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The Second Stiefel-Whitney Class of Small Covers

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Abstract Let $\pi : M^n \longrightarrow P^n$ be an n-dimensional small cover over P^n and $\lambda : \mathcal{F}(P^n) \longrightarrow \mathbb{Z}_2^n$ be its characteristic function. The author uses the symbol $c(\lambda)$ to denote the cardinal number of the image $\mathrm{Im}(\lambda)$. If $c(\lambda) = n + 1$ or $n + 2$, then a necessary and sufficient condition on the existence of spin structure on M^n is given. As a byproduct, under some special conditions, the author uses the second Stiefel-Whitney class to detect when P^n is n-colorable or $(n + 1)$-colorable.

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1 Introduction

The notion of small covers was firstly introduced by M. Davis and T. Januszkiewicz in [4]. An n-dimensional small cover is an n-dimensional smooth closed manifold M^n admitting a $\mathbb{Z}_2^n$-action and its orbit space is an n-dimensional simple convex polytope. The $\mathbb{Z}_2^n$-action on M^n also determines a characteristic function $\lambda : \mathcal{F}(P^n) \longrightarrow \mathbb{Z}_2^n$, where $\mathcal{F}(P^n)$ is the set of all facets of P^n. Conversely, given a simple polytope P^n and a characteristic function $\lambda : \mathcal{F}(P^n) \longrightarrow \mathbb{Z}_2^n$, we can also construct a small cover $M^n(P^n, \lambda)$ over P^n. Davis and Januszkiewicz showed that the $\mathbb{Z}_2$ coefficient equivariant cohomology and cohomology ring structure of a small cover over P^n can be described in terms of polytope P^n and characteristic function λ. In [13], H. Nakayama and Y. Nishimura gave a necessary and sufficient condition on the orientability of small covers. Many further works have also been carried out (eg, see [1], [3], [5], [7–10] and [14]). The motivation of this paper is to characterize the spin structure of small covers in terms of the combinatoric structures of the polytope P^n and the characteristic function λ. The spin structure is very important in differential geometry. It is the foundation of spin geometry. It has many applications to mathematical physics, also to the purely mathematical area, including spin cobordism theory, Atiyah-Singer index theorem and so on. So it is very interesting to consider under what conditions a given oriented manifold admits a spin structure. In [2], A. Borel and F. Hirzebruch proved that a spin structure exists on an oriented vector bundle E if and only if the second Stiefel-Whitney class w_2 of E vanishes.

Let $c(\lambda)$ be the cardinal number of the image $\mathrm{Im}(\lambda)$. In [4], M. Davis and T. Januszkiewicz proved: If $c(\lambda) = n$, then the tangental bundle $TM^n(P^n, \lambda)$ is a trivial bundle. In particular, it is a spin manifold. When $c(\lambda) > n$, we want to know under what condition a small cover is spin. We hope to find the necessary and sufficient condition on the existence of spin structure

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on small covers. Throughout the following, let $\{e_1, e_2, \cdots, e_n\}$ denote the standard basis of $\mathbb{Z}_2^n$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with the i-th entry 1. We mainly consider two cases: $c(\lambda) = n+1$ and $c(\lambda) = n+2$.

The case **I**: $c(\lambda) = n+1$. In this case, without loss of generality, we may assume that $\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_1 + e_2 + \cdots + e_k\}$ $(k > 1)$. We get the following result.

Theorem 1.1 *Let $c(\lambda) = n+1$.*

(i) *If the polytope P^n is n-colorable, then the small cover $M^n(P^n, \lambda)$ is spin if and only if it is orientable (i.e., k is odd).*

(ii) *If the polytope P^n is $(n+1)$-colorable, then the small cover $M^n(P^n, \lambda)$ is spin if and only if $k \equiv 3 \pmod 4$.*

As a byproduct, we also obtain the following theorem.

Theorem 1.2 *Let $c(\lambda) = n+1$ and $k \equiv 1$ or $2 \pmod 4$. Then*

(i) *P^n is n-colorable if and only if $w_2(M^n(P^n, \lambda)) = 0$.*

(ii) *P^n is $(n+1)$-colorable if and only if $w_2(M^n(P^n, \lambda)) \neq 0$.*

Remark 1.1 M. Joswig gave a necessary and sufficient condition on n-colorability of P^n in terms of combinatorics (Theorem 2.1, see Section 2). In Theorem 1.2, under the conditions that $c(\lambda) = n+1$ and $k \equiv 1$ *or* $2 \pmod 4$, we can use the second Stiefel-Whitney class to detect when P^n is n-colorable or $(n+1)$-colorable.

The case **II**: $c(\lambda) = n+2$. Let $K = \{1, 2, \cdots, 8\}$ and $\mathcal{F}^2(P^n)$ be the set of all 2-faces of P^n. Then we can do a partition of $\mathcal{F}^2(P^n)$ (see Section 4). Precisely speaking, there always exists a set $O(P^n, \lambda) \subset K$ such that $\mathcal{F}^2(P^n) = \sqcup_{d \in O(P^n, \lambda)} D_d$, where $D_d \neq \varnothing$ for $\forall d \in O(P^n, \lambda)$, and $D_d = \varnothing$ for $\forall d \in K \backslash O(P^n, \lambda)$. The set $O(P^n, \lambda)$ is uniquely determined by the combinatorial structure of P^n and the characteristic function λ. For $\forall D_d$, $d \in K$, we have a corresponding set $\tilde{D}_d$:

$$\begin{aligned}
\tilde{D}_1 &= \{(x, y, z) \in \mathbb{Z}^3\} \cap E, \\
\tilde{D}_2 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + y \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_3 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + h(z) \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_4 &= \{(x, y, z) \in \mathbb{Z}^3 | y + h(z) \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_5 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + y + h(z) \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_6 &= \{(x, y, z) \in \mathbb{Z}^3 | h(z) \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_7 &= \{(x, y, z) \in \mathbb{Z}^3 | y \equiv 0 \pmod 2\} \cap E, \\
\tilde{D}_8 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) \equiv 0 \pmod 2\} \cap E,
\end{aligned}$$

where

$$\begin{aligned}
&f(x, y) = \frac{x + y + 1}{2}, \\
&h(z) = \frac{z + 1}{2}, \\
&E = \{(x, y, z) \in \mathbb{Z}^3 | x + y \equiv z \equiv 1 \pmod 2 \text{ and } x > 0, z \geq y > 0\}.
\end{aligned}$$

In this case, without loss of generality, we may assume that $\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_1 + e_2 + \cdots + e_{i+j}, e_{i+1} + e_{i+2} + \cdots + e_{i+k}\}$ $(k \geq j)$. Then we have the following theorem.

