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Boundedness and Almost Periodicity of Solutions for a Class of Semilinear Parabolic Equations with Boundary Degeneracy*

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Abstract In this paper the authors investigate the boundedness and almost periodicity of solutions of semilinear parabolic equations with boundary degeneracy. The equations may be weakly degenerate or strongly degenerate on the lateral boundary. The authors prove the existence, uniqueness and global exponential stability of bounded entire solutions, and also establish the existence theorem of almost periodic solutions if the data are almost periodic.

Keywords Almost periodic solution, Bounded entire solution, Boundary degeneracy
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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega$. We consider semilinear parabolic equations of the form

$$\frac{\partial u}{\partial t} - \mathrm{div}(\rho^p(x)\nabla u) + g(x,t,u) = f(x,t) \quad \text{in } \Omega \times \mathbb{R}, \tag{1.1}$$

where $f \in L^\infty(\mathbb{R}; L^2(\Omega))$ is a measurable function in $\Omega \times \mathbb{R}$, $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance of x from the boundary $\partial\Omega$, $0 < p \le 2$ and $p \ne 1$, g is a measurable function in $\Omega \times \mathbb{R} \times \mathbb{R}$ and satisfies some structure conditions.

Equation (1.1) can be used to describe a variety of physical and biological models. For instance, in [4–5] we can find a motivating example of a Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate. It is noted that (1.1) is degenerate on the lateral boundary $\partial\Omega \times \mathbb{R}$. As we know, the well-posed problems for parabolic equations with boundary degeneracy are different from those of common ones. The 1951 paper of Keldys [10] played a significant role in the development of the theory of partial differential equations with boundary degeneracy. Later, Fichera and Oleinik (see [17] and references therein) established general theory on second order elliptic equations with nonnegative characteristic form.

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However, the problem on almost periodic solutions for parabolic equations with boundary degeneracy has as yet received little attention. One objective of this paper is to study the existence of almost periodic solutions for the boundary degenerate parabolic equation (1.1). The time almost periodic dependence reflects the effects of certain "seasonal" variations which are roughly but not exactly periodic. Compared with periodic solutions, almost periodic solutions have more application in physics, and the study is more difficult, since the uniform topology in time in the whole space should be used.

Almost periodic solutions of parabolic equations have been widely investigated over the past 30 years and there have been a great number of results. For example, in [1, 2, 6, 15–16, 19, 21, 23], the authors studied the existence and the longtime behavior of almost periodic solutions for semilinear parabolic equations from the different angle of view. As for the quasilinear parabolic equations, we referrer to [3, 8, 11, 24] and the references therein. It is well known that almost periodic solutions are closely connected with the bounded entire solutions (see [13, 18]). So we also study the boundedness of solutions of (1.1) without almost periodicity assumptions. We will prove the existence, uniqueness and global exponential stability of bounded entire solutions, and establish the existence theorem of almost periodic solutions if the data are almost periodic.

To prescribe and formulate the boundary value condition for parabolic equations with boundary degeneracy reasonably, a local integral form was introduced in [25]. Denote

$$\Sigma_1 = \Big\{x \in \partial\Omega : \text{ There exists } \delta > 0 \text{ such that } \int_{\Omega\cap B_\delta(x)} \frac{1}{\rho^p(x)}\mathrm{d}x < +\infty\Big\},$$
$$\Sigma_2 = \Big\{x \in \partial\Omega : \text{ For any } \delta > 0 \text{ it holds that } \int_{\Omega\cap B_\delta(x)} \frac{1}{\rho^p(x)}\mathrm{d}x = +\infty\Big\},$$

where $B_\delta(x)$ is the ball in $\mathbb{R}^N$ centered at x and with radius δ. We call Σ_1 and Σ_2 the weakly degenerate part and strongly degenerate part of $\partial\Omega$ respectively. It is clear that $\Sigma_1 = \emptyset$ and $\Sigma_2 = \partial\Omega$ if $1 < p \le 2$, whereas $\Sigma_1 = \partial\Omega$ and $\Sigma_2 = \emptyset$ if $0 < p < 1$. We propose the following boundary condition:

$$u(x,t) = 0, \quad (x,t) \in \Sigma_1 \times \mathbb{R}. \tag{1.2}$$

This means that the boundary value condition is imposed only when $0 < p < 1$, and there is no prescription when $1 < p \le 2$. Obviously, it is quite different from the boundary condition of uniformly parabolic equations.

In (1.1), g is a measurable function in $\Omega \times \mathbb{R} \times \mathbb{R}$ satisfying

$$|g(x,t,u) - g(x,t,v)| \le C_0|u - v|, \quad (x,t) \in \Omega \times \mathbb{R},\ u,v \in \mathbb{R} \tag{1.3}$$

with some positive constant C_0, and $g(x,t,\cdot)$ is differentiable at $u = 0$ uniformly in $\Omega \times \mathbb{R}$, i.e.,

$$\lim_{u\to 0} \sup_{\substack{x\in\Omega \\ t\in\mathbb{R}}} \Big|\frac{g(x,t,u) - g(x,t,0)}{u} - \frac{\partial g}{\partial u}(x,t,0)\Big| = 0. \tag{1.4}$$

The paper is organized as follows. In Section 2, we give some notation and definitions, as well as some of our main results. In Section 3, we establish several estimates of weak solutions defined on the half time axis. The main theorems are proved in Section 4.

2 Notation and Main Results

2.1 Notation and definitions

Firstly, we introduce two kinds of weighted Sobolev spaces, which play important roles in our arguments. For $-\infty \le s < t \le +\infty$, denote $Q_s^t = \Omega \times (s,t)$.

Definition 2.1 *Define $W_0^{1,2}(\Omega,\rho^\lambda)$ to be the closure of the set $C_0^\infty(\Omega)$ with respect to the norm*

$$\|u\|_{W_0^{1,2}(\Omega,\rho^\lambda)} = \Big(\int_\Omega \rho^\lambda(x)(|u(x)|^2 + |\nabla u(x)|^2)\mathrm{d}x\Big)^{\frac{1}{2}}, \quad u \in W_0^{1,2}(\Omega,\rho^\lambda).$$

Suppose that $1 \ne \lambda \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary $\partial\Omega$. As for the weighted Sobolev space $W_0^{1,2}(\Omega,\rho^\lambda)$, the following imbedding inequality holds (see [12, p. 53, p. 67]):

$$C_\lambda \int_\Omega \rho^{\lambda-2}(x)|u(x)|^2\mathrm{d}x \le \int_\Omega \rho^\lambda(x)|\nabla u(x)|^2\mathrm{d}x, \quad u \in W_0^{1,2}(\Omega,\rho^\lambda), \tag{2.1}$$

where C_λ is a positive constant depending only on N, Ω and λ.

Definition 2.2 *We denote by $\mathscr{B}_s^t$ the closure of $C_0^\infty(Q_s^t)$ with respect to the norm*

$$\|u\|_{\mathscr{B}_s^t} = \Big(\iint_{Q_s^t} \rho^p(x)(|u(x,\tau)|^2 + |\nabla u(x,\tau)|^2)\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{2}}, \quad u \in \mathscr{B}_s^t.$$

Let $u \in \mathscr{B}_s^t$. By virtue of [25], we know that $u|_{\partial\Omega\times(s,t)} = 0$ in the trace sense if $0 < p < 1$, while there is no trace on $\partial\Omega \times (s,t)$ if $p \ge 1$.

Secondly, we introduce definitions of almost periodic functions.

Definition 2.3 (see [14, p. 1]) *Let X be a Banach space. We say that a function $u(\cdot,t) \in C(\mathbb{R};X)$ is X almost periodic, denoted by $u(\cdot,t) \in AP(X)$, if for any $\varepsilon > 0$, the set*

$$T(\varepsilon,u) = \Big\{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \|u(\cdot,t+\tau) - u(\cdot,t)\|_X < \varepsilon\Big\}$$

is relatively dense, i.e., there is a number $l = l(\varepsilon) > 0$ such that any interval of length l contains at least one number from $T(\varepsilon,u)$.

Definition 2.4 (see [14, p. 33]) *Let $1 \le q < +\infty$. We say that a function $u(\cdot,t) \in L^q_{\mathrm{loc}}(\mathbb{R};X)$ is X almost periodic in the sense of Stepanov, denoted by $u(\cdot,t) \in S^qAP(X)$, if for any $\varepsilon > 0$, the set*

$$T(\varepsilon,u) = \Big\{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \Big(\int_0^1 \|u(\cdot,t+\tau+s) - u(\cdot,t+s)\|_X^q\mathrm{d}s\Big)^{\frac{1}{q}} < \varepsilon\Big\}$$

is relatively dense.

Obviously, if $u(\cdot,t) \in AP(X)$, then $u(\cdot,t) \in S^qAP(X)$, but not vice versa.

At last, we give several definitions of weak solutions to the problem (1.1)–(1.2), which will be referred on different occasions.

Definition 2.5 *Let $-\infty < t_0 < T < +\infty$. We say u a weak solution of the problem* (1.1)–(1.2) *on $Q_{t_0}^T$, if $u \in L^\infty((t_0,T);L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ and for any function $\varphi \in L^\infty((t_0,T);L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_{t_0}^T)$ and $\varphi(\cdot,t_0)|_\Omega = \varphi(\cdot,T)|_\Omega = 0$, the following integral equality holds*:

$$\iint_{Q_{t_0}^T} \Big(- u\frac{\partial \varphi}{\partial t} + \rho^p \nabla u \nabla \varphi + g(x,t,u)\varphi - f\varphi \Big) \mathrm{d}x\mathrm{d}t = 0. \tag{2.2}$$

Definition 2.6 *Let $u_0 \in L^2(\Omega)$. We say u a weak solution of the problem* (1.1)–(1.2) *on $Q_{t_0}^T$ with the initial value $u(x,t_0) = u_0(x)$, if $u \in L^\infty((t_0,T);L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ and for any function $\varphi \in L^\infty((t_0,T);L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_{t_0}^T)$ and $\varphi(\cdot,T)|_\Omega = 0$, the following integral equality holds*:

$$\iint_{Q_{t_0}^T} \Big(- u\frac{\partial \varphi}{\partial t} + \rho^p \nabla u \nabla \varphi + g(x,t,u)\varphi - f\varphi \Big) \mathrm{d}x\mathrm{d}t = \int_\Omega u_0(x)\varphi(x,t_0)\mathrm{d}x. \tag{2.3}$$

A function u is called a weak solution of the problem (1.1)–(1.2) *on $Q_{t_0}^{+\infty}$ with the initial value $u(x,t_0) = u_0(x)$, if for any $T > t_0$, u is a weak solution of the problem* (1.1)–(1.2) *on $Q_{t_0}^T$ with the initial value $u(x,t_0) = u_0(x)$.*

It is easy to see that if u is a weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^T$ with the initial value $u(x,t_0) = u_0(x)$, then it is also a weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^T$, but not vice versa.

