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An Explicit Ladder of Homotopy Categories*

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Abstract For an upper triangular matrix ring, an explicit ladder of height 2 of triangle functors between homotopy categories is constructed. Under certain conditions, the author obtains a localization sequence of homotopy categories of acyclic complexes of injective modules.

Keywords Triangular matrix ring, Homotopy category, Recollememt
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1 Introduction

Let R and S be two rings, and ${}_RM_S$ be an R-S-bimodule. We consider the upper triangular matrix ring

$$\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}.$$

A complex of Λ-modules is written as $\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right)$ with the structure map $\phi^\bullet : M \otimes_S V^\bullet \to X^\bullet$, a chain map of complexes of R-modules. $\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right)$ is called down null-homotopical provided that the complex $V^\bullet$ of S-modules is null-homotopical.

Denote by $\mathbf{K}(\Lambda\text{-Mod})$ the homotopy category of left Λ-modules. We denote by $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ the homotopy category of down null-homotopical complexes of Λ-modules, which is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Mod})$. The homotopy category $\mathbf{K}(R\text{-Mod})$ can be embedded into the triangulated category $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ via a natural triangle functor, which is not a triangle equivalence (see Example 3.1).

Theorem 1.1 *Let Λ be the upper triangular matrix ring. Then there exists a ladder of height* 2,

$$\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \begin{array}{c} \xleftarrow{\ i^*\ } \\ \xrightarrow{\ i_*\ } \\ \xleftarrow{\ i^!\ } \\ \xrightarrow[\ i_2\ ]{} \end{array} \mathbf{K}(\Lambda\text{-Mod}) \begin{array}{c} \xleftarrow{\ j_!\ } \\ \xrightarrow{\ j^*\ } \\ \xleftarrow{\ j_*\ } \\ \xrightarrow[\ j_2\ ]{} \end{array} \mathbf{K}(S\text{-Mod}).$$

We refer to Theorem 3.1 for its proof. We construct the two triangle functors $i^!, i^* :$ $\mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ in Subsections 2.1–2.2. The functor i_* is the inclusion. The triangle functors i_2 and j_2 are constructed in Section 3. The other functors are natural and

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induced from the ones over the module categories. The above ladder induces the ladder of derived categories for the upper triangular matrix ring; see [9, Example 3.4] and compare [11, Corollary 15] and [8, 2.1].

We denote by $\mathbf{K}_{\rm ac}(R\text{-Inj})$ the homotopy category of acyclic complexes of injective left R-modules. If R is left noetherian, then the homotopy category $\mathbf{K}_{\rm ac}(R\text{-Inj})$ is a compactly generated triangulated category such that the corresponding full subcategory consisting of compact objects is triangle equivalent to the singularity category $\mathbf{D}_{\rm sg}(R)$ of R in the sense of [3, 15] (see [13, Corollary 5.4]).

Proposition 1.1 *Let ${}_RM_S$ be an R-S-bimodule such that M_S is a flat right S-module. Let Λ be the upper triangular matrix ring. Then there exists a locolization sequence of homotopy categories*

$$\mathbf{K}_{\rm ac}(S\text{-Inj}) \rightleftarrows \mathbf{K}_{\rm ac}(\Lambda\text{-Inj}) \rightleftarrows \mathbf{K}_{\rm ac}(R\text{-Inj}).$$

This is Proposition 4.1. In the case that S is a semi-simple ring, by the above localization sequence, we have the triangle equivalence

$$\mathbf{K}_{\rm ac}(\Lambda\text{-Inj}) \simeq \mathbf{K}_{\rm ac}(R\text{-Inj}).$$

This triangle equivalence extends [5, Proposition 4.1], which proves that the one-point extensions preserve the singularity categories.

This paper is structured as follows. In Section 2, for the upper triangular matrix ring Λ, we construct the two triangle functors $i^!, i^*$ in Theorem 1.1. In Section 3, we prove the explicit ladder of homotopy categories. In Section 4, we prove the localization sequence of homotopy categories of acyclic complexes of injective modules.

2 The Triangular Matrix Ring and Two Triangle Functors

In this section, for an upper triangular matrix ring, we construct two triangle functors between homotopy categories.

2.1 The triangular matrix ring and a triangle functor

Let R and S be two rings, and ${}_RM_S$ be an R-S-bimodule. We denote by Λ the upper triangular matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$.

A left Λ-module is written as $\left(\begin{smallmatrix} X \\ V \end{smallmatrix}\right)$, where X and V are left R-module and S-module, respectively. It has a structure map $\phi : M \otimes_S V \to X$, which is a morphism of R-modules. A morphism $f : \left(\begin{smallmatrix} X \\ V \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ V' \end{smallmatrix}\right)$ of Λ-modules is a pair $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$, where $\alpha : X \to X'$ is a morphism of R-modules and $\beta : V \to V'$ is a morphism of S-modules satisfying

$$\alpha \circ \phi = \phi' \circ (\mathrm{Id}_M \otimes \beta).$$

For a ring R, we denote by R-Mod the category of left R-modules. Denote by $\mathbf{K}(R\text{-Mod})$ its homotopy category. For the upper triangular matrix ring Λ, we define a triangle functor

$$i^! : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod}).$$

We denote by $\binom{X^\bullet}{V^\bullet}$ the complex $\left(\binom{X^n}{V^n}, \binom{d_X^n}{d_V^n}\right)_{n\in\mathbb{Z}}$ of Λ-modules. The n-th component $\binom{X^n}{V^n}$ has a structure map $\phi^n : M \otimes_S V^n \to X^n$. We denote by $V^\bullet[-1]$ the shifted complex given by $V^\bullet[-1]^n = V^{n-1}$ and $d^n_{V[-1]} = -d_V^{n-1}$ for $n \in \mathbb{Z}$. Let $C^-(V^\bullet) = V^\bullet \oplus V^\bullet[-1]$ be the cone of $\mathrm{Id}_{V^\bullet[-1]}$, whose n-th differential is

$$d^n_{C^-(V)} = \begin{pmatrix} d_V^n & 0 \\ \mathrm{Id}_{V^n} & -d_V^{n-1} \end{pmatrix} : V^n \oplus V^{n-1} \to V^{n+1} \oplus V^n.$$

We define

$$i^! \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} = \begin{pmatrix} X^\bullet \\ V^\bullet \oplus V^\bullet[-1] \end{pmatrix}$$

to be the complex whose n-th component is

$$\begin{pmatrix} X^n \\ V^n \oplus V^{n-1} \end{pmatrix}$$

with the structure map $(\phi^n \quad 0) : M \otimes_S (V^n \oplus V^{n-1}) \to X^n$ and n-th differential is

$$\begin{pmatrix} d_X^n \\ d^n_{C^-(V)} \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \oplus V^{n-1} \end{pmatrix} \to \begin{pmatrix} X^{n+1} \\ V^{n+1} \oplus V^n \end{pmatrix}.$$

Indeed, we have $d_X^n \circ (\phi^n \quad 0) = (\phi^{n+1} \quad 0) \circ (\mathrm{Id}_M \otimes d^n_{C^-(V)})$.

Let $\binom{Y^\bullet}{W^\bullet}$ be another complex of Λ-modules with a structure map $\psi^\bullet : M \otimes_S W^\bullet \to Y^\bullet$. Denote by

$$\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix}_{n\in\mathbb{Z}} : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}$$

a chain map of complexes of Λ-modules. Here,

$$\begin{pmatrix} f^n \\ g^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \end{pmatrix} \to \begin{pmatrix} Y^n \\ W^n \end{pmatrix}$$

is a morphism of Λ-modules.