Theorem 1.3 *Let $c(\lambda) = n + 2$. Then the small cover $M^n(P^n, \lambda)$ is spin if and only if*

$$(i, j, k) \in \cap_{d \in O(P^n,\lambda)} \tilde{D}_d.$$

This paper is organized as follows: In Section 2, we recall some basic definitions and known results; in Section 3, we prove Theorems 1.1–1.2; in Section 4, we give a proof of Theorem 1.3.

2 Preliminary

First, we recall some definitions and results in [4].

Definition 2.1 (cf. [4]) *An n-dimensional convex polytope P^n is called a simple polytope, if precisely n codimension-one faces meet at each vertex.*

Let $\mathcal{F}(P^n)$ be the set of all codimension-one faces of the polytope P^n. If $F_r \in \mathcal{F}(P^n)$, then F_r is called a facet of P^n. Let m be the cardinal number of the set $\mathcal{F}(P^n)$.

Definition 2.2 (cf. [4]) *A simple polytope P^n is called l-colorable, if there exists a function c: $\mathcal{F}(P^n) \to \{1, 2, \cdots, l\}$, such that $c(F_r) \neq c(F_s)$ when $F_r \cap F_s \neq \varnothing$, and l is the minimum integer.*

In [6], M. Joswig proved the following theorem.

Theorem 2.1 (cf. [6, Theorem 16]) *Let P^n be an n-dimensional simple polytope. Then P^n is n-colorable if and only if each 2-face of P^n has even number of vertices.*

Definition 2.3 (cf. [4]) *A function $\lambda : \mathcal{F}(P^n) \to \mathbb{Z}_2^n$ is called a characteristic function, when it satisfies the condition: If $F_1, \cdots, F_n$ are the facets meeting at a vertex of P^n, then $\lambda(F_1), \cdots, \lambda(F_n)$ are linearly independent vectors of $\mathbb{Z}_2^n$.*

If the simple polytope P^n is n-colorable, then we have a characteristic function λ such that $c(\lambda) = n$. If the simple polytope P^n is l-colorable and $l \geq n + 1$, then we may not have a characteristic function λ such that $c(\lambda) = l$. Of course, when $c(\lambda) \geq n + 1$, all elements of $\mathrm{Im}(\lambda)$ are not linearly independent.

Given a simple polytope P^n and a characteristic function $\lambda : \mathcal{F}(P^n) \to \mathbb{Z}_2^n$, we can construct a manifold $M^n(P^n, \lambda)$. For each point $p \in P^n$, let $F(p)$ be the unique face of P^n which contains p in its relative interior. We define an equivalence relation on $P^n \times \mathbb{Z}_2^n$ as follows:

$$(p, g) \sim (q, h) \Leftrightarrow p = q,\ g^{-1}h \in G_{F(p)},$$

where $G_{F(p)}$ is the subgroup generated by $\lambda(F_1), \cdots, \lambda(F_k)$ such that $F(p) = F_1 \cap \cdots \cap F_k$. The quotient space $(P^n \times \mathbb{Z}_2^n)/\sim$ is a manifold, which is denoted by $M^n(P^n, \lambda)$. The manifold $M^n(P^n, \lambda)$ is called a small cover over P^n with characteristic function λ. $\pi = p_1 \circ q^{-1} : M^n(P^n, \lambda) \to P^n$ is induced by the quotient map $q : P^n \times \mathbb{Z}_2^n \to M^n(P^n, \lambda)$ and the projection $p_1 : P^n \times \mathbb{Z}_2^n \to P^n$.

In [4], M. Davis and T. Januszkiewicz proved the following results.

Theorem 2.2 (cf. [4, Theorem 3.1]) *Let $\pi : M^n(P^n, \lambda) \to P^n$ be a small cover over a simple polytope P^n and $b_1(M^n) := \dim_{\mathbb{Z}_2} H_1(M^n; \mathbb{Z}_2)$. Then $b_1(M^n) = m - n$, where m is the cardinal number of the set $\mathcal{F}(P^n)$.*

Theorem 2.3 (cf. [4, Theorem 4.14]) *Let $\pi : M^n(P^n, \lambda) \to P^n$ be a small cover over a simple polytope P^n. Then*

$$H^*(M^n(P^n, \lambda); \mathbb{Z}_2) = \mathbb{Z}_2[v_1, v_2, \cdots, v_m]/(I + J).$$

The symbols $v_1, v_2, \cdots, v_m$ bijectively correspond to $F_1, F_2, \cdots, F_m \in \mathcal{F}(P^n)$ ($|\mathcal{F}(P^n)| = m$), respectively. The ideal I is generated by the monomials

$$v_{s_1} \cdot v_{s_2} \cdot \cdots \cdot v_{s_l},$$

if

$$F_{s_1} \cap F_{s_2} \cap \cdots \cap F_{s_l} = \varnothing.$$

The ideal J is generated by the polynomials

$$\lambda_u = \lambda_{u1} \cdot v_1 + \lambda_{u2} \cdot v_2 + \cdots + \lambda_{um} \cdot v_m, \ 1 \leq u \leq n,$$

where $(\lambda_{uv})_{n\times m}$ denotes the matrix which is determined by the characteristic function λ.

