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Semistable Twisted Holomorphic Chains on Non-Compact Kähler Manifolds*

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Abstract In this paper, the author proves a generalized Donaldson-Uhlenbeck-Yau theorem for twisted holomorphic chain on a non-compact Kähler manifold. As an application, the author obtains a Bogomolov type Chern numbers inequality for semistable twisted holomorphic chain.

Keywords Hermitian metric, Twisted holomorphic chain, Chern numbers inequality
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1 Introduction

Let (M,ω) be a Kähler manifold with complex dimension m. A twisted holomorphic chain consists of a finite number of holomorphic bundles E_i over M and bundle morphisms $\phi_i \in \mathrm{Hom}(E_i\otimes\widetilde{E}_i, E_{i-1})$, where $\{\widetilde{E}_i\}$ is a collection of twisting holomorphic bundles. For simplicity, we denote by $\mathbf{E}=(E_0,E_1,\cdots,E_n)$ the $(n+1)$-tuple of holomorphic bundles E_i over M, by $\phi=(\phi_1,\phi_2,\cdots,\phi_n)$ an n-tuple of bundle holomorphic morphisms $\phi_i\in\mathrm{Hom}(E_i\otimes\widetilde{E}_i,E_{i-1})$ $(1\le i\le n)$, and by $\mathbf{C}=(\mathbf{E},\phi)$ the twisted holomorphic chain. Throughout this paper, we fix an n-tuple $\widetilde{\mathbf{H}}=(\widetilde{H}_1,\cdots,\widetilde{H}_n)$ of Hermitian metrics $\widetilde{H}_i$ on the twisting bundles $\widetilde{E}_i$ $(1\le i\le n)$. Given $\tau=(\tau_0,\tau_1,\cdots,\tau_n)\in R^{n+1}$, we consider the following chain τ-vortex equations

$$\begin{aligned}
&\sqrt{-1}\Lambda_\omega F_{H_0}+\frac{1}{2}\phi_1\circ\phi_1^{*H}=\tau_0\mathrm{Id}_{E_0},\\
&\sqrt{-1}\Lambda_\omega F_{H_i}-\frac{1}{2}(\phi_i^{*H}\circ\phi_i-\phi_{i+1}\circ\phi_{i+1}^{*H})=\tau_i\mathrm{Id}_{E_i},\\
&\sqrt{-1}\Lambda_\omega F_{H_n}-\frac{1}{2}\phi_n^{*H}\circ\phi_n=\tau_n\mathrm{Id}_{E_n},
\end{aligned}\tag{1.1}$$

where $1\le i\le n-1$, Λ_ω denotes the contraction with the Kähler metric ω, H_i is a Hermitian metric on the holomorphic bundle E_i, F_{H_i} is the curvature form of the Chern connection ∇_{H_i} on E_i with respect to the metric H_i, and $\phi_i^{*H}:E_{i-1}\to E_i\otimes\widetilde{E}_i$ is the adjoint morphism of ϕ_i with respect to the Hermitian metrics H_{i-1} on E_{i-1} and $H_i\otimes\widetilde{H}_i$ on $E_i\otimes\widetilde{E}_i$. Moreover, for each $1\le i\le n$, ϕ_i and ϕ_i^{*H} can be seen as morphisms $\phi_i:E_i\to E_{i-1}\otimes\widetilde{E}_i^*$ and $\phi_i^{*H}:E_{i-1}\otimes\widetilde{E}_i^*\to E_i$,

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so $\phi_i^{*H} \circ \phi_i : E_i \to E_i$ makes sense too. In the following, we denote the $(n+1)$-tuple of Hermitian metrics H_i on E_i by $\mathbf{H} = (H_0, H_1, \cdots, H_n)$.

The simplest situation occurs when the chain has a single bundle, no twisting bundles and no bundle holomorphic morphisms, in which case a twisted holomorphic chain is just a holomorphic bundle E, and the chain τ-vortex equation is the Hermitian-Einstein equation. When (M, ω) is a compact Kähler manifold, the Donaldson-Uhlenbeck-Yau theorem states that the stability of holomorphic vector bundle (in the sense of Mumford-Takemoto) implies the solvability of the Hermitian-Einstein equation. This theorem was proved by Narasimhan and Seshadri [23] for compact Riemann surface case, by Donaldson [12–13] for algebraic manifolds and by Uhlenbeck and Yau [25] for general compact Kähler manifolds. The classical Donaldson-Uhlenbeck-Yau theorem has many interesting generalizations (see the references [4–8, 11, 14–15, 18–20, 22, 24] for details). The twisted holomorphic chain and chain τ-vortex equations were introduced and studied by Álvarez-Cónsul and García-Prada [1–3]. They introduced a stability criterion for twisted holomorphic chains, and obtained a generalized Donaldson-Uhlenbeck-Yau theorem, relating the existence of Hermitian metrics satisfying the chain τ-vortex equations (1.1) to the stability of the twisted holomorphic chain. As an application, they (see [2]) also obtained a Bogomolov type Chern numbers inequality for a stable twisted holomorphic chain.

In this paper, we consider the case that the Kähler manifold (M, ω) is not necessarily compact, but satisfies the following three assumptions.

Assumption 1.1 (M, ω) has finite volume.

Assumption 1.2 There exists a smooth exhaustion non-negative function φ on (M, ω) with $\Delta\varphi$ bounded.

Assumption 1.3 There is an increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(x) = x$ for $x > 1$, such that if f is a bounded positive function on (M, ω) with $\Delta f \geq -B$, then

$$\sup_M |f| \leq C(B)\alpha\Big(\int_M |f|\Big).$$

Furthermore, if $\Delta f \geq 0$ then $\Delta f = 0$.

The above assumptions were introduced by Simpson in the paper [24] where he studied the Higgs bundles on some non-compact Kähler manifolds. He also showed that if (M, ω) is a compact Kähler manifold, or (M, ω) is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ and the metric ω is the restriction of a smooth Kähler metric on $\overline{M}$, the above assumptions hold for (M, ω).

Given a Hermitian metric H on a holomorphic vector bundle E over (M, ω), one can define the following Chern numbers of E with respect to the Hermitian metric H by

$$C_1(E, H) = \int_M c_1(E, H) \wedge \frac{\omega^{m-1}}{(m-1)!} = \frac{\sqrt{-1}}{2\pi}\int_M \operatorname{tr} \Lambda_\omega F_H \frac{\omega^m}{m!} \tag{1.2}$$

and

$$\begin{aligned} Ch_2(E,H) &= \int_M \frac{1}{2}(c_1(E,H)^2 - 2c_2(E,H)) \wedge \frac{\omega^{m-2}}{(m-2)!} \\ &= -\frac{1}{8\pi^2}\int_M \mathrm{tr}\,(F_H \wedge F_H) \wedge \frac{\omega^{m-2}}{(m-2)!} \\ &= \frac{1}{8\pi^2}\int_M (|\sqrt{-1}\Lambda_\omega F_H|_H^2 - |F_H|_H^2)\frac{\omega^m}{m!}. \end{aligned} \tag{1.3}$$

By Chern-Weil theory, when (M,ω) is a compact Kähler manifold, we know that the above Chern numbers are independent of the metric H. If M is not compact, the above Chern numbers measured with different metrics need not be equal to a priori.