Definition 2.7 *A function u is called a weak solution of the problem* (1.1)–(1.2) *on $Q_{-\infty}^{+\infty}$, provided that for any $-\infty < s < t < +\infty$, u is a weak solution of the problem* (1.1)–(1.2) *on Q_s^t.*

Definition 2.8 *A function u is called a bounded entire solution of the problem* (1.1)–(1.2), *if u is a weak solution of the problem* (1.1)–(1.2) *on $Q_{-\infty}^{+\infty}$ satisfying*

$$\sup_{t\in\mathbb{R}} \int_\Omega |u(x,t)|^2 \mathrm{d}x + \sup_{t\in\mathbb{R}} \int_\Omega \rho^p |\nabla u(x,t)|^2 \mathrm{d}x + \sup_{t\in\mathbb{R}} \Big\| \frac{\partial u}{\partial s} \Big\|^2_{L^2(Q_t^{t+1})} < +\infty.$$

2.2 Main results

The following notation will be used:

$$\begin{cases} \lambda_p^* = \inf\limits_{0\not\equiv u \in W_0^{1,2}(\Omega,\rho^p)} \dfrac{\displaystyle\int_\Omega \rho^p |\nabla u|^2 \mathrm{d}x}{\displaystyle\int_\Omega |u|^2\mathrm{d}x}, \\ S^* = \sup\limits_{t\in\mathbb{R}} \|f\|^2_{L^2(Q_t^{t+1})} + \sup\limits_{t\in\mathbb{R}} \|g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}. \end{cases} \tag{2.4}$$

If $0 < p \le 2$ and $p \neq 1$, then it follows from (2.1) that

$$\lambda_p^* \ge D^{p-2} C_p^* > 0,$$

where

$$D = \sup_{x\in\Omega} \rho(x), \quad C_p^* = \inf \Big\{ \frac{\displaystyle\int_\Omega \rho^p |\nabla u|^2\mathrm{d}x}{\displaystyle\int_\Omega \rho^{p-2}|u|^2\mathrm{d}x} : u \in W_0^{1,2}(\Omega,\rho^p),\ u \not\equiv 0 \Big\}.$$

We denote by λ_0^* the first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$.

Define the function

$$\sigma(x,t,u)=\begin{cases}\dfrac{g(x,t,u)-g(x,t,0)}{u}, & (x,t)\in Q_{-\infty}^{+\infty},\ 0\neq u\in\mathbb{R},\\ \dfrac{\partial g}{\partial u}(x,t,0), & (x,t)\in Q_{-\infty}^{+\infty},\ u=0.\end{cases} \tag{2.5}$$

Write

$$\sigma^*=\underset{\substack{(x,t)\in\Omega\times\mathbb{R}\\ u\neq v\in\mathbb{R}}}{\operatorname{essinf}}\frac{g(x,t,u)-g(x,t,v)}{u-v}. \tag{2.6}$$

In view of (1.3)–(1.4), we see that $-C_0\le\sigma^*\le\sigma\le C_0$.

Now we state the main results of the paper.

Theorem 2.1 *Suppose that g satisfies* (1.3)–(1.4)*, $S^*<+\infty$ and $\lambda_p^*+\sigma^*>0$. Then the problem* (1.1)–(1.2) *admits uniquely a bounded entire solution u satisfying*

$$\sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{L^2(\Omega)}^2+\sup_{t\in\mathbb{R}}\|\rho^p|\nabla u(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\in\mathbb{R}}\Big\|\frac{\partial u}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2\le CS^*, \tag{2.7}$$

where C is a positive constant depending only on N, p, Ω, σ^* and C_0.

Moreover, if $w(x,t)$ is the weak solution of problem (1.1)–(1.2) *on $Q_{t_0}^{+\infty}$ with the initial value $w(x,t_0)=w_0(x)\in L^2(\Omega)$, then*

$$\|w(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\le\|w_0(\cdot)-u(\cdot,t_0)\|_{L^2(\Omega)}\mathrm{e}^{(\lambda_p^*+\sigma^*)(t_0-t)},\quad t\ge t_0.$$

That is to say, u is globally, exponentially stable.

Theorem 2.2 *In addition to the assumptions of Theorem* 2.1*, if*

$$f(\cdot,t),\ g(\cdot,t,0)\in S^2AP(L^2(\Omega))$$

and $\sigma(\cdot,t,\cdot)\in AP(L^\infty(\Omega\times\mathbb{R}))$, then the unique bounded entire solution

$$u(\cdot,t)\in AP(L^2(\Omega))\cap S^2AP(W_0^{1,2}(\Omega,\rho^p)).$$

Remark 2.1 Under the assumptions of Theorem 2.1, if $f(x,t)$ and $g(x,t,u)$ are ω-periodic in t additionally, i.e., $f(x,t)=f(x,t+\omega)$ for a.e. $(x,t)\in\Omega\times\mathbb{R}$, and $g(x,t,u)=g(x,t+\omega,u)$ for a.e. $(x,t,u)\in\Omega\times\mathbb{R}\times\mathbb{R}$, then the bounded entire solution $u(x,t)$ also is ω-periodic in t.

Remark 2.2 The conclusions of Theorems 2.1–2.2 still hold in the case of $p=0$ (i.e., nondegenerate case). More specially, if $p=0$ and $g(x,t,u)=-C_0u$, then the assumption condition $\lambda_p^*+\sigma^*>0$ turns into $\lambda_0^*-C_0>0$. This condition coincides with that employed in [15] for proving the boundedness of solutions to the linear parabolic equation.

Remark 2.3 Let $p=0$ and $f(x,t)=0$. Then the problem (1.1)–(1.2) is a special case of (5.1) in [20] (more precisely $f(u,\nabla u,x,t)=-g(x,t,u)$). According to [20, Theorem 5.4], it has a unique stable almost periodic solution, provided that $\sigma^*>0$ and (1.1)–(1.2) admits a bounded solution on $Q_{t_0}^{+\infty}$. However, Theorem 2.2 is proved under the assumption $\sigma^*>-\lambda_0^*$ (λ_0^* is a positive constant) and g satisfies (1.3)–(1.4). Therefore, Theorem 2.2 here in some sense generalizes in [20, Theorem 5.4].

3 Estimates of Solutions Defined on the Half Time Axis

For any $f \in L^2(Q_{t_0}^T)$ and $u_0 \in L^2(\Omega)$, there exists uniquely a weak solution to the problem (1.1)–(1.2) on $Q_{t_0}^T$ with the initial value $u(x,t_0) = u_0(x)$ (see [22, Theorem 3.1]). Wang [22] gave some estimates on the solution u. However, the estimates all depend on the length of the time interval (t_0, T), and so the upper bounds in the estimates may tend to $+\infty$ as $T - t_0 \to +\infty$. In order to prove the existence of almost periodic solutions, we need to rebuild some global (in time) estimates for the weak solutions of (1.1)–(1.2) on $Q_{t_0}^{+\infty}$, which are proved by combining the parabolic regularization method and the technique used in [7].

Lemma 3.1 *Suppose $S^* < +\infty$, $\lambda_p^* + \sigma^* > 0$, $u_0 \in L^2(\Omega)$, and assume as well that* (1.3)–(1.4) *are fulfilled. Then there exists uniquely a weak solution u of problem* (1.1)–(1.2) *on $Q_{t_0}^{+\infty}$ with the initial value $u(x,t_0) = u_0(x)$. Furthermore, the solution u satisfies*

$$\sup_{t\ge t_0} \|u(\cdot,t)\|_{L^2(\Omega)}^2 + \sup_{t\ge t_0} \|\rho^p|\nabla u|^2\|_{L^1(Q_t^{t+1})} \le C(\|u_0\|_{L^2(\Omega)}^2 + S^*), \tag{3.1}$$

$$\begin{aligned}&\sup_{t_0\le t\le t_0+1} (t-t_0)\|\rho^p|\nabla u(\cdot,t)|^2\|_{L^1(\Omega)} + \sup_{t\ge t_0+1} \|\rho^p|\nabla u(\cdot,t)|^2\|_{L^1(\Omega)}\\ &\le C(\|u_0\|_{L^2(\Omega)}^2 + S^*),\end{aligned} \tag{3.2}$$

$$\int_{t_0}^{t_0+1} (s-t_0)\Big\|\frac{\partial u(\cdot,s)}{\partial s}\Big\|_{L^2(\Omega)}^2 \mathrm{d}s + \sup_{t\ge t_0+1} \Big\|\frac{\partial u}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2 \le C(\|u_0\|_{L^2(\Omega)}^2 + S^*). \tag{3.3}$$

If $\rho^p|\nabla u_0|^2 \in L^1(\Omega)$ additionally, then

$$\sup_{t\ge t_0} \|\rho^p|\nabla u(\cdot,t)|^2\|_{L^1(\Omega)} + \sup_{t\ge t_0} \Big\|\frac{\partial u}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2 \le C(\|u_0\|_{L^2(\Omega)}^2 + \|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)} + S^*). \tag{3.4}$$

Here and bellow we denote by C a positive constant which only depends on N, p, Ω, σ^* and C_0.

Proof We transform (1.1) into the form

$$\frac{\partial u}{\partial t} - \mathrm{div}(\rho^p(x)\nabla u) + \sigma(x,t,u)u + g(x,t,0) = f(x,t) \quad \text{in } \Omega\times\mathbb{R}. \tag{3.5}$$

Since (3.5) is degenerate on the lateral boundary, we first investigate the corresponding regularized equations.

