Chin. Ann. Math. Ser. B
41(1), 2020, 133–146
DOI: 10.1007/s11401-019-0190-0

Chinese Annals of Mathematics, Series B

Certain Curvature Conditions on *P*-Sasakian Manifolds Admitting a Quater-Symmetric Metric Connection*

Uday Chand DE[1] Peibiao ZHAO[2] Krishanu MANDAL[1]
Yanling HAN[3]

Abstract The authors consider a quarter-symmetric metric connection in a P-Sasakian manifold and study the second order parallel tensor in a P-Sasakian manifold with respect to the quarter-symmetric metric connection. Then Ricci semisymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection is considered. Next the authors study ξ-concircularly flat P-Sasakian manifolds and concircularly semisymmetric P-Sasakian manifolds with respect to the quarter-symmetric metric connection. Furthermore, the authors study P-Sasakian manifolds satisfying the condition $\widetilde{Z}(\xi, Y) \cdot \widetilde{S} = 0$, where $\widetilde{Z}$, $\widetilde{S}$ are the concircular curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Finally, an example of a 5-dimensional P-Sasakian manifold admitting quarter-symmetric metric connection is constructed.

Keywords Quarter-symmetric metric connection, P-Sasakian manifold, Ricci semisymmetric manifold, ξ-Concircularly flat, Concircularly semisymmetric
2000 MR Subject Classification 53C35, 53D40

1 Introduction

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection (see [8]) if the torsion tensor T of the connection $\widetilde{\nabla}$,

$$T(X, Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] \tag{1.1}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a $(1, 1)$ tensor field. If moreover, a quarter-symmetric connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y, Z) = 0,$$

Manuscript received May 27, 2015. Revised June 30, 2017.
[1]Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kol-700019, West Bengal, India. E-mail: uc_de@yahoo.com krishanu.mandal013@gmail.com
[2]Department of Mathematics, Nanjing University of Science and Technology, Nanjing 210094, China. E-mail: pbzhao@njust.edu.cn
[3]Corresponding author. School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China. E-mail: hanyanling1979@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11871275, 11371194).

where $X, Y, Z \in \chi(M)$ are arbitrary vector fields on Mj, then $\widetilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we change ϕX by X, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection (see [23]). Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

A transformation of an n-dimensional Riemannian manifold M is said to be a concircular transformation (see [10, 22]), if it transforms every geodesic circle of M into a geodesic circle. A concircular transformation is always a conformal transformation (see [10]). Here, we mean a geodesic circle by a curve in M whose first curvature is constant and the second curvature is identically zero. Thus, the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see [4]). An important invariant of a concircular transformation is the concircular curvature tensor Z, defined by (see [22])

$$Z(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y], \tag{1.3}$$

where $R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W$ (∇ being the Levi-Civita connection) is the Riemannian curvature tensor and r is the scalar curvature. The importance of concircular transformation and concircular curvature tensor is well known in the differential geometry of certain F-structure such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structure etc. (see [5, 21, 25]).

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$. The notion of semisymmetric, a proper generalization of locally symmetric manifold, is defined by $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó [20].

Quarter-symmetric metric connection in a Riemannian manifold studied by several authors such as Mandal and De [11], Rastogi [15–16], Yano and Imai [24], Mukhopadhyay, Roy and Barua [13], Han et al. [9], Biswas and De [3] and many others. Sular, Özgür and De [19] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Motivated by the above studies in the present paper, we study quarter-symmetric metric connection in a P-Sasakian manifold. The paper is organized as follows. In Section 2, we give a brief account of P-Sasakian manifolds. In Section 3, we discuss the curvature tensor and the Ricci tensor of a P-Sasakian manifold with respect to the quarter-symmetric metric connection. Section 4 is devoted to study the second order parallel tensor in P-sasakian manifolds with respect to the quarter-symmetric metric connection, and prove that the second order parallel tensor is a constant multiple of the metric tensor. In Section 5, we consider a Ricci semisymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection, and in this case we prove that a P-Sasakian manifold is Ricci semisymmetric with respect to the quarter-symmetric metric connection if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection, provided that the characteristic vector field ξ is harmonic. In Section 6, we consider a ξ-concircularly flat P-Sasakian manifold with respect to the quarter-symmetric metric connection, and we show that a P-Sasakian manifold is ξ-concircularly flat with respect to the quarter-symmetric metric connection if and only if its scalar curvature is negative constant with respect to the quarter-symmetric metric connection. Also concircularly semisymmetric P-Sasakian manifolds will be studied in Section 7 and it is proved that in a P-Sasakian manifold, semisymmetry and concircularly semisymmetry are equivalent with respect to the quarter-symmetric metric connection. Next in Section 8, we consider a P-Sasakian manifold satisfying the condition $\widetilde{Z}(\xi, Y) \cdot \widetilde{S} = 0$ with respect to the quarter-symmetric metric connection, and prove that a P-Sasakian manifold satisfies the condition $\widetilde{Z}(\xi, Y) \cdot \widetilde{S} = 0$ with respect to the quarter-symmetric metric connection if and only if the manifold is an η-Einstein manifold provided that the characteristic vector field ξ is harmonic or the scalar curvature is negative constant with respect to the quarter-symmetric metric connection. Finally, we construct an example of a 5-dimensional P-Sasakian manifold admitting quarter-symmetric metric connection, which verifies the Ricci tensor and scalar curvature with respect to the quarter-symmetric metric connection.