Corollary 2.1 (cf. [4, Corollary 6.8]) *Let $\pi : M^n(P^n, \lambda) \to P^n$ be a small cover over a simple convex polytope P^n. Then*

$$w(M^n) = j^* \prod_{r=1}^{m} (1 + v_r),$$

where j^* denotes the projection

$$j^* : \mathbb{Z}_2[v_1, v_2, \cdots, v_m] \to \mathbb{Z}_2[v_1, v_2, \cdots, v_m]/(I + J).$$

Hence, we can know that

$$w_1(M^n) = j^*\Big(\sum_{1\leq r\leq m} v_r\Big);$$
$$w_2(M^n) = j^*\Big(\sum_{1\leq r<s\leq m} v_r \cdot v_s\Big).$$

Proposition 2.1 (cf. [4, Proposition 3.10]) *Let $\pi : M^n(P^n, \lambda) \to P^n$ be a small cover over a simple convex polytope P^n. For each face F of P^n, the class $[M_F]$ is not zero in $H_*(M^n(P^n, \lambda), \mathbb{Z}_2)$, where $M_F := \pi^{-1}(F)$ is a submanifold of $M^n(P^n, \lambda)$.*

If F^l is a l-face (l-dimensional face of P^n), then F^l is the transversal intersection of $(n - l)$ facets. Let v be a vertex of F^l. Then there is an $(n - l)$-face $F^{(n-l)}$ such that $F^l \cap F^{(n-l)} = v$.

Remark 2.1 The 1-dimensional class v_r is Poincaré dual to $[M_{F_r}]$ ($M_{F_r} := \pi^{-1}(F_r)$), where $F_r \in \mathcal{F}(P^n)$. As is well-known, the cup product is Poincaré dual to transversal intersection. Hence, the cup product of the duality of $[M_F]$ with the duality of $[M_{F'}]$ is the duality of $[M_{F\cap F'}]$, if F and F' intersect transversely and zero otherwise. In particular, if $F \cap F' = v$, τ_F and $\tau_{F'}$ are the Poincaré duality of $[M_F]$ and $[M_{F'}]$ respectively, then $\tau_F \cdot \tau_{F'} = \mu_M$, where $\cdot$ represents the cup product, μ_M is the generator of $H^n(M^n(P^n, \lambda), \mathbb{Z}_2)$.

Now we can prove a useful lemma.

Lemma 2.1 *Let $a \in H^2(M^n(P^n, \lambda), \mathbb{Z}_2)$.*
(i) *If $a \cdot \tau_{F^2} = 0$ for $\forall F^2 \in \mathcal{F}^2(P^n)$, then $a = 0$.*
(ii) *If there exists a 2-face $F^2 \in \mathcal{F}^2(P^n)$ such that $a \cdot \tau_{F^2} = \mu_M \neq 0$, then $a \neq 0$.*

Proof (i): If $a \cdot \tau_{F^2} = 0$ for any 2-face F^2, then by Poincaré duality

$$0 = \langle a \cdot \tau_{F^2},\ [M] \rangle = \langle a,\ [M] \cap \tau_{F^2} \rangle = \langle a,\ [M_{F^2}] \rangle,$$

where $[M]$ is the generator of $H_n(M^n, \mathbb{Z}_2)$. Since $\mathbb{Z}_2$ is a field, by Universal Coefficient Theorem, we have

$$H^2(M^n, \mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(H_2(M^n, \mathbb{Z}_2),\ \mathbb{Z}_2).$$

F^2 is arbitrary, hence by Proposition 2.1, it follows that $a = 0$.

(ii): If there exists a 2-face F^2 such that $a \cdot \tau_{F^2} = \mu_M \neq 0$, then obviously, $a \neq 0$.

Here we give two examples to illustrate the ideas.

Example A Let

$$F_1,\ F_2,\ F_3,\ F_4$$

denote the facets of the 3-dimensional simplex Δ^3 and the characteristic function is

$$\lambda : \mathcal{F}(\Delta^3) \longrightarrow \mathbb{Z}_2^3$$

$$\lambda(F_1) = e_1, \quad e_1 = (1, 0, 0),$$

$$\lambda(F_2) = e_2, \quad e_2 = (0, 1, 0),$$

$$\lambda(F_3) = e_3, \quad e_3 = (0, 0, 1),$$

$$\lambda(F_4) = e_1 + e_2 + e_3, \quad e_1 + e_2 + e_3 = (1, 1, 1).$$

The matrix (λ_{uv}) is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the ideal I is

$$\langle v_1 \cdot v_2 \cdot v_3 \cdot v_4 \rangle,$$

and the ideal J is

$$\langle v_1 + v_4, v_2 + v_4, v_3 + v_4 \rangle.$$

Then we have

$$w_1 = v_1 + v_2 + v_3 + v_4 = 4v_1 = 0,$$

namely the small cover $M^3(\Delta^3, \lambda)$ is orientable. Since

$$w_2 = v_1 \cdot v_2 + v_1 \cdot v_3 + v_1 \cdot v_4 + v_2 \cdot v_3 + v_2 \cdot v_4 + v_3 \cdot v_4 = 6v_1^2 = 0,$$

the small cover $M^3(\Delta^3, \lambda)$ is *spin*. In fact, the small cover $M^3(\Delta^3, \lambda)$ is $\mathbb{R}P^3$. We know that the tangent bundle of $\mathbb{R}P^3$ is trivial, so $w_l(\mathbb{R}P^3) = 0$, $1 \leq l \leq 3$.

Example B Let the polytope P^n be I^5 and let $F_1, F_2, \cdots, F_{10}$ be all facets of I^5, where $F_u \cap F_{u+5} = \varnothing$, $1 \le u \le 5$. The characteristic function $\lambda : \mathcal{F}(I^5) \to \mathbb{Z}_2^5$ is defined as follows:

$$\lambda(F_1) = e_1 + e_2 + e_3, \quad \lambda(F_2) = e_2, \quad \lambda(F_3) = e_3, \quad \lambda(F_4) = e_4, \quad \lambda(F_5) = e_5,$$

$$\lambda(F_6) = e_1, \quad \lambda(F_7) = e_2, \quad \lambda(F_8) = e_3, \quad \lambda(F_9) = e_4, \quad \lambda(F_{10}) = e_3 + e_4 + e_5.$$

By computation, we can get $w_1 = 0$, $w_2 = v_1 \cdot v_5 \neq 0$. Hence, the small cover $M^5(I^5, \lambda)$ is not a spin manifold.