Denote

$$C^-(g^\bullet) : V^\bullet \oplus V^\bullet[-1] \to W^\bullet \oplus W^\bullet[-1]$$

such that

$$C^-(g^\bullet)^n = \begin{pmatrix} g^n & 0 \\ 0 & g^{n-1} \end{pmatrix} : V^n \oplus V^{n-1} \to W^n \oplus W^{n-1}.$$

Define

$$i^! \begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} = \begin{pmatrix} f^\bullet \\ C^-(g^\bullet) \end{pmatrix} : \begin{pmatrix} X^\bullet \\ V^\bullet \oplus V^\bullet[-1] \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \oplus W^\bullet[-1] \end{pmatrix}.$$

Lemma 2.1 *We have that* $i^!\binom{f^\bullet}{g^\bullet}$ *is a chain map of complexes of* Λ*-modules.*

Proof Since $\binom{f^n}{g^n}$ is a morphism of Λ-modules, we have

$$f^n \circ (\phi^n \quad 0) = (\psi^n \quad 0) \circ \mathrm{Id}_M \otimes_S \begin{pmatrix} g^n & 0 \\ 0 & g^{n-1} \end{pmatrix}.$$

Then

$$\begin{pmatrix} f^n \\ C^-(g^\bullet)^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \oplus V^{n-1} \end{pmatrix} \to \begin{pmatrix} Y^n \\ W^n \oplus W^{n-1} \end{pmatrix}$$

is a morphism of Λ-modules. The statement follows by direct calculation.

We use “ $\sim$ ” to denote the homotopy equivalence relation.

Lemma 2.2 *If* $\binom{f^\bullet}{g^\bullet} \sim 0$, *then* $i^!\binom{f^\bullet}{g^\bullet} \sim 0$.

Proof Suppose that there exists a morphism

$$\begin{pmatrix} u^n \\ \nu^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \end{pmatrix} \to \begin{pmatrix} Y^{n-1} \\ W^{n-1} \end{pmatrix}$$

of Λ-modules such that

$$\begin{pmatrix} d_Y^{n-1} \\ d_W^{n-1} \end{pmatrix} \circ \begin{pmatrix} u^n \\ \nu^n \end{pmatrix} + \begin{pmatrix} u^{n+1} \\ \nu^{n+1} \end{pmatrix} \circ \begin{pmatrix} d_X^n \\ d_V^n \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix} \tag{2.1}$$

for each $n \in \mathbb{Z}$. Set $\lambda^n = \begin{pmatrix} \nu^n & 0 \\ 0 & -\nu^{n-1} \end{pmatrix} : V^n \oplus V^{n-1} \to W^{n-1} \oplus W^{n-2}$.

Since $\binom{u^n}{\nu^n}$ is a morphism of Λ-modules, we have that

$$\begin{pmatrix} u^n \\ \lambda^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \oplus V^{n-1} \end{pmatrix} \to \begin{pmatrix} Y^{n-1} \\ W^{n-1} \oplus W^{n-2} \end{pmatrix}$$

is a morphism of Λ-modules. Observe that (2.1) implies the following identity

$$\begin{pmatrix} d_Y^{n-1} \\ d_{C^-(W)}^{n-1} \end{pmatrix} \circ \begin{pmatrix} u^n \\ \lambda^n \end{pmatrix} + \begin{pmatrix} u^{n+1} \\ \lambda^{n+1} \end{pmatrix} \circ \begin{pmatrix} d_X^n \\ d_{C^-(V)}^n \end{pmatrix} = \begin{pmatrix} f^n \\ C^-(g^\bullet)^n \end{pmatrix}$$

for each $n \in \mathbb{Z}$, then $i^!\binom{f^\bullet}{g^\bullet} \sim 0$.

We observe that

$$i^! \begin{pmatrix} \mathrm{Id}_{X^\bullet} \\ \mathrm{Id}_{V^\bullet} \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{X^\bullet} \\ \mathrm{Id}_{C^-(V^\bullet)} \end{pmatrix}.$$

Denote by

$$\begin{pmatrix} h^\bullet \\ l^\bullet \end{pmatrix} : \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix} \to \begin{pmatrix} Z^\bullet \\ U^\bullet \end{pmatrix}$$

another chain map of complexes of Λ-modules. We have

$$i^!\left(\begin{pmatrix} h^\bullet \\ l^\bullet \end{pmatrix} \circ \begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} \right) = i^! \begin{pmatrix} h^\bullet \\ l^\bullet \end{pmatrix} \circ i^! \begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix}.$$

By Lemmas 2.1–2.2, we directly have the following consequence.

Proposition 2.1 *Let* Λ *be the upper triangular matrix ring. Then we have that* $i^! : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ *is a functor.*

We will prove that the functor $i^! : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ is a triangle functor. For the upper triangular matrix ring Λ, we observe the natural functor $j^* : \Lambda\text{-Mod} \to S\text{-Mod}$, which sends $\binom{X}{V}$ to V. The functor j^* admits a right adjoint

$$j_* : S\text{-Mod} \to \Lambda\text{-Mod}, \quad V \mapsto \begin{pmatrix} 0 \\ V \end{pmatrix}. \tag{2.2}$$

The corresponding counit $j^* j_* \xrightarrow{\sim} \mathrm{Id}_{S\text{-Mod}}$ is an isomorphism. Then the functor j_* is fully faithful.

The additive functors j^*, j_* induce triangle functors between homotopy categories. We still denote by $j^* : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(S\text{-Mod})$ and $j_* : \mathbf{K}(S\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$. Then (j^*, j_*) is an adjoint pair between homotopy categories with j_* a fully faithful triangle functor.

A complex $\binom{X^\bullet}{V^\bullet}$ of Λ-modules is called down null-homotopical provided that the complex $V^\bullet$ of S-modules is null-homotopical. We denote by $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ the homotopy category of down null-homotopical complexes of Λ-modules. The homotopy category $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Mod})$.

For each $\binom{Y^\bullet}{W^\bullet} \in \mathbf{K}(\Lambda\text{-Mod})$, we have an exact sequence of complexes

$$0 \to \begin{pmatrix} 0 \\ W^\bullet[-1] \end{pmatrix} \to i^! \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix} \xrightarrow{\pi^\bullet} \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix} \to 0 \tag{2.3}$$

such that $\pi^\bullet$ is given by the identity of $Y^\bullet$ and the projection $W^\bullet \oplus W^\bullet[-1] \to W^\bullet$. The above sequence splits in each component.

Lemma 2.3 *Let*

$$\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \in \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}), \quad \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix} \in \mathbf{K}(\Lambda\text{-Mod}).$$

Then the functor $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\big(\binom{X^\bullet}{V^\bullet}, \pi^\bullet\big)$ *induces the following isomorphism*

$$\mathrm{Hom}_{\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, i^! \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big),$$

which is natural in two variables.

Proof Apply the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\big(\binom{X^\bullet}{V^\bullet}, -\big)$ to the sequence (2.3). Observe the following isomorphism

$$\mathrm{Hom}_{\mathbf{K}(S\text{-Mod})}\Big(j^* \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, W^\bullet[-1]\Big) \simeq \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, j_*(W^\bullet[-1])\Big).$$

Since $V^\bullet \sim 0$, the result follows immediately.

By Lemma 2.3, we have the following adjoint pair $(\mathrm{inc}, i^!)$:

$$\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \underset{i^!}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathbf{K}(\Lambda\text{-Mod}) \tag{2.4}$$

between homotopy categories. Here, the functor inc is the inclusion.

Lemma 2.4 (see [10, Lemma 8.3]) *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors between triangulated categories. Suppose that (F, G) is an adjoint pair. Then F is a triangle functor if and only if G is a triangle functor.*

By the above lemma, the functor $i^! : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ is a triangle functor.

2.2 Another triangle functor

For the upper triangular matrix ring Λ, we will define another triange functor

$$i^* : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod}).$$

Recall that we denote by $\binom{X^\bullet}{V^\bullet}$ the complex $\big(\binom{X^n}{V^n}, \binom{d_X^n}{d_V^n}\big)_{n\in\mathbb{Z}}$ of Λ-modules. The n-th component $\binom{X^n}{V^n}$ has the structure map $\phi^n : M \otimes_S V^n \to X^n$ for each $n \in \mathbb{Z}$. We denote by $V^\bullet[1]$ the shifted complex given by $V^\bullet[1]^n = V^{n+1}$ and $d_{V[1]}^n = -d_V^{n+1}$ for $n \in \mathbb{Z}$. We denote

$$d_{C^+(V)}^n = \begin{pmatrix} d_V^n & \mathrm{Id}_{V^{n+1}} \\ 0 & -d_V^{n+1} \end{pmatrix} : V^n \oplus V^{n+1} \to V^{n+1} \oplus V^{n+2},$$

which is the n-th differential of the cone $C^+(V^\bullet) = V^\bullet \oplus V^\bullet[1]$ of $\mathrm{Id}_{V^\bullet}$.

