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Local Strong Solutions for the Compressible Non-Newtonian Models with Density-Dependent Viscosity and Vacuum*

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Abstract The one-dimensional compressible non-Newtonian models are considered in this paper. The extra-stress tensor in our models satisfies a kind of power law structure which was proposed by O. A. Ladyzhenskaya in 1970s. In particular, the viscosity coefficient in our models depends on the density. By using energy-estimate, the authors obtain the existence and uniqueness of local strong solutions for which the density is non-negative.

Keywords Compressible non-Newtonian fluid, Density dependent, Vacuum
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1 Introduction

In this paper, we consider the density-dependent compressible non-Newtonian models in one-dimensional bounded domain

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2)_x-(\mu(\rho)(u_x+|u_x|^{p-2}u_x))_x+P_x=\rho f, \quad (x,t)\in(0,1)\times(0,T),\\ P=A\rho^\gamma, \quad A>0,\ \gamma>1\end{cases} \tag{1.1}$$

with the initial boundary conditions:

$$(\rho,u)|_{t=0}=(\rho_0,u_0), \quad x\in I=[0,1]; \quad u|_{x=0}=u|_{x=1}=0, \quad t\in[0,T], \tag{1.2}$$

where $\mu(\rho)=1+\rho^\alpha$, and the unknown variables $\rho,\ u,\ P$ stand for the fluid density, velocity and pressure, respectively. The constants $A,\ p>2,\ 0<\alpha<1$ are given. For simplicity, it is assumed that $A=1$. External force f and initial value $\rho_0,\ u_0$ satisfy the following regularity conditions:

$$\begin{cases}0\le\rho_0^\alpha\in H^1(I), \quad u_0\in H_0^1(I)\bigcap H^2(I),\\ f\in L^2([0,T];\ L^{\frac{2r}{r-1}}(I))\bigcap H_0^1(I)\bigcap L^\infty([0,T];\ L^2(I)), \quad f_t\in L^\infty([0,T];\ L^2(I)).\end{cases} \tag{1.3}$$

The research of non-Newtonian fluid dynamics involves chemistry, biology, glaciology, geology and other important fields. Its mathematical models have attracted many experts' attention.

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Through studying for the well-posedness of compressible non-Newtonian fluid models, we describe the movement theoretically, and analyze the essential characteristic of non-Newtonian fluid, and also provide theoretical support for the related practical problem.

In 1970s, a new model to study some kinds of non-Newtonian fluid was proposed by Ladyzhenskaya [1], in which the extra-stress tensor $\tau(Du)$ satisfies a kind of power law structure. Further, Málek, Nečas, Rokyta and Růžička summarized and improved power law structure in [2], two of which had been used widely:

$$\tau(Du) = \mu_\infty Du + \mu_0 |Du|^{p-2} Du, \quad \text{Power law};$$
$$\tau(Du) = \mu_\infty Du + \mu_0 (1 + |Du|^2)^{\frac{p-2}{2}} Du, \quad \text{Carreau's law}.$$

In particular, the non-Newtonian model is called Ellis model for $\mu_\infty > 0,\ \mu_0 > 0,\ p > 2$ in Power law structure.

For the incompressible non-Newtonian fluid, the earliest results were obtained by Ladyzhenskaya [1] and Lions [3]. There have been many remarkable researches concerning the well-posedness of the weak solutions in [2, 4–6] and many others.

When density is variable, the models are the compressible non-Newtonian models. The existence of measure-value weak solutions for the models with space dimension $n \geq 2$ was obtained in [7–9]. And Feireisl, Liao, Málek [10] proved global existence of weak solutions for the initial and boundary problem with Carreau's law viscous term. Mamontov [11–12] studied the global existence and regularity estimates of solutions in one and two-dimensional space. Recently, for the models with Carreau's law viscous term, Xu and Yuan [13–14] proved the existence and uniqueness of local strong solutions in one-dimensional space; Fang and Li [15] studied the existence of classical solution for compressible non-Newtonian fluids; Yang and Tong [16] proved the existence and uniqueness of compressible non-Newtonian fluids with Power law viscous term. For more related results, we refer the reader to [17–20]

For the real fluid models, the viscosity coefficient is dependent on the density, so it will be more physically meaningful to study the density-dependent model. In the past decades, there have been a lot of literatures about the mathematical theory of the solutions of compressible Newton fluids with density-dependent viscosity. However, there is little research on the solutions of non-Newtonian fluids with density-dependent viscosity. Fang, Guo and Wang [21] researched the local strong solutions to a compressible non-Newtonian fluid with Carreau's law viscous term and $\mu_\infty = \rho^\alpha,\ \alpha \in (0, 1)$. When $\mu_\infty = 0$ and $0 < \mu_1 \leq \mu_0(\rho) \in C^2([0, \infty)),\ \mu_0(\rho) \leq C\rho^{\beta+1},\ \mu_0'(\rho) \leq C\rho^\beta$ in Carreau's law structure, Chen and Xu [22] proved the existence and uniqueness of the solutions for a class of non-Newtonian fluids.

