)}(x,0,k)$	$\{k \mid \Im(k) \le 0\}$
$\mu_1^{(2)}(x,0,k)$	$\{k \mid \Im(k) \ge 0\}$

For $x = 0$, the domain of the boundedness of μ_1 and μ_2 are as Table 3.

Table 3 Domain of analyticity and bounded for eigenfunctions as $x = 0$.

Eigenfunction	Domain of boundedness
$\mu_1^{(1)}(0,t,k)$	$\{k \mid \Im(4\beta k^3 + 2\alpha k^2) \leq 0\}$
$\mu_1^{(2)}(0,t,k)$	$\{k \mid \Im(4\beta k^3 + 2\alpha k^2) \geq 0\}$
$\mu_2^{(1)}(0,t,k)$	$\{k \mid \Im(4\beta k^3 + 2\alpha k^2) \geq 0\}$
$\mu_2^{(2)}(0,t,k)$	$\{k \mid \Im(4\beta k^3 + 2\alpha k^2) \leq 0\}$

Since the μ_j are solutions of the system of differential equations (2.6a) and (2.6b), they are simply related (in the domains where they are defined)

$$\mu_3(x,t,k) = \mu_2(x,t,k)\mathrm{e}^{-\mathrm{i}(kx+(4\beta k^3+2\alpha k^2)t)\widehat{\sigma}_3}\mu_3(0,0,k), \tag{2.15}$$

$$\begin{aligned}\mu_1(x,t,k) &= \mu_2(x,t,k)\mathrm{e}^{-\mathrm{i}(kx+(4\beta k^3+2\alpha k^2)t)\widehat{\sigma}_3}\mu_1(0,0,k)\\ &= \mu_2(x,t,k)\mathrm{e}^{-\mathrm{i}(kx+(4\beta k^3+2\alpha k^2)t)\widehat{\sigma}_3}(\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)T\widehat{\sigma}_3}\mu_2(0,T,k))^{-1}.\end{aligned} \tag{2.16}$$

Introduce the spectral (2×2-matrix-valued) functions

$$s(k) := \mu_3(0,0,k), \tag{2.17}$$

$$S(k) = S(k;T) := \mu_1(0,0,k) = (\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)T\widehat{\sigma}_3}\mu_2(0,T,k))^{-1}. \tag{2.18}$$

In what follows, $s(k)$ and $S(k)$ will be used to construct a matrix RH problem (more precisely, a family of RH problems parameterized by (x,t)), the solution of which gives the eigenfunctions and hence $q(x,t)$, the solution of the Hirota equation. On the other hand, from (2.12) and (2.17)–(2.18), it follows that $s(k)$ is determined by the initial values of $q(x,t)$, whereas $S(k)$ is determined by the boundary values, namely,

$$s(k) = I - \int_0^\infty \mathrm{e}^{\mathrm{i}ky\widehat{\sigma}_3}(U\mu_3)(y,0,k), \tag{2.19}$$

$$S(k;T) = \Big(I + \int_0^{\mathrm{T}} \mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)\tau\widehat{\sigma}_3}(V\mu_2)(0,\tau,k)\mathrm{d}\tau\Big)^{-1}, \tag{2.20}$$

where $\mu_l(x,0,k),\ l = 1,2,3$ are the solutions of the integral equations

$$\mu_1(0,t,k) = I - \int_t^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)(t-\tau)\widehat{\sigma}_3}(V\mu_1)(0,\tau,k)\mathrm{d}\tau, \tag{2.21}$$

$$\mu_2(0,t,k) = I - \int_0^t \mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)(t-\tau)\widehat{\sigma}_3}(V\mu_2)(0,\tau,k)\mathrm{d}\tau, \tag{2.22}$$

$$\mu_3(x,0,k) = I - \int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-y)\widehat{\sigma}_3}(U\mu_3)(y,0,k)\mathrm{d}y. \tag{2.23}$$

Note that $U(x,0)$ is determined by $q(x,0)$, whereas $V(0,t,k)$ is determined by $q(0,t)$, $q_x(0,t)$ and $q_{xx}(0,t)$.

2.3 The spectral functions

First, let us show that the function $\mu(x,t,k) = \mu_j(x,t,k)$ satisfies the symmetry relations.

Proposition 2.1 *For $j=1,2,3$, the function $\mu(x,t,k)=\mu_j(x,t,k)$ satisfies the symmetry relations*

$$\mu_{11}(x,t,k)=\overline{\mu_{22}(x,t,\overline{k})},\quad \mu_{21}(x,t,k)=\overline{\mu_{12}(x,t,\overline{k})}, \tag{2.24}$$

as well as

$$\begin{aligned}\mu_{11}(x,t,-k)&=\mu_{11}(x,t,k), &\quad \mu_{12}(x,t,-k)&=-\mu_{12}(x,t,k),\\ \mu_{12}(x,t,-k)&=-\mu_{12}(x,t,k), &\quad \mu_{22}(x,t,-k)&=\mu_{22}(x,t,k).\end{aligned}$$

Proof The proof is analogous to that of Proposition 2.1 in [22].