3 The proof of Theorems 1.1–1.2

We consider a simple case. If $c(\lambda) = n$, then the polytope P^n is n-colorable. Let $\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n\}$. We assume that

$$\lambda^{-1}(e_u) = \{F_{k_{u,1}}, F_{k_{u,2}}, \cdots, F_{k_{u,l_u}}\}, \quad 1 \le u \le n,$$

namely, $\lambda^{-1}(e_1), \lambda^{-1}(e_2), \cdots, \lambda^{-1}(e_n)$ is a partition of $\{F_1, F_2, \cdots, F_m\}$ and $\{k_{1,1}, k_{1,2}, \cdots, k_{1,l_1}\}, \cdots, \{k_{n,1}, k_{n,2}, \cdots, k_{n,l_n}\}$ is a partition of $\{1, 2, \cdots, m\}$. Let

$$A_u = v_{k_{u,1}} + v_{k_{u,2}} + \cdots + v_{k_{u,l_u}}, \quad 1 \le u \le n.$$

From the characteristic function λ, we can deduce that the ideal

$$J = \langle A_1, A_2, \cdots, A_n \rangle.$$

We know

$$w_1 = \sum_{s=1}^{m} v_s = \sum_{u=1}^{n} A_u.$$

Because A_u is in the ideal J, namely $A_u = 0$, it implies $w_1 = 0$, so the small cover $M^n(P^n, \lambda)$ is orientable.

If $\lambda(F_s) = \lambda(F_t)$, then $F_s \cap F_t = \varnothing$. We have $v_s \cdot v_t \in I$. Namely, $v_s \cdot v_t = 0$. Therefore,

$$v_{k_{u,l_1}} \cdot v_{k_{u,l_2}} = 0,$$

where $1 \le l_1 < l_2 \le l_u$. Then we have

$$w_2 = \sum_{1 \le s < t \le m} v_s \cdot v_t = \sum_{1 \le u < v \le n} A_u \cdot A_v + \sum_{1 \le u \le n} \sum_{1 \le l_1 < l_2 \le l_u} v_{k_{u,l_1}} \cdot v_{k_{u,l_2}} = \sum_{1 \le u < v \le n} A_u \cdot A_v = 0,$$

namely the small cover $M^n(P^n, \lambda)$ is spin. Therefore, we obtain the following proposition.

Proposition 3.1 (cf. [4, Proposition 6.10]) *If $c(\lambda) = n$, then the small cover $M^n(P^n, \lambda)$ is spin.*

If $c(\lambda) = n + 1$, then the polytope P^n is n-colorable or $(n+1)$-colorable. Without loss of generality, assume that $\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_1 + e_2 + \cdots + e_k\}$. Let the symbol e_{n+1} denote $e_1 + e_2 + \cdots + e_k$. Then we can write

$$\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_{n+1}\}.$$

Furthermore, we assume that

$$\lambda^{-1}(e_u) = \{F_{k_{u,1}}, F_{k_{u,2}}, \cdots, F_{k_{u,l_u}}\}, \quad 1 \le u \le n+1.$$

Let

$$A_u = v_{k_{u,1}} + v_{k_{u,2}} + \cdots + v_{k_{u,l_u}}, \quad 1 \le u \le n+1.$$

As above, we can deduce that the ideal

$$J = \langle A_1 + A_{n+1},\ A_2 + A_{n+1}, \cdots,\ A_k + A_{n+1},\ A_{k+1},\ A_{k+2}, \cdots,\ A_n \rangle.$$

Let

$$C_1 = \{1, 2, \cdots, k, n+1\},$$
$$C_2 = \{k+1, k+2, \cdots, n\}.$$

Then we have

$$\{1, 2, \cdots, n+1\} = C_1 \sqcup C_2.$$

From the ideal J, we get

(1) if $r \in \{1, 2, \cdots, k, n+1\}$, then $A_r = A_{n+1}$;

(2) if $r \in \{k+1, k+2, \cdots, n\}$, then $A_r = 0$.

Since $\lambda^{-1}(e_1)$, $\lambda^{-1}(e_2), \cdots, \lambda^{-1}(e_{n+1})$ form a partition of $\mathcal{F}^2(P^n)$ and $|C_1| = k+1$, we have

$$w_1(M^n) = \sum_{s=1}^{m} v_s = \sum_{u=1}^{n+1} A_u = |C_1| A_{n+1} = (k+1)A_{n+1}.$$

Lemma 3.1 *If $c(\lambda) = n+1$, then $A_{n+1} \neq 0$. Furthermore, the small cover $M^n(P^n, \lambda)$ is orientable if and only if $k \equiv 1 \pmod 2$.*

Proof If $A_{n+1} = 0$, then from the ideal J, we obtain the following equalities:

$$A_1 = A_2 = \cdots = A_k = A_{n+1} = 0$$

and

$$A_{k+1} = A_{k+2} = \cdots = A_n = 0.$$

Then the dimension of $H^1(M^n, \mathbb{Z}_2)$ is $m-n-1$. But by Theorem 2.2 and Universal Coefficient Theorem, we know that the dimension of $H^1(M^n, \mathbb{Z}_2)$ is $m-n$. This is a contradiction, so $A_{n+1} \neq 0$.

By computation, we know $w_1 = (k+1)A_{n+1}$. Hence, $w_1 = 0$ if and only if $k \equiv 1 \pmod 2$.

Now we consider the question, in what situation does the small cover $M^n(P^n, \lambda)$ admit a spin structure?