Let $\eta(x,t,u) \in C^\infty(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R})$ be the standard mollifier (see [9, p. 629]). For $k = 1, 2, \cdots$, $\sigma(\cdot,\cdot,\cdot): \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, define its mollification

$$\begin{aligned}\sigma_k(x,t,u) &= \iint_{\Omega\times\mathbb{R}\times\mathbb{R}} \eta_{\frac1k}(x-x', t-t', u-u')\sigma(x',t',u')\mathrm{d}x'\mathrm{d}t'\mathrm{d}u'\\ &= \iint_{B(0,1)} \eta(x',t',u')\sigma\Big(x-\frac{x'}{k}, t-\frac{t'}{k}, u-\frac{u'}{k}\Big)\mathrm{d}x'\mathrm{d}t'\mathrm{d}u',\end{aligned}$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^{N+2}$ centered at the origin. It is easily verified that $\sigma_k(x,t,u) \in C^\infty(\overline{\Omega}\times\mathbb{R}\times\mathbb{R})$, $\sigma_k \to \sigma$ a.e. in $\Omega\times\mathbb{R}\times\mathbb{R}$ as $k\to+\infty$, and $\sigma^* \le \sigma_k \le C_0$ if we take $k \ge k_0$ (k_0 is a positive constant number). Similarly, we can define $a_k(x)$ and $u_{0k}(x) \in C^\infty(\overline{\Omega})$, $h_k(x,t)$ and $f_k(x,t) \in C^\infty(\overline{\Omega}\times\mathbb{R})$ by using the above convolution operator, which satisfy

$$\rho^p + \frac1k \le a_k \le \rho^p + \frac2k, \quad \|h_k\|_{L^2(Q_s^t)} \le \|g(\cdot,\cdot,0)\|_{L^2(Q_s^t)},$$

$$\|f_k\|_{L^2(Q_s^t)} \le \|f\|_{L^2(Q_s^t)}, \quad \|u_{0k}\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$$

for $k \ge k_0$, and

$$h_k \to g(x,t,0) \quad \text{in } L^2(Q_s^t), \quad f_k \to f \quad \text{in } L^2(Q_s^t), \quad u_{0k} \to u_0 \quad \text{in } L^2(\Omega)$$

as $k \to \infty$ for any $t > s \ge t_0$; further,

$$\|a_k|\nabla u_{0k}|^2\|_{L^1(\Omega)} \le \|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)} + \frac{1}{k},$$

if $\rho^p|\nabla u_0|^2 \in L^1(\Omega)$ additionally. Consider the problem

$$\begin{cases} \dfrac{\partial u_k}{\partial t} - \operatorname{div}(a_k \nabla u_k) + \sigma_k(x,t,u_k)u_k + h_k = f_k & \text{in } Q_{t_0}^{+\infty}, \\ u_k(x,t) = 0 & \text{on } \partial\Omega \times (t_0, +\infty), \\ u_k(x,t_0) = u_{0k}(x) & \text{in } \Omega. \end{cases} \tag{3.6}$$

According to the classical theory on parabolic equations, the problem (3.6) admits a unique classical solution u_k.

(i) Multiply the first equation in (3.6) by u_k and integrate over Ω by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k(\cdot,t)\|_{L^2(\Omega)}^2 + \|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)} + \int_\Omega \sigma_k(x,t,u_k)u_k^2\mathrm{d}x = \int_\Omega (f_k - h_k)u_k\mathrm{d}x. \tag{3.7}$$

Recalling (2.4), we have

$$\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)} \ge \|\rho^p|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)} \ge \lambda_p^*\|u_k(\cdot,t)\|_{L^2(\Omega)}^2.$$

Combine this inequality and (3.7), and then invoke Cauchy's inequality to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_k(\cdot,t)\|_{L^2(\Omega)}^2 + C_1'\|u_k(\cdot,t)\|_{L^2(\Omega)}^2 \le C_2'(\|f_k(\cdot,t)\|_{L^2(\Omega)}^2 + \|h_k(\cdot,t)\|_{L^2(\Omega)}^2), \quad t \ge t_0,$$

where $C_1' = \lambda_p^* + \sigma^* > 0$, $C_2' = 2(\lambda_p^* + \sigma^*)^{-1}$. Multiply the above inequality by $\mathrm{e}^{C_1'\tau}$ and then integrate over $[s,t]$ to obtain

$$\begin{aligned} &\|u_k(\cdot,t)\|_{L^2(\Omega)}^2\mathrm{e}^{C_1't} - \|u_k(\cdot,s)\|_{L^2(\Omega)}^2\mathrm{e}^{C_1's} \\ \le{}& C_2'\int_s^t (\|f_k(\cdot,\tau)\|_{L^2(\Omega)}^2 + \|h_k(\cdot,\tau)\|_{L^2(\Omega)}^2)\mathrm{e}^{C_1'\tau}\mathrm{d}\tau, \quad t > s \ge t_0. \end{aligned} \tag{3.8}$$

Now let $t = s+1$, and consequently

$$\begin{aligned} \|u_k(\cdot,s+1)\|_{L^2(\Omega)}^2 &\le \mathrm{e}^{-C_1'}\|u_k(\cdot,s)\|_{L^2(\Omega)}^2 + C_2'\int_s^{s+1}(\|f_k(\cdot,\tau)\|_{L^2(\Omega)}^2 + \|h_k(\cdot,\tau)\|_{L^2(\Omega)}^2)\mathrm{d}\tau \\ &\le \mathrm{e}^{-C_1'}\|u_k(\cdot,s)\|_{L^2(\Omega)}^2 + C_2'S^*, \quad s \ge t_0. \end{aligned}$$

We therefore see that for any $n \in \mathbb{N}^+$,

$$\|u_k(s+n)\|_{L^2(\Omega)}^2 \le \mathrm{e}^{-nC_1'}\|u_k(\cdot,s)\|_{L^2(\Omega)}^2 + \frac{C_2'S^*}{1-\mathrm{e}^{-C_1'}}, \quad s \ge t_0.$$

Next let $s=t_0$ in (3.8) to get

$$\|u_k(\cdot,t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + C_2' S^*, \quad t_0 \le t \le t_0+1.$$

It follows from the above two inequalities that

$$\sup_{t\ge t_0} \|u_k(\cdot,t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + C_2'S^* + \frac{C_2'S^*}{1-\mathrm{e}^{-C_1'}} \le \|u_0\|^2_{L^2(\Omega)} + CS^*. \tag{3.9}$$

Integrating (3.7) over $[t,t+1]$ $(t\ge t_0)$, recalling $|\sigma_k|\le C_0$, and using (3.9), we conclude that

$$\begin{aligned}\|a_k|\nabla u_k|^2\|_{L^1(Q_t^{t+1})} &= \iint_{Q_t^{t+1}} ((f_k-h_k)u_k - \sigma_k(x,s,u_k)u_k^2)\mathrm{d}x\mathrm{d}s \\ &\quad + \frac{1}{2}\|u_k(\cdot,t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_k(\cdot,t+1)\|^2_{L^2(\Omega)} \\ &\le C(\|u_0\|^2_{L^2(\Omega)} + S^*).\end{aligned}$$

Thus

$$\sup_{t\ge t_0} \|a_k|\nabla u_k|^2\|_{L^1(Q_t^{t+1})} \le C(\|u_0\|^2_{L^2(\Omega)} + S^*). \tag{3.10}$$

(ii) Multiplying the first equation in (3.6) by $\frac{\partial u_k}{\partial t}$ and then integrating over Ω $(t>t_0)$, we get

$$\Big\|\frac{\partial u_k(\cdot,t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)} = \int_\Omega (f_k - h_k - \sigma_k(x,t,u_k)u_k)\frac{\partial u_k}{\partial t}\mathrm{d}x.$$

By Cauchy inequality, we find

$$\begin{aligned}&\Big\|\frac{\partial u_k(\cdot,t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)} \\ &\le 3\|f_k(\cdot,t)\|^2_{L^2(\Omega)} + 3\|h_k(\cdot,t)\|^2_{L^2(\Omega)} + 3C_0^2\|u_k(\cdot,t)\|^2_{L^2(\Omega)}, \quad t>t_0.\end{aligned} \tag{3.11}$$

We continue by multiplying (3.11) by $t-s$ and then integrating over $[s,\tau]$ $(t_0\le s<\tau)$ to find

$$\begin{aligned}&\int_s^\tau (t-s)\Big\|\frac{\partial u_k(\cdot,t)}{\partial t}\Big\|^2_{L^2(\Omega)}\mathrm{d}t + (\tau-s)\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)} \\ &\le 3(\tau-s)(\|f\|^2_{L^2(Q_s^\tau)} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q_s^\tau)} + C_0^2\|u_k\|^2_{L^2(Q_s^\tau)}) + \|a_k|\nabla u_k|^2\|_{L^1(Q_s^\tau)}.\end{aligned} \tag{3.12}$$

Removing the first term on the left-hand side of (3.12) and utilizing (3.9) and (3.10), we can derive

$$\begin{aligned}&\sup_{s\le\tau\le s+2}\{(\tau-s)\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\} \\ &\le 6(\|f\|^2_{L^2(Q_s^{s+2})} + \|g(\cdot,\cdot,0)\|^2_{L^2(Q_s^{s+2})} + C_0^2\|u_k\|^2_{L^2(Q_s^{s+2})}) + \|a_k|\nabla u_k|^2\|_{L^1(Q_s^{s+2})} \\ &\le C(\|u_0\|^2_{L^2(\Omega)} + S^*), \quad \forall s\ge t_0.\end{aligned} \tag{3.13}$$

Therefore

$$\sup_{s+1\le\tau\le s+2}\{\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\}$$

$$
\begin{aligned}
&\le \sup_{s+1\le\tau\le s+2}\{(\tau-s)\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\}\\
&\le \sup_{s\le\tau\le s+2}\{(\tau-s)\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\}\\
&\le C(\|u_0\|^2_{L^2(\Omega)}+S^*), \quad \forall s\ge t_0.
\end{aligned}
$$

From the arbitrary of s, we thereby see that

$$
\sup_{\tau\ge t_0+1}\{\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\}\le C(\|u_0\|^2_{L^2(\Omega)}+S^*).
$$

Now let $s=t_0$ in (3.13) to get that

$$
\sup_{t_0\le\tau\le t_0+1}\{(\tau-t_0)\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\}\le C(\|u_0\|^2_{L^2(\Omega)}+S^*).
$$

We thereupon conclude from the above two inequalities that

$$
\begin{aligned}
&\sup_{t_0\le t\le t_0+1}(t-t_0)\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\ge t_0+1}\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)}\\
&\le C(\|u_0\|^2_{L^2(\Omega)}+S^*).
\end{aligned}\tag{3.14}
$$

Similarly, remove the second term on the left-hand side of (3.12) to deduce

$$
\begin{aligned}
&\int_{t_0}^{t_0+1}(s-t_0)\Big\|\frac{\partial u_k(\cdot,s)}{\partial s}\Big\|^2_{L^2(\Omega)}\mathrm{d}s+\sup_{t\ge t_0+1}\int_t^{t+1}\Big\|\frac{\partial u_k(\cdot,s)}{\partial s}\Big\|^2_{L^2(\Omega)}\mathrm{d}s\\
&\le C(\|u_0\|^2_{L^2(\Omega)}+S^*).
\end{aligned}\tag{3.15}
$$

(iii) The goal next is to show that $u_k\to u$ as $k\to+\infty$, and u is a weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^{+\infty}$ with the initial value $u(x,t_0)=u_0(x)$. So now fix $T>t_0$. Owing to (3.9)–(3.10), there exists a subsequence of $\{u_k\}_{k=1}^\infty$, denoted by itself for convenience, a function $u\in L^\infty((t_0,T);L^2(\Omega))$ and a function $w\in L^2(Q_{t_0}^T)$, such that

$$
u_k\to u \quad \text{weakly in } L^2(Q_{t_0}^T), \quad \sqrt{a_k}\nabla u_k\to w \quad \text{weakly in } L^2(Q_{t_0}^T) \tag{3.16}
$$

as $k\to+\infty$.