For simplicity, we denote

$$\theta_i(\mathbf{H};\tau) = \sqrt{-1}\Lambda_\omega F_{H_i} - \frac{1}{2}(\phi_i^{*H}\circ\phi_i - \phi_{i+1}\circ\phi_{i+1}^{*H}) - \tau_i \mathrm{Id}_{E_i}, \tag{1.4}$$

where we set $\phi_0 = 0$ and $\phi_{n+1} = 0$. We fix the background metrics $\mathbf{K} = (K_0, \cdots, K_n)$ on the chain $\mathbf{C} = (\mathbf{E}, \phi)$, and set the parameters $\tau = (\tau_0, \cdots, \tau_n)$. Making the assumption $\sup\limits_M \big(\sum\limits_{i=0}^n |\theta_i(\mathbf{K};\tau)|_{\mathbf{K}}\big) \leq B$, we define the τ-degree of chain $\mathbf{C}$ with respect to the metric $\mathbf{K}$ to be the real number:

$$2\pi \deg_\tau(\mathbf{C};\mathbf{K}) = \int_M \Big(\sum_{i=0}^n \mathrm{tr}\,\theta_i(\mathbf{K};\tau)\Big)\frac{\omega^m}{m!}. \tag{1.5}$$

Then the τ-slope of chain $\mathbf{C}$ is defined by

$$\mu_\tau(\mathbf{C};\mathbf{K}) = \frac{\deg_\tau(\mathbf{C};\mathbf{K})}{\sum\limits_{i=0}^n \mathrm{rank}\,E_i}. \tag{1.6}$$

The above τ-degree and τ-slope of chain were introduced by Álvarez-Cónsul and García-Prada [1] in the case that (M,ω) is a compact Kähler manifold.

A weakly holomorphic sub-chain of $\mathbf{C}$ is a twisted chain $\mathbf{C}' = (\mathbf{E}', \phi')$ such that E_i' is a saturated sub-sheaf of E_i for each $0 \leq i \leq n$, and $\phi_\alpha \circ (f_\alpha \otimes \mathrm{Id}_{\widetilde{E}_\alpha}) = f_{\alpha-1}\circ\phi_\alpha'$ for each $1 \leq \alpha \leq n$, where $f_i : E_i' \to E_i$ are the inclusion morphisms. When E_i' is a holomorphic sub-bundle of E_i for each $0 \leq i \leq n$, we call $\mathbf{C}'$ a holomorphic sub-chain of $\mathbf{C}$. The weakly holomorphic sub-chain $\mathbf{C}' \hookrightarrow \mathbf{C}$ is called proper if $0 < \sum\limits_{i=0}^n \mathrm{rank}\,E_i' < \sum\limits_{i=0}^n \mathrm{rank}\,E_i$.

If E_i' is a saturated sub-sheaf of E_i, we know that it is a sub-bundle of E_i outside a singularity set Σ_i which is a complex analytic subset in M with complex co-dimension at least 2. The metric K_i induces a metric on E_i' over $M \setminus \Sigma_i$. Let $\pi_i : E_i \to E_i'$ denote the projection onto E_i' with respect to the metric K_i, it is also defined outside Σ_i. The τ-degree and τ-slope of a weakly holomorphic sub-chain $\mathbf{C}'$ with respect to the metric $\mathbf{K}$ are defined by

$$2\pi \deg_\tau(\mathbf{C}', \mathbf{K}) = \int_M \Big(\sum_{i=0}^{n} (\operatorname{tr} \pi_i \circ \theta_i(\mathbf{K}; \tau) - |\overline{\partial}_{E_i \otimes E_i^*} \pi_i|_K^2) - \sum_{\alpha=1}^{n} |\phi_\alpha^\perp|_K^2 \Big) \frac{\omega^m}{m!},$$
$$\mu_\tau(\mathbf{C}', \mathbf{K}) = \frac{\deg_\alpha(\mathbf{C}', \mathbf{K})}{\sum\limits_{i=0}^{n} \operatorname{rank} E_i'}, \tag{1.7}$$

respectively. Here $\phi_\alpha^\perp = \pi_{\alpha-1} \circ \phi_\alpha \circ ((\mathrm{Id}_{E_\alpha} - \pi_\alpha) \otimes \mathrm{Id}_{\widetilde{E}_\alpha})$. The degree of sub-chain defined above is either a real number or $-\infty$, and if the degree is not $-\infty$, then $\pi_i \in L_1^2$ for each $0 \le i \le n$. On the other hand, a straightforward computation shows that

$$\frac{1}{2} \sum_{i=0}^{n} \operatorname{tr} (\pi_i \circ (\phi_i^{*K} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*K})) = - \sum_{\alpha=1}^{n} |\phi_\alpha^\perp|_K^2. \tag{1.8}$$

If (M, ω) is a compact Kähler manifold, by Chern-Weil theory and formula (1.8), the degree $\deg_\tau(\mathbf{C}', \mathbf{K})$ is a holomorphic invariant which is independent of the metric $\mathbf{K}$, in fact we have

$$\deg_\tau(\mathbf{C}', \mathbf{K}) = \sum_{i=0}^{n} (\deg(E_i') - \tau_i \operatorname{rank} E_i'), \tag{1.9}$$

where $\deg(E_i')$ is just the degree of the sheaf E_i'.

Definition 1.1 *Let $\mathbf{C} = (\mathbf{E}, \phi)$ be a twisted holomorphic chain over (M, ω), $\tau = (\tau_0, \tau_1, \cdots, \tau_n) \in R^{n+1}$, and $\mathbf{K} = (K_0, K_1, \cdots, K_n)$ be an $(n+1)$-tuple of Hermitian metrics on chain $\mathbf{C}$. We say that the twisted holomorphic chain $\mathbf{C}$ is analytic τ-(semi)stable with respect to the metric $\mathbf{K}$ if for all proper weakly holomorphic sub-chain $\mathbf{C}' \hookrightarrow \mathbf{C}$, we have*

$$\mu_\tau(\mathbf{C}', \mathbf{K}) < (\le) \mu_\tau(\mathbf{C}, \mathbf{K}). \tag{1.10}$$

The above analytic τ-(semi)stability was introduced by Wang and Zhang [26]. It is independent of the background metrics $\mathbf{K} = (K_0, \cdots, K_n)$ when M is compact. They proved that the analytic τ- stability implies the solvability of the chain τ-vortex equations (1.1) for twisted holomorphic chain on some non-compact Kähler manifolds which satisfy the Assumptions 1.1–1.3. In this paper, we consider the semi-stable case. We prove that if the twisted holomorphic chain is analytic τ-semi-stable then the above chain τ-vortex equations (1.1) admit an approximate solution in L^2-norm sense. Using this result, we can obtain a Bogomolov type Chern numbers inequality for analytic τ-semi-stable twisted holomorphic chain.

Theorem 1.1 *Let (M, ω) be a Kähler manifold which is not necessarily compact, but satisfies the above Assumptions* 1.1–1.3*, and* $(\mathbf{E}, \phi)$ *be a twisted holomorphic chain over M with an $(n+1)$-tuple of Hermitian metrics $\mathbf{K}$ satisfying $\sup\limits_M \big(\sum\limits_{i=0}^{n} |\Lambda_\omega F_{K_i}|_K \big) < \infty$, $\sup\limits_M \big(\sum\limits_{i=1}^{n} |\phi_i|_K^2 \big) < \infty$ and $\deg_\tau(\mathbf{C}; \mathbf{K}) = 0$. Suppose $\mathbf{C} = (\mathbf{E}, \phi)$ is analytic τ-semi-stable with respect to the initial metrics $\mathbf{K}$. Then the chain τ-vortex equations* (1.1) *admit an approximate solution in L^2-norm sense, i.e., for any small positive ε, there is an $(n+1)$-tuple of Hermitian metrics*

