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Exact Boundary Controllability for the Spatial Vibration of String with Dynamical Boundary Conditions*

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Abstract This paper deals with the spatial vibration of an elastic string with masses at the endpoints. The authors derive the corresponding quasilinear wave equation with dynamical boundary conditions, and prove the exact boundary controllability of this system by means of a constructive method with modular structure.

Keywords Spatial vibration of a string, Exact boundary controllability, Dynamical boundary condition
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1 Introduction

In [9] we introduced a quasilinear wave equation with dynamical boundary conditions to describe the lateral vibration of an elastic string with masses on the corresponding ends. The exact boundary controllability of this system was realized using the constructive method given in [1, 3–4].

In this paper, we will consider the spatial vibration in $\mathbb{R}^3$ of an elastic string with masses on the ends. The model is shown in Section 2 (see [2, 5–6, 8]). The main target of this paper is to establish controllability results for an elastic string governed by a system of quasilinear wave equations with dynamical boundary conditions. The main results are shown in Section 3 and proved in Section 5, which are based on the theory of semi-global C^2 solution to the corresponding non-local mixed problem (see Section 4).

2 Spatial Vibration of an Elastic String

In order to consider the dynamical behavior of a single elastic string in three-dimensional space, which is not limited to small deformation, we first establish its dynamical equations.

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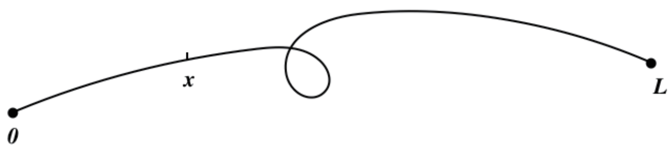

Figure 1 String in natural state.

As shown in Figure 1, when a string is in a uniform, stress-free natural state, we parameterize it by its rest arc length x with $x \in [0, L]$, L being the natural length of the string.

The position at time t of the point corresponding to the parameter x will be denoted by $Y = Y(t, x)$, where $Y = (Y_1, Y_2, Y_3)$ is a vector, and $Y(0, x) = Y_0(x)$ describes the initial position of the string.

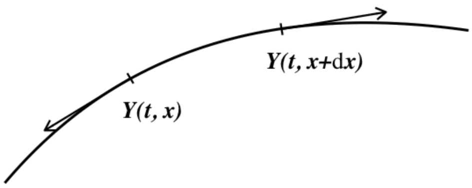

Figure 2 Tension on a small part of the string.

Consider a small part of the string, which is placed on the interval corresponding to the rest arc length $x \in [x, x + \mathrm{d}x]$, hence $Y(t, x)$ and $Y(t, x + \mathrm{d}x)$ denote the positions of the endpoints of this small part at time t (see Figure 2):

$$Y = Y(t, y), \quad y \in [x, x + \mathrm{d}x].$$

Without external force, this small part of string is subjected to the tensions given from both sides and the D'Alembert force of the movement. Since the string is soft and does not resist bending, the tension must follow the tangential direction, which can be taken as $S\frac{Y_x}{|Y_x|}$ with

$$S = S(|Y_x|)$$

being a function of the extension. Hence the resultant of the tension is given by

$$\Big(S(|Y_x|)\frac{Y_x}{|Y_x|}\Big)\Big|_x^{x+\mathrm{d}x} = \frac{\partial}{\partial x}\Big(S(|Y_x|)\frac{Y_x}{|Y_x|}\Big)\mathrm{d}x. \tag{2.1}$$

Assume that the linear density of the string in the natural state is 1, then the mass on $[x, x + \mathrm{d}x]$ is $\mathrm{d}x$, and the D'Alembert force of this part of string is

$$-Y_{tt}\mathrm{d}x.$$

Hence, $Y = Y(t,x)$ should satisfy

$$Y_{tt} - \Big(S(|Y_x|)\frac{Y_x}{|Y_x|}\Big)_x = 0. \tag{2.2}$$

Since $Y = Y(t,x)$ is a vector-valued function, (2.2) is a system of second-order quasilinear partial differential equations containing three equations, which describe the motion of an elastic string in space.

For the function $S(r)$, we make the following physically meaningful assumptions:

(S1) $S(r)$ is a given C^2 function of r for $r \geq 1$,

(S2) $S(r) > 0,\ \forall r > 1$, and $S(1) = 0$,

and the following assumption suitable to the mathematical arguments:

(S3) $S'(r) > \frac{S(r)}{r},\ \forall r > 1$.