We will show that there exists a subsequence of $\{u_k\}_{k=1}^\infty$, denoted by itself for convenience, such that

$$
u_k\to u \quad \text{strongly in } L^1(Q_{t_0}^T). \tag{3.17}
$$

To see this, fix a positive integer $\overline{m}>\frac{1}{T-t_0}$ satisfying $\big\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\frac{1}{\overline{m}}\big\}\ne\emptyset$. For any integer $m\ge\overline{m}$, denote

$$
\Omega_m=\Big\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\frac{1}{m}\Big\},\quad Q^{(m)}=\Omega_m\times\Big(t_0+\frac{1}{m},T\Big).
$$

It follows from the embedding theorem, the estimates (3.9)–(3.10) and (3.15) that there exists a subsequence of $\{u_k\}_{k=1}^{+\infty}$, denoted by $\{u_{k_{\overline{m}}(l)}\}_{l=1}^{+\infty}$, such that

$$
u_{k_{\overline{m}}(l)}\to u \quad \text{in } L^2(Q^{(\overline{m})}) \text{ as } l\to+\infty.
$$

Similarly, for $m \geq \overline{m}+1$, there exists a subsequence of $\{u_{k_{m-1}(l)}\}_{l=1}^{+\infty}$, denoted by $\{u_{k_m(l)}\}_{l=1}^{+\infty}$, such that

$$u_{k_m(l)} \to u \quad \text{in } L^2(Q^{(m)}) \quad \text{as } l \to +\infty. \tag{3.18}$$

Applying Hölder's inequality, we have

$$\begin{aligned}\Big(\iint_{Q_{t_0}^T \setminus Q^{(m)}} |u_k - u| \mathrm{d}x\mathrm{d}t\Big)^2 &\leq \text{meas}(Q_{t_0}^T \setminus Q^{(m)}) \iint_{Q_{t_0}^T \setminus Q^{(m)}} |u_k - u|^2 \mathrm{d}x\mathrm{d}t \\ &\leq 2\text{meas}(Q_{t_0}^T \setminus Q^{(m)}) \iint_{Q_{t_0}^T} (|u_k|^2 + |u|^2) \mathrm{d}x\mathrm{d}t \\ &\leq C(T-t_0)(\|u_0\|_{L^2(\Omega)}^2 + S^*) \text{meas}(Q_{t_0}^T \setminus Q^{(m)}) \to 0 \end{aligned} \tag{3.19}$$

as $m \to +\infty$. Give $\varepsilon > 0$. Owing to (3.19), there exists a positive integer $m_0 \geq \overline{m}+1$ such that

$$\iint_{Q_{t_0}^T \setminus Q^{(m_0)}} |u_k - u| \mathrm{d}x\mathrm{d}t < \frac{\varepsilon}{2}, \quad k = 1, 2, \cdots. \tag{3.20}$$

Due to (3.18), there exists a positive integer l_0 such that for the so fixed m_0 and any $l \geq l_0$,

$$\iint_{Q^{(m_0)}} |u_{k_{m_0}(l)} - u| \mathrm{d}x\mathrm{d}t < \frac{\varepsilon}{2}.$$

Therefore, for any $m \geq l_0 + m_0$, we get from the above inequality and (3.20) that

$$\begin{aligned}\iint_{Q_{t_0}^T} |u_{k_m(m)} - u| \mathrm{d}x\mathrm{d}t = &\iint_{Q_{t_0}^T \setminus Q^{(m_0)}} |u_{k_m(m)} - u| \mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q^{(m_0)}} |u_{k_m(m)} - u| \mathrm{d}x\mathrm{d}t < \varepsilon.\end{aligned}$$

Hence

$$\lim_{m \to +\infty} \iint_{Q_{t_0}^T} |u_{k_m(m)} - u| \mathrm{d}x\mathrm{d}t = 0,$$

and the assertion (3.17) holds.

Clearly, (3.17) implies that there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$u_k \to u \quad \text{a.e. in } Q_{t_0}^T. \tag{3.21}$$

By (3.10), there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$\nabla u_k \to \nabla u \quad \text{weakly in } L^2(\Omega_m \times (t_0, T))$$

for sufficiently large m, and so

$$\sqrt{a_k} \nabla u_k \to \rho^{\frac{p}{2}} \nabla u \quad \text{weakly in } \ L^2(\Omega_m \times (t_0, T)).$$

Thus $w = \rho^{\frac{p}{2}} \nabla u$ a.e. in $\Omega_m \times [t_0, T]$. From the arbitrariness of m we see that $w = \rho^{\frac{p}{2}} \nabla u$ a.e. in $Q_{t_0}^T$. Hence

$$\sqrt{a_k} \nabla u_k \to \rho^{\frac{p}{2}} \nabla u \quad \text{weakly in } L^2(Q_{t_0}^T). \tag{3.22}$$

For any $\varphi \in C^1(\overline{Q_{t_0}^T})$ satisfying $\varphi(x,t) = 0$ for x near $\partial\Omega$ or $t = T$, multiply (3.6) by φ and then integrate by parts over $Q_{t_0}^T$ to get

$$\begin{aligned}&\iint_{Q_{t_0}^T} \Big(- u_k \frac{\partial \varphi}{\partial t} + a_k \nabla u_k \nabla \varphi + \sigma_k(x,t,u_k) u_k \varphi \Big) \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_{t_0}^T} (f_k - h_k)\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_{0k}(x)\varphi(x,t_0)\mathrm{d}x. \end{aligned} \tag{3.23}$$

We claim that

$$\lim_{k\to+\infty} \iint_{Q_{t_0}^T} (\sigma_k(x,t,u_k)u_k - \sigma(x,t,u)u)\varphi \mathrm{d}x\mathrm{d}t = 0. \tag{3.24}$$

In fact, by Hölder's inequality and the continuity of $\sigma(x,t,u)$ with respect to u,

$$\begin{aligned}&|\sigma_k(x,t,u_k(x,t)) - \sigma_k(x,t,u(x,t))|^2 \\ &\le \iint_{B(0,1)} \eta(x',t',u') \Big| \sigma\Big(x - \frac{x'}{k}, t - \frac{t'}{k}, u_k - \frac{u'}{k}\Big) - \sigma\Big(x - \frac{x'}{k}, t - \frac{t'}{k}, u - \frac{u'}{k}\Big)\Big|^2 \mathrm{d}x'\mathrm{d}t'\mathrm{d}u' \\ &\to 0 \quad \text{a.e. in } Q_{t_0}^T. \end{aligned}$$

So we obtain from the dominated convergence theorem that

$$\lim_{k\to+\infty} \|\sigma_k(x,t,u_k(x,t)) - \sigma_k(x,t,u(x,t))\|_{L^2(Q_{t_0}^T)} = 0,$$

which together with (3.9) leads to

$$\lim_{k\to+\infty} \iint_{Q_{t_0}^T} (\sigma_k(x,t,u_k(x,t)) - \sigma_k(x,t,u(x,t)))u_k\varphi \mathrm{d}x\mathrm{d}t = 0.$$

Similarly, we deduce

$$\lim_{k\to+\infty} \iint_{Q_{t_0}^T} (\sigma_k(x,t,u(x,t)) - \sigma(x,t,u(x,t)))u_k\varphi \mathrm{d}x\mathrm{d}t = 0$$

by Hölder's inequality and the dominated convergence theorem. Recalling that $u_k \to u$ strongly in $L^1(Q_{t_0}^T)$, we derive

$$\lim_{k\to+\infty} \iint_{Q_{t_0}^T} \sigma(x,t,u(x,t))(u_k - u)\varphi \mathrm{d}x\mathrm{d}t = 0.$$

Whence assertion (3.24) follows from the above three limits. Letting $k \to +\infty$ in (3.23), and using (3.16), (3.22) and (3.24), one gets

$$\begin{aligned}&\iint_{Q_{t_0}^T} \Big(- u \frac{\partial \varphi}{\partial t} + \rho^p \nabla u \nabla \varphi + \sigma(x,t,u)u\varphi \Big) \mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_{t_0}^T} (f - g(x,t,0))\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x)\varphi(x,t_0)\mathrm{d}x. \end{aligned}$$

For any function $\varphi \in L^\infty((t_0,T);L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_{t_0}^T)$ and $\varphi(\cdot,T)|_\Omega = 0$, the above integral equality still holds after an approximate procedure. Therefore from the arbitrary of T,

u is a weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^{+\infty}$ with the initial value $u(x,t_0)=u_0(x)$. Observe that the estimates (3.1)–(3.3) follow easily from (3.9)–(3.10) and (3.14)–(3.15).

The uniqueness can be proved by the Holmgren method, which is similar to the proof of [22, Theorem 3.1], and we omit the details.