2 *P*-Sasakian Manifolds

Let M be an n-dimensional differentiable manifold with a $(1,1)$-type tensor field ϕ, a characteristic vector field ξ and a 1-form η such that

$$\phi^2 X = X - \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0. \tag{2.1}$$

Then (ϕ, ξ, η) is called an almost paracontact structure and M is an almost paracontact manifold. Moreover, if M admits a Riemannian metric g such that

$$g(\xi, X) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

then (ϕ, ξ, η, g) is called almost paracontact metric structure and M is an almost paracontact metric manifold (see [17]). If (ϕ, ξ, η, g) satisfy the following equations:

$$\begin{aligned} &\mathrm{d}\eta = 0, \quad \nabla_X \xi = \phi X, \\ &(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \end{aligned} \tag{2.3}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold (see [1]). Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly an SP-Sasakian manifold (see [18]) if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.4}$$

Also in a P-Sasakian manifold the following relations hold (see [1, 14]):

$$S(X,\xi) = -(n-1)\eta(X), \quad Q\xi = -(n-1)\xi, \tag{2.5}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.6}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$\eta(R(X,Y)\xi) = 0 \tag{2.9}$$

for any vector fields $X, Y, Z \in \chi(M)$, where R is the $(1,3)$-type Riemannian curvature tensor, S is the $(0,2)$-type Ricci tensor and Q is the Ricci operator defined by

$$g(QX,Y) = S(X,Y).$$

P-Sasakian manifolds were studied by several authors such as De et al. [6], Yildiz et al. [26], Desmukh and Ahmed [7], Matsumoto, Ianus and Mihai [12], Özgür [14], Adati and Miyazawa [2] and many others.

An almost paracontact Riemannian manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies the the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold.

3 Curvature Tensor of a *P*-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Let $\widetilde{\nabla}$ be a linear connection and ∇ be a Levi-Civita connection of an almost paracontact metric M such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + U(X,Y), \tag{3.1}$$

where U is a $(1,1)$-type tensor. For $\widetilde{\nabla}$ a quarter-symmetric metric connection in M, we have (see [8])

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.3}$$

From (1.2) and (3.3) we get

$$T'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \tag{3.4}$$

By (1.2), (3.4) and (3.2), we have

$$U(X,Y) = \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{3.5}$$

Therefore a quarter-symmetric metric connection $\widetilde{\nabla}$ in a P-Sasakian manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{3.6}$$

Let $\widetilde{R}$ and R be the curvature tensors with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ respectively. Then we have from [11] that

$$\begin{aligned}\widetilde{R}(X,Y)U = {} & R(X,Y)U + 3g(\phi X, U)\phi Y - 3g(\phi Y, U)\phi X \\ & + \eta(U)[\eta(X)Y - \eta(Y)X] \\ & - [\eta(X)g(Y,U) - \eta(Y)g(X,U)]\xi, \end{aligned} \tag{3.7}$$

and

$$\widetilde{S}(Y,U) = S(Y,U) + 2g(Y,U) - (n+1)\eta(Y)\eta(U) - 3\,\mathrm{trace}\,\phi\; g(\phi Y, U), \tag{3.8}$$

where

$$\widetilde{R}(X,Y)U = \widetilde{\nabla}_X\widetilde{\nabla}_Y U - \widetilde{\nabla}_Y\widetilde{\nabla}_X U - \widetilde{\nabla}_{[X,Y]}U$$

and $\widetilde{S}$ and S are the Ricci tensors of the connections $\widetilde{\nabla}$ and ∇, respectively. In [11], Mandal and De proved the following theorem.