The fact that U and V are traceless together with the asymptotics of $\mu_j,\ j=1,2,3$ imply $\det\mu_j(x,t,k)=1$ for $j=1,2,3$. Thus

$$\det s(k)=\det S(k;T)=1. \tag{2.25}$$

It follows from Proposition 2.1 that

$$\begin{aligned}s_{11}(k)&=\overline{s_{22}(\overline{k})}, &\quad s_{21}(k)&=\overline{s_{12}(\overline{k})},\\ S_{11}(k)&=\overline{S_{22}(\overline{k})}, &\quad S_{21}(k)&=\overline{S_{12}(\overline{k})}.\end{aligned}$$

Then we may use the following notation for s and S :

$$s(k)=\begin{pmatrix}\overline{a(\overline{k})} & b(k)\\ \overline{b(\overline{k})} & a(k)\end{pmatrix},\quad S(k)=\begin{pmatrix}\overline{A(\overline{k})} & B(k)\\ \overline{B(\overline{k})} & A(k)\end{pmatrix}. \tag{2.26}$$

The symmetries of Proposition 2.1 imply that $a(k)$ and $A(k)$ are even functions of k, whereas $b(k)$ and $B(k)$ are odd functions of k, that is

$$\begin{aligned}a(-k)&=a(k), &\quad b(-k)&=-b(k),\\ A(-k)&=A(k), &\quad B(-k)&=-B(k).\end{aligned} \tag{2.27}$$

The definitions of $\mu_j(0,t,k),\ j=1,2$, and of $\mu_2(x,0,k)$ imply that these functions have larger domains of boundedness,

$$\mu_1(0,t,k)=(\mu_1^{(24)}(0,t,k),\mu_1^{(13)}(0,t,k)), \tag{2.28}$$

$$\mu_2(0,t,k)=(\mu_2^{(13)}(0,t,k),\mu_2^{(24)}(0,t,k)), \tag{2.29}$$

$$\mu_2(x,0,k)=(\mu_2^{(12)}(x,0,k),\mu_2^{(34)}(x,0,k)). \tag{2.30}$$

According to the definition of $s(k),S(k)$, we can get

$$\begin{pmatrix}b(k)\\ a(k)\end{pmatrix}=\mu_3^{(12)}(0,0,k),\quad \begin{pmatrix}-\mathrm{e}^{-2\mathrm{i}(4\beta k^3+2\beta k^2)T}B(k)\\ \overline{A(\overline{k})}\end{pmatrix}=\mu_2^{(24)}(0,T,k), \tag{2.31}$$

where the vectors $\mu_3^{(12)}(x,0,k)$ and $\mu_2^{(24)}(0,t,k)$ satisfy the following ODEs:

$$\partial_x\mu_3^{(12)}(x,0,k)+2\mathrm{i}k\begin{pmatrix}1&0\\0&0\end{pmatrix}\mu_3^{(12)}(x,0,k)=U(x,0)\mu_3^{(12)}(x,0,k), \tag{2.32}$$

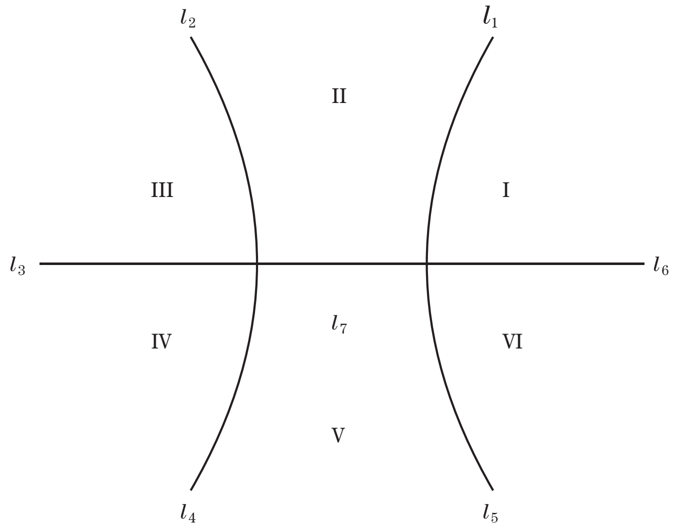

Figure 2 The contour $\bigcup_{i=1}^{7} l_i$.

where $\Im(k) \geq 0,\ 0 < x < \infty$ and $\lim_{x\to\infty} \mu_3^{(12)}(x,0,k) = (0,1)^{\mathrm{T}}$, and

$$\partial_t \mu_2^{(24)}(0,t,k) + 2\mathrm{i}(4\beta k^3 + 2\alpha k^2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_2^{(24)}(0,t,k) = V(0,t,k)\mu_2^{(24)}(0,t,k), \tag{2.33}$$

where $\Im(4\beta k^3 + 2\alpha k^2) \leq 0,\ 0 < t < T$ and $\lim_{t\to 0} \mu_2^{(24)}(0,t,k) = (0,1)^{\mathrm{T}}$.

Let us summarize the properties of the spectral functions:

(1) $a(k)$ and $b(k)$ are defined for $\{k \in \mathbb{R} \mid \Im k \geq 0\}$ and analytic for $\{k \in \mathbb{R} \mid \Im k > 0\}$;

(2) $a(k)\overline{a(\overline{k})} - b(k)\overline{b(\overline{k})} = 1,\ k \in \mathbb{R}$;

(3) $a(k) = 1 + O\big(\frac{1}{k}\big)$, $b(k) = O\big(\frac{1}{k}\big)$, as $k \to \infty$;

(4) $A(k)$ and $B(k)$ are entire functions bounded for $\{k \in \mathbb{C} \mid \Im(4\beta k^3 + 2\alpha k^2) \geq 0\}$. If $T = \infty$, the functions $A(k)$ and $B(k)$ are defined only for $\{k \in \mathbb{C} \mid \Im(4\beta k^3 + 2\alpha k^2) \geq 0\}$.