(iv) If $\rho^p|\nabla u_0|^2\in L^1(\Omega)$ additionally, integrating (3.11) over $[t_0,\tau]$ and applying (3.9), we have

$$\begin{aligned}
&\Big\|\frac{\partial u_k}{\partial s}\Big\|^2_{L^2(Q_{t_0}^\tau)}+\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\\
&\le \|a_k|\nabla u_{0k}|^2\|_{L^1(\Omega)}+3\|f_k\|^2_{L^2(Q_{t_0}^\tau)}+3\|h_k\|^2_{L^2(Q_{t_0}^\tau)}+3C_0^2\|u_k\|^2_{L^2(Q_{t_0}^\tau)}\\
&\le \|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)}+3\|f\|^2_{L^2(Q_{t_0}^\tau)}+3\|g(\cdot,\cdot,0)\|^2_{L^2(Q_{t_0}^\tau)}\\
&\quad +C(\tau-t_0)(\|u_0\|^2_{L^2(\Omega)}+S^*)+\frac{1}{k}.
\end{aligned}\tag{3.25}$$

Hence remove the first term on the left-hand side of the above inequality to find that

$$\sup_{t_0\le\tau\le t_0+1}\|a_k|\nabla u_k(\cdot,\tau)|^2\|_{L^1(\Omega)}\le C(\|u_0\|^2_{L^2(\Omega)}+\|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)}+S^*)+\frac{1}{k},$$

which together with (3.14) implies

$$\sup_{t\ge t_0}\|a_k|\nabla u_k(\cdot,t)|^2\|_{L^1(\Omega)}\le C(\|u_0\|^2_{L^2(\Omega)}+\|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)}+S^*)+\frac{1}{k}.\tag{3.26}$$

Similarly, removing the second term on the left-hand side of (3.25), we get

$$\int_{t_0}^{t_0+2}\Big\|\frac{\partial u_k(\cdot,t)}{\partial t}\Big\|^2_{L^2(\Omega)}\mathrm{d}t\le C(\|u_0\|^2_{L^2(\Omega)}+\|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)}+S^*)+\frac{1}{k},$$

which together with (3.15) leads to

$$\sup_{t\ge t_0}\Big\|\frac{\partial u_k}{\partial s}\Big\|^2_{L^2(Q_t^{t+1})}\le C(\|u_0\|^2_{L^2(\Omega)}+\|\rho^p|\nabla u_0|^2\|_{L^1(\Omega)}+S^*)+\frac{1}{k}.$$

Consequently (3.4) follows from (3.26) and the above inequality. The proof is complete.

Lemma 3.2 *Assume that g satisfies* (1.3)–(1.4), $S^*<+\infty$, $\lambda_p^*+\sigma^*>0$, $u_0\in L^2(\Omega)$, *and u is the weak solution of the problem* (1.1)–(1.2) *on $Q_{t_0}^{+\infty}$ with the initial value $u(x,t_0)=u_0(x)$. Let $\tau\in\mathbb{R}$, $v_0\in L^2(\Omega)$, and $v(x,t)$ be the weak solution of the following problem:*

$$\begin{cases}\dfrac{\partial v(x,t)}{\partial t}-\operatorname{div}(\rho^p(x)\nabla v(x,t))+g(x,t+\tau,v(x,t))\\ =f(x,t+\tau) & \text{in } Q_{t_0}^{+\infty},\\ v(x,t)=0 & \text{on } \Sigma_1\times(t_0,+\infty),\\ v(x,t_0)=v_0(x) & \text{in } \Omega.\end{cases}\tag{3.27}$$

Then we have

$$\begin{aligned}\sup_{t\ge t_0}\|v(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\le C\Big(&\|v_0-u_0\|^2_{L^2(\Omega)}+\sup_{t\in\mathbb{R}}\|f(\cdot,\cdot+\tau)-f(\cdot,\cdot)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|g(\cdot,\cdot+\tau,0)-g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}\Big)\end{aligned}$$

$$+ C(\|u_0\|^2_{L^2(\Omega)} + S^*) \sup_{t\in\mathbb{R}} \|\sigma(\cdot, t+\tau, \cdot) - \sigma(\cdot, t, \cdot)\|^2_{L^\infty(\Omega\times\mathbb{R})}. \tag{3.28}$$

Proof Let a_k, σ_k, h_k, f_k, u_{0k} and u_k be as that in Lemma 3.1. Suppose that now v_k is the classical solution of the problem

$$\begin{cases} \dfrac{\partial v_k}{\partial t} - \operatorname{div}(a_k \nabla v_k) + \sigma_k(x, t+\tau, v_k)v_k + h_k(x, t+\tau) \\ = f_k(x, t+\tau) & \text{in } Q_{t_0}^{+\infty}, \\ v_k(x,t) = 0 & \text{on } \partial\Omega \times (t_0, +\infty), \\ v_k(x, t_0) = v_{0k}(x) & \text{in } \Omega, \end{cases} \tag{3.29}$$

where $v_{0k} \in C^\infty(\overline{\Omega})$, $v_{0k} \to v_0$ strongly in $L^2(\Omega)$ and $\|v_{0k}\|_{L^2(\Omega)} \le \|v_0\|_{L^2(\Omega)}$. From the proof of Lemma 3.1 we can deduce the estimates (3.1)–(3.3) for the solution v_k, and so that there exists $\{k_i\}_{i=1}^\infty \subset \{k\}_{k=1}^\infty$, and a function $v \in L^\infty((t_0, +\infty); L^2(\Omega)) \cap \mathscr{B}_{t_0}^T$ for any $t_0 < T < +\infty$, such that

$$\begin{cases} v_{k_i} \to v & \text{weakly in } L^2(Q_{t_0}^T), \\ v_{k_i} \to v & \text{strongly in } L^1(Q_{t_0}^T), \\ v_{k_i} \to v & \text{a.e. in } Q_{t_0}^T, \\ \sqrt{a_{k_i}}\nabla v_{k_i} \to \rho^{\frac{p}{2}} \nabla v & \text{weakly in } L^2(Q_{t_0}^T) \end{cases}$$

as $i \to +\infty$. One can easily conclude that v is the weak solution of the problem (3.27) owing to the above convergence processes. Moreover, in the space $L^\infty((t_0, +\infty); L^2(\Omega))$, the weak solution of the problem (3.27) is unique.

Combining (3.6) and (3.29), we have

$$\begin{cases} \dfrac{\partial(v_k - u_k)}{\partial t} - \operatorname{div}(a_k \nabla (v_k - u_k)) + [\sigma_k(x, t+\tau, v_k)v_k - \sigma_k(x, t, u_k)u_k] \\ = h_k(x,t) - h_k(x, t+\tau) + f_k(x, t+\tau) - f_k(x,t) & \text{in } Q_{t_0}^{+\infty}, \\ v_k(x,t) - u_k(x,t) = 0 & \text{on } \partial\Omega \times (t_0, +\infty), \\ v_k(x, t_0) - u_k(x, t_0) = v_{0k}(x) - u_{0k}(x) & \text{in } \Omega. \end{cases}$$

Multiplying the first equation by $v_k - u_k$ and integrating over Ω by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_k(\cdot,t) - u_k(\cdot,t)\|^2_{L^2(\Omega)} + \|a_k|\nabla(v_k - u_k)|^2\|_{L^1(\Omega)} \\ &+ \int_\Omega (\sigma_k(x, t+\tau, v_k)v_k - \sigma_k(x, t, u_k)u_k)(v_k - u_k)\mathrm{d}x \\ = &\int_\Omega (h_k(x,t) - h_k(x, t+\tau) + f_k(x, t+\tau) - f_k(x,t))(v_k - u_k)\mathrm{d}x, \quad t \ge t_0. \end{aligned} \tag{3.30}$$

Recalling the definitions of $\sigma_k(x,t,u)$ and σ^*, we compute as $u \ne v$ that

$$\begin{aligned} &\frac{\sigma_k(x,t,v)v - \sigma_k(x,t,u)u}{v-u} \\ = &\iint_{B(0,1)} \eta \frac{\sigma\Big(x - \frac{x'}{k}, t - \frac{t'}{k}, v - \frac{u'}{k}\Big)\Big(v - \frac{u'}{k}\Big) - \sigma\Big(x - \frac{x'}{k}, t - \frac{t'}{k}, u - \frac{u'}{k}\Big)\Big(u - \frac{u'}{k}\Big)}{v-u} \mathrm{d}x'\mathrm{d}t'\mathrm{d}u' \end{aligned}$$

$$
+\iint_{B(0,1)} \eta u' \frac{\sigma\Big(x-\frac{x'}{k},t-\frac{t'}{k},v-\frac{u'}{k}\Big)-\sigma\Big(x-\frac{x'}{k},t-\frac{t'}{k},u-\frac{u'}{k}\Big)}{k(v-u)}\mathrm{d}x'\mathrm{d}t'\mathrm{d}u'
\geq \sigma^* - \frac{2C_0}{k|v-u|}.
$$

Consequently,

$$
\begin{aligned}
&\int_\Omega (\sigma_k(x,t+\tau,v_k)v_k - \sigma_k(x,t+\tau,u_k)u_k)(v_k-u_k)\mathrm{d}x \\
&\geq \sigma^*\|v_k(\cdot,t)-u_k(\cdot,t)\|^2_{L^2(\Omega)} - \frac{2C_0}{k}\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^1(\Omega)} \\
&\geq \sigma^*\|v_k(\cdot,t)-u_k(\cdot,t)\|^2_{L^2(\Omega)} - \frac{2C_0\sqrt{\operatorname{meas}\Omega}}{k}\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^2(\Omega)}.
\end{aligned}
$$

Invoking (3.9) and applying the Hölder inequality, we derive

$$
\begin{aligned}
&\int_\Omega (\sigma_k(x,t+\tau,u_k)-\sigma_k(x,t,u_k))u_k(v_k-u_k)\mathrm{d}x \\
&\geq -\|u_k(\cdot,t)\|_{L^2(\Omega)}\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^2(\Omega)}\sup_{t\in\mathbb{R}}\|\sigma_k(\cdot,t+\tau,\cdot)-\sigma_k(\cdot,t,\cdot)\|_{L^\infty(\Omega\times\mathbb{R})} \\
&\geq -C(\|u_0\|_{L^2(\Omega)}+\sqrt{S^*})\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^2(\Omega)}\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|_{L^\infty(\Omega\times\mathbb{R})}.
\end{aligned}
$$

Then it follows from the above two inequalities that

$$
\begin{aligned}
&\int_\Omega (\sigma_k(x,t+\tau,v_k)v_k - \sigma_k(x,t,u_k)u_k)(v_k-u_k)\mathrm{d}x \\
&\geq \sigma^*\|v_k(\cdot,t)-u_k(\cdot,t)\|^2_{L^2(\Omega)} - \frac{2C_0\sqrt{\operatorname{meas}\Omega}}{k}\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^2(\Omega)} \\
&\quad - C(\|u_0\|_{L^2(\Omega)}+\sqrt{S^*})\|v_k(\cdot,t)-u_k(\cdot,t)\|_{L^2(\Omega)}\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|_{L^\infty(\Omega\times\mathbb{R})}.
\end{aligned}
$$

Combine (3.30) and the above inequality, and then utilize the same arguments as to the proof of (3.9) to find

$$
\begin{aligned}
&\sup_{t\geq t_0}\|v_k(\cdot,t)-u_k(\cdot,t)\|^2_{L^2(\Omega)} \\
&\leq C\Big(\|v_0-u_0\|^2_{L^2(\Omega)}+\sup_{t\in\mathbb{R}}\|f(\cdot,\cdot+\tau)-f(\cdot,\cdot)\|^2_{L^2(Q_t^{t+1})} \\
&\quad +\sup_{t\in\mathbb{R}}\|g(\cdot,\cdot+\tau,0)-g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}+\frac{1}{k^2}\Big) \\
&\quad + C(\|u_0\|^2_{L^2(\Omega)}+S^*)\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|^2_{L^\infty(\Omega\times\mathbb{R})}.
\end{aligned}
\tag{3.31}
$$

Taking $k=k_i$ and passing to limits as $i\to+\infty$ in (3.31), we thereby obtain the estimate (3.28). The proof is complete.