If $\lambda(F_s) = \lambda(F_t)$, then $F_s \cap F_t = \varnothing$. We have $v_s \cdot v_t \in I$. Namely, $v_s \cdot v_t = 0$.

$$w_2(M) = \sum_{1 \le s < t \le m} v_s \cdot v_t = \sum_{1 \le u < v \le n+1} A_u \cdot A_v.$$

By the identities in (1) and (2), we can get

$$w_2(M) = \sum_{1 \le u < v \le n+1} A_u \cdot A_v = \binom{|C_1|}{2} A_{n+1}^2 = \frac{k(k+1)}{2} A_{n+1}^2.$$

Lemma 3.2 *If $c(\lambda) = n+1$, then $w_2(M) = \frac{k(k+1)}{2} A_{n+1}^2$.*

Now we choose an arbitrary 2-dimensional face F^2. Since P^n is simple, F^2 is the intersection of $n-2$ facets and the images under λ of the $(n-2)$ facets give $n-2$ vectors. We denote the set of the $n-2$ vectors by $\mathrm{Im}(F^2)$. Each vertex of F^2 is the intersection of F^2 with other two facets. Let $\mathcal{F}_s = \lambda^{-1}(e_s)$, $s \in \{1, \cdots, n+1\}$. We can give a partition of the vertices of F^2:

$$\begin{aligned}
B_1(F^2) &= \{X \cap Y \cap F^2 | X \in \mathcal{F}_p, Y \in \mathcal{F}_q;\ \forall p, q \in C_1, (p < q) \text{ with } e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)\},\\
B_2(F^2) &= \{X \cap Y \cap F^2 | X \in \mathcal{F}_p, Y \in \mathcal{F}_q;\ \forall p \in C_1,\ \forall q \in C_2 \text{ with } e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)\},\\
B_3(F^2) &= \{X \cap Y \cap F^2 | X \in \mathcal{F}_p, Y \in \mathcal{F}_q;\ \forall p, q \in C_2, (p < q) \text{ with } e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)\}.
\end{aligned}$$

We use the symbol $g_t(F^2)$ to denote the cardinal number of the set $B_t(F^2)$ $(1 \le t \le 3)$, and use τ_{F^2} to denote the Poincaré duality of $[M_{F^2}]$. Actually, this defines a function

$$g_t : \mathcal{F}^2(P^n) \longrightarrow \mathbb{Z},$$

where $\mathcal{F}^2(P^n)$ is the set of all 2-faces of P^n. Note that when $p \neq q$ and $e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)$, by definitions of A_p, A_q and Poincaré duality (see Remark 2.1), we have

$$\begin{aligned}
A_p \cdot A_q \cdot \tau_{F^2} &= (v_{k_{p,1}} + v_{k_{p,2}} + \cdots + v_{k_{p,l_p}}) \cdot (v_{k_{q,1}} + v_{k_{q,2}} + \cdots + v_{k_{q,l_q}}) \cdot \tau_{F^2}\\
&= \#\{X \cap Y \cap F^2 | X \in \mathcal{F}_p, Y \in \mathcal{F}_q\} \cdot \mu_M,
\end{aligned}$$

where μ_M is the generator of $H^n(M, \mathbb{Z}_2)$. By definition of set B_1, we get

$$\sum_{p,q \in C_1, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} \equiv g_1(F^2) \mu_M \pmod 2.$$

Since $|\mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)| = 3$, one may suppose that $\mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2) = \{e_{p_1}, e_{q_1}, e_{r_1}\}$.

(i) If $p_1, q_1, r_1 \in C_1$, then by the identity $A_r = A_{n+1}$ in (1), we have

$$\sum_{p,q \in C_1, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} = \binom{3}{2} \cdot A_{n+1}^2 \cdot \tau_{F^2} = 3 \cdot A_{n+1}^2 \cdot \tau_{F^2}.$$

(ii) If $p_1, q_1 \in C_1$ and $r_1 \in C_2$, then

$$\sum_{p,q \in C_1, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} = \binom{2}{2} \cdot A_{n+1}^2 \cdot \tau_{F^2} = A_{n+1}^2 \cdot \tau_{F^2}.$$

(iii) If $p_1 \in C_1$, $q_1, r_1 \in C_2$ or $p_1, q_1, r_1 \in C_2$, then

$$\sum_{p,q \in C_1, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} = 0,$$

and $g_1(F^2) = 0$ $(B_1(F^2) = \varnothing)$. Naturally, we have $A_{n+1}^2 \cdot \tau_{F^2} = 0 = g_1(F^2)\mu_M$.

From (i)–(iii), we can infer that:

$$\sum_{p,q \in C_1, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda) \backslash \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} \equiv A_{n+1}^2 \cdot \tau_{F^2} \equiv g_1(F^2) \mu_M \pmod 2.$$

By the identity $A_r = 0$ in (2) and the sets B_2, B_3, we have

$$0 \equiv \sum_{p\in C_1, q\in C_2}^{e_p, e_q \in \mathrm{Im}(\lambda)\setminus \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} \equiv g_2(F^2)\mu_M \pmod 2,$$

$$0 \equiv \sum_{p,q\in C_2, p<q}^{e_p, e_q \in \mathrm{Im}(\lambda)\setminus \mathrm{Im}(F^2)} A_p \cdot A_q \cdot \tau_{F^2} \equiv g_3(F^2)\mu_M \pmod 2.$$

It follows that

$$g_2(F^2) \equiv 0 \pmod 2,$$
$$g_3(F^2) \equiv 0 \pmod 2.$$

Let $f_0(F^2)$ be the number of the vertices of F^2, then we know

$$g_1(F^2) + g_2(F^2) + g_3(F^2) = f_0(F^2).$$

Since $g_2(F^2) \equiv g_3(F^2) \equiv 0 \pmod 2$, we can obtain

$$g_1(F^2) \equiv f_0(F^2) \pmod 2.$$

Lemma 3.3 *If $c(\lambda) = n+1$, then*
(i) *P^n is n-colorable if and only if $A_{n+1}^2 = 0$;*
(ii) *P^n is $(n+1)$-colorable if and only if $A_{n+1}^2 \neq 0$.*

Proof (i) By Theorem 2.1, if P^n is n-colorable, then $g_1(F^2) \equiv f_0(F^2) \equiv 0 \pmod 2$. For any τ_{F^2}, $A_{n+1}^2 \cdot \tau_{F^2} = g_1(F^2) \cdot \mu_M = 0$. From Lemma 2.1, it follows that $A_{n+1}^2 = 0$.