The aim of this paper is to study density-dependent compressible non-Newtonian fluids with Power law viscous term, which makes the viscosity term to be with the stronger nonlinearity. It should be pointed out that the viscosity coefficient $\mu(\rho)$ is more general, therefore, it is more difficult to establish uniform estimates.

The main results of this paper are as follows.

Theorem 1.1 *Assume that* $p > 2$*, and the initial values* (ρ_0, u_0) *satisfy* (1.3) *and the* *compatibility condition*

$$-[\mu(\rho_0)(u_{0x}+|u_{0x}|^{p-2}u_{0x})]_x+P_x(\rho_0)=\sqrt{\rho_0}g \quad for\ a.e.\ x\in I \tag{1.4}$$

with some $g\in L^2(I)$*. Then there exists a time* $T_*\in(0,+\infty)$ *and a unique strong solution* (ρ,u) *to the initial boundary problem* (1.1)–(1.2) *such that*

$$\begin{aligned}&0\le\rho,\rho^\alpha\in C([0,T];H^1(I)),\quad \rho_t\in C([0,T];L^2(I)),\quad \sqrt{\rho}u_t\in L^\infty(0,T;L^2(I)),\\&u\in C([0,T];H_0^1(I))\cap L^\infty(0,T;H^2(I)),\quad u_t\in L^2(0,T;H_0^1(I)).\end{aligned}\tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, we obtain the existence of local strong solutions to the problem (1.1)–(1.2) with positive density. We complete the proof of Theorem 1.1 in Section 3.

2 A Local Existence for Positive Density

In this section, we assume that ρ_0 is a smooth function and there exists a positive number δ $(0<\delta\ll 1)$ such that $\rho_0\ge\delta$. And we prove the local existence of strong solutions with positive initial densities to the problem (1.1)–(1.2).

2.1 Uniform estimates of the approximate solutions

To prove the theorem, we first construct a sequence of the approximate solutions inductively as follows:

(i) First define $u^0=0$, and assume that $u^{k-1}\in C([0,T];\ W_0^{1,p}(I))\cap L^\infty([0,T];\ H^2(I))$ was defined for $k\ge 1$.

(ii) According to the classical existence theorem of first order hyperbolic conservation law (see [23]) and the parabolic equation (see [24]), we can obtain the unique smooth solution (ρ^k,u^k), satisfying the following approximate system:

$$\rho_t^k+u^{k-1}\rho_x^k+u_x^{k-1}\rho^k=0, \tag{2.1}$$

$$\rho^ku_t^k+\rho^ku^{k-1}u_x^k-(\mu(\rho^k)(u_x^k+|u_x^k|^{p-2}u_x^k))_x+P_x(\rho^k)=\rho^kf \tag{2.2}$$

with the initial and boundary conditions

$$(\rho^k,u^k)|_{t=0}=(\rho_0,u_0),\quad x\in[0,1],\quad u^k|_{x=0}=u^k|_{x=1}=0,\quad t\in[0,T]. \tag{2.3}$$

Here ρ_0 is a smooth function, and $u_0\in H_0^1\cap H^2$ satisfies the compatibility condition (1.4).

Let $K\ge 1$ be a fixed large integer, and let us introduce an auxiliary function $\Phi_K(t)$ defined by

$$\Phi_K(t)=\max_{1\le k\le K}\sup_{0\le s\le t}(1+\|u^k(s)\|_{W_0^{1,p}(I)}+\|\rho^k(s)\|_{H^1(I)}+\|(\rho^k)^\alpha(s)\|_{H^1(I)}+\|\sqrt{\rho^k}u_t^k\|_{L^2(I)}).$$

Firstly, we have the lemma as follows.