The initial condition is given by

$$t = 0:\ Y = Y_0(x), \quad Y_t = Y_1(x). \tag{2.3}$$

At the initial time, we assume that the string is stretched, namely, Y_0 satisfies

$$|Y_0'(x)| > 1, \quad 0 \leq x \leq L, \tag{2.4}$$

and we set

$$r_0 = \min_{0 \leq x \leq L} |Y_0'(x)| > 1, \tag{2.5}$$

where $Y_0 \in (C^2[0,L])^3,\ Y_1 \in (C^1[0,L])^3$ with small norms $\||Y_0'| - r_0\|_{C^1[0,L]}$ and $\|Y_1\|_{(C^1[0,L])^3}$, such that the conditions of C^2 compatibility at the points $(t,x) = (0,0)$ and $(0,L)$ are satisfied, respectively.

Consider this elastic string with unit masses at the endpoints, which can be described by the dynamical boundary conditions as follows:

$$x = 0:\ Y_{tt} = S(|Y_x|)\frac{Y_x}{|Y_x|} + h(t), \tag{2.6}$$

$$x = L:\ Y_{tt} = -S(|Y_x|)\frac{Y_x}{|Y_x|} + \overline{h}(t), \tag{2.7}$$

where $h = (h_1, h_2, h_3)^{\mathrm{T}}(t)$, $\overline{h} = (\overline{h}_1, \overline{h}_2, \overline{h}_3)^{\mathrm{T}}(t)$ are both C^0 vector-valued functions of t, part of which can be used to be controls on the endpoints.

Remark 2.1 Assumption (S3) can be improved as

(S3') $S'(r) > 0,\ \forall r > 1$,

and the conclusion in this paper is still valid, but the controllability time must be suitably modified.

3 Exact Boundary Controllability

In the previous section, we have established the spatial vibration model of an elastic string with end-masses by the second-order partial differential equations (2.2) with dynamical boundary conditions (2.6)–(2.7) (see [2, 5–6, 8]). We now discuss the exact boundary controllability of this system.

For $T > 0$ and any given final condition

$$t = T: \; Y = \overline{Y}_0(x), \quad Y' = \overline{Y}_1(x), \tag{3.1}$$

where $\overline{Y}_0 \in (C^2[0,L])^3$, $\overline{Y}_1 \in (C^1[0,L])^3$ with small norms $\||\overline{Y}_0'| - r_0\|_{C^1[0,L]}$ and $\|\overline{Y}_1\|_{(C^1[0,L])^3}$, we will establish the local exact boundary controllability around $r_0 > 1$ as follows.

Theorem 3.1 (Two-Sided Controllability) *Let*

$$T > L\sqrt{\frac{r_0}{S(r_0)}}, \quad r_0 > 1. \tag{3.2}$$

For any given initial data (Y_0, Y_1) *and final data* $(\overline{Y}_0, \overline{Y}_1)$ *with small*

$$\|(|Y_0'| - r_0, Y_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$$

and

$$\|(|\overline{Y}_0'| - r_0, \overline{Y}_1)\|_{C^1[0,L]\times(C^1[0,L])^3},$$

there exist boundary controls $h = (h_1, h_2, h_3)$ *and* $\overline{h} = (\overline{h}_1, \overline{h}_2, \overline{h}_3)$ *with small norms*

$$\|h\|_{(C^0[0,T])^3} \text{ and } \|\overline{h}\|_{(C^0[0,T])^3},$$

such that the mixed initial-boundary value problem (2.2)–(2.3) *and* (2.6)–(2.7) *admits a unique* C^2 *solution* $Y = Y(t,x)$ *with small norm* $\||Y_x| - r_0, Y_t\|_{C^1(\mathcal{R}(T))\times(C^1(\mathcal{R}(T)))^3}$ *on the domain* $\mathcal{R}(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$*, which exactly satisfies the final condition* (3.1) *and* $|Y_x| > 1,\ \forall (t,x) \in \mathcal{R}(T)$.