(ii) If P^n is $(n+1)$-colorable, then by Theorem 2.1, there exists a 2-face F^2 such that $f_0(F^2) \equiv 1 \pmod 2$. Namely, $g_1(F^2) \equiv 1 \pmod 2$. We know $A_{n+1}^2 \cdot \tau_{F^2} = g_1(F^2) \cdot \mu_M = \mu_M \neq 0$. Hence, by Lemma 2.1, we have $A_{n+1}^2 \neq 0$.

If $c(\lambda) = n+1$, then P^n is n-colorable or $(n+1)$-colorable. So the sufficiency is obvious.

When $k \equiv 1 \pmod 4$ or $k \equiv 2 \pmod 4$, one knows $w_2(M^n) = A_{n+1}^2$ by Lemma 3.2. It is immediate that Theorem 1.2 follows from Lemma 3.3.

Proof of Theorem 1.1 From Lemma 3.1, we know $w_1(M) = 0$ if and only if $k \equiv 1 \pmod 2$. By Lemma 3.2, we have $w_2(M^n) = \frac{k(k+1)}{2} A_{n+1}^2$.

(i) P^n is n-colorable if and only if $A_{n+1}^2 = 0$. Hence, the small cover $M^n(P^n, \lambda)$ is spin if and only if it is orientable.

(ii) P^n is $(n+1)$-colorable if and only if $A_{n+1}^2 \neq 0$. So the small cover $M^n(P^n, \lambda)$ is spin if and only if $k \equiv 1 \pmod 2$ and $\frac{k(k+1)}{2} \equiv 0 \pmod 2$. Namely, $k \equiv 3 \pmod 4$.

4 The proof of Theorem 1.3

If $c(\lambda) = n+2$, then the simple polytope can be n-colorable, $(n+1)$-colorable or $(n+2)$-colorable. Without loss of generality, assume that $\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_1 + e_2 + \cdots + e_{i+j}, e_{i+1} + e_{i+2} + \cdots + e_{i+k}\}$ $(k \geq j)$. Let the symbols e_{n+1}, e_{n+2} denote $e_1 + e_2 + \cdots + e_{i+j}$, $e_{i+1} + e_{i+2} + \cdots + e_{i+k}$ respectively. Then we can write

$$\mathrm{Im}(\lambda) = \{e_1, e_2, \cdots, e_n, e_{n+1}, e_{n+2}\}.$$

Furthermore, we assume that

$$\lambda^{-1}(e_u) = \{F_{k_{u,1}}, F_{k_{u,2}}, \cdots, F_{k_{u,l_u}}\}, \quad 1 \le u \le n+2.$$

Let

$$A_u = v_{k_{u,1}} + v_{k_{u,2}} + \cdots + v_{k_{u,l_u}}, \quad 1 \le u \le n+2.$$

We can deduce that the ideal $J = \langle A_1 + A_{n+1}, A_2 + A_{n+1}, \cdots, A_i + A_{n+1}, A_{i+1} + A_{n+1} + A_{n+2}, A_{i+2} + A_{n+1} + A_{n+2}, \cdots, A_{i+j} + A_{n+1} + A_{n+2}, A_{i+j+1} + A_{n+2}, A_{i+j+2} + A_{n+2}, \cdots, A_{i+k} + A_{n+2}, A_{i+k+1}, \cdots, A_n\rangle$.

Let

$$\begin{aligned} C_1 &= \{1, 2, \cdots, i, n+1\}, \\ C_2 &= \{i+1, i+2, \cdots, i+j\}, \\ C_3 &= \{i+j+1, i+j+2, \cdots, i+k, n+2\}, \\ C_4 &= \{i+k+1, i+k+2, \cdots, n\}. \end{aligned}$$

Then we have

$$\{1, 2, \cdots, n+2\} = C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4,$$

giving a partition of $\{1, 2, \cdots, n+2\}$. From the ideal J, we have the following four identities:

(3) if $r \in C_1$, then $A_r = A_{n+1}$;

(4) if $r \in C_2$, then $A_r = A_{n+1} + A_{n+2}$;

(5) if $r \in C_3$, then $A_r = A_{n+2}$;

(6) if $r \in C_4$, then $A_r = 0$.

Since $|C_1| = i+1$, $|C_2| = j$ and $|C_3| = k - j + 1$ $(k \ge j)$, we have

$$\begin{aligned} w_1(M) &= \sum_{r=1}^{m} v_r = \sum_{u=1}^{n+2} A_u \\ &= |C_1|A_{n+1} + |C_2|(A_{n+1} + A_{n+2}) + |C_3|A_{n+2} \text{ (by identities in (3)–(6))} \\ &= (i+1)A_{n+1} + j(A_{n+1} + A_{n+2}) + (k-j+1)A_{n+2} \\ &= (i+j+1)A_{n+1} + (k+1)A_{n+2}. \end{aligned}$$

Let $E = \{(x, y, z) \in \mathbb{Z}^3 | x + y \equiv z \equiv 1 \pmod 2 \text{ and } x > 0, z \ge y > 0\}$.

Lemma 4.1 *If $c(\lambda) = n+2$, then $A_{n+1} \ne 0$, $A_{n+2} \ne 0$ and $A_{n+1} + A_{n+2} \ne 0$. Furthermore, the small cover $M^n(P^n, \lambda)$ is orientable if and only if $(i, j, k) \in E$.*

Proof It is a similar argument way to Lemma 3.1.