Lemma 2.1 *Assume that* (ρ^k, u^k) *is the smooth solution of* (2.1)–(2.2). *There exists* $T \in (0,\ \infty)$, *such that*

$$\|u^k_{xx}\|_{L^2(I)} \le C\Phi_K^{\gamma+3}(t), \tag{2.4}$$

$$\|\rho^k_t\|_{L^2(I)} + \|(\rho^k)^\alpha_t\|_{L^2(I)} \le C\Phi_K^2(t) \tag{2.5}$$

for $1 \le k \le K,\ t \in [0, T]$, *where* C *is independent of* $\delta,\ k$.

Proof From (2.2), we have

$$[\mu(\rho^k)(u^k_x + |u^k_x|^{p-2}u^k_x)]_x = \rho^k u^k_t + \rho^k u^{k-1}u^k_x + P_x(\rho^k) - \rho^k f \tag{2.6}$$

and

$$[\mu(\rho^k)(u^k_x + |u^k_x|^{p-2}u^k_x)]_x = \mu_x(\rho^k)(u^k_x + |u^k_x|^{p-2}u^k_x) + \mu(\rho^k)(u^k_{xx} + (p-1)|u^k_x|^{p-2}u^k_{xx}), \tag{2.7}$$

$$|\mu(\rho^k)(u^k_{xx} + (p-1)|u^k_x|^{p-2}u^k_{xx})| \ge |u^k_{xx}|(1 + |u^k_x|^{p-2}). \tag{2.8}$$

Combining (2.6)–(2.8), we have

$$\begin{aligned}|u^k_{xx}| &\le \frac{1}{1+|u^k_x|^{p-2}}|\rho^k u^k_t + \rho^k u^{k-1}u^k_x + P_x(\rho^k) - \rho^k f| + \frac{1}{1+|u^k_x|^{p-2}}|\mu_x(\rho^k)(u^k_x + |u^k_x|^{p-2}u^k_x)|\\ &\le |\rho^k u^k_t + \rho^k u^{k-1}u^k_x + P_x(\rho^k) - \rho^k f| + |\mu_x(\rho^k)u^k_x|.\end{aligned}$$

Applying Gagliardo-Nirenberg inequation, we have

$$\begin{aligned}\|u^k_{xx}\|_{L^2(I)} &\le \|\rho^k\|^{\frac12}_{L^\infty(I)}\|\sqrt{\rho^k}u^k_t\|_{L^2(I)} + \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^\infty(I)}\|u^k_x\|_{L^2(I)} + \|P_x(\rho^k)\|_{L^2(I)}\\ &\quad + \|\rho^k\|_{L^\infty(I)}\|f\|_{L^2(I)} + \|[(\rho^k)^\alpha]_x\|_{L^2(I)}\|u^k_x\|_{L^\infty(I)}\\ &\le \|\rho^k\|^{\frac12}_{L^\infty(I)}\|\sqrt{\rho^k}u^k_t\|_{L^2(I)} + \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^\infty(I)}\|u^k_x\|_{L^2(I)} + \|P_x(\rho^k)\|_{L^2(I)}\\ &\quad + \|\rho^k\|_{L^\infty(I)}\|f\|_{L^2(I)} + \|[(\rho^k)^\alpha]_x\|_{L^2(I)}\|u^k_x\|^{\frac{p}{p+2}}_{L^p(I)}\|u^k_{xx}\|^{\frac{2}{p+2}}_{L^2(I)}\\ &\le \|\rho^k\|^{\frac12}_{L^\infty(I)}\|\sqrt{\rho^k}u^k_t\|_{L^2(I)} + \|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty(I)}\|u^k_x\|_{L^2(I)} + \|P_x(\rho^k)\|_{L^2(I)}\\ &\quad + \|\rho^k\|_{L^\infty(I)}\|f\|_{L^2(I)} + \frac{p}{p+2}\|[(\rho^k)^\alpha]_x\|^{\frac{p+2}{p}}_{L^2(I)}\|u^k_x\|_{L^p(I)} + \frac12\|u^k_{xx}\|_{L^2(I)}\\ &\le C\Phi_K^{\gamma+3}(t) + \frac12\|u^k_{xx}\|_{L^2(I)}.\end{aligned} \tag{2.9}$$

So we obtain

$$\|u^k_{xx}\|_{L^2(I)} \le C\Phi_K^{\gamma+3}(t).$$

Multiplying (2.1) by $\alpha(\rho^k)^{\alpha-1}$, we get that

$$[(\rho^k)^\alpha]_t + [(\rho^k)^\alpha]_x u^{k-1} + \alpha(\rho^k)^\alpha u^{k-1}_x = 0. \tag{2.10}$$