Theorem 3.1 *For a P-Sasakian manifold (M,g) with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$, we have*

(a) *the curvature tensor $\widetilde{R}$ is given by* (3.7),

(b) *the Ricci tensor $\widetilde{S}$ is symmetric,*

(c) $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 0$,

(d) $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(Y,X,Z,W) = 0$,

(e) $\widetilde{R}(X,Y,Z,W) = \widetilde{R}(Z,W,X,Y)$,

(f) $\widetilde{S}(Y,\xi) = -2(n-1)\eta(Y)$,

where $X,Y,Z,W \in \chi(M)$.

Again by contraction of (3.8) we have

$$\widetilde{r} = r + n - 1 - 3(\mathrm{trace}\,\phi)^2.$$

Hence we can state the following theorem.

Theorem 3.2 *Let M be an n-dimensional P-Sasakian manifold which admits quarter-symmetric metric connection $\widetilde{\nabla}$. Then the scalar curvature $\widetilde{r}$ with respect to $\widetilde{\nabla}$ and scalar curvature with respect to Levi-Civita connection are related by the following relation*

$$\widetilde{r} = r + n - 1 - 3(\mathrm{trace}\,\phi)^2.$$

By making use of (2.7)–(2.8) and (2.1) in (3.7) we obtain

$$\widetilde{R}(\xi, Y)U = 2[\eta(U)Y - g(U, Y)\xi] \tag{3.9}$$

and

$$\widetilde{R}(X, Y)\xi = 2[\eta(X)Y - \eta(Y)X], \tag{3.10}$$

where $X, Y \in \chi(M)$.

4 The Second Order Parallel Tensor in *P*-Sasakian Manifolds with Respect to the Quarter-Symmetric Metric Connection

Definition 4.1 *A tensor T of second order is said to be a second order parallel tensor if $\nabla T = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.*

Let α be a second order parallel tensor with respect to the quarter-symmetric metric connection, that is, $\widetilde{\nabla}\alpha = 0$. Then it follows that

$$\alpha(\widetilde{R}(W, X)Y, U) + \alpha(Y, \widetilde{R}(W, X)U) = 0 \tag{4.1}$$

for any vector fields $X, Y, U, W \in \chi(M)$.

Substituting $W = Y = U = \xi$ into (4.1) gives

$$\alpha(\xi, \widetilde{R}(\xi, X)\xi) = 0,$$

which gives by virtue of (3.9) that

$$\alpha(X, \xi) - g(X, \xi)\alpha(\xi, \xi) = 0. \tag{4.2}$$

Differentiating (4.2) covariantly along Y, we get

$$\begin{aligned}&[g(\widetilde{\nabla}_Y X, \xi) + g(X, \widetilde{\nabla}_Y \xi)]\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\widetilde{\nabla}_Y \xi, \xi)\\ &- [\alpha(\widetilde{\nabla}_Y X, \xi) + \alpha(X, \widetilde{\nabla}_Y \xi)] = 0.\end{aligned} \tag{4.3}$$

Putting $X = \widetilde{\nabla}_Y X$ in (4.2) we obtain

$$\alpha(\widetilde{\nabla}_Y X, \xi) - g(\widetilde{\nabla}_Y X, \xi)\alpha(\xi, \xi) = 0. \tag{4.4}$$

In view of (4.3) and (4.4) it follows that

$$g(X, \widetilde{\nabla}_Y \xi)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\widetilde{\nabla}_Y \xi, \xi) - \alpha(X, \widetilde{\nabla}_Y \xi) = 0. \tag{4.5}$$

By (3.6) it follows from (4.5) that

$$g(X, \phi Y)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\phi Y, \xi) - \alpha(X, \phi Y) = 0. \tag{4.6}$$

Replacing X by ϕY in (4.2) and using $\eta \circ \phi = 0$ give

$$\alpha(\phi Y, \xi) = 0. \tag{4.7}$$

From (4.6)–(4.7) we have

$$g(X, \phi Y)\alpha(\xi, \xi) - \alpha(X, \phi Y) = 0. \tag{4.8}$$

Replacing Y by ϕY in (4.8) and using (4.2) and (2.1) yield

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y). \tag{4.9}$$

Differentiating (4.9) covariantly with respect to $\widetilde{\nabla}$ along any vector field on M, it can be easily seen that $\alpha(\xi, \xi)$ is constant. Hence we can state the following proposition.