$\mathbf{H}_\varepsilon = (H_{0,\varepsilon}, H_{1,\varepsilon}, \cdots, H_{n,\varepsilon})$ *with* $|\overline{\partial}_{E_i}(K_i^{-1}H_{i,\varepsilon})|_{K_i} \in L^2(M,\omega)$, $H_{i,\varepsilon}$ *and* K_i *mutually bounded, and* $\sup\limits_M |\Lambda_\omega F_{H_{i,\varepsilon}}|_{H_{i,\varepsilon}} < \infty$ *for each* $i = 0, 1, \cdots, n$, *such that*

$$\int_M \sum_{i=0}^n |\theta_i(\mathbf{H}_\varepsilon; \tau)|^2_{H_\varepsilon} \frac{\omega^m}{m!} \le \varepsilon. \tag{1.11}$$

Moreover, assume that $\sum\limits_{i=1}^n |\nabla_K^{1,0}\phi_i|_K$ *and* $\sum\limits_{i=1}^n |\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}|_{\widetilde{H}_i}$ *both belong to* $L^2(M,\omega)$, *and* $\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}$ *is positive semi-definite for each* $i = 1, \cdots, n$, *where* $\widetilde{H}_i$ *are fixed Hermitian metrics on the twisting bundles* $\widetilde{E}_i$. *If* $\mathbf{C} = (\mathbf{E}, \phi)$ *is analytic* τ*-semi-stable with respect to the metrics* $\mathbf{K}$, *then the following Bogomolov type Chern numbers inequality holds*:

$$\sum_{i=0}^n \tau_i C_1(E_i, K_i) \ge 2\pi \sum_{i=0}^n Ch_2(E_i, K_i). \tag{1.12}$$

When the Kähler manifold (M, ω) is compact, the above Chern numbers inequality (1.12) was proved by Álvarez-Cónsul and García-Prada [2] for τ-stable twisted holomorphic chain. There are many results on the existence of approximate solution of Hermitian-Einstein equation on semi-stable holomorphic bundle and semi-stable Higgs bundle (see references [9–10, 16–17, 21] for details). Theorem 1.1 extends the above results to the non-compact case. The difficult part of Theorem 1.1 is to prove the existence of approximate solution of the chain τ-vortex equations (1.1). We will use the heat flow method and follow the argument used by Li and Zhang [21] in the Higgs bundles case. Even though the global approach is similar, some key estimates require new inputs because the base manifold M is not necessarily compact. The paper is organized as follows. In Section 2, we recall some estimates and preliminaries which will be used in the proof of main theorem. In Section 3, we prove the existence of approximate solution of the chain τ-vortex equations (1.1) and deduce the Bogomolov type Chern numbers inequality for analytic τ-semi-stable twisted holomorphic chain.

2 Preliminaries on Twisted Holomorphic Chain

Let $\mathbf{C} = (\mathbf{E}, \phi)$ be a twisted holomorphic chain on a Kähler manifold (M, ω), and $\mathbf{K} = (K_0, K_1, \cdots, K_n)$ be an $(n+1)$-tuple of Hermitian metrics on chain $\mathbf{C}$. We study the following evolution equations of $(n+1)$-tuple of Hermitian metrics $\mathbf{H}(t) = (H_0(t), H_1(t), \cdots, H_n(t))$ on $\mathbf{C}$ with initial metrics $\mathbf{H}(0) = \mathbf{K}$,

$$\begin{aligned}
H_0^{-1}\frac{\partial H_0}{\partial t} &= -2\Big(\sqrt{-1}\Lambda_\omega F_{H_0} + \frac{1}{2}\phi_1 \circ \phi_1^{*H} - \tau_0 \mathrm{Id}_{E_0}\Big),\\
H_i^{-1}\frac{\partial H_i}{\partial t} &= -2\Big(\sqrt{-1}\Lambda_\omega F_{H_i} - \frac{1}{2}(\phi_i^{*H} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H}) - \tau_i \mathrm{Id}_{E_i}\Big),\\
H_n^{-1}\frac{\partial H_n}{\partial t} &= -2\Big(\sqrt{-1}\Lambda_\omega F_{H_n} - \frac{1}{2}\phi_n^{*H} \circ \phi_n - \tau_n \mathrm{Id}_{E_n}\Big),
\end{aligned} \tag{2.1}$$

where $1 \le i \le n-1$. We first recall some basic estimates for the heat flow (2.1).

Proposition 2.1 (see [26, Propositions 2.1–2.2]) *Let $\mathbf{H}(t) = (H_0(t), H_1(t), \cdots, H_n(t))$ be a solution of the heat flow* (2.1). *Then*

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\sum_{i=0}^{n} \mathrm{tr}\,\theta_i(\mathbf{H}(t);\tau) = 0, \tag{2.2}$$

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\Big(\sum_{i=0}^{n} |\theta_i(\mathbf{H}(t);\tau)|^2_{H_i(t)}\Big) \geq 0, \tag{2.3}$$

and

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)\Big(\sum_{i=1}^{n} |\phi_i|^2_{H(t)}\Big) \geq 2\sum_{i=1}^{n} |\partial_H \phi_i|^2_H + C_1\Big(\sum_{i=1}^{n} |\phi_i|^2_{H(t)}\Big)^2 \\ - \max_{1\leq i\leq n}\{|\tau_i - \tau_{i-1}|\}\Big(\sum_{i=1}^{n} |\phi_i|^2_{H(t)}\Big),\end{aligned} \tag{2.4}$$

where C_1 is a positive constant depending only on $\{\mathrm{rank}(E_i)\}_{i=0}^n$ *and* $\{\mathrm{rank}(\widetilde{E}_i)\}_{i=0}^n$.

We recall the following existence of long time solution of the heat flow (2.1).

Proposition 2.2 (see [26, Proposition 3.6]) *Let (M, ω) be a Kähler manifold satisfying the Assumptions* 1.1–1.3, *and let $\mathbf{C} = (\mathbf{E}, \phi)$ be a twisted holomorphic chain over M. Suppose the initial $(n+1)$-tuple of Hermitian metrics $\mathbf{K}$ satisfy $\sup\limits_M\big(\sum\limits_{i=0}^{n} |\theta_i(\mathbf{K};\tau)|_{K_i}\big) = B_1 < \infty$. Then there is a unique solution $\mathbf{H}(t)$ to the heat equations* (2.1) *with $\mathbf{H}(0) = \mathbf{K}$ such that $\sum\limits_{i=0}^{n}\sup\limits_M(\mathrm{tr}\,(K_i^{-1}H_i) + \mathrm{tr}\,(H_i^{-1}K_i)) < \infty$ on each finite interval of time. For this solution, we have $\sup\limits_M\big(\sum\limits_{i=0}^{n} |\theta_i(\mathbf{H};\tau)|_H\big) \leq \sup\limits_M\big(\sum\limits_{i=0}^{n} |\theta_i(\mathbf{K};\tau)|_K\big)$. Furthermore, if $\Phi^2(\mathbf{K}) = \sup\limits_M\big(\sum\limits_{i=1}^{n} |\phi_i|^2_K\big) = B_2 < \infty$, then the solution $\mathbf{H}(t)$ must satisfy $\Phi^2(\mathbf{H}(t)) = \sup\limits_M\big(\sum\limits_{i=1}^{n} |\phi_i|^2_{H(t)}\big) \leq \max\{B_2, B_3\}$ for all $t \geq 0$, where B_3 is a positive constant depending only on* $\mathrm{rank}(E_i)$, $\mathrm{rank}(\widetilde{E}_i)$ *and* τ_i.