Applying (2.1) and (2.10) we deduce that

$$\|[(\rho^k)]_t\|_{L^2(I)} \le \|\rho^k_x(t)\|_{L^2(I)}\|u^{k-1}(t)\|_{L^\infty(I)} + \|\rho^k\|_{L^\infty(I)}\|u^{k-1}_x\|_{L^2(I)} \le \Phi_K^2(t),$$

$$\|[(\rho^k)^\alpha]_t\|_{L^2(I)} \le \|[(\rho^k)^\alpha(I)]_x(t)\|_{L^2(I)}\|u^{k-1}(t)\|_{L^\infty(I)} + \alpha\|(\rho^k)^\alpha\|_{L^\infty(I)}\|u^{k-1}_x\|_{L^2(I)} \le \Phi_K^2(t).$$

This completes the proof of Lemma 2.1.

Next, we estimate the first term of $\Phi_K(t)$.

Lemma 2.2 *Assume that* (ρ^k, u^k) *is the smooth solution of* (2.1)–(2.2). *There exists* $T \in (0,\ \infty)$, *such that*

$$\int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2(I)}^2 \mathrm{d}s + \|u_x^k\|_{L^p(I)}^p \le C\Big(1 + \int_0^t \Phi_K^{2p(\gamma+1)+4}(s)\mathrm{d}s\Big)$$

for all $t \in [0, T]$.

Proof Multiplying (2.2) by u_t^k and integrating over $[0,1]$, we have

$$\begin{aligned} &\|\sqrt{\rho^k} u_t^k\|_{L^2(I)}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mu(\rho^k)|u_x^k|^2\mathrm{d}x + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mu(\rho^k)|u_x^k|^p\mathrm{d}x \\ \le\ & \frac{1}{2}\int_0^1 [\mu(\rho^k)]_t |u_x^k|^2\mathrm{d}x + \frac{1}{p}\int_0^1 [\mu(\rho^k)]_t |u_x^k|^p\mathrm{d}x \\ & - \int_0^1 [\rho^k u^{k-1} u_x^k u_t^k + P_t u_x - (P u_x)_t - \rho^k f u_t^k]\mathrm{d}x. \end{aligned} \tag{2.11}$$

Integrating (2.11) over $[0, t]$, we obtain

$$\begin{aligned} &\int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2(I)}^2\mathrm{d}s + \int_0^1 |u_x^k|^2\mathrm{d}x + \int_0^1 |u_x^k|^p\mathrm{d}x \\ \le\ & C + \int_0^t \|(\rho^k)_t^\alpha\|_{L^2(I)}\|u_{xx}^k\|_{L^2(I)}\|u_x^k\|_{L^p(I)}\mathrm{d}s + \int_0^t \|(\rho^k)_t^\alpha\|_{L^2(I)}\|u_{xx}^k\|_{L^2(I)}^{\frac{p}{2}}\|u_x^k\|_{L^p(I)}\mathrm{d}s \\ & + \int_0^t \|P_x(\rho^k)\|_{L^2(I)}\|u_x^k\|_{L^2(I)}\|u^{k-1}\|_{L^\infty(I)}\mathrm{d}s + \int_0^t \|\sqrt{\rho^k} f\|_{L^2(I)}^2\mathrm{d}s \\ & + \frac{1}{2}\|P(\rho^k)\|_{L^2(I)}^2 + \frac{1}{2}\int_0^1 |u_x^k|^2\mathrm{d}x + \gamma\int_0^t \|P(\rho^k)\|_{L^\infty(I)}\|u_x^{k-1}\|_{L^2(I)}\|u_x^k\|_{L^2(I)}\mathrm{d}s \\ & + \int_0^t \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^\infty(I)}^2\|u_x^k\|_{L^2(I)}^2\mathrm{d}s + \frac{1}{2}\int_0^t \|\sqrt{\rho^k} u_t\|_{L^2(I)}^2\mathrm{d}s. \end{aligned} \tag{2.12}$$