Proposition 4.1 *On a P-Sasakian manifold, admitting a quarter-symmetric metric connection, a second order symmetric parallel tensor with respect to the quarter-symmetric metric connection is a constant multiple of the associated metric tensor.*

Suppose that the Ricci tensor $\widetilde{S}$ with respect to the quarter-symmetric metric connection is parallel in a P-Sasakian manifold. Since $\widetilde{S}$ is symmetric, from Proposition 4.1 it follows that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection. Converse is obvious.

Therefore we can state the following theorem.

Theorem 4.1 *In a P-Sasakian manifold admitting quarter-symmetric metric connection, the Ricci tensor $\widetilde{S}$ of $\widetilde{\nabla}$ is parallel with respect to the quarter-symmetric metric connection if and only if the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.*

5 Ricci Semisymmetric *P*-Sasakian Manifolds with Respect to the Quarter-Symmetric Metric Connection

In this section we investigate about Ricci semisymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection, that is, the curvature tensor satisfies the condition

$$(\widetilde{R}(X, Y) \cdot \widetilde{S})(U, V) = 0,$$

where $\widetilde{R}(X, Y)$ denotes the derivation of the tensor algebra at each point of the manifold. This implies

$$\widetilde{S}(\widetilde{R}(X, Y)U, V) + \widetilde{S}(U, \widetilde{R}(X, Y)V) = 0. \tag{5.1}$$

Using (3.8) we have from (5.1) that

$$S(\widetilde{R}(X, Y)U, V) + 2g(\widetilde{R}(X, Y)U, V) - (n + 1)\eta(\widetilde{R}(X, Y)U)\eta(V)$$

$$- 3\,\mathrm{trace}\,\phi\; g(\widetilde{R}(X,Y)U,\phi V) + S(\widetilde{R}(X,Y)V,U) + 2g(\widetilde{R}(X,Y)V,U)$$
$$- (n+1)\eta(\widetilde{R}(X,Y)V)\eta(U) - 3\,\mathrm{trace}\,\phi\; g(\widetilde{R}(X,Y)V,\phi U) = 0. \tag{5.2}$$

Substituting $X = U = \xi$ into (5.2) and using (3.9), (2.8) and (2.5), we have

$$\begin{aligned} &2[S(Y,V) + (n-1)\eta(Y)\eta(V)] + 4[g(Y,V) - \eta(Y)\eta(V)] - 6\,\mathrm{trace}\,\phi\; g(Y,\phi V)\\ &+ 2(n-1)[g(Y,V) - \eta(Y)\eta(V)] + 4[\eta(Y)\eta(V) - g(Y,V)]\\ &- 2(n+1)[\eta(Y)\eta(V) - g(Y,V)] = 0. \end{aligned} \tag{5.3}$$

This implies

$$S(V,Y) = -2ng(V,Y) + (n+1)\eta(V)\eta(Y) + 3\,\mathrm{trace}\,\phi\; g(\phi V,Y). \tag{5.4}$$

Therefore, if $\mathrm{trace}\,\phi = 0$, that is, if the characteristic vector field ξ is harmonic, then the manifold is an η-Einstein manifold. Conversely, let the relation (5.4) hold. Then by (3.8) we have

$$\widetilde{S}(V,Y) = -2(n-1)g(V,Y). \tag{5.5}$$

Therefore, it is clear that

$$\widetilde{S}(\widetilde{R}(X,Y)U,V) + \widetilde{S}(U,\widetilde{R}(X,Y)V) = 0, \tag{5.6}$$

that is, $\widetilde{R}\cdot\widetilde{S} = 0$. This leads to the following theorem.

Theorem 5.1 *A P-Sasakian manifold admitting quarter-symmetric metric connection is Ricci semisymmetric with respect to quarter-symmetric metric connection if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection, provided that the characteristic vector field ξ is harmonic.*

6 ξ-Concircularly Flat P-Sasakian Manifolds with Respect to the Quarter-Symmetric Metric Connection

ξ-conformally flat K-contact manifolds were studied by Zhen et al. [27]. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, the conformal curvature tensor C is a map,

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus \{\xi_p\}.$$

An almost contact metric manifold M^n is called ξ-conformally flat if the projection of the image of C onto $\{\xi_p\}$ is zero (see [27]). Analogous to the definition of ξ-conformally flat almost contact metric manifold, we define ξ-concircularly flat P-Sasakian manifolds.