For each $a > 0$, we denote the compact subset $\{x \in M \mid \varphi(x) \leq a\}$ by M_a. Since the exhaustion function φ is smooth, Sard theorem tells us that ∂M_a is smooth for almost each a. When the boundary ∂M_a is smooth, we consider the Dirichlet boundary problem and the Neumann boundary problem of the heat flow (2.1). Then we have the long time solutions (see [26, Theorem 3.5]), i.e., there exist two families of $(n+1)$-tuple of Hermitian metrics $\mathbf{H}_a(t) = (H_{0,a}(t), \cdots, H_{n,a}(t))$ and $\widetilde{\mathbf{H}}_a(t) = (\widetilde{H}_{0,a}(t), \cdots, \widetilde{H}_{n,a}(t))$ such that

$$h_{i,a}^{-1}(t)\frac{\partial h_{i,a}(t)}{\partial t} = -2\theta_i(\mathbf{H}_a(t);\tau), \quad h_{i,a}(t)|_{t=0} = \mathrm{Id}_{E_i}, \quad h_{i,a}|_{\partial M_a} = \mathrm{Id}_{E_i}|_{\partial M_a} \tag{2.5}$$

and

$$\widetilde{h}_{i,a}^{-1}(t)\frac{\partial \widetilde{h}_{i,a}(t)}{\partial t} = -2\theta_i(\widetilde{\mathbf{H}}_a(t);\tau), \quad \widetilde{h}_{i,a}(t)|_{t=0} = \mathrm{Id}_{E_i}, \quad \frac{\partial}{\partial \nu}\widetilde{h}_{i,a}(t)|_{\partial M_a} = 0 \tag{2.6}$$

for all $t \in (0, +\infty)$ and $0 \leq i \leq n$, where $h_{i,a} = K_i^{-1}H_{i,a}$ and $\widetilde{h}_{i,a} = K_i^{-1}\widetilde{H}_{i,a}$. According to the maximum principle and the assumption $\sup\limits_{M}\big(\sum\limits_{i=0}^{n}|\theta_i(\mathbf{K};\tau)|_{K_i}\big)=B_1<\infty$, we have

$$\sup_{M_a}\Big(\sum_{i=0}^{n}|\theta_i(\mathbf{H}_a(t);\tau)|^2_{H_{i,a}(t)}\Big)\le B_1 \tag{2.7}$$

and

$$\sup_{M_a}\Big(\sum_{i=0}^{n}|\theta_i(\widetilde{\mathbf{H}}_a(t);\tau)|^2_{\widetilde{H}_{i,a}(t)}\Big)\le B_1 \tag{2.8}$$

for all $t\ge 0$ and $a>0$. Wang and Zhang proved that (see [26, Proposition 3.6]), by passing to a subsequence $a\to+\infty$, $\mathbf{H}_a(t)\to\mathbf{H}(t)$ (or $\widetilde{\mathbf{H}}_a(t)\to\mathbf{H}(t)$) in local C^∞-topology over any compact subset of $M\times[0,+\infty)$, and $\mathbf{H}(t)$ is just the solution of the evolution equations (2.1). By the Dirichlet boundary condition, we know that $\theta_i(\mathbf{H}_a(t);\tau)|_{\partial M_a}=0$ for all $t>0$. Using (2.3) and Stokes theorem, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_a}\sum_{i=0}^{n}|\theta_i(\mathbf{H}_a(t);\tau)|^2_{H_{i,a}(t)}\frac{\omega^m}{m!}\le 0 \tag{2.9}$$

for all $t>0$ and $a>0$. Now we prove the following monotonicity of $\int_M\sum\limits_{i=0}^{n}|\theta_i(\mathbf{H}(t);\tau)|^2_{H_i(t)}\frac{\omega^m}{m!}$.

Lemma 2.1 *Let* $\mathbf{H}(t)$ *be the long time solution of the heat equations* (2.1) *which is obtained in Proposition* 2.2*, then*

$$\int_M\sum_{i=0}^{n}|\theta_i(\mathbf{H}(t_1);\tau)|^2_{H_i(t_1)}\frac{\omega^m}{m!}\ge\int_M\sum_{i=0}^{n}|\theta_i(\mathbf{H}(t_2);\tau)|^2_{H_i(t_2)}\frac{\omega^m}{m!} \tag{2.10}$$

for all $0<t_1\le t_2$.

Proof We prove it by contradiction. If not, then there exist $0<t_1<t_2$ such that

$$\int_M\sum_{i=0}^{n}|\theta_i(\mathbf{H}(t_1);\tau)|^2_{H_i(t_1)}\frac{\omega^m}{m!}+\varepsilon_0\le\int_M\sum_{i=0}^{n}|\theta_i(\mathbf{H}(t_2);\tau)|^2_{H_i(t_2)}\frac{\omega^m}{m!} \tag{2.11}$$

for some positive constant ε_0. We choose a compact subset $\Omega\subset M$ such that

$$B_1\cdot\mathrm{Vol}(M\setminus\Omega)<\frac{1}{4}\varepsilon_0. \tag{2.12}$$

Since $\mathbf{H}_a(t)\to\mathbf{H}(t)$ in local C^∞-topology as $a\to+\infty$, we have

$$\Big|\int_\Omega\sum_{i=0}^{n}(|\theta_i(\mathbf{H}(t_1);\tau)|^2_{H_i(t_1)}-|\theta_i(\mathbf{H}_a(t_1);\tau)|^2_{H_{i,a}(t_1)})\frac{\omega^m}{m!}\Big|\le\frac{1}{8}\varepsilon_0 \tag{2.13}$$

and

$$\Big|\int_\Omega\sum_{i=0}^{n}(|\theta_i(\mathbf{H}(t_2);\tau)|^2_{H_i(t_2)}-|\theta_i(\mathbf{H}_a(t_2);\tau)|^2_{H_{i,a}(t_2)})\frac{\omega^m}{m!}\Big|\le\frac{1}{8}\varepsilon_0 \tag{2.14}$$

for some large a such that $\Omega \subset M_a$. By (2.7) and (2.12)–(2.14), we obtain

$$\begin{aligned}\int_{M_a}\sum_{i=0}^{n}(|\theta_i(\mathbf{H}_a(t_1);\tau)|^2_{H_{i,a}(t)})\frac{\omega^m}{m!} &\le \int_{\Omega}\sum_{i=0}^{n}(|\theta_i(\mathbf{H}_a(t_1);\tau)|^2_{H_{i,a}(t)})\frac{\omega^m}{m!} + B_1\cdot \mathrm{Vol}(M\setminus\Omega)\\ &\le \int_{\Omega}\sum_{i=0}^{n}(|\theta_i(\mathbf{H}(t_1);\tau)|^2_{H_i(t_1)})\frac{\omega^m}{m!} + \frac{3}{8}\varepsilon_0\\ &\le \int_{M}\sum_{i=0}^{n}(|\theta_i(\mathbf{H}(t_2);\tau)|^2_{H_i(t_2)})\frac{\omega^m}{m!} - \frac{5}{8}\varepsilon_0\\ &\le \int_{M_a}\sum_{i=0}^{n}(|\theta_i(\mathbf{H}_a(t_2);\tau)|^2_{H_{i,a}(t_2)})\frac{\omega^m}{m!} - \frac{1}{4}\varepsilon_0. \qquad (2.15)\end{aligned}$$

This contradicts (2.9).