We estimate $\|P(\rho^k)\|_{L^2(I)}^2$ as follows:

$$\begin{aligned} \|P(\rho^k)\|_{L^2(I)}^2 &= \int_0^1 |P(\rho_0)|^2\mathrm{d}x + 2\int_0^t\int_0^1 P(\rho^k)(P(\rho^k))'(-\rho_x^k u^{k-1} - \rho^k u_x^{k-1})\mathrm{d}x\mathrm{d}s \\ &\le C + C\int_0^t \|P(\rho^k)\|_{L^\infty(I)}\|\rho^k(s)\|_{L^\infty(I)}^{\gamma-1}\|\rho^k(s)\|_{H^1(I)}\|u^{k-1}\|_{H^1(I)}\mathrm{d}s \\ &\le C\Big(1 + \int_0^t \Phi_K^{2\gamma+1}(s)\mathrm{d}s\Big). \end{aligned} \tag{2.13}$$

Using Lemma 2.1 and (2.13), we can obtain

$$\int_0^t \|\sqrt{\rho^k} u_t^k\|_{L^2(I)}^2\mathrm{d}s + \|u_x^k\|_{L^p(I)}^p \le C\Big(1 + \int_0^t \Phi_K^{2p(\gamma+1)+4}(s)\mathrm{d}s\Big). \tag{2.14}$$

The lemma is proved.

Then, we will estimate $\|\rho^k(t)\|_{H^1(I)}, \|(\rho^k)^\alpha(t)\|_{H^1(I)}$.

Lemma 2.3 *Assume that* (ρ^k, u^k) *is the solution of* (2.1)–(2.2). *We have*

$$\|\rho^k(t)\|_{H^1(I)} + \|(\rho^k)^\alpha(t)\|_{H^1(I)} \le C\exp\Big\{\int_0^t \Phi_K^{\gamma+3}\mathrm{d}s\Big\} \tag{2.15}$$

for any $t \in [0, T]$.

Proof Multiplying (2.1) by ρ^k and integrating over $(0,1)$, there exists $T\in(0,\infty)$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1|\rho^k(t)|^2\mathrm{d}x+\int_0^1(\rho^k u^{k-1})_x\rho^k(t)\mathrm{d}x=0. \tag{2.16}$$

Then, we can estimate the left-hand side of (2.16) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1|\rho^k(t)|^2\mathrm{d}x\le\int_0^1|\rho^k|^2|u_x^{k-1}|(t)\mathrm{d}x\le C\|u_{xx}^{k-1}\|_{L^2}\|\rho^k\|_{L^2}^2. \tag{2.17}$$

Differentiating (2.1) with respect to x, and multiplying the resultant equation by ρ_x^k, and integrating it over $(0,1)$ on x, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1|\rho_x^k(t)|^2\mathrm{d}x&=-\int_0^1\Big[\frac{3}{2}u_x^{k-1}(\rho_x^k)^2+\rho^k\rho_x^k u_{xx}^{k-1}\Big]\mathrm{d}x\\&\le C(\|u_{xx}^{k-1}\|_{L^2(I)}\|\rho_x^k\|_{L^2(I)}^2+\|\rho^k\|_{L^\infty(I)}\|\rho_x^k\|_{L^2(I)}\|u_{xx}^{k-1}\|_{L^2(I)}).\end{aligned} \tag{2.18}$$

Combining (2.17)–(2.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^k(t)\|_{H^1(I)}^2\le C\|\rho^k\|_{H^1(I)}^2\|u_{xx}^{k-1}\|_{L^2(I)}. \tag{2.19}$$

From Gronwall's inequality, we get

$$\|\rho^k(t)\|_{H^1(I)}\le\|\rho_0\|_{H^1(I)}^2\exp\Big(C\int_0^t\|u_{xx}^{k-1}\|_{L^2(I)}\mathrm{d}s\Big)\le C\exp\Big(\int_0^t\Phi_K^{\gamma+3}\mathrm{d}s\Big). \tag{2.20}$$

Applying (2.10), we can obtain the estimate $\|(\rho^k)^\alpha(t)\|_{H^1(I)}$ using the same method for the estimate of $\|\rho^k(t)\|_{H^1(I)}$ as follows:

$$\|(\rho^k)^\alpha(t)\|_{H^1(I)}\le C\exp\Big(\int_0^t\Phi_K^{\gamma+3}\mathrm{d}s\Big). \tag{2.21}$$

The lemma is proved.

Following that, we will estimate $\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}^2$.