Definition 6.1 *A manifold M is said to be ξ-concircularly flat if the following relation holds*:

$$Z(X,Y)\xi = 0 \tag{6.1}$$

for any vector fields $X, Y \in \chi(M)$, and Z is the concircular curvature tensor defined by (1.3).

In this section, we study ξ-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (1.3), we have

$$\widetilde{R}(X,Y)\xi - \frac{\widetilde{r}}{n(n-1)}[g(Y,\xi)X - g(X,\xi)Y] = 0 \tag{6.2}$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

Using (3.10) and (6.2) we get

$$\Big[2 + \frac{\widetilde{r}}{n(n-1)}\Big][\eta(X)Y - \eta(Y)X] = 0. \tag{6.3}$$

Since $\eta(X)Y - \eta(Y)X \neq 0$, then we have

$$\widetilde{r} = -2n(n-1). \tag{6.4}$$

Conversely, if the relation (6.4) holds. Therefore, using (3.10) we obtain

$$\widetilde{R}(X,Y)\xi - \frac{\widetilde{r}}{n(n-1)}[g(Y,\xi)X - g(X,\xi)Y] = 0, \tag{6.5}$$

that is, $\widetilde{Z}(X,Y)\xi = 0$.

By the above discussions we can state the following theorem.

Theorem 6.1 *A P-Sasakian manifold admitting quarter-symmetric metric connection is ξ-concircularly flat with respect to quarter-symmetric metric connection if and only if the scalar curvature is negative constant with respect to quarter-symmetric metric connection.*

7 Concircularly Semisymmetric *P*-Sasakian Manifolds with Respect to the Quarter-Symmetric Metric Connection

In this section, we deal with concircularly semisymmetric P-Sasakian manifold with respect to the quarter-symmetric metric connection.

Definition 7.1 *A P-Sasakian manifold M is said to be concircularly semisymmetric with respect to the quarter-symmetric metric connection if $\widetilde{R} \cdot \widetilde{Z} = 0$ holds.*

Now $(\widetilde{R}(X,Y) \cdot \widetilde{Z})(U,V)W = 0$ implies

$$\begin{aligned}&\widetilde{R}(X,Y)\widetilde{Z}(U,V)W - \widetilde{Z}(\widetilde{R}(X,Y)U,V)W \\ &- \widetilde{Z}(U,\widetilde{R}(X,Y)V)W - \widetilde{Z}(U,V)\widetilde{R}(X,Y)W = 0.\end{aligned} \tag{7.1}$$

In view of (1.3) and (7.1), we have

$$\widetilde{R}(X,Y)\widetilde{R}(U,V)W-\widetilde{R}(\widetilde{R}(X,Y)U,V)W$$
$$-\widetilde{R}(U,\widetilde{R}(X,Y)V)W-\widetilde{R}(U,V)\widetilde{R}(X,Y)W=0, \tag{7.2}$$

that is, $\widetilde{R}\cdot\widetilde{R}=0$. This leads to the following theorem.

Theorem 7.1 *In a P-Sasakian manifold admitting quarter-symmetric metric connection, semisymmetry and concircularly semisymmetry with respect to $\widetilde{\nabla}$ are equivalent.*

In [11] Mandal and De proved that if a P-Sasakian manifold is semisymmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection. In this position, we can state the following corollary.

Corollary 7.1 *If a P-Sasakian manifold admits quarter-symmetric metric connection is concircularly semisymmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.*

8 P-Sasakian Manifolds Satisfying the Condition $\widetilde{Z}(\xi,Y)\cdot\widetilde{S}=0$ with Respect to the Quarter-Symmetric Metric Connection

In this section, we consider a P-Sasakian manifold M satisfying the condition $\widetilde{Z}(\xi,Y)\cdot\widetilde{S}=0$ with respect to the quarter-symmetric metric connection. Now $\widetilde{Z}(\xi,Y)\cdot\widetilde{S}=0$ implies

$$\widetilde{S}(\widetilde{Z}(\xi,Y)U,V)+\widetilde{S}(U,\widetilde{Z}(\xi,Y)V)=0 \tag{8.1}$$

for any vector fields $Y,U,V\in\chi(M)$. Using (1.3) and (3.9), we obtain from (8.1) that

$$\Big[2+\frac{\widetilde{r}}{n(n-1)}\Big][\eta(U)\widetilde{S}(Y,V)-g(Y,U)\widetilde{S}(\xi,V)$$
$$+\eta(V)\widetilde{S}(Y,U)-g(Y,V)\widetilde{S}(\xi,U)]=0. \tag{8.2}$$