We define

$$i^*\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} = \begin{pmatrix} X^\bullet \oplus (M \otimes_S V^\bullet[1]) \\ V^\bullet \oplus V^\bullet[1] \end{pmatrix}$$

to be the complex whose n-th component is $\left(\begin{smallmatrix} X^n \oplus (M \otimes_S V^{n+1}) \\ V^n \oplus V^{n+1} \end{smallmatrix}\right)$ with the structure map

$$\begin{pmatrix} \phi^n & 0 \\ 0 & \mathrm{Id}_{M \otimes_S V^{n+1}} \end{pmatrix} : M \otimes_S (V^n \oplus V^{n+1}) \to X^n \oplus (M \otimes_S V^{n+1}),$$

and n-th differential is

$$\begin{pmatrix} p^n \\ d^n_{C^+(V)} \end{pmatrix} : \begin{pmatrix} X^n \oplus (M \otimes_S V^{n+1}) \\ V^n \oplus V^{n+1} \end{pmatrix} \to \begin{pmatrix} X^{n+1} \oplus (M \otimes_S V^{n+2}) \\ V^{n+1} \oplus V^{n+2} \end{pmatrix},$$

where

$$p^n = \begin{pmatrix} d^n_X & \phi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes d^{n+1}_V \end{pmatrix}.$$

In fact, we have

$$p^n \circ \begin{pmatrix} \phi^n & 0 \\ 0 & \mathrm{Id}_{M \otimes_S V^{n+1}} \end{pmatrix} = \begin{pmatrix} \phi^{n+1} & 0 \\ 0 & \mathrm{Id}_{M \otimes_S V^{n+2}} \end{pmatrix} \circ (\mathrm{Id}_M \otimes d^n_{C^+(V)})$$

and $p^{n+1} \circ p^n = 0$ for each $n \in \mathbb{Z}$.

Let $\left(\begin{smallmatrix} Y^\bullet \\ W^\bullet \end{smallmatrix}\right)$ be another complex of Λ-modules with a structure map $\psi^\bullet : M \otimes_S W^\bullet \to Y^\bullet$. We write the n-th differential of $i^*\left(\begin{smallmatrix} Y^\bullet \\ W^\bullet \end{smallmatrix}\right)$ as

$$\begin{pmatrix} q^n \\ d^n_{C^+(W)} \end{pmatrix} : \begin{pmatrix} Y^n \oplus (M \otimes_S W^{n+1}) \\ W^n \oplus W^{n+1} \end{pmatrix} \to \begin{pmatrix} Y^{n+1} \oplus (M \otimes_S W^{n+2}) \\ W^{n+1} \oplus W^{n+2} \end{pmatrix},$$

where

$$q^n = \begin{pmatrix} d^n_Y & \psi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes d^{n+1}_W \end{pmatrix}.$$

For a chain map

$$\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix}_{n \in \mathbb{Z}} : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}$$

of complexes of Λ-modules, we define

$$i^*\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} = \begin{pmatrix} f^\bullet \vee g^\bullet \\ C^+(g^\bullet) \end{pmatrix} : \begin{pmatrix} X^\bullet \oplus (M \otimes_S V^\bullet[1]) \\ V^\bullet \oplus V^\bullet[1] \end{pmatrix} \to \begin{pmatrix} Y^\bullet \oplus (M \otimes_S W^\bullet[1]) \\ W^\bullet \oplus W^\bullet[1] \end{pmatrix}$$

such that

$$\begin{pmatrix} (f^\bullet \vee g^\bullet)^n \\ C^+(g^\bullet)^n \end{pmatrix} : \begin{pmatrix} X^n \oplus (M \otimes_S V^{n+1}) \\ V^n \oplus V^{n+1} \end{pmatrix} \to \begin{pmatrix} Y^n \oplus (M \otimes_S W^{n+1}) \\ W^n \oplus W^{n+1} \end{pmatrix}$$

is given by

$$(f^\bullet \vee g^\bullet)^n = \begin{pmatrix} f^n & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+1} \end{pmatrix}, \quad C^+(g^\bullet)^n = \begin{pmatrix} g^n & 0 \\ 0 & g^{n+1} \end{pmatrix}.$$

Lemma 2.5 *We have that* $i^*\left(\begin{smallmatrix} f^\bullet \\ g^\bullet \end{smallmatrix}\right)$ *is a chain map of complexes of* Λ*-modules.*

Proof Since $\binom{f^n}{g^n}$ is a morphism of Λ-modules, we observe that

$$\begin{pmatrix} f^n & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+1} \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \mathrm{Id}_{M\otimes_S V^{n+1}} \end{pmatrix}$$

$$= \begin{pmatrix} \psi^n & 0 \\ 0 & \mathrm{Id}_{M\otimes_S W^{n+1}} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_M \otimes g^n & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+1} \end{pmatrix}.$$

This implies that $\binom{(f^\bullet \vee g^\bullet)^n}{C^+(g^\bullet)^n}$ is a morphism of Λ-modules.

It remains to prove

$$\begin{pmatrix} q^n \\ d^n_{C^+(W)} \end{pmatrix} \circ \begin{pmatrix} (f^\bullet \vee g^\bullet)^n \\ C^+(g^\bullet)^n \end{pmatrix} = \begin{pmatrix} (f^\bullet \vee g^\bullet)^{n+1} \\ C^+(g^\bullet)^{n+1} \end{pmatrix} \circ \begin{pmatrix} p^n \\ d^n_{C^+(V)} \end{pmatrix}$$

for each $n \in \mathbb{Z}$. We have that

$$\begin{aligned} q^n \circ (f^\bullet \vee g^\bullet)^n &= \begin{pmatrix} d^n_Y & \psi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes d^{n+1}_W \end{pmatrix} \begin{pmatrix} f^n & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} d^n_Y \circ f^n & \psi^{n+1} \circ (\mathrm{Id}_M \otimes g^{n+1}) \\ 0 & -\mathrm{Id}_M \otimes (d^{n+1}_W \circ g^{n+1}) \end{pmatrix} \\ &= \begin{pmatrix} f^{n+1} \circ d^n_X & f^{n+1} \circ \phi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes (g^{n+2} \circ d^{n+1}_V) \end{pmatrix} \\ &= \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+2} \end{pmatrix} \begin{pmatrix} d^n_X & \phi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes d^{n+1}_V \end{pmatrix} \\ &= (f^\bullet \vee g^\bullet)^{n+1} \circ p^n. \end{aligned}$$

Since $g^\bullet : V^\bullet \to W^\bullet$ is a chain map, we have $d^n_{C^+(W)} \circ C^+(g^\bullet)^n = C^+(g^\bullet)^{n+1} \circ d^n_{C^+(V)}$ by direct calculation.

Lemma 2.6 *If* $\binom{f^\bullet}{g^\bullet} \sim 0$, *then* $i^*\binom{f^\bullet}{g^\bullet} \sim 0$.

Proof Suppose that there exists a morphism

$$\begin{pmatrix} \eta^n \\ \theta^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \end{pmatrix} \to \begin{pmatrix} Y^{n-1} \\ W^{n-1} \end{pmatrix}$$

of Λ-modules such that

$$\begin{pmatrix} d^{n-1}_Y \\ d^{n-1}_W \end{pmatrix} \circ \begin{pmatrix} \eta^n \\ \theta^n \end{pmatrix} + \begin{pmatrix} \eta^{n+1} \\ \theta^{n+1} \end{pmatrix} \circ \begin{pmatrix} d^n_X \\ d^n_V \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix} \tag{2.5}$$

for each $n \in \mathbb{Z}$. Take

$$\Phi^n = \begin{pmatrix} \eta^n & 0 \\ 0 & -\mathrm{Id}_M \otimes \theta^{n+1} \end{pmatrix} : X^n \oplus (M \otimes_S V^{n+1}) \to Y^{n-1} \oplus (M \otimes_S W^n)$$

and

$$\Psi^n = \begin{pmatrix} \theta^n & 0 \\ 0 & -\theta^{n+1} \end{pmatrix} : V^n \oplus V^{n+1} \to W^{n-1} \oplus W^n.$$

Since $\binom{\eta^n}{\theta^n}$ is a morphism of Λ-modules, we have that

$$\begin{pmatrix} \Phi^n \\ \Psi^n \end{pmatrix} : \begin{pmatrix} X^n \oplus (M \otimes_S V^{n+1}) \\ V^n \oplus V^{n+1} \end{pmatrix} \to \begin{pmatrix} Y^{n-1} \oplus (M \otimes_S W^n) \\ W^{n-1} \oplus W^n \end{pmatrix}$$

is a morphism of Λ-modules.