(5) $A(k)\overline{A(\overline{k})} - B(k)\overline{B(\overline{k})} = 1,\ k \in \mathbb{C}$ ($4\beta k^3 + 2\alpha k^2 \in \mathbb{R}$ if $T = \infty$).

(6) $A(k) = 1 + O\big(\frac{1}{k}\big),\ B(k) = O\big(\frac{1}{k}\big),\ k \to \infty,\ \Im(4\beta k^3 + 2\alpha k^2) \geq 0.$

All of these properties follow from the analyticity and boundedness of $\mu_3(x,0,k)$ and $\mu_1(0,t,k)$, from the conditions of unit determinant, and from the large k asymptotics of these eigenfunctions. Regarding $B(k)$ we note that $B(k) = B(k;T)$, where

$$B(k;t) = -\mathrm{e}^{2\mathrm{i}(4\beta k^3 + 2\alpha k^2)t}(\mu_2^{(24)}(0,t,k))_1,$$

from which it immediately follows that $B(k;t)$ is an entire function of k bounded for $\Im(4\beta k^3 + 2\alpha k^2) \leq 0$.

2.4 The RH problem

Equations (2.15) and (2.16) can be rewritten in a form expressing the jump condition of a 2×2 RH problem. This involves only tedious but straightforward algebraic manipulations (see [1] for details). The final form is

$$M_-(x,t,k)=M_+(x,t,k)J(x,t,k),\quad k\in\{k\mid \Im(4\beta k^3+2\alpha k^2)=0\}\equiv\bigcup_{i=1}^{6} l_i, \tag{2.34}$$

where $l_i,\ i=1,\cdots,6$ are defined in Figure 2, and $M_\pm(x,t,k)$ are the limit values (as k approaches $\{k\mid \Im(4\beta k^3+2\alpha k^2)=0$ from $\Omega_\pm\}$, see Figure 3) of $M(x,t,k)$ defined as

$$M(x,t,k)=\begin{cases}\left(\mu_3^{(1)},\dfrac{\mu_1^{(2)}}{\overline{d(\overline{k})}}\right), & k\in \mathrm{IV}\cup\mathrm{VI},\\ \left(\mu_3^{(1)},\dfrac{\mu_2^{(2)}}{\overline{a(\overline{k})}}\right), & k\in V,\\ \left(\dfrac{\mu_1^{(1)}}{d(k)},\mu_3^{(2)}\right), & k\in \mathrm{I}\cup\mathrm{III},\\ \left(\dfrac{\mu_2^{(1)}}{a(k)},\mu_3^{(2)}\right), & k\in \mathrm{II},\end{cases} \tag{2.35}$$

where

$$d(k)=a(k)\overline{A(\overline{k})}-b(k)\overline{B(\overline{k})},\quad k\in\mathrm{I}\cup\mathrm{III}, \tag{2.36}$$

and

$$J(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ \Gamma(k)\mathrm{e}^{2\mathrm{i}\theta(k)} & 1\end{pmatrix}, & k\in l_1\cup l_2,\\ \begin{pmatrix}1 & -\overline{\Gamma(\overline{k})}\mathrm{e}^{-2\mathrm{i}\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in l_4\cup l_5,\\ \begin{pmatrix}1 & -\overline{\Gamma(\overline{k})}\mathrm{e}^{-2\mathrm{i}\theta(k)}\\ 0 & 1\end{pmatrix} & \\ \quad\times\begin{pmatrix}1-|\gamma(k)|^2 & \gamma(k)\mathrm{e}^{-2\mathrm{i}\theta(k)}\\ -\overline{\gamma(k)}\mathrm{e}^{2\mathrm{i}\theta(k)} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \Gamma(k)\mathrm{e}^{2\mathrm{i}\theta(k)} & 1\end{pmatrix}, & k\in l_3\cup l_6,\\ \begin{pmatrix}1-|\gamma(k)|^2 & \gamma(k)\mathrm{e}^{-2\mathrm{i}\theta(k)}\\ -\overline{\gamma(k)}\mathrm{e}^{2\mathrm{i}\theta(k)} & 1\end{pmatrix}, & k\in l_7\end{cases} \tag{2.37}$$

with

$$\begin{aligned}\gamma(k)&=\frac{b(k)}{\overline{a}(k)}, && k\in\mathbb{R},\\ \Gamma(k)&=\frac{B(\overline{k})}{a(k)d(k)}, && k\in l_3\cup l_4\cup l_5\cup l_6,\\ \theta(k)&=kx+(4\beta k^3+2\alpha k^2)t.\end{aligned}$$

The matrix $M(x,t,k)$ defined in (2.35) is in general a meromorphic function of k in $\mathbb{C}\setminus\{\Im(4\beta k^3+2\alpha k^2)=0\}$. The possible poles of M are generated by the zeros of $a(k)$, $d(k)$, and by the complex conjugates of these zeros.