Since $\mathrm{tr}\,\widetilde{h}_{i,a}$ satisfies Neumann boundary condition on ∂M_a, it holds that

$$\int_{M_a}\sqrt{-1}\Lambda_\omega(\overline{\partial}\partial\mathrm{tr}\,\widetilde{h}_{i,a})\frac{\omega^m}{m!}=0. \tag{2.16}$$

From the estimates (2.8), we see that $\widetilde{h}_{i,a}$ and $|\Lambda_\omega F_{\widetilde{H}_{i,a}}|$ are uniformly bounded for finite time intervals. Integrating the following identity

$$\Lambda_\omega\mathrm{tr}\left(\widetilde{h}_{i,a}\circ(F_{\widetilde{H}_{i,a}}-F_{K_i})+(\overline{\partial}_{E_i}\widetilde{h}_{i,a})\circ\widetilde{h}_{i,a}^{-1}\circ\partial_{K_i}\widetilde{h}_{i,a}\right)=\Lambda_\omega(\overline{\partial}\partial\mathrm{tr}\,\widetilde{h}_{i,a}) \tag{2.17}$$

over M_a, we deduce

$$\int_{M_a}|(\overline{\partial}_{E_i}\widetilde{h}_{i,a})\circ\widetilde{h}_{i,a}^{-\frac{1}{2}}|^2_{(\widetilde{H}_{i,a},\omega)}\frac{\omega^m}{m!}\le\widehat{C} \tag{2.18}$$

for finite time intervals, where $\widehat{C}$ is a constant independent of a. From this estimate, it follows that $|\overline{\partial}_{E_i}(K_i^{-1}H_i(t))|_{K_i}\in L^2(M,\omega)$ for all $t\ge 0$.

Applying $\frac{\partial}{\partial\nu}$ to both sides of the heat equation (2.6), we know that $\sum\limits_{i=0}^{n}\mathrm{tr}\,\theta_i(\widetilde{\mathbf{H}}_a(t);\tau)$ satisfies the corresponding Neumann boundary condition. So (2.2) and Stokes theorem imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_a}\sum_{i=0}^{n}\mathrm{tr}\,\theta_i(\widetilde{\mathbf{H}}_a(t);\tau)\frac{\omega^m}{m!}=0 \tag{2.19}$$

for all $t\ge 0$ and $a>0$. Since $\widetilde{\mathbf{H}}_a(t)\to\mathbf{H}(t)$ in local C^∞-topology as $a\to+\infty$, by a similar argument as that in Lemma 2.1, we get that $\int_M\sum\limits_{i=0}^{n}\mathrm{tr}\,\theta_i(\mathbf{H}(t);\tau)\frac{\omega^m}{m!}$ is independent of t. Hence we obtain the following lemma.

Lemma 2.2 *Let* $\mathbf{H}(t)$ *be the long time solution of the heat equations* (2.1) *which is obtained in Proposition* 2.2*, then*

$$|\overline{\partial}_{E_i}(K_i^{-1}H_i(t))|_{K_i}\in L^2(M,\omega) \tag{2.20}$$

and

$$2\pi\deg_\tau(\mathbf{C};\mathbf{H}(t))=\int_M\sum_{i=0}^{n}\mathrm{tr}\,\theta_i(\mathbf{H}(t);\tau)\frac{\omega^m}{m!}=2\pi\deg_\tau(\mathbf{C};\mathbf{K}) \tag{2.21}$$

for all $t\ge 0$.

3 Proof of Theorem 1.1

In this section, we will use the heat flow method to prove that the analytic τ-semi-stability implies that the chain τ-vortex equations (1.1) admit an approximate solution in L^2-norm sense and deduce the Bogomolov type Chern numbers inequality (1.12). Before giving the detailed proof, we need to recall some notations. Let K be a fixed Hermitian metric on a holomorphic bundle E over M, and denote

$$\mathcal{S}_K(E) = \{s \in \Omega^0(M, \mathrm{End}(E)) \mid s^{*K} = s\}. \tag{3.1}$$

Given $\rho \in C^\infty(R, R)$, $\Psi \in C^\infty(R \times R, R)$, $s \in \mathcal{S}_K(E)$, $p \in \Omega^0(M, \mathrm{End}(E))$, we define $\rho(s)$ and $\Psi[s](p)$ as follows. At each point x on M, let $\{e_i\}_{i=1}^r$ be a unitary basis with respect to the metric K, such that $s(e_i) = \delta_i e_i$, and $\{e_i^*\}_{i=1}^r$ be the dual basis for $\{e_i\}_{i=1}^r$, then $p \in \Omega^0(M, \mathrm{End}(E))$ can be written as $p = \Sigma p_{ij} e_i^* \otimes e_j$. We set

$$\rho(s)(e_i) = \rho(\delta_i)e_i \tag{3.2}$$

and

$$\Psi[s](p) = \Sigma \Psi(\delta_i, \delta_j) p_{ij} e_i^* \otimes e_j. \tag{3.3}$$

Let's recall Donaldson's functional defined on the space $\mathscr{P}_0$ of Hermitian metrics on the bundle E (see [24, Section 5] for details),

$$\mathcal{M}_E(K, H) = \int_{M \setminus \Sigma} \mathrm{tr}\,(s\sqrt{-1}\Lambda_\omega F_K) + \langle \Psi(s)(\overline{\partial}_E s), \overline{\partial}_E s\rangle_K \frac{\omega^m}{m!}, \tag{3.4}$$

where $\Psi(x, y) = (x - y)^{-2}(\mathrm{e}^{y-x} - (y - x) - 1)$, and $\exp s = K^{-1}H$. We recall the modified Donaldson's functional of two $(n+1)$-tuple of Hermitian metrics $\mathbf{K} = (K_0, \cdots, K_n)$ and $\mathbf{H} = (H_0, \cdots, H_n)$ on the twisted holomorphic chain $\mathbf{C}$,

$$\begin{aligned}\mathcal{M}_{\mathbf{C},\alpha}(\mathbf{K}, \mathbf{H}) = {}& \sum_{i=0}^{n} \mathcal{M}_{E_i}(K_i, H_i) + \sum_{i=1}^{n} \int_M (|\phi_i|_H^2 - |\phi_i|_K^2) \frac{\omega^m}{m!} \\ & - 2\sum_{i=0}^{n} \int_M \alpha_i \mathrm{tr}\,(\log(K_i^{-1}H_i)) \frac{\omega^m}{m!},\end{aligned} \tag{3.5}$$

where $\alpha = (\alpha_0, \cdots, \alpha_n) \in R^{n+1}$. From the estimates in Proposition 2.2 and Lemma 2.2, it is easy to see that $H_i(t)$ belongs to the definition space $\mathscr{P}_0$ and $\mathcal{M}_{\mathbf{C},\tau}(\mathbf{K}, \mathbf{H}(t))$ is well defined for the evolving $(n+1)$-tuple of Hermitian metrics $\mathbf{H}(t)$ along the heat flow (2.1) (for every $t \geq 0$). Furthermore, we have the following lemma.

Lemma 3.1 (see [26, Lemma 7.1]) *Let* $\mathbf{H}(t)$ *be a solution of the heat flow* (2.1) *with an initial* $(n+1)$*-tuple of Hermitian metrics* $\mathbf{K}$ *which satisfy the same conditions in Theorem* 1.1*, then*

$$\mathcal{M}_{\mathbf{C},\tau}(\mathbf{H}(t_1), \mathbf{H}(t_3)) = \mathcal{M}_{\mathbf{C},\tau}(\mathbf{H}(t_1), \mathbf{H}(t_2)) + \mathcal{M}_{\mathbf{C},\tau}(\mathbf{H}(t_2), \mathbf{H}(t_3)) \tag{3.6}$$

for all $0 \le t_1 \le t_2 \le t_3$, *and*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{\mathbf{C},\tau}(\mathbf{K},\mathbf{H}(t))\\ &= -\int_M \sum_{i=0}^{n} |2\sqrt{-1}\Lambda_\omega F_{H_i(t)} - (\phi_i^{*H(t)} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H(t)}) - 2\tau_i \mathrm{Id}_{E_i}|^2_{H_i(t)} \frac{\omega^m}{m!}\end{aligned} \tag{3.7}$$

for all $t \ge 0$.