Substituting $U=\xi$ into (8.2) and using Theorem 3.1, we have

$$\Big[2+\frac{\widetilde{r}}{n(n-1)}\Big][\widetilde{S}(Y,V)+2(n-1)g(Y,V)]=0. \tag{8.3}$$

With the help of (3.8) and (8.3), we obtain

$$\Big[2+\frac{\widetilde{r}}{n(n-1)}\Big][S(Y,V)+2ng(Y,V)-(n+1)\eta(Y)\eta(V)$$
$$-3\,\mathrm{trace}\,\phi\; g(\phi Y,V)]=0. \tag{8.4}$$

This implies either

$$\widetilde{r}=-2n(n-1) \tag{8.5}$$

or

$$S(Y,V) = -2ng(Y,V) + (n+1)\eta(Y)\eta(V) + 3\,\mathrm{trace}\,\phi\; g(\phi Y,V). \tag{8.6}$$

Therefore the manifold M is an η-Einstein manifold provided that the characteristic vector field ξ is harmonic.

Conversely, if we take $\widetilde{r} = -2n(n-1)$, then

$$\widetilde{Z}(\xi,Y)U = 0. \tag{8.7}$$

Thus it is clear that $\widetilde{Z}(\xi,Y)\cdot\widetilde{S} = 0$ for any vector fields $Y \in \chi(M)$.

Also assume that the relation (8.6) holds. Then using (8.6) and (3.8) yields

$$\widetilde{S}(Y,V) = -2(n-1)g(Y,V). \tag{8.8}$$

Hence it is easily shown that

$$\widetilde{Z}(\xi,Y)\cdot\widetilde{S} = 0. \tag{8.9}$$

Thus we can state the following theorem.

Theorem 8.1 *A P-Sasakian manifold admitting quarter-symmetric metric connection $\widetilde{\nabla}$ satisfies the condition $\widetilde{Z}(\xi,Y)\cdot\widetilde{S} = 0$ if and only if either the manifold is η-Einstein provided that the characteristic vector field ξ is harmonic or the scalar curvature is negative constant with respect to quarter-symmetric metric connection.*

9 Example of a 5-Dimensional *P*-Sasakian Manifold Admitting Quarter-Symmetric Metric Connection

We consider the 5-dimensional manifold $M \le \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$.

We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\quad e_2 = \frac{\partial}{\partial y},\quad e_3 = \frac{\partial}{\partial z},\quad e_4 = \frac{\partial}{\partial u},\quad e_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_i,e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j,\ i,j = 1,2,3,4,5. \end{cases}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z,e_5)$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_3, \quad \phi(e_4) = e_4, \quad \phi(e_5) = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5) = 1, \quad \phi^2 Z = Z - \eta(Z)e_5$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$$

for any vector fields $Z, U \in \chi(M)$. Thus for $e_5 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M.

Then we have

$$\begin{aligned} &[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = e_1, \\ &[e_2, e_3] = [e_2, e_4] = 0, \quad [e_2, e_5] = e_2, \\ &[e_3, e_4] = 0, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_4. \end{aligned} \tag{9.1}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula as

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).$$

Taking $e_5 = \xi$ and using (9.1), we get

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1, \\ &\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2, \\ &\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3, \\ &\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4, \\ &\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0. \end{aligned}$$