Theorem 3.2 (One-Sided Controllability) *Let*

$$T > 2L\sqrt{\frac{r_0}{S(r_0)}}, \quad r_0 > 1. \tag{3.3}$$

For any given initial data (Y_0, Y_1) *and final data* $(\overline{Y}_0, \overline{Y}_1)$ *with small* $\|(|Y_0'| - r_0, Y_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$ *and* $\|(|\overline{Y}_0'| - r_0, \overline{Y}_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$*, and for any given boundary condition* (2.6) *on* $x = 0$ *with* $h \equiv 0$*, such that the conditions of* C^2 *compatibility are satisfied at the points* $(t,x) = (0,0)$ *and* $(T,0)$*, respectively, there exist boundary controls* $\overline{h} = (\overline{h}_1, \overline{h}_2, \overline{h}_3)$ *with small norm* $\|\overline{h}\|_{(C^0[0,T])^3}$ *on* $x = L$*, such that the mixed initial-boundary value problem* (2.2)–(2.3) *and* (2.6)–(2.7) *admits a unique* C^2 *solution* $Y = Y(t,x)$ *with small norm* $\||Y_x| - r_0, Y_t\|_{C^1(\mathcal{R}(T))\times(C^1(\mathcal{R}(T)))^3}$ *on the domain* $\mathcal{R}(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$*, which exactly satisfies the final condition* (3.1) *and* $|Y_x| > 1,\ \forall (t,x) \in \mathcal{R}(T)$.

4 Existence and Uniqueness of Semi-global C^2 Solution

In order to obtain the exact boundary controllability for system (2.2) with dynamical boundary conditions (2.6)–(2.7), we should first prove the existence and uniqueness of semi-global C^2 solution $Y = Y(t,x)$ to the corresponding initial-boundary value problem.

To this end, we reduce the second-order system (2.2) to a first-order quasilinear hyperbolic system. Let

$$u = Y_x, \quad v = Y_t. \tag{4.1}$$

System (2.2) can be reduced to the following first-order quasilinear system:

$$\begin{cases} u_t - v_x = 0, \\ v_t - \Big(\dfrac{S(r)}{r} u\Big)_x = 0, \end{cases} \tag{4.2}$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}},\ v = (v_1, v_2, v_3)^{\mathrm{T}}$ and $r = |u|$.

Lemma 4.1 *Under assumptions* (S1)–(S3) *for* $S(r)$, *condition* $|Y_x| > 1$ *guarantees that* (4.2) *is a quasilinear hyperbolic system.*

Proof Let

$$U = (u_1, u_2, u_3, v_1, v_2, v_3)^{\mathrm{T}}, \quad f(r) = \frac{S(r)}{r}.$$

System (4.2) can be rewritten as

$$U_t - AU_x = 0, \tag{4.3}$$

where

$$A = \left(\begin{array}{ccc:ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hdashline -f - \dfrac{f' u_1^2}{|u|} & -\dfrac{f' u_1 u_2}{|u|} & -\dfrac{f' u_1 u_3}{|u|} & 0 & 0 & 0 \\ -\dfrac{f' u_1 u_2}{|u|} & -f - \dfrac{f' u_2^2}{|u|} & -\dfrac{f' u_2 u_3}{|u|} & 0 & 0 & 0 \\ -\dfrac{f' u_1 u_3}{|u|} & -\dfrac{f' u_2 u_3}{|u|} & -f - \dfrac{f' u_3^2}{|u|} & 0 & 0 & 0 \end{array}\right). \tag{4.4}$$

It is easy to see that the characteristic equation of the first-order system (4.3) is

$$0 = |\lambda \mathrm{I} - A| = \Big(\lambda^2 - \frac{S(r)}{r}\Big)^2 (\lambda^2 - S'(r)). \tag{4.5}$$

Noting assumptions (S1)–(S3), when $r > 1$, we get the real eigenvalues

$$\lambda_1 = -\sqrt{S'(r)} < \lambda_2 = \lambda_3 = -\sqrt{\frac{S(r)}{r}} < \lambda_4 = \lambda_5 = \sqrt{\frac{S(r)}{r}} < \lambda_6 = \sqrt{S'(r)}, \tag{4.6}$$