It remains to prove

$$\begin{pmatrix} q^{n-1} \\ d^{n-1}_{C^+(W)} \end{pmatrix} \circ \begin{pmatrix} \Phi^n \\ \Psi^n \end{pmatrix} + \begin{pmatrix} \Phi^{n+1} \\ \Psi^{n+1} \end{pmatrix} \circ \begin{pmatrix} p^n \\ d^n_{C^+(V)} \end{pmatrix} = \begin{pmatrix} (f^\bullet \vee g^\bullet)^n \\ C^+(g^\bullet)^n \end{pmatrix}$$

for each $n \in \mathbb{Z}$. We have

$$\begin{aligned}
&q^{n-1} \circ \Phi^n + \Phi^{n+1} \circ p^n \\
&= \begin{pmatrix} d_Y^{n-1} & \psi^n \\ 0 & -\mathrm{Id}_M \otimes d_W^n \end{pmatrix} \begin{pmatrix} \eta^n & 0 \\ 0 & -\mathrm{Id}_M \otimes \theta^{n+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \eta^{n+1} & 0 \\ 0 & -\mathrm{Id}_M \otimes \theta^{n+2} \end{pmatrix} \begin{pmatrix} d_X^n & \phi^{n+1} \\ 0 & -\mathrm{Id}_M \otimes d_V^{n+1} \end{pmatrix} \\
&= \begin{pmatrix} d_Y^{n-1} \circ \eta^n + \eta^{n+1} \circ d_X^n & \eta^{n+1} \circ \phi^{n+1} - \psi^n \circ (\mathrm{Id}_M \otimes \theta^{n+1}) \\ 0 & \mathrm{Id}_M \otimes (d_W^n \circ \theta^{n+1} + \theta^{n+2} \circ d_V^{n+1}) \end{pmatrix} \\
&= \begin{pmatrix} f^n & 0 \\ 0 & \mathrm{Id}_M \otimes g^{n+1} \end{pmatrix} \\
&= (f^\bullet \vee g^\bullet)^n.
\end{aligned}$$

Here, the second last equality uses (2.5). Similarly, we have $d^{n-1}_{C^+(W)} \circ \Psi^n + \Psi^{n+1} \circ d^n_{C^+(V)} = C^+(g^\bullet)^n$.

We observe that

$$i^* \begin{pmatrix} \mathrm{Id}_{X^\bullet} \\ \mathrm{Id}_{V^\bullet} \end{pmatrix} = \mathrm{Id}_{i^*\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right)}$$

and that i^* preserves the composition of morphisms in the category of complexes of Λ-modules. By Lemmas 2.5–2.6 we directly have the following consequence.

Proposition 2.2 *Let Λ be the upper triangular matrix ring. Then we have that $i^* : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ is a functor.*

The functor $j^* : \Lambda\text{-Mod} \to S\text{-Mod}$ admits a left adjoint

$$j_! : S\text{-Mod} \to \Lambda\text{-Mod}$$

which sends V to $\left(\begin{smallmatrix} M \otimes_S V \\ V \end{smallmatrix}\right)$ with the structure map the identity of $M \otimes_S V$. Observe that the corresponding unit $\mathrm{Id}_{S\text{-Mod}} \xrightarrow{\sim} j^* j_!$ is an isomorphism. Then the functor $j_!$ is fully faithful.

The additive functor $j_!$ induces a triangle functor $\mathbf{K}(S\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$, still denoted by $j_!$. We have that $(j_!, j^*)$ is an adjoint pair between homotopy categories with $j_!$ a fully faithful triangle functor.

For each $\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) \in \mathbf{K}(\Lambda\text{-Mod})$, we have an exact sequence of complexes

$$0 \to \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \xrightarrow{\iota^\bullet} i^* \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to j_!(V^\bullet[1]) \to 0 \tag{2.6}$$

such that $\iota^\bullet$ is given by

$$X^\bullet \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} X^\bullet \oplus (M \otimes_S V^\bullet[1]), \quad V^\bullet \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} V^\bullet \oplus V^\bullet[1].$$

The above sequence splits in each component.

Lemma 2.7 *Let*

$$\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \in \mathbf{K}(\Lambda\text{-Mod}), \quad \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix} \in \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod}).$$

Then we have the following isomorphism induced by $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(\iota^\bullet, \left(\begin{smallmatrix} Y^\bullet \\ W^\bullet \end{smallmatrix}\right))$:

$$\mathrm{Hom}_{\mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})}\Big(i^* \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big),$$

which is natural in two variables.

Proof Apply $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(-, \left(\begin{smallmatrix} Y^\bullet \\ W^\bullet \end{smallmatrix}\right))$ to the sequence (2.6). We have the isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(j_!(V^\bullet[1]), \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(S\text{-Mod})}\Big(V^\bullet[1], j^* \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big).$$

Since $W^\bullet \sim 0$, the statement follows directly.

By the above lemma, we have the following adjoint pair (i^*, inc)

$$\mathbf{K}(\Lambda\text{-Mod}) \underset{\mathrm{inc}}{\overset{i^*}{\rightleftarrows}} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod}) \tag{2.7}$$

of homotopy categories. By Lemma 2.4, $i^* : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})$ is a triangle functor.

3 An Explicit Ladder of Homotopy Categories

In this section, we prove the explicit ladder of height 2 in the introduction.

Recall that a diagram of triangle functors between triangulated categories

$$\mathcal{T}' \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow[\;i^!\;]{} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow[\;j_*\;]{} \end{array} \mathcal{T}''$$

forms a recollement (see [6, Section 2]), provided that the following conditions are satisfied:

(R1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) The three functors i_*, $j_!$ and j_* are fully faithful;

(R3) $\mathrm{Im}\, i_* = \mathrm{Ker}\, j^*$.

Here, $\mathrm{Im}\, i_*$ and $\mathrm{Ker}\, j^*$ are essential image and kernel of i_* and j^*, respectively. The definition of recollement is equivalent to that given in [2, 1.4].

Remark 3.1 We mention that in the above recollement $\mathrm{Im}\, j_! = \mathrm{Ker}\, i^*$ and $\mathrm{Im}\, j_* = \mathrm{Ker}\, i^!$.

A ladder (see [9, Section 3]) of height 2 is a diagram of triangle functors between triangulated categories

$$\mathcal{T}' \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow{\;i^!\;} \\ \xrightarrow[\;i_2\;]{} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow{\;j_*\;} \\ \xrightarrow[\;j_2\;]{} \end{array} \mathcal{T}''$$

such that any three consecutive rows form a recollement. There are two recollements in a ladder of height 2.

Let Λ be the upper triangular matrix ring. Recall that the homotopy category $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Mod})$. Set i_* to be the inclusion $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$.

Proposition 3.1 *Let Λ be the upper triangular matrix ring. Then there exists a recollement of homotopy categories*

$$\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \overset{\xleftarrow{i^*}}{\underset{\xleftarrow[i^!]{}}{\xrightarrow{i_*}}} \mathbf{K}(\Lambda\text{-Mod}) \overset{\xleftarrow{j_!}}{\underset{\xleftarrow[j_*]{}}{\xrightarrow{j^*}}} \mathbf{K}(S\text{-Mod}).$$

Proof By (2.4) and (2.7), $(i_*, i^!)$ and (i^*, i_*) are adjoint pairs. Recall that $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs between homotopy categories with $j_!, j_* : \mathbf{K}(S\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ fully faithful functors. We observe that $\operatorname{Ker} j^* = \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$. The statement follows directly.