Proposition 3.1 *Let* $\mathbf{H}(t)$ *be a solution of the heat flow* (2.1) *with an initial* $(n+1)$*-tuple of Hermitian metrics* $\mathbf{K}$ *which satisfy the same conditions in Theorem* 1.1. *If the twisted holomorphic chain* $\mathbf{C} = (\mathbf{E}, \phi)$ *is analytic* τ*-semi-stable with respect to the initial metrics* $\mathbf{K}$*, then*

$$\int_M \sum_{i=0}^{n} |2\sqrt{-1}\Lambda_\omega F_{H_i(t)} - (\phi_i^{*H(t)} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H(t)}) - 2\tau_i \mathrm{Id}_{E_i}|^2_{H_i(t)} \frac{\omega^m}{m!} \to 0 \tag{3.8}$$

as $t \to +\infty$.

Proof Set $h_i(t) = K_i^{-1}H_i(t) = \exp(s_i(t))$ for all $0 \le i \le n$. By a direct calculation, we derive

$$\begin{aligned}&\frac{\partial}{\partial t}\log(\mathrm{tr}\, h_i(t) + \mathrm{tr}\, h_i^{-1}(t))\\ &= \frac{\mathrm{tr}\Big(h_i(t)\cdot h_i^{-1}(t)\dfrac{\partial h_i(t)}{\partial t}\Big) - \mathrm{tr}\Big(h_i^{-1}(t)\dfrac{\partial h_i(t)}{\partial t}\cdot h_i^{-1}(t)\Big)}{\mathrm{tr}\, h_i(t) + \mathrm{tr}\, h_i^{-1}(t)}\\ &\le |2\sqrt{-1}\Lambda_\omega F_{H_i(t)} - (\phi_i^{*H(t)} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H(t)}) - 2\tau_i \mathrm{Id}_{E_i}|_{H_i(t)}.\end{aligned} \tag{3.9}$$

From Proposition 2.2, we see that $\sup\limits_M |2\sqrt{-1}\Lambda_\omega F_{H_i(t)} - (\phi_i^{*H(t)} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H(t)}) - \tau_i \mathrm{Id}_{E_i}|_{H_i(t)}$ is bounded independent of t. So there exists a constant $\widetilde{C}_1$ such that

$$\sup_M \log(\mathrm{tr}\, h_i(t) + \mathrm{tr}\, h_i^{-1}(t)) \le \log(2\mathrm{rank}(E_i)) + \widetilde{C}_1 t \tag{3.10}$$

for all $0 \le i \le n$ and $t \ge 0$. On the other hand, by [26, Corollary 2.8], we have

$$\begin{aligned}&\Delta \lg\Big(\sum_{i=0}^{n}(\mathrm{tr}\,(h_i) + \mathrm{tr}\,(h_i^{-1}))\Big)\\ &\ge -\Big(\sum_{i=0}^{n}|2\sqrt{-1}\Lambda_\omega F_{K_i} - (\phi_i^{*K} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*K}) - 2\tau_i \mathrm{Id}_{E_i}|_{K_i}\Big)\\ &\quad -\Big(\sum_{i=0}^{n}|2\sqrt{-1}\Lambda_\omega F_{H_i} - (\phi_i^{*H} \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^{*H}) - 2\tau_i \mathrm{Id}_{E_i}|_{H_i}\Big).\end{aligned} \tag{3.11}$$

Due to the Assumption 1.3, there exist two constants $\widetilde{C}_2$ and $\widetilde{C}_3$ such that

$$\Big\| \lg\Big(\sum_{i=0}^{n}(\mathrm{tr}\,(h_i) + \mathrm{tr}\,(h_i^{-1}))\Big)\Big\|_{L^\infty}$$

$$\leq \widetilde{C}_2\Big(\int_M \lg\Big(\sum_{i=0}^{n}(\operatorname{tr}(h_i)+\operatorname{tr}(h_i^{-1}))\Big)\frac{\omega^m}{m!}+\widetilde{C}_3\Big). \tag{3.12}$$

On the other hand, one can check that

$$\lg\Big(\frac{1}{2\sum\limits_{i=0}^{n} r_i}\sum_{i=0}^{n}(\operatorname{tr}h_i+\operatorname{tr}h_i^{-1})\Big)\leq\sum_{i=0}^{n}|s_i|_{K_i}=\sum_{i=0}^{n}|s_i|_{H_i}$$

$$\leq\Big(\sum_{i=0}^{n} r_i^{\frac{1}{2}}\Big)\lg\sum_{i=0}^{n}(\operatorname{tr}h_i+\operatorname{tr}h_i^{-1}), \tag{3.13}$$

where $r_i=\operatorname{rank}E_i$. So there exist positive constants $\widetilde{C}_4$ and $\widetilde{C}_5$ such that

$$\sum_{i=0}^{n}\|s_i(t)\|_{L^\infty}\leq\widetilde{C}_4\Big(\sum_{i=0}^{n}\|s_i(t)\|_{L^1}\Big)+\widetilde{C}_5 \tag{3.14}$$

for all $t\geq 0$.

The monotonicity of $\int_M\sum\limits_{i=0}^{n}|\theta_i(\mathbf{H}(t);\tau)|^2_{H_i(t)}\frac{\omega^m}{m!}$ yields that

$$\lim_{t\to+\infty}\int_M\sum_{i=0}^{n}|2\sqrt{-1}\Lambda_\omega F_{H_i(t)}-(\phi_i^{*H(t)}\circ\phi_i-\phi_{i+1}\circ\phi_{i+1}^{*H(t)})-2\tau_i\mathrm{Id}_{E_i}|^2_{H_i(t)}\frac{\omega^m}{m!}=C^*. \tag{3.15}$$

Now we prove (3.8) by contradiction. If $C^*>0$, based on (3.7), we get

$$\mathcal{M}_{\mathbf{C},\tau}(\mathbf{K},\mathbf{H}(t))\leq -C^*t \tag{3.16}$$

for all $0<t_0\leq t$. Then it is clear that (3.10), (3.13) and (3.16) imply

$$\liminf_{t\to+\infty}\frac{-\mathcal{M}_{\mathbf{C},\tau}(\mathbf{K},\mathbf{H}(t))}{\sum\limits_{i=0}^{n}\|s_i(t)\|_{L^1}}\geq\frac{C^*}{\widetilde{C}_6}, \tag{3.17}$$

where $\widetilde{C}_6$ is a positive constant. According to the definition of modified Donaldson's functional (3.5) and (3.16), we know that there must exists a sequence $t_j\to+\infty$ such that

$$\sum_{i=0}^{n}\|s_i(t_j)\|_{L^1}\to+\infty. \tag{3.18}$$

Let $E=E_0\oplus E_1\cdots\oplus E_n$ be the direct sum of holomorphic bundles E_i, $K=K_0\oplus K_1\cdots\oplus K_n$ and $H=H_0\oplus H_1\cdots\oplus H_n$ be the induced Hermitian metrics on E. Denote $h=h_0\oplus h_1\cdots\oplus h_n$ and $s=s_0\oplus s_1\cdots\oplus s_n\in\operatorname{End}(E)$. The morphisms $\phi_i: E_i\otimes\widetilde{E}_i\to E_{i-1}$ induce a section $\widetilde{\phi}=\bigoplus\limits_{i=1}^{n}\phi_i$ of the bundle $\bigoplus\limits_{i=1}^{n}\operatorname{Hom}(E_i\otimes\widetilde{E}_i,E_{i-1})$ (or $\bigoplus\limits_{i=1}^{n}\operatorname{Hom}(E_i,E_{i-1}\otimes\widetilde{E}_i^*)$). Then we define the endomorphisms $\widetilde{\phi}^{*H}\circ\widetilde{\phi}=\sum\limits_{i=1}^{n}\phi_i^{*H}\circ\phi_i$ and $\widetilde{\phi}\circ\widetilde{\phi}^{*H}=\sum\limits_{i=1}^{n}\phi_i\circ\phi_i^{*H}\in\operatorname{End}(E)$. In the following, we denote by $\pi_i: E\to E$ the projection onto the sub-bundle E_i with respect to the initial Hermitian metric $K = K_0 \oplus K_1 \cdots \oplus K_n$. The heat flow (2.1) can be rewritten as the following:

$$H^{-1}\frac{\partial H}{\partial t} = -2\Big(\sqrt{-1}\Lambda_\omega F_H - \frac{1}{2}(\widetilde{\phi}^{*H} \circ \widetilde{\phi} - \widetilde{\phi} \circ \widetilde{\phi}^{*H}) - \sum_{i=0}^{n} \tau_i \pi_i\Big). \tag{3.19}$$