Similarly, if $\lambda(F_s) = \lambda(F_t)$, then $F_s \cap F_t = \varnothing$. We have $v_s \cdot v_t \in I$. Namely, $v_s \cdot v_t = 0$. Hence,

$$w_2(M^n) = \sum_{1 \le s < t \le m} v_s \cdot v_t = \sum_{1 \le u < v \le n+2} A_u \cdot A_v.$$

By the identities in (3)–(6), we have

$$\begin{aligned} w_2(M^n) =&\binom{|C_1|}{2}A_{n+1}^2 + \binom{|C_2|}{2}(A_{n+1}+A_{n+2})^2 \\ &+ \binom{|C_3|}{2}A_{n+2}^2 + |C_1||C_2|A_{n+1}(A_{n+1}+A_{n+2}) \\ &+ |C_1||C_3|A_{n+1}A_{n+2} + |C_2||C_3|(A_{n+1}+A_{n+2})A_{n+2} \\ =&\frac{i(i+1)}{2}A_{n+1}^2 + \frac{j(j-1)}{2}(A_{n+1}+A_{n+2})^2 \\ &+ \frac{(k-j+1)(k-j)}{2}A_{n+2}^2 + (i+1)jA_{n+1}(A_{n+1}+A_{n+2}) \\ &+ (i+1)(k-j+1)A_{n+1}A_{n+2} + j(k-j+1)(A_{n+1}+A_{n+2})A_{n+2} \\ =&\frac{(i+j)(i+j+1)}{2}A_{n+1}^2 + \frac{k(k+1)}{2}A_{n+2}^2 \\ &+ [(i+1)(k+1) + j(k-j+1)]A_{n+1}A_{n+2}. \end{aligned}$$

If $w_1(M^n) = 0$, then $i + j \equiv 1 \pmod 2$ and $k \equiv 1 \pmod 2$. Hence,

$$w_2(M^n) = \frac{i+j+1}{2}A_{n+1}^2 + \frac{k+1}{2}A_{n+2}^2 + jA_{n+1}\cdot A_{n+2}.$$

Lemma 4.2 *If $c(\lambda) = n + 2$ and $w_1(M^n) = 0$, then*

$$w_2(M^n) = \frac{i+j+1}{2}A_{n+1}^2 + \frac{k+1}{2}A_{n+2}^2 + jA_{n+1}\cdot A_{n+2}.$$

Now we choose an arbitrary 2-dimensional face F^2. Since P^n is simple, F^2 is the intersection of $n-2$ facets and the images under λ of the $(n-2)$ facets give $n-2$ vectors. We denote the set of the $n-2$ vectors by $\mathrm{Im}(F^2)$. Each vertex of F^2 is the intersection of F^2 with other two facets. Let $\mathcal{F}_s = \lambda^{-1}(e_s)$, $s \in \{1, 2, \cdots, n+2\}$. We define three subsets formed by vertices of F^2 as follows:

$$\begin{aligned} B_4(F^2,s,t) &= \{X\cap Y\cap F^2 | X\in\mathcal{F}_s, Y\in\mathcal{F}_t;\ s,t\in C_1, (s<t) \text{ with } e_s, e_t \in \mathrm{Im}(\lambda)\backslash\mathrm{Im}(F^2)\}, \\ B_5(F^2,s,t) &= \{X\cap Y\cap F^2 | X\in\mathcal{F}_s, Y\in\mathcal{F}_t;\ s\in C_1,\ t\in C_3 \text{ with } e_s, e_t \in \mathrm{Im}(\lambda)\backslash\mathrm{Im}(F^2)\}, \\ B_6(F^2,s,t) &= \{X\cap Y\cap F^2 | X\in\mathcal{F}_s, Y\in\mathcal{F}_t;\ s,t\in C_3, (s<t) \text{ with } e_s, e_t \in \mathrm{Im}(\lambda)\backslash\mathrm{Im}(F^2)\}. \end{aligned}$$

Let $\tilde{g}_4(F^2,s,t)$ denote the cardinal number of the set $B_4(F^2,s,t)$ and τ_{F^2} denote the Poincaré duality of $[M_{F^2}]$. Actually, we can use $\tilde{g}_4(F^2,s,t)$ to define a function

$$g_4 : \mathcal{F}^2(P^n) \longrightarrow \mathbb{Z}_2 = \{0,1\},$$

where $\mathcal{F}^2(P^n)$ is the set of all 2-faces of P^n, such that if $\tilde{g}_4(F^2,s,t)$ $(s<t)$ is even, then $g_4(F^2) = 0$, and if $\tilde{g}_4(F^2,s,t)$ is odd, then $g_4(F^2) = 1$.

We claim that g_4 is well-defined. For this, it suffices to show that for each $F^2 \in \mathcal{F}^2(P^n)$, the odevity of $g_4(F^2,s,t)$ does not depend on the choices of s, t.

If there exists another pair s_1, $t_1 \in C_1$ $(s_1 < t_1)$ with e_{s_1}, $e_{t_1} \in \mathrm{Im}(\lambda)\backslash\mathrm{Im}(F^2)$, then by the identity $A_r = A_{n+1}$ in (3), we have

$$\begin{aligned} A_s\cdot A_t\cdot\tau_{F^2} &= A_{n+1}^2\cdot\tau_{F^2}, \\ A_{s_1}\cdot A_{t_1}\cdot\tau_{F^2} &= A_{n+1}^2\cdot\tau_{F^2}. \end{aligned}$$

Similar to the argument of the case: $c(\lambda) = n+1$. By Poincaré duality (see Remark 2.1), it follows that

$$A_s \cdot A_t \cdot \tau_{F^2} = \tilde{g}_4(F^2, s, t)\mu_M,$$
$$A_{s_1} \cdot A_{t_1} \cdot \tau_{F^2} = \tilde{g}_4(F^2, s_1, t_1)\mu_M.$$

Hence,

$$\tilde{g}_4(F^2, s, t) \equiv \tilde{g}_4(F^2, s_1, t_1) \pmod 2.$$

In a similar way as above, we can also use $B_5(F^2, s, t)$, $B_6(F^2, s, t)$ to define the functions g_5, $g_6 : \mathcal{F}^2(P^n) \longrightarrow \mathbb{Z}_2$, respectively.