Let $\binom{X}{V}$ be a left Λ-module. We use the adjoint isomorphism

$$\operatorname{Hom}_R(M \otimes_S V, X) \cong \operatorname{Hom}_S(V, \operatorname{Hom}_R(M, X))$$

to give a description of left Λ-modules. Then the left Λ-module $\binom{X}{V}$ has an adjoint structure map $\phi : V \to \operatorname{Hom}_R(M, X)$, which is an S-morphism. A Λ-morphism from $\binom{X}{V}$ to $\binom{X'}{V'}$ is a pair of morphisms $\binom{\alpha}{\beta}$ with $\alpha : X \to X'$ an R-morphism and $\beta : V \to V'$ an S-morphism such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & \operatorname{Hom}_R(M, X) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \operatorname{Hom}_R(M,\alpha)} \\ V' & \xrightarrow{\phi'} & \operatorname{Hom}_R(M, X'). \end{array}$$

Define a functor

$$j_2 : \Lambda\text{-Mod} \to S\text{-Mod}$$

such that $j_2\binom{X}{V} = \operatorname{Ker}\phi$. Recall the functor $j_* : S\text{-Mod} \to \Lambda\text{-Mod}$ (see (2.2)). Observe that for any $V \in S\text{-Mod}$: $\binom{Y}{W} \in \Lambda\text{-Mod}$, we have the following isomorphism:

$$\operatorname{Hom}_{\Lambda\text{-Mod}}\Big(j_*(V), \begin{pmatrix} Y \\ W \end{pmatrix}\Big) \cong \operatorname{Hom}_{S\text{-Mod}}\Big(V, j_2\begin{pmatrix} Y \\ W \end{pmatrix}\Big). \tag{3.1}$$

Then (j_*, j_2) is an adjoint pair. Similarly, the functor j_2 induces a triangle functor $\mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(S\text{-Mod})$, still denoted by j_2. The induced functors (j_*, j_2) is an adjoint pair of homotopy categories.

Denote $q : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})/\operatorname{Im} j_*$. We will define a functor $\lambda : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$. Let $\binom{X^\bullet}{V^\bullet}$ be a complex of Λ-modules with a structure map $\phi^\bullet : V^\bullet \to \operatorname{Hom}_R(M, X^\bullet)$. Define $\lambda\binom{X^\bullet}{V^\bullet}$ to be the cone of the canonical chain map $\binom{0}{\operatorname{Ker}\phi^\bullet} \to \binom{X^\bullet}{V^\bullet}$. We have the following triangle in the homotopy category $\mathbf{K}(\Lambda\text{-Mod})$:

$$\begin{pmatrix} 0 \\ \operatorname{Ker}\phi^\bullet \end{pmatrix} \to \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \lambda\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} 0 \\ \operatorname{Ker}\phi^\bullet[1] \end{pmatrix}. \tag{3.2}$$

Let $\binom{Y^\bullet}{W^\bullet}$ be another complex of Λ-modules with a structure map $\psi^\bullet : W^\bullet \to \operatorname{Hom}_R(M, Y^\bullet)$. For a chain map

$$\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}$$

of complexes of Λ-modules, we define

$$\lambda\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} : \begin{pmatrix} 0 \\ \operatorname{Ker}\phi^\bullet[1] \end{pmatrix} \oplus \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} 0 \\ \operatorname{Ker}\psi^\bullet[1] \end{pmatrix} \oplus \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}$$

to be the chain map given by $g^\bullet[1] : \operatorname{Ker}\phi^\bullet[1] \to \operatorname{Ker}\psi^\bullet[1]$ and

$$\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}.$$

Lemma 3.1 *Let*

$$\begin{pmatrix} f^\bullet \\ g^\bullet \end{pmatrix} : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}$$

be a chain map of Λ-modules. If $\left(\begin{smallmatrix} f^\bullet \\ g^\bullet \end{smallmatrix}\right) \sim 0$, then $\lambda\left(\begin{smallmatrix} f^\bullet \\ g^\bullet \end{smallmatrix}\right) \sim 0$.

Proof Suppose that for any $n \in \mathbb{Z}$, there exists a morphism of Λ-modules

$$\begin{pmatrix} u^n \\ \nu^n \end{pmatrix} : \begin{pmatrix} X^n \\ V^n \end{pmatrix} \to \begin{pmatrix} Y^{n-1} \\ W^{n-1} \end{pmatrix}$$

such that

$$\begin{pmatrix} d_Y^{n-1} \\ d_W^{n-1} \end{pmatrix} \circ \begin{pmatrix} u^n \\ \nu^n \end{pmatrix} + \begin{pmatrix} u^{n+1} \\ \nu^{n+1} \end{pmatrix} \circ \begin{pmatrix} d_X^n \\ d_V^n \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix}. \tag{3.3}$$

Let $\phi^n : V^n \to \operatorname{Hom}_R(M, X^n)$ and $\psi^n : W^n \to \operatorname{Hom}_R(M, Y^n)$ be adjoint structure maps of $\left(\begin{smallmatrix} X^n \\ V^n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} Y^n \\ W^n \end{smallmatrix}\right)$, respectively. If $x \in \operatorname{Ker}\phi^{n+1}$, then $\nu^{n+1}(x) \in \operatorname{Ker}\psi^n$. In fact, we have $(\psi^n \circ \nu^{n+1})(x) = \operatorname{Hom}_R(M, u^{n+1})(\phi^{n+1}(x)) = 0$.

Take

$$\varrho^n = \begin{pmatrix} s^n & 0 \\ 0 & t^n \end{pmatrix} : \begin{pmatrix} 0 \\ \operatorname{Ker}\phi^{n+1} \end{pmatrix} \oplus \begin{pmatrix} X^n \\ V^n \end{pmatrix} \to \begin{pmatrix} 0 \\ \operatorname{Ker}\psi^n \end{pmatrix} \oplus \begin{pmatrix} Y^{n-1} \\ W^{n-1} \end{pmatrix},$$

where $s^n = \left(\begin{smallmatrix} 0 \\ -\nu^{n+1} \end{smallmatrix}\right)$ and $t^n = \left(\begin{smallmatrix} u^n \\ \nu^n \end{smallmatrix}\right)$. Since s^n and t^n are morphisms of Λ-modules, ϱ^n is a morphism of Λ-modules. By (3.3), ϱ^n gives the null-homotopy $\lambda\left(\begin{smallmatrix} f^\bullet \\ g^\bullet \end{smallmatrix}\right) \sim 0$.

By the above lemma, $\lambda\left(\begin{smallmatrix} f^\bullet \\ g^\bullet \end{smallmatrix}\right)$ is well-defined. Observe that λ preserves the identity map and the composition of morphisms in the category of complexes of Λ-modules. Then $\lambda : \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{K}(\Lambda\text{-Mod})$ is a functor. Observe that $(\lambda \circ j_*)(V^\bullet) \sim 0$ for any complex $V^\bullet$ of S-modules. Then we have the following functor:

$$\lambda : \mathbf{K}(\Lambda\text{-Mod})/\operatorname{Im} j_* \to \mathbf{K}(\Lambda\text{-Mod}).$$

Here, $\mathbf{K}(\Lambda\text{-Mod})/\operatorname{Im} j_*$ is the Verdier quotient.

Recall that (j_*, j_2) is an adjoint pair. Let $U^\bullet$ be a complex of S-modules. Apply the functor $\operatorname{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(j_*(U^\bullet), -)$ to the triangle (3.2). Then we have

$$\operatorname{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(j_*(U^\bullet), \lambda\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}\Big) = 0.$$

(q, λ) is an adjoint pair and λ is fully faithful; compare [16, Lemma 1.3]. In fact, there are isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathbf{K}(\Lambda\text{-Mod})/\operatorname{Im} j_*}\Big(q\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big) &\cong \operatorname{Hom}_{\mathbf{K}(\Lambda\text{-Mod})/\operatorname{Im} j_*}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \lambda\begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}\Big(\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}, \lambda\begin{pmatrix} Y^\bullet \\ W^\bullet \end{pmatrix}\Big). \end{aligned}$$

Set i_2 to be the composition

$$\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \xrightarrow{\rm inc} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow{q} \mathbf{K}(\Lambda\text{-Mod})/{\rm Im}j_* \xrightarrow{\lambda} \mathbf{K}(\Lambda\text{-Mod}).$$

Since $q \circ {\rm inc}$ is the quasi-inverse of $\mathbf{K}(\Lambda\text{-Mod})/{\rm Im}j_* \xrightarrow{i^!} \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ and (q, λ) is an adjoint pair $(i^!, i_2)$,

$$\mathbf{K}(\Lambda\text{-Mod}) \underset{i_2}{\overset{i^!}{\rightleftarrows}} \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$$

is an adjoint pair and the right adjoint functor i_2 is fully faithful.

The following is the main result of this paper.