4 Proofs of Main Results

Based on Lemmas 3.1–3.2, we now prove the main results in this paper.

Proof of Theorem 2.1 For any $l \in \mathbb{N}^+$, one gets from Lemma 3.1 that the problem (1.1)–(1.2) on $Q_{-l}^{+\infty}$ with the initial value $u(x,-l)=0$ admits uniquely a weak solution, denoted by $u^{(l)}(x,t)$, which satisfies

$$\sup_{t\geq -l}\|u^{(l)}(\cdot,t)\|_{L^2(\Omega)}^2+\sup_{t\geq -l}\|\rho^p|\nabla u^{(l)}(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\geq -l}\Big\|\frac{\partial u^{(l)}}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2\leq CS^*, \tag{4.1}$$

where C is a constant independent of l.

Fix $t_0=-1$. Due to (4.1), we can derive by using the arguments as in the proof of (3.16)–(3.17) and (3.21)–(3.22) that there exists a subsequence $\{u^{(l_1(i))}\}_{i=1}^{+\infty}\subset\{u^{(l)}\}_{l=1}^{+\infty}$, and the limit function $u_{(1)}\in L^\infty((-1,+\infty);L^2(\Omega))\cap\mathscr{B}_s^T$ for any $-1<s<T<+\infty$, such that

$$\begin{cases} u^{(l_1(i))}\to u_{(1)} & \text{weakly in } L^2(Q_s^T),\\ u^{(l_1(i))}\to u_{(1)} & \text{strongly in } L^1(Q_s^T),\\ u^{(l_1(i))}\to u_{(1)} & \text{a.e. in } Q_s^T,\\ \rho^{\frac{p}{2}}\nabla u^{(l_1(i))}\to \rho^{\frac{p}{2}}\nabla u_{(1)} & \text{weakly in } L^2(Q_s^T)\end{cases}$$

as $i\to+\infty$. It follows from (4.1) that

$$\sup_{t\geq -1}\|u_{(1)}(\cdot,t)\|_{L^2(\Omega)}^2+\sup_{t\geq -1}\|\rho^p|\nabla u_{(1)}(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\geq -1}\Big\|\frac{\partial u_{(1)}}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2\leq CS^*.$$

By repeating the process above, we see that for $t_0=-j$, $j=2,3,\cdots$, there exists a family of subsequences $\{u^{(l_j(i))}\}_{i=1}^{+\infty}\subset\{u^{(l_{j-1}(i))}\}_{i=1}^{+\infty}\subset\cdots\subset\{u^{(l_1(i))}\}_{i=1}^{+\infty}$ (where $l_j(i)\geq j$ for $i=1,2,\cdots$), and a function sequence $u_{(j)}\in L^\infty((-j,+\infty);L^2(\Omega))\cap\mathscr{B}_s^T$ for any $-j<s<T<+\infty$, such that

$$\begin{cases} u^{(l_j(i))}\to u_{(j)} & \text{weakly in } L^2(Q_s^T),\\ u^{(l_j(i))}\to u_{(j)} & \text{strongly in } L^1(Q_s^T),\\ u^{(l_j(i))}\to u_{(j)} & \text{a.e. in } Q_s^T,\\ \rho^{\frac{p}{2}}\nabla u^{(l_j(i))}\to \rho^{\frac{p}{2}}\nabla u_{(j)} & \text{weakly in } L^2(Q_s^T)\end{cases} \tag{4.2}$$

as $i\to+\infty$, and

$$\sup_{t\geq -j}\|u_{(j)}(\cdot,t)\|_{L^2(\Omega)}^2+\sup_{t\geq -j}\|\rho^p|\nabla u_{(j)}(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\geq -j}\Big\|\frac{\partial u_{(j)}}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2\leq CS^*. \tag{4.3}$$

Notice that $u_{(j)}(x,t)=u_{(j-1)}(x,t)$ in $\Omega\times(-(j-1),+\infty)$ $(j=2,3,\cdots)$. That is to say, $u_{(j)}$ is the extension of $u_{(j-1)}$ to $\Omega\times(-j,+\infty)$. Define $u:\Omega\times\mathbb{R}\to\mathbb{R}$ as follows:

$$u(x,t)=u_{(j)}(x,t)\quad \text{if } (x,t)\in\Omega\times(-j,+\infty),$$

where $j=1,2,\cdots$.

Let us show that u is a weak solution of the problem (1.1)–(1.2) on $Q_{-\infty}^{+\infty}$. By (4.3), we have

$$\sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{L^2(\Omega)}^2+\sup_{t\in\mathbb{R}}\|\rho^p|\nabla u(\cdot,t)|^2\|_{L^1(\Omega)}+\sup_{t\in\mathbb{R}}\Big\|\frac{\partial u}{\partial s}\Big\|_{L^2(Q_t^{t+1})}^2\leq CS^*. \tag{4.4}$$

This implies that $u\in L^\infty(\mathbb{R};L^2(\Omega))\cap\mathscr{B}_s^T$ for any $-\infty<s<T<+\infty$. For any $-\infty<s<T<+\infty$, there exists a $j_0\in\mathbb{N}^+$ such that $(s,T)\subset(-j_0,+\infty)$. From Definition 2.5, we have that for any function $\varphi \in C^1(\overline{Q_s^T})$ satisfying $\varphi(x,t)=0$ for x near $\partial\Omega$, $\varphi(\cdot,T)|_\Omega=\varphi(\cdot,s)|_\Omega=0$, the following integral equality holds:

$$\iint_{Q_s^T}\Big[-u^{(l_{j_0}(i))}\frac{\partial\varphi}{\partial t}+\rho^p\nabla u^{(l_{j_0}(i))}\nabla\varphi+(\sigma(x,t,u^{(l_{j_0}(i))})u^{(l_{j_0}(i))} + g(x,t,0)-f)\varphi\Big]\mathrm{d}x\mathrm{d}t=0. \tag{4.5}$$

In virtue of (4.2), we derive from the dominated convergence theorem and the continuity of $\sigma(x,t,u)$ with respect to u that

$$\begin{aligned}&\iint_{Q_s^T}(\sigma(x,t,u^{(l_{j_0}(i))})u^{(l_{j_0}(i))}-\sigma(x,t,u_{(j_0)})u_{(j_0)})\varphi\mathrm{d}x\mathrm{d}t\\ =&\iint_{Q_s^T}\sigma(x,t,u^{(l_{j_0}(i))})(u^{(l_{j_0}(i))}-u_{(j_0)})\varphi\mathrm{d}x\mathrm{d}t\\ &+\iint_{Q_s^T}(\sigma(x,t,u^{(l_{j_0}(i))})-\sigma(x,t,u_{(j_0)}))u_{(j_0)}\varphi\mathrm{d}x\mathrm{d}t\\ \to&\ 0\quad \text{as } i\to+\infty.\end{aligned} \tag{4.6}$$

Letting $i\to\infty$ with (4.2) and (4.6), one gets from (4.5) that

$$\iint_{Q_s^T}\Big(-u_{(j_0)}\frac{\partial\varphi}{\partial t}+\rho^p\nabla u_{(j_0)}\nabla\varphi+g(x,t,u_{(j_0)})\varphi-f\varphi\Big)\mathrm{d}x\mathrm{d}t=0.$$

For any function $\varphi\in L^\infty((s,T);L^2(\Omega))\cap\mathscr{B}_s^T$ with $\frac{\partial\varphi}{\partial t}\in L^2(Q_s^T)$ and $\varphi(\cdot,s)|_\Omega=\varphi(\cdot,T)|_\Omega=0$, the above integral equality still holds after an approximate procedure. Note that $u=u_{(j_0)}$ on $\Omega\times(-j_0,+\infty)$ and $Q_s^T\subset\Omega\times(-j_0,+\infty)$. Therefore

$$\iint_{Q_s^T}\Big(-u\frac{\partial\varphi}{\partial t}+\rho^p\nabla u\nabla\varphi+g(x,t,u)\varphi-f\varphi\Big)\mathrm{d}x\mathrm{d}t=0.$$

From the arbitrary of s and T, we see that u is the weak solution of the problem (1.1)–(1.2) on $Q_{-\infty}^{+\infty}$.