By the identity in (3), we can get

$$A^2_{n+1} \cdot \tau_{F^2} \equiv A_s \cdot A_t \cdot \tau_{F^2} \equiv \tilde{g}_4(F^2, s, t)\mu_M \equiv g_4(F^2)\mu_M \pmod 2.$$

Similarly, by the identities in (3) and (5), we have the following:

$$A_{n+1} \cdot A_{n+2} \cdot \tau_{F^2} \equiv g_5(F^2)\mu_M \pmod 2,$$
$$A^2_{n+2} \cdot \tau_{F^2} \equiv g_6(F^2)\mu_M \pmod 2.$$

Moreover, by Lemma 4.2, we know

$$w_2(M^n) = \frac{i+j+1}{2} A^2_{n+1} + \frac{k+1}{2} A^2_{n+2} + j A_{n+1} \cdot A_{n+2}.$$

Therefore, we obtain

$$\begin{aligned} w_2 \cdot \tau_{F^2} &= \frac{i+j+1}{2} \cdot A^2_{n+1} \cdot \tau_{F^2} + \frac{k+1}{2} \cdot A^2_{n+2} \cdot \tau_{F^2} + j \cdot A_{n+1} \cdot A_{n+2} \cdot \tau_{F^2} \\ &= \Big[\frac{i+j+1}{2} \cdot g_4(F^2) + j \cdot g_5(F^2) + \frac{k+1}{2} \cdot g_6(F^2)\Big]\mu_M. \end{aligned}$$

This gives the following lemma.

Lemma 4.3 *If $c(\lambda) = n+2$ and $w_1(M^n) = 0$, then*

$$w_2 \cdot \tau_{F^2} = \Big[\frac{i+j+1}{2} \cdot g_4(F^2) + j \cdot g_5(F^2) + \frac{k+1}{2} \cdot g_6(F^2)\Big]\mu_M.$$

Assume that

$$\begin{aligned} D_1 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_5(F^2) = g_6(F^2) = 0\}, \\ D_2 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_5(F^2) = 1,\ g_6(F^2) = 0\}, \\ D_3 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_6(F^2) = 1,\ g_5(F^2) = 0\}, \\ D_4 &= \{F^2 \in \mathcal{F}^2(P^n) | g_5(F^2) = g_6(F^2) = 1,\ g_4(F^2) = 0\}, \\ D_5 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_5(F^2) = g_6(F^2) = 1\}, \\ D_6 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_5(F^2) = 0,\ g_6(F^2) = 1\}, \\ D_7 &= \{F^2 \in \mathcal{F}^2(P^n) | g_4(F^2) = g_6(F^2) = 0,\ g_5(F^2) = 1\}, \\ D_8 &= \{F^2 \in \mathcal{F}^2(P^n) | g_5(F^2) = g_6(F^2) = 0,\ g_4(F^2) = 1\}. \end{aligned}$$

Obviously, $\mathcal{F}^2(P^n) = \cup_{d\in K} D_d$, where $K = \{1, 2, \cdots, 8\}$. For each set D_d $(1 \le d \le 8)$, we can give a corresponding set $\tilde{D}_d$ as follows:

$$\begin{aligned}
\tilde{D}_1 &= \{(x, y, z) \in \mathbb{Z}^3\} \cap E,\\
\tilde{D}_2 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + y \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_3 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + h(z) \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_4 &= \{(x, y, z) \in \mathbb{Z}^3 | y + h(z) \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_5 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) + y + h(z) \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_6 &= \{(x, y, z) \in \mathbb{Z}^3 | h(z) \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_7 &= \{(x, y, z) \in \mathbb{Z}^3 | y \equiv 0 \pmod 2\} \cap E,\\
\tilde{D}_8 &= \{(x, y, z) \in \mathbb{Z}^3 | f(x, y) \equiv 0 \pmod 2\} \cap E,
\end{aligned}$$

where

$$\begin{aligned}
f(x, y) &= \frac{x + y + 1}{2},\\
h(z) &= \frac{z + 1}{2}.
\end{aligned}$$

Lemma 4.4 *Assume that $D_d \neq \varnothing$ for some $d \in K$. If $F^2 \in D_d$, then $w_2 \cdot \tau_{F^2} = 0$ if and only if $(i, j, k) \in \tilde{D}_d$.*

Proof We only give the proof of the case $d = 2$, since the arguments of other cases are similar. We would like to leave them as exercises to readers.

Assume that $d = 2$. By Lemma 4.3, we have

$$w_2(M^n) \cdot \tau_{F^2} = \Big[\frac{i + j + 1}{2} \cdot g_4(F^2) + j \cdot g_5(F^2) + \frac{k + 1}{2} \cdot g_6(F^2)\Big]\mu_M.$$

If $F^2 \in D_2$, then $g_4(F^2) = g_5(F^2) = 1,\ g_6(F^2) = 0$. Hence, we have

$$w_2 \cdot \tau_{F^2} = \Big[\frac{i + j + 1}{2} + j\Big]\mu_M.$$

We can infer that $w_2 \cdot \tau_{F^2} = 0$ if and only if $\frac{i+j+1}{2} + j \equiv 0 \pmod 2$, namely $(i, j, k) \in \tilde{D}_2$.

We know $\mathcal{F}^2(P^n) = \cup_{d\in K} D_d$, where $K = \{1, 2, \cdots, 8\}$. Hence, for each $F^2 \in \mathcal{F}^2(P^n)$, there exists a unique $d \in K$ such that $F^2 \in D_d$. So there is uniquely a function

$$\alpha : \mathcal{F}^2(P^n) \longrightarrow K$$

such that for each $F^2 \in \mathcal{F}^2(P^n)$, $F^2 \in D_{\alpha(F^2)}$. Let $O(P^n, \lambda) = \mathrm{Im}(\alpha) \subset K$. Then we have $\mathcal{F}^2(P^n) = \sqcup_{d\in O(P^n,\lambda)} D_d$. Namely, if $d \in O(P^n, \lambda)$, then $D_d \neq \varnothing$, and if $d \in K \backslash O(P^n, \lambda)$, then $D_d = \varnothing$. Since the sets D_d $(1 \le d \le 8)$ are determined by the combinatorial structure of P^n and the characteristic function λ. Hence, we can know that the set $O(P^n, \lambda)$ is uniquely determined by the combinatorial structure of P^n and the characteristic function λ.

Proof of Theorem 1.3 From Lemma 2.1, one knows that $w_2 = 0$ if and only if $w_2 \cdot \tau_{F^2} = 0$ for $\forall F^2 \in \mathcal{F}^2(P^n)$. By Lemma 4.4, the conclusion holds.

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