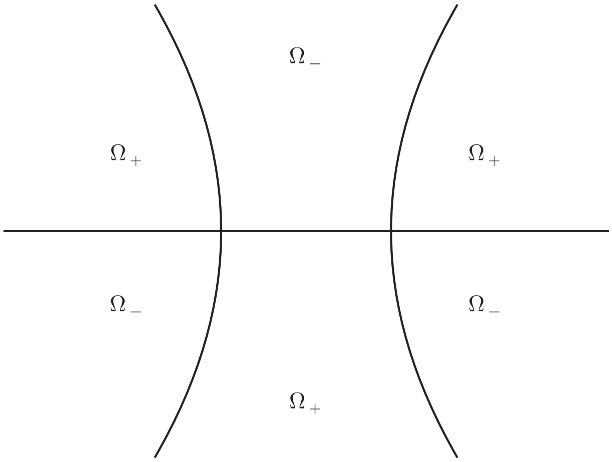

Figure 3 Domain Ω_+ and Ω_-.

Assumption 2.1 We assume that

(1) $a(k)$ has n simple zeros $\{k_j\}_i^n$, $n = n_1+2n_2$, where $\{k_j\}_1^{n_1} \in \text{II}$, $\{\mathrm{k_j}\}_{\mathrm{n_1+1}}^{\mathrm{n_1+n_2}} \in \text{I}$, $\mathrm{k_{n_1+n_2+j}} \in$ III, $\mathrm{j} = 1, \cdots, \mathrm{n_2}$.

(2) $d(k)$ has $2\Lambda_1$ simple zeros $\{\lambda_j\}_1^{2\Lambda_1}$, such that $\lambda_j \in \text{I}$, $j = 1, \cdots, \Lambda_1$, $\lambda_{\Lambda_1+j} \in \text{III}$.

(3) None of the zeros of $a(k)$ in $\text{III} \cup \text{I}$ coincides with a zero of $d(k)$.

In order to evaluate the associated residues, we introduce the notation $[A]_1$ (respectively, $[A]_2$) for the first (respectively, second) column of a 2×2 matrix A and we also write $\dot{a}(k) = \frac{\mathrm{d}a}{\mathrm{d}k}$. The associated residue formulae are the following:

$$\mathrm{Res}_{k_j}\big[M(x,t,k)\big]_1 = \frac{\mathrm{e}^{2\mathrm{i}\theta(k_j)}}{\dot{a}(k_j)b(k_j)}\big[M(x,t,k_j)\big]_2, \quad j = 1, \cdots, n_1, \tag{2.38}$$

$$\mathrm{Res}_{\overline{k}_j}\big[M(x,t,k)\big]_2 = \frac{\mathrm{e}^{-2\mathrm{i}\theta(\overline{k}_j)}}{\overline{\dot{a}(k_j)b(k_j)}}\big[M(x,t,\overline{k}_j)\big]_1, \quad j = 1, \cdots, n_1, \tag{2.39}$$

and

$$\mathrm{Res}_{\lambda_j}[M(x,t,k)]_1 = \frac{\overline{B(\overline{\lambda_j})}\mathrm{e}^{2\mathrm{i}\theta(\lambda_j)}}{\dot{d}(\lambda_j)b(\lambda_j)}[M(x,t,\lambda_j)]_2, \quad j = 1, \cdots, \Lambda_1, \tag{2.40}$$

$$\mathrm{Res}_{\overline{\lambda}_j}[M(x,t,k)]_2 = \frac{B(\overline{\lambda_j})\mathrm{e}^{-2\mathrm{i}\theta(\overline{\lambda}_j)}}{\overline{\dot{d}(\lambda_j)b(\lambda_j)}}[M(x,t,\overline{\lambda}_j)]_1, \quad j = 1, \cdots, \Lambda_1, \tag{2.41}$$

where $\theta(k_j) = k_j x + (4\beta k_j^3 + 2\alpha k_j^2)t$ and note that the only zeros of $a(k)$, which generate poles of M, are those in II. We can using the analogous method in [21–22] to obtain (2.38)–(2.41).

2.5 The global relation

The initial and boundary values of a solution of the Hirota equation are not independent. It turns out that the relations between the initial and boundary values of the solution can be expressed in a surprisingly simple form in term of the corresponding spectral functions.

Evaluating (2.15) at $x=0,\ t=T$, we find

$$\mu_3(0,T,k)=\mu_2(0,T,k)\mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)T\widehat{\sigma}_3}\mu_3(0,0,k). \tag{2.42}$$

According to the definition of $s(k),S(k)$ in (2.17) and (2.18), and using (2.12) to evaluate $\mu_3(0,T,k)$, (2.42) becomes

$$-I+(S(k,T))^{-1}s(k)+\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)T\widehat{\sigma}_3}\int_0^\infty \mathrm{e}^{\mathrm{i}ky\widehat{\sigma}_3}(U\mu_3)(y,T,k)\mathrm{d}y=0. \tag{2.43}$$

The $(1,2)$ coefficient of this equation is

$$a(k)B(k)-A(k)b(k)=\mathrm{e}^{2\mathrm{i}(4\beta k^3+2\alpha k^2)T}c^+(k),\quad \Im k>0 \tag{2.44}$$

for $T<\infty$, where

$$c^+(k)=c^+(k;T)=\int_0^\infty \mathrm{e}^{2\mathrm{i}ky\widehat{\sigma}_3}(U\mu_3)_{12}(y,T,k)\mathrm{d}y \tag{2.45}$$

is a function analytic for $\Im k>0$ which is $O\big(\frac{1}{k}\big)$ as $k\to\infty$. (2.45) is the global relation.