and the corresponding left eigenvectors can be taken as

$$\begin{aligned} & l_1(u) = (\sqrt{S'(r)} u; u), \\ & l_2(u) = \Big(-\sqrt{\frac{S(r)}{r}} u_2, \sqrt{\frac{S(r)}{r}} u_1, 0; -u_2, u_1, 0\Big), \\ & l_3(u) = \Big(-\sqrt{\frac{S(r)}{r}} u_3, 0, \sqrt{\frac{S(r)}{r}} u_1; -u_3, 0, u_1\Big), \\ & l_4(u) = \Big(-\sqrt{\frac{S(r)}{r}} u_2, \sqrt{\frac{S(r)}{r}} u_1, 0; u_2, -u_1, 0\Big), \\ & l_5(u) = \Big(-\sqrt{\frac{S(r)}{r}} u_3, 0, \sqrt{\frac{S(r)}{r}} u_1; u_3, 0, u_1\Big), \\ & l_6(u) = (\sqrt{S'(r)} u; -u) \end{aligned} \tag{4.7}$$

with $u = (u_1, u_2, u_3)$, which compose a complete set of left eigenvectors. Hence (4.2) is a hyperbolic system.

Noting (4.1), the initial condition (2.3) can be rewritten as

$$t = 0: \ (u, v) = (\phi, \psi) \triangleq (Y_0', Y_1), \quad 0 \le x \le L. \tag{4.8}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}, \psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}} \in (C^1[0, L])^3$ with small norms $\||\phi| - r_0\|_{C^1[0,L]}$ and $\|\psi\|_{(C^1[0,L])^3}$. In particular, at the initial moment, by (2.5) we have

$$|\phi(x)| \ge r_0 > 1, \quad 0 \le x \le L. \tag{4.9}$$

Meanwhile, the dynamical boundary conditions (2.6)–(2.7) can be correspondingly replaced by the following non-local boundary conditions:

$$x = 0: \ v(t, 0) = \psi(0) + \int_0^t S(|u(\tau, 0)|) \frac{u(\tau, 0)}{|u(\tau, 0)|} \, \mathrm{d}\tau + \int_0^t h(\tau) \, \mathrm{d}\tau, \tag{4.10}$$

$$x = L: \ v(t, L) = \psi(L) - \int_0^t S(|u(\tau, L)|) \frac{u(\tau, L)}{|u(\tau, L)|} \, \mathrm{d}\tau + \int_0^t \overline{h}(\tau) \, \mathrm{d}\tau. \tag{4.11}$$

Thus, the original forward mixed problem is reduced to a mixed problem for a first-order quasilinear hyperbolic system associated with related non-local boundary conditions. Moreover, assume that for the original problem (2.2)–(2.3) and (2.6)–(2.7). The conditions of C^2 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively. Then the conditions of C^1 compatibility at the points $(t, x) = (0, 0)$ and $(0, L)$ are satisfied, respectively, for the forward problem (4.2), (4.8) and (4.10)–(4.11). With a similar method as in [9], we can get the existence of semi-global C^1 solution to the forward problem (4.2), (4.8) and (4.10)–(4.11). Then we have the following lemma.

Lemma 4.2 *Under the assumptions given in Section* 1, *for any given* $T > 0$, *suppose that* $\|(|Y_0'| - r_0, Y_1)\|_{C^1[0,L] \times (C^1[0,L])^3}$, $\|h\|_{(C^0[0,T])^3}$ *and* $\|\overline{h}\|_{(C^0[0,T])^3}$ *are small enough* (*depending on* T), *and the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (0, 0)$ *and* $(0, L)$, *respectively. Then, the forward mixed initial-boundary value problem* (2.2)–(2.3) *and* (2.6)–(2.7) *admits a unique semi-global* C^2 *solution* $Y = Y(t, x)$ *with small norm*

$$\|(|Y_x| - r_0, Y_t)\|_{C^1[0,L] \times (C^1[0,L])^3}$$

on the domain $\mathcal{R}(T) = \{(t, x) \mid 0 \le t \le T, \ 0 \le x \le L\}$.

Similarly, the final conditions (3.1) can be rewritten as

$$t = T: \ (u, v) = (\Phi, \Psi) \triangleq (\overline{Y}_0', \overline{Y}_1), \quad 0 \le x \le L, \tag{4.12}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^{\mathrm{T}}, \ \Psi = (\Psi_1, \Psi_2, \Psi_3)^{\mathrm{T}} \in (C^1[0, L])^3$ with small norms $\||\Phi| - r_0\|_{C^1[0,L]}$ and $\|\Psi\|_{(C^1[0,L])^3}$. We have the following lemma.