Set $l_j = \|s(t_j)\|_{L^1}$ and $u_j = l_j^{-1}s(t_j) \in \mathrm{End}(E)$. From (3.18), we see that $l_j \to \infty$ as $j \to \infty$. Using (3.14), we obtain

$$\|u_j\|_{L^1} = 1, \quad \|u_j\|_{L^\infty} \le \widetilde{C}_7 \tag{3.20}$$

for all j, where $\widetilde{C}_7$ is a positive constant.

Combining the formula (2.21) and the initial assumption $\deg_\tau(\mathbf{C}; K) = 0$, we obtain

$$\begin{aligned}
&\int_M \operatorname{tr} s(t_j)\frac{\omega^m}{m!} = \int_M \sum_{i=0}^{n} \operatorname{tr} s_i(t_j)\frac{\omega^m}{m!} \\
&= \int_0^{t_j}\int_M \sum_{i=0}^{n} \frac{\partial \operatorname{tr} s_i(t)}{\partial t}\frac{\omega^m}{m!}\mathrm{d}t = \int_0^{t_j}\int_M \sum_{i=0}^{n} \frac{\partial \det(K_i^{-1}H_i(t))}{\partial t}\frac{\omega^m}{m!}\mathrm{d}t \\
&= -2\int_0^{t_j}\int_M \sum_{i=0}^{n} \operatorname{tr}\left(\theta_i(\mathbf{H}(t),\tau)\right)\frac{\omega^m}{m!}\mathrm{d}t = -2\int_0^{t_i} \deg_\tau(\mathbf{C}; K)\mathrm{d}t = 0.
\end{aligned} \tag{3.21}$$

So it holds that

$$\int_M \operatorname{tr} u_j \frac{\omega^m}{m!} = 0. \tag{3.22}$$

By (3.20), (3.22) and a similar argument as that in [26, Proposition 7.2], we have that, by choosing a subsequence which also is denoted by u_j, $u_j \rightharpoonup u_\infty$ weakly in L_1^2 as $j \to \infty$. The limit u_∞ satisfies $\|u_\infty\|_{L^1} = 1$, $\int_M \operatorname{tr} u_\infty \frac{\omega^m}{m!} = 0$ and

$$\|u_\infty\|_{L^\infty} \le \widetilde{C}_7. \tag{3.23}$$

Moreover, we deduce

$$\begin{aligned}
&\int_M \Big\langle u_j, 2\sqrt{-1}\Lambda_\omega F_K - 2\sum_{i=0}^{n}\tau_i\pi_i\Big\rangle_K \frac{\omega^m}{m!} + 2\int_M \langle \Psi[u_j](\overline{\partial}_E u_j), \overline{\partial}_E u_j\rangle_K \frac{\omega^m}{m!} \\
&\le l_j^{-1}\Big(\int_M \Big\langle s(t_j), 2\sqrt{-1}\Lambda_\omega F_K - 2\sum_{i=0}^{n}\tau_i\phi_i\Big\rangle_K \frac{\omega^m}{m!} + 2\int_M \langle \Psi[s(t_j)](\overline{\partial}_E s(t_j)), \overline{\partial}_E s(t_j)\rangle_K \frac{\omega^m}{m!}\Big) \\
&\le \frac{\mathcal{M}_{\mathbf{C},\tau}(\mathbf{K}, \mathbf{H}(t_j))}{\sum_{i=0}^{n}\|s_i(t_j)\|_{L^1}} + l_j^{-1}\Big(\int_M (|\widetilde{\phi}|^2_{H(t_j)} - |\widetilde{\phi}|^2_K)\frac{\omega^m}{m!}\Big).
\end{aligned}$$

Applying (3.17) and the same discussion as that in [24, Lemma 5.4], we get

$$\int_M \Big\langle u_\infty, \sqrt{-1}\Lambda_\omega F_K - \sum_{i=0}^{n}\tau_i\pi_i\Big\rangle_K + \langle \Psi[u_\infty](\overline{\partial}_{E^*\otimes E} u_\infty), \overline{\partial}_{E^*\otimes E} u_\infty\rangle_K \frac{\omega^m}{m!} \le -\frac{C^*}{\widetilde{C}_6} \tag{3.24}$$

and

$$\int_M \left\langle u_\infty, \sqrt{-1}\Lambda_\omega F_K - \sum_{i=0}^{n} \tau_i \pi_i \right\rangle_K + \langle \zeta[u_\infty](\overline{\partial}_{E^*\otimes E} u_\infty), \overline{\partial}_{E^*\otimes E} u_\infty \rangle_K \frac{\omega^m}{m!} \le -\frac{C^*}{\widetilde{C}_6} \tag{3.25}$$

for any positive function $\zeta \in C^\infty(R\times R, R)$ which satisfies $\zeta(x,y) \le (x-y)^{-1}$ whenever $x > y$.

From (3.25) and the same argument in [26, Lemma 7.3], it follows that the eigenvalues of u_∞ are constant almost everywhere. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_l$ denote the distinct eigenvalues of u_∞. Since $\|u_\infty\|_{L^1} = 1$ and $\int_M \operatorname{tr} u_\infty \frac{\omega^m}{m!} = 0$, we must have $l \ge 2$. For $1 \le \beta < l$, define $P_\beta : R \to R$ such that

$$P_\beta(x) = \begin{cases} 1, & \text{if } x \le \lambda_\beta, \\ 0, & \text{if } x \ge \lambda_{\beta+1}. \end{cases} \tag{3.26}$$

Set

$$\pi'_\beta = P_\beta(u_\infty), \tag{3.27}$$

where the notation is the same as the one of (3.2). Based on [26, Proposition 7.4], we obtain

(1) $\pi'_\beta \in L^2_1(\mathcal{S}_K(E))$;

(2) $\pi'^2_\beta = \pi'_\beta = \pi'^{*K}_\beta$;

(3) $(\mathrm{Id} - \pi'_\beta)\overline{\partial}_{E^*\otimes E}(\pi'_\beta) = 0$ almost everywhere;

(4) $\|(\mathrm{Id} - \pi'_\beta) \circ \widetilde{\phi} \circ \pi'_\beta\|^2_{L^2} = 0$.