3 The Spectral Functions

The analysis of Section 2 motivates the following definitions for the spectral functions.

Definition 3.1 (The Spectral Functions $a(k)$ and $b(k)$) *Given $q_0(x)\in S(\mathbb{R}^+)$, we define the map*

$$\mathbb{S}:\{q_0(x)\}\to\{a(k),b(k)\},$$

by

$$\begin{pmatrix} b(k)\\ a(k)\end{pmatrix}=[\mu_3(0,k)]_2,\quad \Im k\geq 0,$$

where $\mu_3(x,k)$ is the unique solution of the Volterra linear integral equation

$$mu_3(x,k)=I+\int_\infty^x \mathrm{e}^{\mathrm{i}k(x'-x)\widehat{\sigma}_3}U(x',0,k)\mu_3(x',k)\mathrm{d}x',$$

and $U(x,0,k)$ is given in terms of $q_0(x)$ by (2.4).

Proposition 3.1 *The spectral functions $a(k)$ and $b(k)$ have the properties*:

(i) *$a(k)$ and $b(k)$ are analytic for* $\mathrm{Im}\,k>0$ *and continuous and bounded for* $\mathrm{Im}k\geq 0$.

(ii) $a(k)=1+O\big(\frac{1}{k}\big),\ b(k)=O\big(\frac{1}{k}\big),\ k\to\infty,\ \Im\,k\geq 0.$

(iii) $a(k)\overline{a(\overline{k})}-b(k)\overline{b(\overline{k})}=1,\ k\in\mathbb{R}.$

(iv) $a(-k)=a(k),\ b(-k)=-b(k),\ \Im\,k\geq 0.$

(v) *The map* $\mathbb{Q}:\{a(k),b(k)\}\mapsto\{q_0(x)\}$, *inverse to* $\mathbb{S}$, *is defined by*

$$q_0(x)=2\mathrm{i}\lim_{k\to\infty}(kM^{(x)}(x,k))_{12}, \tag{3.1}$$

where $M^{(x)}(x,k)$ *is the unique solution of the following RH problem*:

(1) $M^{(x)}(x,k)=\begin{cases}M_-^{(x)}(x,k), & \Im k\le 0,\\ M_+^{(x)}(x,k), & \Im k\ge 0\end{cases}$ *is a sectionally meromorphic function.*

(2) $M_-^{(x)}(x,k)=M_+^{(x)}(x,k)J^{(x)}(x,k),\ k\in\mathbb{R}$, *where*

$$J^{(x)}(x,k)=\begin{pmatrix}1 & -\dfrac{b(k)}{\overline{a(\overline{k})}}\mathrm{e}^{-2\mathrm{i}kx}\\ \dfrac{\overline{b(\overline{k})}}{a(k)}\mathrm{e}^{2\mathrm{i}kx} & \dfrac{1}{a(k)\overline{a(\overline{k})}}\end{pmatrix},\quad k\in\mathbb{R}. \tag{3.2}$$

(3) $M^{(x)}(x,k)=I+O\big(\frac{1}{k}\big),\ k\to\infty.$

(4) $a(k)$ *has* n *simple zeros* $\{k_j\}_i^n$, $n=n_1+2n_2$, *where* $\{k_j\}_1^{n_1}\in\mathrm{II}$, $\{k_j\}_{n_1+1}^{n_1+n_2}\in\mathrm{I}, k_{n_1+n_2+j}\in\mathrm{III}$, $j=1,\cdots,n_2$.

(5) *The first column of* $M_+^{(x)}$ *has simple poles at* $k=k_j$, $j=1,\cdots,n$, *and the second column of* $M_-^{(x)}$ *has simple poles at* $k=\overline{k}_j$, $j=1,\cdots,n$. *The associated residues are given by*

$$\operatorname*{Res}_{\zeta_j}[M^{(x)}(x,k)]_1=\frac{\mathrm{e}^{2\mathrm{i}k_j^2x}}{\dot{a}(k_j)b(k_j)}[M^{(x)}(x,k_j)]_2, \tag{3.3}$$

$$\operatorname*{Res}_{\overline{k}_j}[M^{(x)}(x,k)]_2=\frac{\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2x}}{\overline{\dot{a}(k_j)b(k_j)}}[M^{(x)}(x,\overline{k}_j)]_1. \tag{3.4}$$

(vi) *We have*

$$\mathbb{S}^{-1}=\mathbb{Q}.$$

Proof (i)–(iv) follow from the definition. The derivation of (v) and (vi), we can follow Appendix A [21, Appendix A].

Definition 3.2 (The Spectral Functions $A(k)$ and $B(k)$) *Let* $g_0(t)$ *and* $g_1(t)$ *be smooth functions. The map*

$$\widetilde{\mathbb{S}}:\{g_0(t),g_1(t),g_2(t)\}\to\{A(k),B(k)\}$$

is defined by

$$\begin{pmatrix}B(k)\\A(k)\end{pmatrix}=[\mu_1(0,k)]_2,$$

where $\mu_1(t,k)$ *is the unique solution of Volterra linear integral equation*

$$\mu_1(t,k)=I+\int_T^t\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)(t'-t)\widehat{\sigma}_3}V(0,t',k)\mu_1(t',k)\mathrm{d}t',$$

and $V(0,t,k)$ *is given in terms of* $\{g_0(t),g_1(t),g_2(t)\}$ *by* (2.5).