Theorem 3.1 *Let Λ be the upper triangular matrix ring. Then there exists an explicit ladder of height 2 of homotopy categories*

$$\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \\ \xrightarrow[i_2]{} \end{array} \mathbf{K}(\Lambda\text{-Mod}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \\ \xrightarrow[j_2]{} \end{array} \mathbf{K}(S\text{-Mod}).$$

Proof By Proposition 3.1 and Remark 3.1, the result follows immediately.

We observe the natural triangle functor

$$T : \mathbf{K}(R\text{-Mod}) \to \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod}), \quad X^\bullet \mapsto \begin{pmatrix} X^\bullet \\ 0 \end{pmatrix}.$$

This triangle functor is fully faithful, but is not a triangle equivalence (see Exmaple 3.1).

Recall that the derived category $\mathbf{D}(R\text{-Mod})$ is the localization of $\mathbf{K}(R\text{-Mod})$ with respect to the class of quasi-isomorphisms in $\mathbf{K}(R\text{-Mod})$.

Observe that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} X^\bullet \\ 0 \end{pmatrix} \to \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \tag{3.4}$$

is a quasi-isomorphism in $\mathbf{K}(\Lambda\text{-Mod})$ whenever $\begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \in \mathbf{K}(\Lambda\text{-Mod})$ and $V^\bullet$ is acyclic. We denote by Σ the class of quasi-isomorphisms in $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ and by $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})[\Sigma^{-1}]$ the localization of $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ with respect to Σ.

Lemma 3.2 *The triangle functor $T : \mathbf{K}(R\text{-Mod}) \to \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$ induces a triangle equivalence*

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{\sim} \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})[\Sigma^{-1}].$$

Proof Let $s^\bullet : \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to \begin{pmatrix} Y^\bullet \\ 0 \end{pmatrix}$ be a quasi-isomorphism in $\mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})$. Observe that $V^\bullet$ is acyclic. Then by (3.4) there exists a quasi-isomorphism

$$f^\bullet = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} X^\bullet \\ 0 \end{pmatrix} \to \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix}$$

such that $s^\bullet \circ f^\bullet$ is quasi-isomorphism. Then the functor

$$\mathbf{K}(R\text{-Mod})/\mathbf{K}_{\rm ac}(R\text{-Mod}) \to \mathbf{K}_{\rm dnh}(\Lambda\text{-Mod})[\Sigma^{-1}]$$

induced by T is fully faithful. By (3.4) the functor is dense.

We still denote the equivalence $\mathbf{D}(R\text{-Mod}) \xrightarrow{\sim} \mathbf{K}_{\text{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}]$ by T. We observe the triangle functor $H : \mathbf{K}_{\text{dnh}}(\Lambda\text{-Mod}) \to \mathbf{K}(R\text{-Mod})$ such that $H\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) = X^\bullet$. By (3.4), the induced functor

$$H : \mathbf{K}_{\text{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \to \mathbf{D}(R\text{-Mod})$$

is the quasi-inverse of T.

It is well known that for an upper triangular matrix ring, there is a recollement of derived categories; compare [11, Corollary 15] and [8, 2.1]. We prove that the ladder in Theorem 3.1 induces the ladder of derived categories; compare [9, Example 3.4].

We denote by $\mathbf{K}_{\boldsymbol{p}}(S\text{-Mod})$ the full subcategory of homotopically projective complexes of $\mathbf{K}(S\text{-Mod})$. Recall that the quotient functor $\mathbf{K}(S\text{-Mod}) \to \mathbf{D}(S\text{-Mod})$ induces an equivalence $\mathbf{K}_{\boldsymbol{p}}(S\text{-Mod}) \xrightarrow{\sim} \mathbf{D}(S\text{-Mod})$. We denote by $\boldsymbol{p} : \mathbf{D}(S\text{-Mod}) \to \mathbf{K}_{\boldsymbol{p}}(S\text{-Mod})$, which is a left adjoint to the quotient functor (see [12, Theorem 8.1.2]).

Observe that j^* and j_* preserve acyclic complexes. We have the induced functors

$$\mathbf{D}(\Lambda\text{-Mod}) \xrightarrow{j^*} \mathbf{D}(S\text{-Mod}), \quad \mathbf{D}(S\text{-Mod}) \xrightarrow{j_*} \mathbf{D}(\Lambda\text{-Mod}).$$

We denote by $\mathbf{L}j_!$ the left derived functor of $j_!$, which is the composition

$$\mathbf{D}(S\text{-Mod}) \xrightarrow{\boldsymbol{p}} \mathbf{K}_{\boldsymbol{p}}(S\text{-Mod}) \xrightarrow{j_!} \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{D}(\Lambda\text{-Mod}).$$

Lemma 3.3 *$(\mathbf{L}j_!, j^*)$ and (j^*, j_*) of induced functors are adjoint pairs. Moreover, $\mathbf{L}j_!$ and j_* are fully faithful.*

Proof We have

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(j_!(P^\bullet), N^\bullet) \simeq \mathrm{Hom}_{\mathbf{K}(S\text{-Mod})}(P^\bullet, j^*(N^\bullet)) = 0,$$

whenever $P^\bullet \in \mathbf{K}_{\boldsymbol{p}}(S\text{-Mod})$ and $N^\bullet$ is acyclic. We conclude that $j_!$ preserves homotopically projective complexes. Using this we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(\Lambda\text{-Mod})}(\mathbf{L}j_!(X^\bullet), Y^\bullet) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(j_!\boldsymbol{p}(X^\bullet), Y^\bullet)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(S\text{-Mod})}(\boldsymbol{p}(X^\bullet), j^*(Y^\bullet))\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S\text{-Mod})}(X^\bullet, j^*(Y^\bullet)).\end{aligned}$$

Thus $(\mathbf{L}j_!, j^*)$ is an adjoint pair.

The corresponding unit

$$\eta : \mathrm{Id}_{\mathbf{D}(S\text{-Mod})} \xrightarrow{\sim} j^* \circ \mathbf{L}j_!$$

is an isomorphism. In fact, $\eta_{X^\bullet} : X^\bullet \to \boldsymbol{p}(X^\bullet)$ is the right roof

$$X^\bullet \Longleftarrow \boldsymbol{p}(X^\bullet) \xrightarrow{\mathrm{Id}} \boldsymbol{p}(X^\bullet)$$

for each $X^\bullet \in \mathbf{D}(S\text{-Mod})$, which is an isomorphism in $\mathbf{D}(S\text{-Mod})$. This implies that $\mathbf{L}j_!$ is fully faithful.

By [15, Lemma 1.2], (j^*, j_*) of induced functors is an adjoint pair. The induced functor j_* is fully faithful; compare [6, Lemma 2.1].

We denote by $\mathbf{L}i^*$ the left derived functor of i^*, which is the composition

$$\mathbf{D}(\Lambda\text{-Mod}) \xrightarrow{\boldsymbol{p}} \mathbf{K}_{\boldsymbol{p}}(\Lambda\text{-Mod}) \xrightarrow{i^*} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod}) \to \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \xrightarrow{H} \mathbf{D}(R\text{-Mod}).$$

Denote by $\mathbf{K}_{\boldsymbol{i}}(\Lambda\text{-Mod})$ the full subcategory of homotopically injective complexes of $\mathbf{K}(\Lambda\text{-Mod})$. We denote $\boldsymbol{i}: \mathbf{D}(\Lambda\text{-Mod}) \to \mathbf{K}_{\boldsymbol{i}}(\Lambda\text{-Mod})$, which is a right adjoint to the quotient functor (see [12, Theorem 8.1.2]). Denote by $\mathbf{R}i_*$ the right derived functor of i_*, which is the composition

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{T} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \xrightarrow{\boldsymbol{i}} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow{i^!} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod}) \xrightarrow{i_*} \mathbf{K}(\Lambda\text{-Mod}) \to \mathbf{D}(\Lambda\text{-Mod}).$$

Since i_* and $i^!$ preserve acyclic complexes, we have the induced functors

$$\mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \xrightarrow{i_*} \mathbf{D}(\Lambda\text{-Mod})$$

and

$$\mathbf{D}(\Lambda\text{-Mod}) \xrightarrow{i^!} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}].$$

By [15, Lemma 1.2], $(i_*, i^!)$ of induced functors is an adjoint pair such that the left adjoint functor i_* is fully faithful; compare [6, Lemma 2.1].