Using the above equations in (3.6) yields

$$\begin{aligned} &\widetilde{\nabla}_{e_1} e_1 = -2e_5, \quad \widetilde{\nabla}_{e_1} e_2 = 0, \quad \widetilde{\nabla}_{e_1} e_3 = 0, \quad \widetilde{\nabla}_{e_1} e_4 = 0, \quad \widetilde{\nabla}_{e_1} e_5 = 2e_1, \\ &\widetilde{\nabla}_{e_2} e_1 = 0, \quad \widetilde{\nabla}_{e_2} e_2 = -2e_5, \quad \widetilde{\nabla}_{e_2} e_3 = 0, \quad \widetilde{\nabla}_{e_2} e_4 = 0, \quad \widetilde{\nabla}_{e_2} e_5 = 2e_2, \\ &\widetilde{\nabla}_{e_3} e_1 = 0, \quad \widetilde{\nabla}_{e_3} e_2 = 0, \quad \widetilde{\nabla}_{e_3} e_3 = -2e_5, \quad \widetilde{\nabla}_{e_3} e_4 = 0, \quad \widetilde{\nabla}_{e_3} e_5 = 2e_3, \\ &\widetilde{\nabla}_{e_4} e_1 = 0, \quad \widetilde{\nabla}_{e_4} e_2 = 0, \quad \widetilde{\nabla}_{e_4} e_3 = 0, \quad \widetilde{\nabla}_{e_4} e_4 = -2e_5, \quad \widetilde{\nabla}_{e_4} e_5 = 2e_4, \\ &\widetilde{\nabla}_{e_5} e_1 = 0, \quad \widetilde{\nabla}_{e_5} e_2 = 0, \quad \widetilde{\nabla}_{e_5} e_3 = 0, \quad \widetilde{\nabla}_{e_5} e_4 = 0, \quad \widetilde{\nabla}_{e_5} e_5 = 0. \end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_1 = e_3, \quad R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1,e_4)e_1 = e_4, \quad R(e_1,e_4)e_4 = -e_1, \quad R(e_1,e_5)e_1 = e_5, \quad R(e_1,e_5)e_5 = -e_1,$$

$$R(e_2,e_3)e_2 = e_3, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_2,e_4)e_2 = e_4, \quad R(e_2,e_4)e_4 = -e_2,$$

$$R(e_2,e_5)e_2 = e_5, \quad R(e_2,e_5)e_5 = -e_2, \quad R(e_3,e_4)e_3 = e_4, \quad R(e_3,e_4)e_4 = -e_3,$$

$$R(e_3,e_5)e_3 = e_5, \quad R(e_3,e_5)e_5 = -e_3, \quad R(e_4,e_5)e_4 = e_5, \quad R(e_4,e_5)e_5 = -e_4.$$

$$\widetilde{R}(e_1,e_2)e_1 = 4e_2, \quad \widetilde{R}(e_1,e_2)e_2 = -4e_1, \quad \widetilde{R}(e_1,e_3)e_1 = 4e_3,$$

$$\widetilde{R}(e_1,e_3)e_3 = -4e_1, \quad \widetilde{R}(e_1,e_4)e_1 = 4e_4, \quad \widetilde{R}(e_1,e_4)e_4 = -4e_1,$$

$$\widetilde{R}(e_1,e_5)e_1 = 2e_5, \quad \widetilde{R}(e_1,e_5)e_5 = -2e_1, \quad \widetilde{R}(e_2,e_3)e_2 = 4e_3,$$

$$\widetilde{R}(e_2,e_3)e_3 = -4e_2, \quad \widetilde{R}(e_2,e_4)e_2 = 4e_4, \quad \widetilde{R}(e_2,e_4)e_4 = -4e_2,$$

$$\widetilde{R}(e_2,e_5)e_2 = 2e_5, \quad \widetilde{R}(e_2,e_5)e_5 = -2e_2, \quad \widetilde{R}(e_3,e_4)e_3 = 4e_4,$$

$$\widetilde{R}(e_3,e_4)e_4 = -4e_3, \quad \widetilde{R}(e_3,e_5)e_3 = 2e_5, \quad \widetilde{R}(e_3,e_5)e_5 = -2e_3,$$

$$\widetilde{R}(e_4,e_5)e_4 = 2e_5, \quad \widetilde{R}(e_4,e_5)e_5 = -2e_4.$$

Using the above expressions of curvature tensor, we get

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4,$$

$$\widetilde{S}(e_1,e_1) = \widetilde{S}(e_2,e_2) = \widetilde{S}(e_3,e_3) = \widetilde{S}(e_4,e_4) = -14, \quad \widetilde{S}(e_5,e_5) = -8.$$

Therefore $r = -20$ and $\widetilde{r} = -64$.

Thus, Theorem 3.2 is verified.

Acknowledgement The fourth author would like to thank Professor P. B. Zhao and U. C. De for their help.

References

[1] Adati, T. and Matsumoto, K., On conformally recurrent and conformally symmetric P-Sasakian manifolds, *TRU Math.*, **13**, 1977, 25–32.