Proposition 3.2 *The spectral functions* $A(\zeta)$ *and* $B(\zeta)$ *have the following properties*:

(i) $A(k)$ *and* $B(k)$ *are entire functions bounded for* $\Im(4\beta k^3+2\alpha k^2)\ge 0$. *If* $T=\infty$, *the functions* $A(k)$ *and* $B(k)$ *are defined only for* $\Im(4\beta k^3+2\alpha k^2)\ge 0$.

(ii) $A(k)=1+O\big(\frac{1}{k}\big),\ B(k)=O\big(\frac{1}{k}\big),\ k\to\infty,\ \Im\,(4\beta k^3+2\alpha k^2)\geq 0.$

(iii) $A(k)\overline{A(\overline{k})}-B(k)\overline{B(\overline{k})}=1,\ k\in\mathbb{C},\ ((4\beta k^3+2\alpha k^2)\in\mathbb{R}$ *if* $T=\infty)$.

(iv) *The map* $\widetilde{\mathbb{Q}}:\{A(\zeta),B(\zeta)\}\mapsto\{g_0(t),g_1(t),g_2(t)\}$, *inverse to* $\widetilde{\mathbb{S}}$, *is defined by the solution of an appropriate RH problem as follows:*

$$\begin{aligned}
&g_0(t)=2\mathrm{i}m_{12}^{(1)}(t),\\
&g_1(t)=4(m^{(2)})_{12}(t)+2\mathrm{i}g_0(t)(m^{(1)})_{22}(t),\\
&g_2(t)=g_0^3(t)+8\mathrm{i}(m^{(3)})_{12}(t)+4g_0(t)(m^{(2)})_{22}(t)+2\mathrm{i}g_1(t)(m^{(1)})_{22}(t),
\end{aligned}\tag{3.5}$$

where the functions $m^{(1)}(t),\ m^{(2)}(t)$, *and* $m^{(3)}(t)$ *are determined by the asymptotic expansion*

$$M^{(t)}(t,k)=I+\frac{m^{(1)}(t)}{k}+\frac{m^{(2)}(t)}{k^2}+\frac{m^{(3)}(t)}{k^3}+O\Big(\frac{1}{k^4}\Big),\quad k\to\infty,$$

where $M^{(t)}(t,k)$ *is the unique solution of the following RH problem:*

(1) $M^{(t)}(t,k)=\begin{cases}M_-^{(t)}(t,k), & \Im\,(4\beta k^3+2\alpha k^2)\leq 0,\\ M_+^{(t)}(t,k), & \Im\,(4\beta k^3+2\alpha k^2)\geq 0\end{cases}$
is a sectionally meromorphic function.

(2) $M_-^{(t)}(t,k)=M_+^{(t)}(t,k)J^{(t)}(t,k),\ (4\beta k^3+2\alpha k^2)\in\mathbb{R}$, *where*

$$J^{(t)}(t,k)=\begin{pmatrix}1 & -\dfrac{B(k)}{\overline{A(\overline{k})}}\mathrm{e}^{-2\mathrm{i}(4\beta k^3+2\alpha k^2)t},\\ \dfrac{\overline{B(\overline{k})}}{A(k)}\mathrm{e}^{2\mathrm{i}(4\beta k^3+2\alpha k^2)t} & \dfrac{1}{A(k)\overline{A(\overline{k})}}\end{pmatrix},\quad (4\beta k^3+2\alpha k^2)\in\mathbb{R}.\tag{3.6}$$

(3)

$$M^{(t)}(t,k)=I+O\Big(\frac{1}{k}\Big),\ k\to\infty.$$

(4) *We assume that* $A(k)$ *has* $2N$ *simple zeros* $\{z_j\}_{j=1}^{2N}$ *such that* $\Im\,(4\beta z_j^3+2\alpha z_j^2)>0$, $j=1,\cdots,2N$.

(5) *The first column of* $M_+^{(t)}$ *has simple poles at* $z=z_j,\ j=1,\cdots,2N$, *and the second column of* $M_-^{(t)}$ *has simple poles at* $z=\overline{z}_j,\ j=1,\cdots,2N$. *The associated residues are given by*

$$\operatorname*{Res}_{z_j}[M^{(t)}(t,\zeta)]_1=\frac{\mathrm{e}^{4iz_j^4t}}{\dot{A}(z_j)B(z_j)}[M^{(t)}(t,z_j)]_2,\quad j=1,\cdots,2N,\tag{3.7}$$

$$\operatorname*{Res}_{\overline{z}_j}[M^{(t)}(x,\zeta)]_2=\frac{\mathrm{e}^{-4i\overline{z}_j^4t}}{\overline{\dot{A}(z_j)B(z_j)}}[M^{(t)}(t,\overline{z}_j)]_1,\quad j=1,\cdots,2N.\tag{3.8}$$

(vi) *We have*

$$\widetilde{\mathbb{S}}^{-1}=\widetilde{\mathbb{Q}}.$$

Proof (i)–(iv) follow from the definition. The derivation of (vi), we can follow [21, Appendix A].