Now we set about proving the uniqueness of bounded entire solutions. Firstly, we claim that if u is a bounded entire solution of the problem (1.1)–(1.2), then for any $-\infty<t_1<t_2<+\infty$ the following integral equality

$$\begin{aligned}&\iint_{Q_{t_1}^{t_2}}\Big(-u\frac{\partial\varphi}{\partial t}+\rho^p\nabla u\nabla\varphi+g(x,t,u)\varphi-f\varphi\Big)\mathrm{d}x\mathrm{d}t\\ =&\int_\Omega u(x,t_1)\varphi(x,t_1)\mathrm{d}x-\int_\Omega u(x,t_2)\varphi(x,t_2)\mathrm{d}x\end{aligned} \tag{4.7}$$

holds for any function $\varphi\in L^\infty((t_1,t_2);L^2(\Omega))\cap\mathscr{B}_{t_1}^{t_2}$ with $\frac{\partial\varphi}{\partial t}\in L^2(Q_{t_1}^{t_2})$. To see this, we select $\varphi_k\in C^1(\overline{Q_{t_1}^{t_2}})$ $(k=1,2,\cdots)$ satisfying $\varphi_k(x,t)=0$ for x near $\partial\Omega$ such that

$$\begin{aligned}&\varphi_k\to\varphi\quad\text{strongly in } L^2(Q_{t_1}^{t_2})\cap\mathscr{B}_{t_1}^{t_2},\quad \frac{\partial\varphi_k}{\partial t}\to\frac{\partial\varphi}{\partial t}\quad\text{strongly in } L^2(Q_{t_1}^{t_2}),\\ &\varphi_k(\cdot,t)\to\varphi(\cdot,t)\quad\text{weakly in } L^2(\Omega)\text{ for any } t\in(t_1,t_2)\end{aligned} \tag{4.8}$$

as $k \to +\infty$. Choose $\theta(s) \in C_0^\infty(\mathbb{R})$ such that $\theta(s) \geq 0$ for $s \in \mathbb{R}$, $\theta(s) = 0$ for $|s| > 1$, $\int_{\mathbb{R}} \theta(s)\mathrm{d}s = 1$. For $h > 0$, define $\theta_h(s) = \frac{1}{h}\theta\big(\frac{s}{h}\big)$ and

$$\eta_h(t) = \int_{t-t_2+2h}^{t-t_1-2h} \theta_h(s)\mathrm{d}s.$$

Then

$$\eta_h(t) \in C_0^\infty(t_1, t_2), \quad \lim_{h\to 0^+} \eta_h(t) = 1 \quad \text{for } t \in (t_1, t_2).$$

Choosing $\varphi = \varphi_k(x,t)\eta_h(t)$ in (2.2) with t_0, T being replaced by t_1, t_2 respectively, we obtain

$$\iint_{Q_{t_1}^{t_2}} \Big[-u\frac{\partial \varphi_k}{\partial t}\eta_h + u\varphi_k(\theta_h(t-t_2+2h) - \theta_h(t-t_1-2h))\Big]\mathrm{d}x\mathrm{d}t + \iint_{Q_{t_1}^{t_2}} \rho^p \nabla u \nabla \varphi_k \eta_h \mathrm{d}x\mathrm{d}t + \iint_{Q_{t_1}^{t_2}} (g(x,t,u) - f)\varphi_k\eta_h \mathrm{d}x\mathrm{d}t = 0. \tag{4.9}$$

Note that

$$\begin{aligned}
&\Big| \iint_{Q_{t_1}^{t_2}} u\varphi_k\theta_h(t-t_2+2h)\mathrm{d}x\mathrm{d}t - \int_\Omega u(x,t_2)\varphi_k(x,t_2)\mathrm{d}x\Big| \\
=& \Big| \iint_{Q_{t_2-3h}^{t_2-h}} (u(x,t)\varphi_k(x,t) - u(x,t_2)\varphi_k(x,t_2))\theta_h(t-t_2+2h)\mathrm{d}x\mathrm{d}t\Big| \\
\leq& \sup_{t_2-3h<t<t_2-h} \int_\Omega |u(x,t)\varphi_k(x,t) - u(x,t_2)\varphi_k(x,t_2)|\mathrm{d}x \\
\leq& \sup_{t_2-3h<t<t_2-h} \int_\Omega |u(x,t) - u(x,t_2)||\varphi_k(x,t)|\mathrm{d}x \\
&+ \sup_{t_2-3h<t<t_2-h} \int_\Omega |u(x,t_2)||\varphi_k(x,t) - \varphi_k(x,t_2)|\mathrm{d}x.
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
&\sup_{t_2-3h<t<t_2-h} \int_\Omega |u(x,t) - u(x,t_2)||\varphi_k(x,t)|\mathrm{d}x \\
\leq& \sup_{t_2-3h<t<t_2-h} \iint_{Q_t^{t_2}} \Big|\frac{\partial u(x,s)}{\partial s}\Big||\varphi_k(x,t)|\mathrm{d}x\mathrm{d}s \\
\leq& \sqrt{3h}\sqrt{\operatorname{meas}\Omega}\,\Big\|\frac{\partial u}{\partial s}\Big\|_{L^2(Q_{t_1}^{t_2})}\|\varphi_k\|_{C(\overline{\Omega}\times[t_1,t_2])}
\end{aligned}$$

and

$$\sup_{t_2-3h<t<t_2-h} \int_\Omega |u(x,t_2)||\varphi_k(x,t) - \varphi_k(x,t_2)|\mathrm{d}x \leq \|u(\cdot,t_2)\|_{L^2(\Omega)} \sup_{x\in\overline{\Omega}} \operatorname*{osc}_{t\in[t_2-3h,t_2]} \varphi_k(x,t).$$

Hence it follows from the above arguments that for any given $k \in \mathbb{N}^+$,

$$\iint_{Q_{t_1}^{t_2}} u\varphi_k\theta_h(t-t_2+2h)\mathrm{d}x\mathrm{d}t \to \int_\Omega u(x,t_2)\varphi_k(x,t_2)\mathrm{d}x \quad \text{as } h \to 0^+.$$

Likewise we have

$$\iint_{Q_{t_1}^{t_2}} u\varphi_k\theta_h(t-t_1-2h)\mathrm{d}x\mathrm{d}t \to \int_\Omega u(x,t_1)\varphi_k(x,t_1)\mathrm{d}x \quad \text{as } h \to 0^+.$$

Letting $h \to 0^+$ and then $k \to +\infty$ in (4.9) yield (4.7).

Let $u_1(x,t),\ u_2(x,t)$ be two bounded entire solutions of the problem (1.1)–(1.2). For any function $\varphi \in L^\infty((t_0,t);L^2(\Omega)) \cap \mathscr{B}_{t_0}^t$ with $\frac{\partial\varphi}{\partial s} \in L^2(Q_{t_0}^t)$, it follows from (4.7) that

$$\iint_{Q_{t_0}^t} \Big(-(u_1-u_2)\frac{\partial\varphi}{\partial s} + \rho^p \nabla(u_1-u_2)\nabla\varphi + (g(x,s,u_1)-g(x,s,u_2))\varphi \Big)\mathrm{d}x\mathrm{d}s$$
$$= \int_\Omega (u_1(x,t_0)-u_2(x,t_0))\varphi(x,t_0)\mathrm{d}x - \int_\Omega (u_1(x,t)-u_2(x,t))\varphi(x,t)\mathrm{d}x.$$

Letting $\varphi = u_1 - u_2$ in the above equality, we derive from the arbitrariness of t_0 and t that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)} + \|\rho^p|\nabla(u_1(\cdot,t)-u_2(\cdot,t))|^2\|_{L^1(\Omega)}$$
$$+ \int_\Omega (g(x,t,u_1(x,t))-g(x,t,u_2(x,t)))(u_1(x,t)-u_2(x,t))\mathrm{d}x = 0 \quad \text{a.e. } t \in \mathbb{R}.$$

Then we employ (2.4) and the definition of σ^* to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)} + 2(\lambda_p^*+\sigma^*)\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)} \le 0 \quad \text{a.e. } t \in \mathbb{R}, \tag{4.10}$$

which implies that $\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)}$ is decreasing. If the stated conclusion of the uniqueness is not true, there would exist $t_1 \in \mathbb{R}$ and positive number δ such that

$$\|u_1(\cdot,t_1)-u_2(\cdot,t_1)\|^2_{L^2(\Omega)} \ge \delta > 0.$$

Integrating (4.10) over $[t_2,t_1]$ $(t_2 < t_1)$, we have

$$\|u_1(\cdot,t_1)-u_2(\cdot,t_1)\|^2_{L^2(\Omega)} + 2(\lambda_p^*+\sigma^*)\int_{t_2}^{t_1}\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t$$
$$\le \|u_1(\cdot,t_2)-u_2(\cdot,t_2)\|^2_{L^2(\Omega)}.$$

It follows from the above two inequalities that

$$\delta + 2(\lambda_p^*+\sigma^*)\|u_1(\cdot,t_1)-u_2(\cdot,t_1)\|^2_{L^2(\Omega)}(t_1-t_2) \le \|u_1(\cdot,t_2)-u_2(\cdot,t_2)\|^2_{L^2(\Omega)},$$

since $\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)}$ is decreasing. Consequently

$$\|u_1(\cdot,t_2)-u_2(\cdot,t_2)\|^2_{L^2(\Omega)} \to +\infty \quad \text{as } t_2 \to -\infty.$$

However this conclusion is at variance with the definition of bounded entire solution.

It remains to prove the stability of the solution $u(x,t)$. Suppose that $w(x,t)$ is the weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^{+\infty}$ with the initial condition $w(x,t_0) = w_0(x) \in L^2(\Omega)$. Owing to (3.3), we might not expect $w(x,t)-u(x,t)$ to be a test function in $Q_{t_0}^T$. We will try to get the stability of $u(x,t)$ with the help of $v(x,t)$, which is the weak solution of the problem (1.1)–(1.2) on $Q_{t_0}^{+\infty}$ with the initial value $v(x,t_0) = v_0(x) \in W_0^{1,2}(\Omega)$. For any given $\varepsilon > 0$, we can select $v_0(x)$ such that

$$\|w_0 - v_0\|_{L^2(\Omega)} < \varepsilon.$$

In a way similar to the proof of (4.10), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)} + 2(\lambda_p^*+\sigma^*)\|v(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)} \le 0 \quad \text{a.e. } t \ge t_0.$$

Consequently

$$\|v(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\le\|v_0(\cdot)-u(\cdot,t_0)\|^2_{L^2(\Omega)}e^{2(\lambda_p^*+\sigma^*)(t_0-t)},\quad t\ge t_0,$$

which leads to

$$\begin{aligned}&\|w(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\\ \le&\|w(\cdot,t)-v(\cdot,t)\|_{L^2(\Omega)}+\|v(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\\ \le&\|w(\cdot,t)-v(\cdot,t)\|_{L^2(\Omega)}+\|v_0(\cdot)-u(\cdot,t_0)\|_{L^2(\Omega)}e^{(\lambda_p^*+\sigma^*)(t_0-t)},\quad t\ge t_0.\end{aligned}$$

Noting that

$$\sup_{t\ge t_0}\|w(\cdot,t)-v(\cdot,t)\|^2_{L^2(\Omega)}\le C\|w_0-v_0\|^2_{L^2(\Omega)}\le C\varepsilon^2,$$

according to Lemma 3.2, we derive from the above two inequalities that

$$\begin{aligned}\|w(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}&\le\sqrt{C}\varepsilon+\|v_0(\cdot)-u(\cdot,t_0)\|_{L^2(\Omega)}e^{(\lambda_p^*+\sigma^*)(t_0-t)}\\ &\le(\sqrt{C}+1)\varepsilon+\|w_0(\cdot)-u(\cdot,t_0)\|_{L^2(\Omega)}e^{(\lambda_p^*+\sigma^*)(t_0-t)},\quad t\ge t_0.\end{aligned}$$

From the arbitrariness of ε we have

$$\|w(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}\le\|w_0(\cdot)-u(\cdot,t_0)\|_{L^2(\Omega)}e^{(\lambda_p^*+\sigma^*)(t_0-t)},\quad t\ge t_0.$$

The proof is complete.