Lemma 4.3 *Under the assumptions given in Section* 1, *for any given* $T > 0$, *suppose that* $\|(|\overline{Y}_0'| - r_0, \overline{Y}_1)\|_{C^1[0,L] \times (C^1[0,L])^3}$, $\|h\|_{(C^0[0,T])^3}$, $\|\overline{h}\|_{(C^0[0,T])^3}$ *are small enough* (*depending on* T), *and the conditions of* C^2 *compatibility are satisfied at the points* $(t, x) = (T, 0)$ *and* (T, L), *respectively. Then, the backward mixed initial-boundary value problem* (2.2), (2.6)–(2.7) *and* (3.1) *admits a unique semi-global* C^2 *solution* $Y = Y(t, x)$ *with small norm* $\|(|Y_x| - r_0, Y_t)\|_{C^1(\mathcal{R}(T)) \times (C^1(\mathcal{R}(T)))^3}$ *on the domain* $\mathcal{R}(T) = \{(t, x) \mid 0 \le t \le T, \ 0 \le x \le L\}$.

Remark 4.1 The semi-global C^2 solution $Y = Y(t,x)$ given in Lemmas 4.2–4.3 is always keeping $|Y_x| > 1$ on the domain $\mathcal{R}(T)$, which means that the string is always in extension.

Remark 4.2 (Hidden Regularity) For the semi-global C^2 solution $Y = Y(t,x)$ given in Lemma 4.2 (or Lemma 4.3), if $h(t) \equiv 0$, or more generally, $h \in (C^1[0,T])^3$ with small $C^1[0,T]$ norm, there is a hidden regularity on $x = 0$ that $Y(\cdot,0) \in (C^3[0,T])^3$ with small norm $\|(|Y_x(\cdot,0)| - r_0, Y_t(\cdot,0))\|_{C^1[0,T]\times(C^2[0,T])^3}$.

Remark 4.3 If the boundary condition on $x = 0$ is not of dynamical type, namely, the corresponding boundary condition can be taken as any one of the following boundary conditions:

$$x = 0: \ Y = h(t) \quad \text{(Dirichlet Type)}, \tag{4.13a}$$

$$x = 0: \ Y_x = h(t) \quad \text{(Neumann Type)}, \tag{4.13b}$$

$$x = 0: \ Y_x - bY = h(t) \quad \text{(Third Type)}, \tag{4.13c}$$

or the boundary condition (2.7) on $x = L$ is replaced by any one of the following boundary conditions:

$$x = L: \ Y = \overline{h}(t), \tag{4.14a}$$

$$x = L: \ Y_x = \overline{h}(t), \tag{4.14b}$$

$$x = L: \ Y_x + \overline{b}Y = \overline{h}(t), \tag{4.14c}$$

where $b, \overline{b}$ are positive constants, then the conclusions of Lemmas 4.2–4.3 are still valid. It is worth mentioning that the solution $Y = Y(t,x)$ loses its hidden regularity on the boundary without dynamical boundary conditions.

5 Proof of Theorem 3.1 and Theorem 3.2

In order to prove Theorems 3.1–3.2, by means of the constructive method with modular structure given in [1, 3–4, 9], it suffices to prove the following.

Theorem 5.1 *Under the assumptions given in Theorem* 3.1. *For any given initial data* (Y_0, Y_1) *and final data* $(\overline{Y}_0, \overline{Y}_1)$ *with small norms* $\|(|Y_0'| - r_0, Y_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$ *and* $\|(|\overline{Y}_0'| - r_0, \overline{Y}_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$, *the spatial vibration system* (2.2) *admits a* C^2 *solution* $Y = Y(t,x)$ *with small norm* $\|(|Y_x| - r_0, Y_t)\|_{C^1(\mathcal{R}(T))\times(C^1(\mathcal{R}(T)))^3}$ *on the domain* $\mathcal{R}(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$, *which exactly satisfies the initial condition* (2.3) *and the final condition* (3.1).