4 The Riemann-Hilbert Problem

Theorem 4.1 *Let $q_0(x) \in S(\mathbb{R}^+)$. Suppose that the set $\{g_0(t), g_1(t), g_2(t)\}$ of smooth functions is such that the associated spectral functions $s(k)$ and $S(k)$ satisfy the global relation* (2.45) *if $T < \infty$ and if $T = \infty$ the global relation is replaced by*

$$B(k)a(k) - A(k)b(k) = 0, \quad k \in \mathrm{V}.$$

Assume that the possible zeros $\{k_j\}_1^n$ of $a(k)$ and $\{k_j\}_1^{2\Lambda_1}$ of $d(k)$ are as in Assumption 2.1. *Define $M(x,t,k)$ as the solution of the following 2×2 matrix RH problem:*

(1) *M is sectionally meromorphic in $k \in \mathbb{C} \setminus \{(4\beta k^3 + 2\alpha k^2) \in \mathbb{R}\}$.*

(2) *The first column of M has simple poles at $k = \overline{k}_j$, $j = 1, \cdots, n_1$, and at $k = \overline{k}_j$, $j = 1, \cdots, \Lambda_1$. The associated residues satisfy the relations* (2.38)–(2.41)*.*

(3) *M satisfies the jump condition*

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad (4\beta k^3 + 2\alpha k^2) \in \mathbb{R},$$

where M is M_- for $\Im(4\beta k^3 + 2\alpha k^2) \le 0$, is M_+ for $\Im(4\beta k^3 + 2\alpha k^2) \ge 0$, and J is defined in (2.37)*, see Figure* 2.

(4) *$M(x,t,k) = I + O\left(\frac{1}{k}\right)$, $k \to \infty$.*

Then $M(x,t,k)$ exists and is unique.

Define $q(x,t)$ in terms of $M(x,t,k)$ by

$$q(x,t) = 2\mathrm{i} \lim_{k \to \infty} (kM(x,t,k))_{12}. \tag{4.1}$$

The $q(x,t)$ solves the Hirota equation. Furthermore,

$$q(x,0) = q_0(x), \quad q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t), \quad q_{xx}(0,t) = g_2(t).$$

Proof In the case when $a(k)$ and $d(k)$ have no zeros, the unique solvability is a consequence of the following vanishing lemma.

Lemma 4.1 (Vanishing Lemma) *The Riemann-Hilbert problem in Theorem* 4.1 *with the vanishing boundary condition*

$$M(x,t,z) \to 0, \quad \textit{as } z \to \infty$$

has only the zero solution.

Proof The proof is an analogue of [22, Lemma 4.2]

If $a(z)$ and $d(z)$ have zeros this singular RH problem can be mapped to a regular one coupled with a system of algebraic equations (see [23]). Moreover, it follows from standard arguments using the dressing method (see [24–25]) that if M solves the above RH problem and $q(x,t)$ is defined by (4.1), then $q(x,t)$ solves the Hirota equation (1.1).

Proof of $q(x,0) = q_0(x)$. Define $M^{(x)}(x,k)$ by

$$M^{(x)} = M(x,0,k), \quad k \in D_1 \cup D_3 \cup D_4 \cup D_6,$$

$$M^{(x)} = M(x,0,k)J_1(x,0,k), \quad k \in D_2, \tag{4.2}$$
$$M^{(x)} = M(x,0,k)J_3^{-1}(x,0,k), \quad k \in D_5.$$

We first discuss the case where $a(z)$ and $d(z)$ have no zeros. Then the function $M^{(x)}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$. Furthermore,

$$M_-^{(x)}(x,k) = M_+^{(x)}(x,k)J^{(x)}(x,k), \quad k \in \mathbb{R},$$
$$M^{(x)}(x,k) = I + \Big(\frac{1}{k}\Big), \quad k \to \infty,$$

where $J^{(x)}(x,k)$ is defined in (3.2). Thus according to (3.1),

$$q_0(x) = 2\mathrm{i} \lim_{k\to\infty} (kM^{(x)}(x,k))_{12}.$$

Comparing this with (4.1) evaluated at $t = 0$, we conclude that $q_0(x) = q(x,0)$.

We now discuss the case when the sets $\{z_j\}$ and $\{k_j\}$ are not empty. The first column of $M(x,t,k)$ has poles at $\{k_j\}_{n_1+1}^{n_1+2n_2}$ for $k \in \mathrm{I} \cup \mathrm{III}$ and has poles at $\{\lambda_j\}_1^{2\Lambda_1}$ for $\zeta \in \mathrm{I} \cup \mathrm{III}$. On the other hand, the first column of $M^{(x)}(x,k)$ should have poles at $\{k_j\}_{n_1+1}^{n_1+2n_2}$. Now we shall show that the transformation defined by (4.2) maps the former poles to the latter ones. Since $M^{(x)} = M(x,0,k)$ for $k \in \mathrm{I} \cup \mathrm{III}$, $M^{(x)}$ has poles at $\{k_j\}_{n_1+1}^{n_1+2n_2}$ with the correct residue condition. Letting $M = (M_1, M_2)$, (4.2) can be written as

$$M^{(x)}(x,k) = (M_1(x,0,k) - \Gamma(k)\mathrm{e}^{-2\mathrm{i}kx}M_2(x,0,k), M_2(x,0,k)). \tag{4.3}$$