Proof of Theorem 2.2 We claim that for any $\tau\in\mathbb{R}$,

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\|u(\cdot,t+\tau)-u(\cdot,t)\|^2_{L^2(\Omega)}\le C(S^*+1)\Big(\sup_{t\in\mathbb{R}}\|f(\cdot,\cdot+\tau)-f(\cdot,\cdot)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|g(\cdot,\cdot+\tau,0)-g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|^2_{L^\infty(\Omega\times\mathbb{R})}\Big).\end{aligned}\tag{4.11}$$

To confirm this, for any $l\in\mathbb{N}^+$ suppose that $v^{(l)}(x,t)$ is a weak solution of the following problem:

$$\begin{cases}\dfrac{\partial v^{(l)}}{\partial t}-\operatorname{div}(\rho^p(x)\nabla v^{(l)})+g(x,t+\tau,v^{(l)})=f(x,t+\tau) & \text{in } Q_{-l}^{+\infty},\\ v^{(l)}(x,t)=0 & \text{on } \Sigma_1\times(-l,+\infty),\\ v^{(l)}(x,-l)=0 & \text{in } \Omega.\end{cases}\tag{4.12}$$

Let $u^{(l)}(x,t)$, $u^{(l_j(i))}(x,t)$ be as that in the proof of Theorem 2.1. Without loss of generality, we can select a family of subsequences $\{v^{(l_j(i))}\}_{i=1}^{+\infty}\subset\{v^{(l_{j-1}(i))}\}_{i=1}^{+\infty}\subset\cdots\subset\{v^{(l_1(i))}\}_{i=1}^{+\infty}\subset\{v^{(l)}\}_{l=1}^{+\infty}$, and a function sequence $v_{(j)}\in L^\infty((-j,+\infty);L^2(\Omega))\cap\mathscr{B}_s^T$ for any $-j<s<T<+\infty$, such that

$$\begin{cases}v^{(l_j(i))}\to v_{(j)} & \text{weakly in } L^2(Q_s^T),\\ v^{(l_j(i))}\to v_{(j)} & \text{strongly in } L^1(Q_s^T),\\ v^{(l_j(i))}\to v_{(j)} & \text{a.e. in } Q_s^T,\\ \rho^{\frac{p}{2}}\nabla v^{(l_j(i))}\to\rho^{\frac{p}{2}}\nabla v_{(j)} & \text{weakly in } L^2(Q_s^T)\end{cases}$$

as $i \to +\infty$, and

$$\sup_{t\ge -j}\|v_{(j)}(\cdot,t)\|^2_{L^2(\Omega)} + \sup_{t\ge -j}\|\rho^p|\nabla v_{(j)}(\cdot,t)|^2\|_{L^1(\Omega)} + \sup_{t\ge -j}\left\|\frac{\partial v_{(j)}}{\partial s}\right\|^2_{L^2(Q_t^{t+1})} \le CS^*.$$

Since $v_{(j)}(x,t) = v_{(j-1)}(x,t)$ in $\Omega\times(-(j-1),+\infty)$ $(j=2,3,\cdots)$, we can define $v:\Omega\times\mathbb{R}\to\mathbb{R}$ as follows:

$$v(x,t) = v_{(j)}(x,t) \quad \text{if } (x,t)\in\Omega\times(-j,+\infty),$$

where $j = 1,2,\cdots$. We thereupon conclude from the the proof of Theorem 2.1 that $v(x,t)$ is a bounded entire solution of the following problem:

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} - \operatorname{div}(\rho^p(x)\nabla v(x,t)) + g(x,t+\tau,v(x,t)) = f(x,t+\tau) & \text{in } Q_{-\infty}^{+\infty}, \\ v(x,t) = 0 & \text{on } \Sigma_1\times\mathbb{R}. \end{cases} \tag{4.13}$$

In view of Lemma 3.2 we have

$$\begin{aligned}\sup_{t\ge -l_j(i)}\|v^{(l_j(i))}(\cdot,t) - u^{(l_j(i))}(\cdot,t)\|^2_{L^2(\Omega)} \le C(S^*+1)\Big(&\sup_{t\in\mathbb{R}}\|f(\cdot,\cdot+\tau)-f(\cdot,\cdot)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|g(\cdot,\cdot+\tau,0)-g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|^2_{L^\infty(\Omega\times\mathbb{R})}\Big).\end{aligned}$$

Note that $\{v^{(l_j(i))}\}_{i=1}^{+\infty}$ and $\{u^{(l_j(i))}\}_{i=1}^{+\infty}$ are common subsequences of $\{v^{(l_{j-1}(i))}\}_{i=1}^{+\infty}$ and $\{u^{(l_{j-1}(i))}\}_{i=1}^{+\infty}$ respectively. Owing to the above inequality we thereby obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}}\|v(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)} \le C(S^*+1)\Big(&\sup_{t\in\mathbb{R}}\|f(\cdot,\cdot+\tau)-f(\cdot,\cdot)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|g(\cdot,\cdot+\tau,0)-g(\cdot,\cdot,0)\|^2_{L^2(Q_t^{t+1})}\\ &+\sup_{t\in\mathbb{R}}\|\sigma(\cdot,t+\tau,\cdot)-\sigma(\cdot,t,\cdot)\|^2_{L^\infty(\Omega\times\mathbb{R})}\Big).\end{aligned}$$

Obviously $u(x,t+\tau)$ also is the bounded entire solution of problem (4.13). This implies $v(x,t) = u(x,t+\tau)$ for a.e. $(x,t)\in Q_{-\infty}^{+\infty}$ due to the uniqueness of bounded entire solution of problem (4.13), whence the assertion (4.11) holds.

Now we are in position to prove the almost periodicity of $u(x,t)$. Since $f(\cdot,t),\ g(\cdot,t,0)\in S^2AP(L^2(\Omega))$ and $\sigma(\cdot,t,\cdot)\in AP(L^\infty(\Omega\times\mathbb{R}))$, we conclude from (4.11) that $u(\cdot,t)\in C(\mathbb{R};L^2(\Omega))$. Letting

$$\tau\in T(\varepsilon,f)\cap T(\varepsilon,g(\cdot,\cdot,0))\cap T(\varepsilon,\sigma), \tag{4.14}$$

we deduce from (4.11) that

$$\sup_{t\in\mathbb{R}}\|u(\cdot,t+\tau)-u(\cdot,t)\|^2_{L^2(\Omega)} \le 3C(S^*+1)\varepsilon^2. \tag{4.15}$$

This implies

$$T(\varepsilon,f)\cap T(\varepsilon,g(\cdot,\cdot,0))\cap T(\varepsilon,\sigma)\subseteq T(\sqrt{3C(S^*+1)}\varepsilon,u).$$

It is well known that the intersection of two relatively dense sets is also relatively dense, hence $u(\cdot,t)\in AP(L^2(\Omega))$. Furthermore, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t+\tau)-u(\cdot,t)\|^2_{L^2(\Omega)}+\|\rho^{\frac{p}{2}}\nabla(u(\cdot,t+\tau)-u(\cdot,t))\|^2_{L^2(\Omega)}\\ =&-\int_\Omega(\sigma(x,t+\tau,u(x,t+\tau))u(x,t+\tau)-\sigma(x,t,u(x,t))u(x,t))(u(x,t+\tau)-u(x,t))\mathrm{d}x\\ &+\int_\Omega(g(x,t,0)-g(x,t+\tau,0)+f(x,t+\tau)-f(x,t))(u(x,t+\tau)-u(x,t))\mathrm{d}x. \qquad (4.16)\end{aligned}$$

Recalling the definition of σ^* and (4.14)–(4.15), we compute

$$\begin{aligned}&\int_\Omega(\sigma(x,t+\tau,u(x,t+\tau))u(x,t+\tau)-\sigma(x,t,u(x,t))u(x,t))(u(x,t+\tau)-u(x,t))\mathrm{d}x\\ =&\int_\Omega(\sigma(x,t+\tau,u(x,t+\tau))u(x,t+\tau)-\sigma(x,t+\tau,u(x,t))u(x,t))(u(x,t+\tau)-u(x,t))\mathrm{d}x\\ &+\int_\Omega(\sigma(x,t+\tau,u(x,t))-\sigma(x,t,u(x,t)))u(x,t)(u(x,t+\tau)-u(x,t))\mathrm{d}x\\ \geq&-3C|\sigma^*|(S^*+1)\varepsilon^2-\sqrt{3}(S^*+1)C\varepsilon^2. \qquad (4.17)\end{aligned}$$

But it follows from Hölder's inequality and (4.11) that

$$\begin{aligned}&\int_\Omega(g(x,t,0)-g(x,t+\tau,0)+f(x,t+\tau)-f(x,t))(u(x,t+\tau)-u(x,t))\mathrm{d}x\\ \leq&\sqrt{3C(S^*+1)}\varepsilon(\|g(x,t+\tau,0)-g(x,t,0)\|_{L^2(\Omega)}+\|f(x,t+\tau)-f(x,t)\|_{L^2(\Omega)}). \qquad (4.18)\end{aligned}$$

Integrating (4.16) over $[t,t+1]$ and utilizing (4.15) and (4.17)–(4.18), we obtain

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\|\rho^{\frac{p}{2}}\nabla(u(\cdot,s+\tau)-u(\cdot,s))\|^2_{L^2(\Omega)}\mathrm{d}s\leq C'\varepsilon^2,$$

where C' is a positive constant which depends only on N, p, Ω, σ^*, C_0 and S^*. Thus $u(\cdot,t)\in S^2AP(W_0^{1,2}(\Omega,\rho^p))$. The proof is compete.

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