Theorem 5.2 *Under the assumptions given in Theorem* 3.2 *and Remark* 4.2. *For any given initial data* (Y_0, Y_1) *and final data* $(\overline{Y}_0, \overline{Y}_1)$ *with small norms* $\|(|Y_0'| - r_0, Y_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$ *and* $\|(|\overline{Y}_0'| - r_0, \overline{Y}_1)\|_{C^1[0,L]\times(C^1[0,L])^3}$, *and for any given boundary function h with small norm* $\|h\|_{(C^0[0,T])^3}$ *on* $x = 0$, *suppose that the conditions of* C^2 *compatibility are satisfied at the points* $(t,x) = (0,0)$ *and* $(T,0)$, *respectively. The system* (2.2) *with the boundary condition* (2.6) *admits a* C^2 *solution* $Y = Y(t,x)$ *with small norm* $\|(|Y_x| - r_0, Y_t)\|_{C^1(\mathcal{R}(T))\times(C^1(\mathcal{R}(T)))^3}$ *on the domain* $\mathcal{R}(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$, *which exactly satisfies the initial condition* (2.3) *and the final condition* (3.1).

The C^2 solution $Y = Y(t,x)$ required in Theorems 5.1–5.2 can be constructed just as in the proof shown in [10], and with the following lemma.

Lemma 5.1 *Under assumptions* (S1)–(S3) *for* $S(r)$, *we have*

$$\det\Big(\frac{\partial}{\partial u}\Big(\frac{S(|u|)u}{|u|}\Big)\Big|_{|u|=r_0}\Big)\neq 0, \tag{5.1}$$

where $\frac{\partial}{\partial u}\big(\frac{S(|u|)u}{|u|}\big)\big|_{|u|=r_0}$ *denotes the Jacobi matrix of* $\frac{S(|u|)u}{|u|}$ *with respect to* $(u_1,u_2,u_3)^{\rm T}$ *when* $|u|=r_0$.

Proof Let

$$f(|u|)=\frac{S(|u|)}{|u|}. \tag{5.2}$$

We have

$$\frac{\partial(f(|u|)u_i)}{\partial u_j}=f(|u|)\delta_{ij}+f'(|u|)\frac{u_iu_j}{|u|}. \tag{5.3}$$

Then, we have

$$\begin{aligned}\det\Big(\frac{\partial}{\partial u}\Big(\frac{S(|u|)u}{|u|}\Big)\Big)&=f^2(f+f'|u|)\\&=\Big(\frac{S(|u|)}{|u|}\Big)^2\Big(\frac{S(|u|)}{|u|}+\Big(\frac{S(|u|)}{|u|}\Big)'|u|\Big)\\&=\frac{S^2(|u|)S'(|u|)}{|u|^3}.\end{aligned} \tag{5.4}$$

Noting $r_0>1$ and when $|u|>1$, by (S2)–(S3) we have

$$S(|u|)>0,\quad S'(|u|)>0,$$

and we get (5.1) immediately.

Finally, we briefly describe the constructive method of the solution required in Theorem 5.2 to get the one-sided boundary controllability.

Proof of Theorem 5.2 By (3.3) and the continuity, there exists an $\varepsilon_0>0$ so small that

$$T>2L\sup_{||u|-r_0|\le\varepsilon_0}\sqrt{\frac{|u|}{S(|u|)}}. \tag{5.5}$$

Let

$$T_1=L\sup_{||u|-r_0|\le\varepsilon_0}\sqrt{\frac{|u|}{S(|u|)}}. \tag{5.6}$$

(i) We first consider the forward mixed initial-boundary value problem for system (2.2) with the initial condition (2.3), the boundary condition (2.6) and the artificial boundary condition as follows:

$$x=L: Y=q(t),$$

where $q=(q_1,q_2,q_3)(t)$ is any given C^2 vector-valued function of t with small $(C^2[0,T_1])^3$ norm, such that the conditions of C^2 compatibility are satisfied at the point $(t,x)=(0,L)$. By

Lemma 4.2 and Remark 4.3, on the domain $\mathcal{R}_f = \{(t,x) \mid 0 \le t \le T_1,\ 0 \le x \le L\}$, there exists a unique C^2 solution $Y = Y_f(t,x)$ satisfying

$$||Y_{fx}| - r_0| \le \varepsilon_0 \quad \text{on } \mathcal{R}_f.$$

(ii) Similarly, we consider the backward mixed initial-boundary value problem for system (2.2) with the final condition (3.1), the boundary condition (2.6) and the following artificial boundary condition:

$$x = L : Y = \overline{q}(t),$$

where $\overline{q} = (\overline{q}_1, \overline{q}_2, \overline{q}_3)(t)$ is any given C^2 vector-valued function of t with small $(C^2[T - T_1, T])^3$ norm, such that the conditions of C^2 compatibility are satisfied at the point(T, L). By Lemma 4.3 and Remark 4.3, on the domain $\mathcal{R}_b = \{(t,x) \mid T - T_1 \le t \le T,\ 0 \le x \le L\}$, there exists a unique C^2 solution $Y = Y_b(t,x)$ satisfying

$$||Y_{bx}| - r_0| \le \varepsilon_0 \quad \text{on } \mathcal{R}_b.$$

(iii) From the above construction on Y_f and Y_b, we can determine the corresponding value of (Y, Y_t, Y_{tt}, Y_x) on $x = 0$:

$$(Y, Y_t, Y_{tt}, Y_x) = \begin{cases} (a(t), a'(t), a''(t), \overline{a}(t)), & 0 \le t \le T_1, \\ (b(t), b'(t), b''(t), \overline{b}(t)), & T - T_1 \le t \le T. \end{cases} \tag{5.7}$$

Next, we first find $c(t) \in (C^3[0,T])^3$ with small $C^3[0,T]$ norm, such that

$$c(t) = \begin{cases} a(t), & 0 \le t \le T_1, \\ b(t), & T - T_1 \le t \le T. \end{cases} \tag{5.8}$$

Thus, on $x = 0$ we get $(Y, Y_t, Y_{tt}) = (c(t), c'(t), c''(t))$.

Noting (5.1) given in Lemma 5.1, by the Implicit Function Theorem, the boundary condition (2.6) can be uniquely rewritten by

$$x = 0 :\ Y_x = \widetilde{\mathbf{G}}(t, Y_{tt}), \tag{5.9}$$

where $\widetilde{\mathbf{G}}$ is a C^1 vector-valued function.

Set

$$\overline{c}(t) = \widetilde{\mathbf{G}}(t, c''(t)). \tag{5.10}$$

Noting the hidden regularity given in Remark 4.2, we have $\overline{c}(t) \in (C^1[0,T])^3$ with small norm $\||\overline{c}| - r_0\|_{C^1[0,T]}$, and

$$\overline{c}(t) = \begin{cases} \overline{a}(t), & 0 \le t \le T_1, \\ \overline{b}(t), & T - T_1 \le t \le T. \end{cases} \tag{5.11}$$

Thus, $(u_{tt}, u_x) = (c''(t), \overline{c}(t))$ satisfies the boundary condition (2.6) on the whole interval $[0,T]$.

(iv) We now change the status of t and x, and consider the rightward mixed initial-boundary value problem for system (2.2) with the initial condition

$$x = 0 :\ Y = c(t), \quad Y_x = \overline{c}(t), \quad 0 \le t \le T \tag{5.12}$$

and the following boundary conditions of Dirichlet type:

$$t = 0: \; Y = Y_0(x), \quad 0 \le x \le L, \tag{5.13}$$

$$t = T: \; Y = \overline{Y}_0(x), \quad 0 \le x \le L. \tag{5.14}$$

Thus, there exists a unique C^2 solution $Y = Y(t, x)$ on the domain $\mathcal{R}(T) = \{(t, x) \mid 0 \le t \le T, \; 0 \le x \le L\}$, and

$$||Y_x| - r_0| < \varepsilon_0 \quad \text{on } \mathcal{R}(T).$$

(v) Obviously, the C^2 solutions $Y = Y(t, x)$ and $Y_f = Y_f(t, x)$ satisfy simultaneously the same system (2.2), the same initial condition (5.12) and the same boundary condition (5.13). Noting the choice of T_1 given by (5.6), the domain

$$\left\{(t, x) \,\middle|\, 0 \le t \le \frac{T_1}{L}(L - x), \; 0 \le x \le L\right\} \tag{5.15}$$

is included inside the maximum determinate domain of the corresponding rightward one-sided mixed initial boundary value problem. By uniqueness of C^2 solutions to the one-sided mixed initial boundary value problem (see [1, 7]), $Y \equiv Y_f$ on the domain (5.15), in particular, on the interval $0 \le x \le L$ on the x-axis. Hence, $Y = Y(t, x)$ satisfies the initial condition (2.3). In a similar manner we obtain that $Y = Y(t, x)$ satisfies the final condition (3.1).

Thus, we obtain a solution $Y = Y(t, x)$ satisfying all the requirements of Theorem 5.2.

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