The residue condition (2.40) at λ_j implies that $M^{(x)}$ has no poles at λ_j. On the other hand, (4.3) shows that $M^{(x)}$ has poles at $\{\zeta_j\}_{2n_1+1}^{2n}$ with residues given by

$$\operatorname*{Res}_{k_j}[M^{(x)}(x,k)]_1 = -\operatorname*{Res}_{k_j}\Gamma(k)\mathrm{e}^{-2\mathrm{i}kx}[M^{(x)}(x,k_j)]_2, \quad j = 2n_1+1, \cdots, 2n.$$

Using the definition of $\Gamma(k)$ and the equation $d(k_j) = -b(k_j)\overline{B(\overline{k}_j)}$, this becomes the residue condition of (3.3). Similar considerations apply to $\overline{k}_j$ and $\overline{\lambda}_j$.

Proof of $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$ and $q_{xx}(0,t) = g_2(t)$. Recall that M satisfies (2.35) and jump condition (2.37). Let $M^{(t)}(t,k)$ be defined by

$$M^{(t)}(t,\zeta) = M(0,t,\zeta)G(t,\zeta), \tag{4.4}$$

where G is given by $G^{(j)}$ for $k \in D_j$, $j = 1, \cdots, 6$. Suppose that we can find matrices $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ holomorphic for $\operatorname{Im} k > 0$ (and continuous for $\operatorname{Im} k \geq 0$), matrices $G^{(4)}$, $G^{(5)}$ and $G^{(6)}$ holomorphic for $\operatorname{Im} k < 0$ (and continuous for $\operatorname{Im} k \leq 0$), which tend to I as $k \to \infty$, and which satisfy

$$\begin{aligned}
J_1(0,t,k)G^{(2)}(t,k) &= G^{(1)}(t,k)J^{(t)}(t,k), \quad k \in \ell_1,\\
J_1(0,t,k)G^{(2)}(t,k) &= G^{(3)}(t,k)J^{(t)}(t,k), \quad k \in \ell_2,\\
J_3(0,t,k)G^{(5)}(t,k) &= G^{(4)}(t,k)J^{(t)}(t,k), \quad k \in \ell_4,\\
J_3(0,t,k)G^{(5)}(t,k) &= G^{(6)}(t,k)J^{(t)}(t,k), \quad k \in \ell_5,\\
J_4(0,t,k)G^{(4)}(t,k) &= G^{(3)}(t,k)J^{(t)}(t,k), \quad k \in \ell_3,\\
J_4(0,t,k)G^{(6)}(t,k) &= G^{(1)}(t,k)J^{(t)}(t,k), \quad k \in \ell_6,
\end{aligned} \tag{4.5}$$

where $J^{(t)}$ is the jump matrix defined in (3.6). Then, since $J_2 = J_3 J_4^{-1} J_1$, it follows that

$$J_2(0,t,k)G^{(5)}(t,k) = G^{(2)}(t,k)J^{(t)}(t,k), \quad k \in \ell_7,$$

and (2.35) and (4.4) imply that $M^{(t)}$ satisfies the RH problem defined in Proposition 3.2. If the sets $\{k_j\}$ and $\{\lambda_j\}$ are empty, this immediately yields the desired result. We claim that such $G^{(j)}$ matrices are

$$G^{(1)} = G^{(3)} = \begin{pmatrix} \dfrac{a(k)}{A(k)} & c^{+}(k)\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)(T-t)} \\ 0 & \dfrac{A(k)}{a(k)} \end{pmatrix}, \tag{4.6}$$

$$G^{(4)} = G^{(6)} = \begin{pmatrix} \dfrac{\overline{A(\bar{k})}}{\overline{a(\bar{k})}} & 0 \\ \overline{c^{+}(\bar{k})}\mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)(T-t)} & \dfrac{\overline{a(\bar{k})}}{\overline{A(\bar{k})}} \end{pmatrix}, \tag{4.7}$$

$$G^{(2)} = \begin{pmatrix} d(k) & \dfrac{-b(k)\mathrm{e}^{-\mathrm{i}(4\beta k^3+2\alpha k^2)t}}{\overline{A(\bar{k})}} \\ 0 & \dfrac{1}{d(k)} \end{pmatrix}, \quad G^{(5)} = \begin{pmatrix} \dfrac{1}{\overline{d(\bar{k})}} & 0 \\ \dfrac{-\overline{b(\bar{k})}\mathrm{e}^{\mathrm{i}(4\beta k^3+2\alpha k^2)}}{A(k)} & \overline{d(\bar{k})} \end{pmatrix}. \tag{4.8}$$

We omit the verification that these $G^{(j)}$ matrices fulfill the requirements (4.5) as well as the verification of the residue conditions in the case of non-empty sets $\{k_j\}$ and $\{\lambda_j\}$, since analogous arguments can be found in the proof of Theorem 4.1 in [21].

Remark 4.1 (1) If we take $\alpha = 0,\ \beta = 1$, then we can yield the same conclusion as Fokas, Its and Sung in [21].

(2) If we take $\alpha = 1,\ \beta = 0$, then we can yield the same conclusion as Boutet de Monvel, Fokas and Shepelsky in [11].

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