Lemma 3.4 *The functor $\mathbf{R}i_*$ is natural isomorphic to the composition*

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{T} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \xrightarrow{i_*} \mathbf{D}(\Lambda\text{-Mod}).$$

Proof We observe that $X^\bullet \cong i^!(X^\bullet) \cong (i^! \circ \boldsymbol{i})(X^\bullet)$ in $\mathbf{D}(\Lambda\text{-Mod})$ for any $X^\bullet \in \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})$.

Lemma 3.5 $(\mathbf{L}i^*, \mathbf{R}i_*)$ *of induced functors is an adjoint pair.*

We denote by $\mathbf{R}j_2$ the right derived functor of j_2, which is the composition

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{\boldsymbol{i}} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow{j_2} \mathbf{K}(S\text{-Mod}) \to \mathbf{D}(S\text{-Mod}).$$

Denote by $\mathbf{R}i_2$ the right derived functor of i_2, which is the following composition:

$$\mathbf{D}(R\text{-Mod}) \xrightarrow{T} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}] \xrightarrow{\boldsymbol{i}} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow{i^!} \mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod}) \xrightarrow{i_2} \mathbf{K}(\Lambda\text{-Mod}) \longrightarrow \mathbf{D}(\Lambda\text{-Mod}).$$

We have the ladder of derived categories for the upper triangular matrix ring; compare [9, Example 3.4].

Corollary 3.1 *Let Λ be the upper triangular matrix ring. Then there exists a ladder of height* 2,

$$\mathbf{D}(R\text{-Mod}) \begin{array}{c} \xleftarrow{\mathbf{L}i^*} \\ \xrightarrow{\mathbf{R}i_*} \\ \xleftarrow{H\circ i^!} \\ \xrightarrow[\mathbf{R}i_2]{} \end{array} \mathbf{D}(\Lambda\text{-Mod}) \begin{array}{c} \xleftarrow{\mathbf{L}j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \\ \xrightarrow[\mathbf{R}j_2]{} \end{array} \mathbf{D}(S\text{-Mod}).$$

Proof By Lemma 3.4, $(\mathbf{R}i_*, H\circ i^!)$ is an adjoint pair and $\mathbf{R}i_*$ is fully faithful.

We have

$$\operatorname{Ker} j^* = \left\{ \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \in \mathbf{D}(\Lambda\text{-Mod}) \;\middle|\; V^\bullet \cong 0 \text{ in } \mathbf{D}(S\text{-Mod}) \right\}$$

and

$$\operatorname{Im} \mathbf{R}i_* = \left\{ \begin{pmatrix} X^\bullet \\ 0 \end{pmatrix} \in \mathbf{D}(\Lambda\text{-Mod}) \right\}.$$

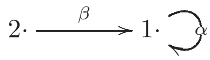
By (3.4), we obtain $\operatorname{Ker} j^* = \operatorname{Im} \mathbf{R}i_*$. By Lemmas 3.3–3.5, the first three rows is a recollement.

Observe that for any $X^\bullet \in \mathbf{D}(\Lambda\text{-Mod})$, we have $i^!(X^\bullet) \cong (i^! \circ \boldsymbol{p})(X^\bullet)$ in $\mathbf{K}_{\mathrm{dnh}}(\Lambda\text{-Mod})[\Sigma^{-1}]$. We can directly check that $(H\circ i^!, \mathbf{R}i_2)$ and $(j_*, \mathbf{R}j_2)$ are adjoint pairs. Then $\mathbf{R}i_2$ is fully faithful. By Remark 3.1, the last three rows is a recollement.

Example 3.1 Let Q be the following quiver with one vertex and one loop.

Let k be a field. Denote by $A = kQ/J^2$ the corresponding algebra with radical square zero. Indeed, its Jacobson radical $\mathrm{rad}A = kQ_1$ satisfying $(\mathrm{rad}A)^2 = 0$. Set $I = D(A_A)$.

Let Q' be the following quiver:



Then the corresponding algebra $A' = kQ'/J^2$ with radical square zero is the one-point extension of A by $k\alpha$. In fact, we have $A' = \left(\begin{smallmatrix} A & k\alpha \\ 0 & k \end{smallmatrix}\right)$ with $\mathrm{rad}A \cdot k\alpha = 0$. Denote by $I_1 = D(e_1A')$ and $I_2 = D(e_2A')$ the corresponding indecomposable injective A'-modules. Then $I_1 = \left(\begin{smallmatrix} I \\ \mathrm{Hom}_A(k\alpha, I) \end{smallmatrix}\right)$ is a left A'-module via the natural evaluation map $k\alpha \otimes_k \mathrm{Hom}_A(k\alpha, I) \to I$, and $I_2 = \left(\begin{smallmatrix} 0 \\ k \end{smallmatrix}\right)$.

Recall from [14, Definition 2.4] the injective Leavitt complex of a finite quiver without sinks. Denote by $\mathcal{I}^\bullet_{Q'}$ the injective Leavitt complex of Q'. By [14, Lemma 2.10], the canonical map $Z^n_{\mathcal{I}_{Q'}} \to \mathcal{I}^n_{Q'}$ is an injective envelope for each $n \in \mathbb{Z}$. Recall from [13, Appendix B] the notion of homotopically minimal complex. By Lemma B.1 of Appendix B in [13], $\mathcal{I}^\bullet_{Q'}$ is homotopically minimal.

We observe that $\mathcal{I}^\bullet_{Q'} \in \mathbf{K}_{\mathrm{dnh}}(A'\text{-Mod})$. Suppose that $\mathcal{I}^\bullet_{Q'} \cong \left(\begin{smallmatrix} X^\bullet \\ 0 \end{smallmatrix}\right)$ in the category $\mathbf{K}_{\mathrm{dnh}}(A'\text{-Mod})$ for some $X^\bullet \in \mathbf{K}(A\text{-Mod})$. Let $f^\bullet : \mathcal{I}^\bullet_{Q'} \to \left(\begin{smallmatrix} X^\bullet \\ 0 \end{smallmatrix}\right)$ and $g^\bullet : \left(\begin{smallmatrix} X^\bullet \\ 0 \end{smallmatrix}\right) \to \mathcal{I}^\bullet_{Q'}$ be chain maps such that $(g^\bullet \circ f^\bullet) \sim \mathrm{Id}_{\mathcal{I}^\bullet_{Q'}}$. Since $\mathcal{I}^\bullet_{Q'}$ is homotopically minimal, $g^\bullet \circ f^\bullet$ is an isomorphism of complexes. Then there exists a decomposition $\left(\begin{smallmatrix} X^\bullet \\ 0 \end{smallmatrix}\right) = \mathcal{I}^\bullet_{Q'} \oplus H^\bullet$ of complexes. We have

$$\mathcal{I}^n_{Q'} \oplus H^n = \begin{pmatrix} X^n \\ 0 \end{pmatrix} \tag{3.5}$$

for $n \in \mathbb{Z}$. However,

$$\mathcal{I}^0_{Q'} = I_1 \oplus I_1 \oplus I_2 \oplus I_2 = \begin{pmatrix} I \oplus I \\ \mathrm{Hom}_A(k\alpha, I) \oplus \mathrm{Hom}_A(k\alpha, I) \oplus k \oplus k \end{pmatrix}.$$

This is a contradiction to (3.5). This implies that the embedding triangle functor $T : \mathbf{K}(A\text{-Mod}) \to \mathbf{K}_{\mathrm{dnh}}(A'\text{-Mod})$ is not dense, thus it is not a triangle equivalence.

4 A Localization Sequence of Homotopy Categories

In this section, we prove the localization sequence of functors between homotopy categories of acyclic complexes of injective modules.

Recall that a diagram of triangle functors between triangulated categories

$$\mathcal{T}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{T}''$$

is a localization sequence (see [13, Definition 3.1]), provided that the following conditions are satisfied:

(R1) $(i_*, i^!)$ and (j^*, j_*) are adjoint pairs;

(R2) The two functors i_* and j_* are fully faithful;

(R3) $\operatorname{Im} i_* = \operatorname{Ker} j^*$.

For the upper triangular matrix ring Λ, the following lemma is well known; compare [4, Lemma 3.1], [16, Lemma 1.2] and [1, III, Propositions 2.3, 2.5(c)].