[2] Adati, T. and Miyazawa, T., Some properties of P-Sasakian manifolds, *TRU Math.*, **13**, 1977, 33–42.

[3] Biswas, S. C. and De, U. C., Quarter-symmetric metric connection in SP-Sasakian manifold, *Common. Fac. Sci. Univ. Ank. Al.*, **46**, 1997, 49–56.

[4] Blair, D. E., Inversion Theory and Conformal Mapping, Stud. Math. Libr., **9**, Amer. Math. Soc., Provindence, RI, 2000.

[5] Blair, D. E., Kim, J. S. and Tripathi, M. M., On the concircular curvature tensor of a contact metric manifold, *J. Korean Math. Soc.*, **42**, 2005, 883–892.

[6] De, U. C., Özgür, C., Arslan, K., et al., On a type of P-Sasakian manifolds, *Mathematica Balkanica*, **22**, 2008, 25–36.

[7] Desmukh, S. and Ahmed, S., Para Sasakian manifolds isometrically immersed in spaces of constant curvature, *Kyungpook J. Math.*, **20**, 1980, 112–121.

[8] Golab, S., On semi-symmetric and quarter-symmetric linear connections, *Tensor* (*N.S.*), **29**, 1975, 249–254.

[9] Han, Y., Yun, H. T. and Zhao, P., Some invariants of quarter-symmetric metric connections under the projective transformation, *Filomat*, **27**(4), 2013, 679–691.

[10] Kuhnel, W., Conformal transformations between einstein spaces, Conform. Geometry, Aspects Math., **12**, Vieweg, Braunschweig, 1988, 105–146.

[11] Mandal, K. and De, U. C., Quarter-symmetric metric connection in a P-Sasakian manifold, *Analele Universitatii de Vest Timisoara, Seria Matematica Informatica*, **53**, 2015, 137–150.

[12] Matsumoto, K., Ianus, S. and Mihai, I., On P-Sasakian manifolds which admit certain tenso fields, *Publ. Math. Debrecen*, **33**, 1986, 61–65.

[13] Mukhopadhyay, S., Roy, A. K. and Barua, B., Some properties of quarter-symmetric metric connection on a Riemannian manifold, *Soochow J. Math.*, **17**, 1991, 205–211.

[14] Özgür, C., On a class of Para-Sasakian manifolds, *Turkish J. Math.*, **29**, 2005, 249–257.

[15] Rastogi, S. C., On quarter-symmetric metric connection, *C. R. Acad. Sci. Bulgar*, **31**, 1978, 811–814.

[16] Rastogi, S. C., On quarter-symmetric metric connection, *Tensor* (*N.S.*), **44**, 1987, 133–141.

[17] Satō, I., On a structure similar to the almost contact structure, *Tensor* (*N.S.*), **30**, 1976, 219–224.

[18] Satō, I. and Matsumoto, K., On P-Sasakian manifolds satisfying certain conditions, *Tensor* (*N.S.*), **33**, 1979, 173–178.

[19] Sular, S., Özgür, C. and De, U. C., Quarter-symmetric metric connection in a Kenmotsu manifold, *SUT J. of Math.*, **44**, 2008, 297–306.

[20] Szabó, Z. I., Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, I, the local version, *J. Differential. Geom.*, **17**, 1982, 531–582.

[21] Tashiro, Y., Complete Riemannian manifolds and some vector fields, *Trans. Amer. Math. Soc.*, **117**, 1965, 251–275.

[22] Yano, K., Concircular geometry I, Concircular transformation, *Proc. Imp. Acad. Tokyo*, **16**, 1940, 195–200.

[23] Yano, K., On semi-symmetric metric connection, *Rev. Roumaine Math. Pures Appl.*, **15**, 1970, 1579–1586.

[24] Yano, K. and Imai, T., Quarter-symmetric metric connections and their curvature tensors, *Tensor* (*N.S.*), **38**, 1982, 13–18.

[25] Yano, K. and Kon, M., Structures on Manifolds, **40**, World Scientific, Singapore, 1984.

[26] Yildiz, A., Turan, M. and Acet, B. E., On Para-Sasakian Manifolds, DPU Fen Bilimleri Enstitusu Dergis, **24**, 2011, 27–34.

[27] Zhen, G., Cabrerizo, J. L., Fernández, L. M. and Fernández, M., On ξ-conformally flat contact metric manifolds, *Indian J. Pure Appl. Math.*, **28**, 1997, 725–734.