Set $\pi'_{\beta i} = \pi_i \circ \pi'_\beta \circ i_{E_i}$ and $\phi'_{\beta i} = \phi_i\big|_{\pi'_{\beta i}}$. By Uhlenbeck and Yau's regularity theorem of L^2_1-subbundle (see [25]), we know that $\pi'_{\beta i}$ represents a saturated sub-sheaf $E'_{\beta i}$ of E_i. On the other hand, property (4) implies that

$$\phi_i \circ \pi'_{\beta i} \otimes \mathrm{Id}_{\widetilde{E}_i} = \pi'_{\beta(i-1)} \circ \phi_i \circ \pi'_{\beta i} \otimes \mathrm{Id}_{\widetilde{E}_i}. \tag{3.28}$$

So those $(E'_{\beta i}, \phi'_{\beta i})$ determine a sequence of proper weakly holomorphic sub-chain $\mathbf{C}_\beta$ of $\mathbf{C} = (\mathbf{E}, \phi)$. Define

$$Q(\tau) := \lambda_l \deg_\tau(\mathbf{C}, \mathbf{K}) - \sum_{\beta=1}^{l-1} (\lambda_{\beta+1} - \lambda_\beta) \deg_\tau(\mathbf{C}_\beta, \mathbf{K}). \tag{3.29}$$

Due to $u_\infty = \lambda_l \mathrm{Id}_E - \sum_{\beta=1}^{l-1} (\lambda_{\beta+1} - \lambda_\beta)\pi_\beta$ and $\int_{M\setminus\Sigma} \operatorname{tr} u_\infty \frac{\omega^m}{m!} = 0$, we have

$$\lambda_l \mathrm{rank}(E) - \sum_{\beta=1}^{l-1} (\lambda_{\beta+1} - \lambda_\beta) \mathrm{rank}(\pi'_\beta) = 0. \tag{3.30}$$

Then it holds that

$$Q(\tau) = \sum_{\beta=1}^{l-1} \mathrm{rank}(\pi'_\beta)(\lambda_{\beta+1} - \lambda_\beta)(\mu_\tau(\mathbf{C}, \mathbf{K}) - \mu_\tau(\mathbf{C}_\beta, \mathbf{K})). \tag{3.31}$$

Set $\widetilde{\zeta}=\sum\limits_{\beta=1}^{l-1}(\lambda_{\beta+1}-\lambda_\beta)(\mathrm{d}P_\beta)^2$. It is easy to see that $\widetilde{\zeta}(\lambda,\mu)\le(\lambda-\mu)^{-1}$ for $\lambda>\mu$. From the formulas (7.30)–(7.31) in [26] and (3.25), we get

$$\begin{aligned}Q(\tau)&=\frac{1}{2\pi}\int_M\Big\langle u_\infty,\sqrt{-1}\Lambda_\omega F_K-\sum_{i=0}^n\tau_i\pi_i\Big\rangle_K\\&\quad+\langle\widetilde{\zeta}[u_\infty](\overline{\partial}_{E^*\otimes E}u_\infty),\overline{\partial}_{E^*\otimes E}u_\infty\rangle_K\frac{\omega^m}{m!}\\&\le-\frac{C^*}{\widetilde{C}_6}<0.\end{aligned}\tag{3.32}$$

On the other hand, the semi-stability and (3.31) imply $Q(\tau)\ge0$, which derives a contradiction. So we get $C^*=0$ and this concludes the proof of Proposition 3.1.

Proof of Theorem 1.1 By the estimates in Section 2, we know that $|\overline{\partial}_{E_i}(K_i^{-1}H_i(t))|_{K_i}\in L^2(M,\omega)$, $H_i(t)$ and K_i mutually bounded, and $\sup\limits_M|\Lambda_\omega F_{H_i(t)}|_{H_i(t)}<\infty$ for each $i=0,1,\cdots,n$ and $t\ge0$. So (3.17) implies the existence of an approximate solution of the chain τ-vortex equations (1.1) in L^2-norm sense. Now we only need to prove the Bogomolov type Chern numbers inequality (1.12).

From the assumption $\overline{\partial}_{E_i^*\otimes\widetilde{E}_i^*\otimes E_{i-1}}\phi_i=0$, it follows that

$$\begin{aligned}\int_M\sum_{i=1}^n|\nabla_H^{1,0}\phi_i|_H^2\frac{\omega^m}{m!}&=\sum_{i=0}^n\int_M\mathrm{Re}\langle\sqrt{-1}\Lambda_\omega F_{H_i},\phi_{i+1}\circ\phi_{i+1}^{*H}-\phi_i^{*H}\circ\phi_i\rangle_H\frac{\omega^m}{m!}\\&\quad-\sum_{i=1}^n\int_M\mathrm{Re}\langle\phi_i\circ(\mathrm{Id}_{E_i}\otimes\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}),\phi_i\rangle_H\frac{\omega^m}{m!}.\end{aligned}\tag{3.33}$$

Since the proof of the above equality can be found in [26, Proposition 8.2], we omit it. A straightforward computation shows that

$$\begin{aligned}&-\sum_{i=0}^n8\pi^2Ch_2(E_i,H_i)+\sum_{i=0}^n4\pi\tau_iC_1(E_i,H_i)\\&=\sum_{i=1}^n\|\partial_H\phi_i\|_{L^2}^2+\sum_{i=0}^n\Big(\|F_{H_i}\|_{L^2}^2+\Big\|\frac12(\phi_{i+1}\circ\phi_{i+1}^{*H}-\phi_i^{*H}\circ\phi_i)-\tau_i\mathrm{Id}_{E_i}\Big\|_{L^2}^2\Big)\\&\quad+\sum_{i=1}^n\int_M\langle\phi_i\circ(\mathrm{Id}_{E_i}\otimes\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}),\phi_i\rangle_H\frac{\omega^m}{m!}\\&\quad-\sum_{i=0}^n\Big\|\sqrt{-1}\Lambda_\omega F_{H_i}+\frac12(\phi_{i+1}\circ\phi_{i+1}^{*H}-\phi_i^{*H}\circ\phi_i)-\tau_i\mathrm{Id}_{E_i}\Big\|_{L^2}^2.\end{aligned}\tag{3.34}$$

The assumption that $\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}$ is positive semi-definite for each $i=1,\cdots,n$ yields that

$$\begin{aligned}&-\sum_{i=0}^n8\pi^2Ch_2(E_i,H_i)+\sum_{i=0}^n4\pi\tau_iC_1(E_i,H_i)\\&\ge-\sum_{i=0}^n\Big\|\sqrt{-1}\Lambda_\omega F_{H_i}+\frac12(\phi_{i+1}\circ\phi_{i+1}^{*H}-\phi_i^{*H}\circ\phi_i)-\tau_i\mathrm{Id}_{E_i}\Big\|_{L^2}^2.\end{aligned}\tag{3.35}$$

Noting the assumption that $0 \leq \sqrt{-1}\partial\overline{\partial}\varphi \leq C\omega$, $\sum\limits_{i=1}^{n} |\nabla_K^{1,0}\phi_i|_K$ and $\sum\limits_{i=1}^{n} |\sqrt{-1}\Lambda_\omega F_{\widetilde{H}_i}|_{\widetilde{H}_i}$ both belong to $L^2(M,\omega)$, we have

$$Ch_2(E_i, H_i(t)) \geq Ch_2(E_i, K_i) \tag{3.36}$$

and

$$C_1(E_i, H_i(t)) = C_1(E_i, K_i) \tag{3.37}$$

for every $t \geq 0$. The proof of the above two formulas can be found in [26, Theorem 8.3]. Combining (3.35)–(3.37), we obtain

$$\begin{aligned} & -\sum_{i=0}^{n} 8\pi^2 Ch_2(E_i, K_i) + \sum_{i=0}^{n} 4\pi\tau_i C_1(E_i, K_i) \\ & \geq -\sum_{i=0}^{n} \Big\| \sqrt{-1}\Lambda_\omega F_{H_i(t)} + \frac{1}{2}(\phi_{i+1} \circ \phi_{i+1}^{*H(t)} - \phi_i^{*H(t)} \circ \phi_i) - \tau_i \mathrm{Id}_{E_i} \Big\|_{L^2}^2 \end{aligned} \tag{3.38}$$

for all $t \geq 0$. So (3.17) implies (1.12). This completes the proof of Theorem 1.1.

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