Lemma 4.1 *Let $\binom{X}{V}$ be a left Λ-module.*

(1) *If $\binom{X}{V}$ is an injective Λ-module, then X is an injective R-module.*

(2) *For any left R-module Y, we have a natural isomorphism*

$$\operatorname{Hom}_{R\text{-Mod}}(X, Y) \simeq \operatorname{Hom}_{\Lambda\text{-Mod}}\left(\begin{pmatrix} X \\ V \end{pmatrix}, \begin{pmatrix} Y \\ \operatorname{Hom}_R(M, Y) \end{pmatrix}\right),$$

where $\binom{Y}{\operatorname{Hom}_R(M,Y)}$ is a left Λ-module via the natural evaluation map $M \otimes_S \operatorname{Hom}_R(M, Y) \to Y$. In particular, $\binom{Y}{\operatorname{Hom}_R(M,Y)}$ is an injective Λ-module if and only if Y is an injective R-module.

(3) *Let W be an S-module. Then $\binom{0}{W}$ is an injective Λ-module if and only if W is an injective S-module.*

We denote by R-Inj the category of injective left R-modules. By the above lemma, we have additive functors

$$F : \Lambda\text{-Inj} \to R\text{-Inj}, \quad \begin{pmatrix} X \\ V \end{pmatrix} \mapsto X$$

and

$$G : R\text{-Inj} \to \Lambda\text{-Inj}, \quad Y \mapsto \begin{pmatrix} Y \\ \operatorname{Hom}_R(M, Y) \end{pmatrix}$$

such that (F, G) is an adjoint pair. We observe that for an injective Λ-module $\binom{X}{V}$, the morphism $\binom{X}{V} \to (G \circ F)\binom{X}{V}$ given by the corresponding unit is split epic.

We denote by $\mathbf{K}(R\text{-Inj})$ the homotopy category of complexes of injective left R-modules, which is a triangulated subcategory of $\mathbf{K}(R\text{-Mod})$. The additive functors F and G induce triangle functors $\widetilde{F} : \mathbf{K}(\Lambda\text{-Inj}) \to \mathbf{K}(R\text{-Inj})$ and $\widetilde{G} : \mathbf{K}(R\text{-Inj}) \to \mathbf{K}(\Lambda\text{-Inj})$. We have that $(\widetilde{F}, \widetilde{G})$ is an adjoint pair between homotopy categories

$$\mathbf{K}(\Lambda\text{-Inj}) \underset{\widetilde{G}}{\overset{\widetilde{F}}{\rightleftarrows}} \mathbf{K}(R\text{-Inj}). \tag{4.1}$$

In what follows, let R, S be two rings and ${}_RM_S$ an R-S-bimodule such that M_S is a flat right S-module. We consider the corresponding upper triangular matrix ring Λ.

Lemma 4.2 *Let ${}_RM_S$ be an R-S-bimodule. Then M is a flat right S-module if and only if $\mathrm{Hom}_R(M, I)$ is an injective left S-module for any injective left R-module I.*

Denote by $\mathbf{K}_{\rm ac}(R\text{-Inj})$ the full subcategory of $\mathbf{K}(R\text{-Inj})$ formed by acyclic complexes of injective left R-modules. For $X^\bullet \in \mathbf{K}_{\rm ac}(R\text{-Inj})$, we have $\widetilde{G}(X^\bullet) \in \mathbf{K}(\Lambda\text{-Inj})$. By Lemma 4.2, Lemma 4.1(3) and the construction of $i^!$,

$$(i^! \circ \widetilde{G})(X^\bullet) = i^! \begin{pmatrix} X^\bullet \\ \mathrm{Hom}_R(M, X^\bullet) \end{pmatrix} = \begin{pmatrix} X^\bullet \\ \mathrm{Hom}_R(M, X^\bullet) \oplus \mathrm{Hom}_R(M, X^\bullet)[-1] \end{pmatrix}$$

belongs to $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$. Let $G' = i^! \circ \widetilde{G}$. We have the triangle functor

$$G' : \mathbf{K}_{\rm ac}(R\text{-Inj}) \to \mathbf{K}_{\rm ac}(\Lambda\text{-Inj}).$$

Since the additive functor $F : \Lambda\text{-Inj} \to R\text{-Inj}$ preserves exact sequences, we have a triangle functor

$$F' : \mathbf{K}_{\rm ac}(\Lambda\text{-Inj}) \to \mathbf{K}_{\rm ac}(R\text{-Inj}).$$

By the adjoint pairs $(\widetilde{F}, \widetilde{G})$ and (2.4), we have the following consequence immediately.

Corollary 4.1 *Let ${}_RM_S$ be an R-S-bimodule such that M_S is a flat right S-module. Let Λ be the upper triangular matrix ring. Then (F', G') is an adjoint pair.*

The counit $\varepsilon : F' \circ G' \to \mathrm{Id}_{\mathbf{K}_{\rm ac}(R\text{-Inj})}$ is the identity. In fact, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{K}_{\rm ac}(R\text{-Inj})}((F' \circ G')(Y^\bullet), Y^\bullet) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Inj})}(G'(Y^\bullet), \widetilde{G}(Y^\bullet)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})}(G'(Y^\bullet), G'(Y^\bullet)).\end{aligned}$$

By Lemma 2.3, $\mathrm{Id}_{Y^\bullet}$ is sent to $\mathrm{Id}_{G'(Y^\bullet)}$ by the two isomorphisms. This implies that G' is fully faithful.

Recall that

$$\mathrm{Ker}\, F' = \left\{ \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \in \mathbf{K}_{\rm ac}(\Lambda\text{-Inj}) \;\middle|\; F' \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \cong 0 \text{ in } \mathbf{K}_{\rm ac}(R\text{-Inj}) \right\}$$

is a triangulated subcategory of $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$.

Lemma 4.3 *We have*

$$\mathrm{Ker}\, F' = \left\{ \left(\begin{pmatrix} 0 \\ W^\bullet \end{pmatrix} \right) \in \mathbf{K}_{\rm ac}(\Lambda\text{-Inj}) \right\}.$$

Proof Suppose $\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) \in \mathrm{Ker}\, F'$. Recall that the morphism $\left(\begin{smallmatrix} X^n \\ V^n \end{smallmatrix}\right) \to (G \circ F)\left(\begin{smallmatrix} X^n \\ V^n \end{smallmatrix}\right)$ given by the unit of (F, G) is split epic for each $n \in \mathbb{Z}$. Then we have an exact sequence of complexes

$$0 \to \begin{pmatrix} 0 \\ W^\bullet \end{pmatrix} \to \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to (\widetilde{G} \circ \widetilde{F}) \begin{pmatrix} X^\bullet \\ V^\bullet \end{pmatrix} \to 0,$$

which splits in each component. Since $F'\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) \cong 0$, we have $\widetilde{F}\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) \cong 0$. Then we have $\left(\begin{smallmatrix} X^\bullet \\ V^\bullet \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} 0 \\ W^\bullet \end{smallmatrix}\right)$ in $\mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$.

By the above lemma, we observe the triangle equivalence $\mathbf{K}_{\rm ac}(S\text{-Inj}) \xrightarrow{\sim} \mathrm{Ker}\, F'$ sending $W^\bullet$ to $\left(\begin{smallmatrix} 0 \\ W^\bullet \end{smallmatrix}\right)$. Denote by i the composition of the equivalence and the inclusion $\mathrm{Ker}\, F' \to \mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$.

Proposition 4.1 *Let ${}_RM_S$ be an R-S-bimodule such that M_S is a flat right S-module. Let Λ be the upper triangular matrix ring. There exists a localization sequence of homotopy categories*

$$\mathbf{K}_{\mathrm{ac}}(S\text{-Inj}) \underset{i_\rho}{\overset{i}{\rightleftarrows}} \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathbf{K}_{\mathrm{ac}}(R\text{-Inj}).$$

Proof Recall from Corollary 4.1 that (F', G') is an adjoint pair. Then the quotient functor $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \to \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})/\mathrm{Ker}\, F'$ is left adjoint to the composition

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})/\mathrm{Ker}\, F' \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(R\text{-Inj}) \xrightarrow{G'} \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}).$$

Here, the triangle equivalence in the above composition follows from [7, Proposition 1.3]. By [13, Lemma 3.2(3)], the result follows immediately.

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