Lemma 2.4 *Assume that (ρ^k,u^k) is the solution of (2.1)–(2.2), and satisfies compatibility condition (1.4). There exists $T\in(0,\infty)$, such that*

$$\begin{aligned}&\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}^2+\int_0^t\|u_{xt}^k\|_{L^2(I)}^2\mathrm{d}s+\int_0^t\int_0^1|u_x^k|^{p-2}|u_{xt}^k|^2\mathrm{d}x\mathrm{d}s\\&\le C\Big(1+\int_\tau^t\Phi_K^{2p(\gamma+3)}(s)\mathrm{d}s\Big)\end{aligned} \tag{2.22}$$

for $1\le k\le K,\ t\in[0,T]$, where C is independent of $\delta,\ k$.

Proof Differentiating (2.2) with respect to t, multiplying this by u_t^k and integrating over $(0,1)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\rho^k|u_t^k|^2\mathrm{d}x+\int_0^1\mu(\rho^k)[(u_{xt}^k)^2+(p-1)|u_x^k|^{p-2}(u_{xt}^k)^2]\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_0^1 \mu_t(\rho^k)(u_x^k+|u_x^k|^{p-2}u_x^k)u_{xt}^k\mathrm{d}x + \int_0^1(\rho^k u^{k-1})_x[(u_t^k)^2 - u^{k-1}u_x^k u_t^k - fu_t^k]\mathrm{d}x\\
&\quad + \int_0^1 P_t(\rho^k)u_{xt}^k\mathrm{d}x - \int_0^1 \rho^k u_t^{k-1}u_t^k u_x^k\mathrm{d}x + \int_0^1 \rho^k f_t u_t^k\mathrm{d}x\\
&\le \int_0^1 |P_x(\rho^k)||u^k||u_{xt}^k|\mathrm{d}x + \gamma\int_0^1 |P(\rho^k)||u_x^k||u_{xt}^k|\mathrm{d}x + 2\int_0^1 |\rho^k||u^{k-1}||u_t^k||u_{xt}^k|\mathrm{d}x\\
&\quad + \int_0^1 |\rho^k||u^{k-1}|^2|u_x^k||u_{xt}^k|\mathrm{d}x + \int_0^1 |\rho^k||u^{k-1}||u_t^k||u_x^{k-1}||u_x^k|\mathrm{d}x + \int_0^1 |\rho^k||u^{k-1}|^2|u_t^k||u_{xx}^k|\mathrm{d}x\\
&\quad + \int_0^1 |\rho^k||u_t^k||u_t^{k-1}||u_x^k|\mathrm{d}x + \int_0^1 |\rho^k||u^{k-1}||u_t^k||f_x|\mathrm{d}x + \int_0^1 |\rho^k||u^{k-1}||u_{xt}^k||f|\mathrm{d}x\\
&\quad + \int_0^1 |\rho^k||f_t||u_t^k|\mathrm{d}x + \int_0^1 |[(\rho^k)^\alpha]_t||u_x^k||u_{xt}^k|\mathrm{d}x + \int_0^1 |[(\rho^k)^\alpha]_t||u_x^k|^{p-1}|u_{xt}^k|\mathrm{d}x\\
&= \sum_{j=1}^{12}\mathrm{I}_j. \qquad (2.23)
\end{aligned}
$$

Applying Young's inequality and Sobolev inequality, we obtain

$$
\begin{aligned}
\mathrm{I}_1 &\le \|P_x(\rho^k)\|_{L^2(I)}\|u^k\|_{L^\infty(I)}\|u_{xt}^k\|_{L^2(I)} \le C_\varepsilon\Phi_K^{2\gamma+2}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_2 &\le \|P(\rho^k)\|_{L^\infty(I)}\|u_x^k\|_{L^p(I)}\|u_{xt}^k\|_{L^2(I)} \le C_\varepsilon\Phi_K^{2\gamma+2}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_3 &\le \|\rho^k\|_{L^\infty(I)}^{\frac12}\|u^{k-1}\|_{L^\infty(I)}\|\sqrt{\rho^k}u_t^k\|_{L^2(I)}\|u_{xt}^k\|_{L^2(I)} \le \Phi_K^{\frac52}(t)\|u_{xt}^k\|_{L^2(I)}\\
&\le C_\varepsilon\Phi_K^5(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_4 &\le \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^6(I)}^2\|u_{xt}^k\|_{L^2(I)}\|u_x^k\|_{L^6(I)} \le C_\varepsilon\Phi_K^{2\gamma+12}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_5 &\le \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^6(I)}\|u_t^k\|_{L^3(I)}\|u_x^{k-1}\|_{L^3(I)}\|u_t^k\|_{L^6(I)} \le C_\varepsilon\Phi_K^{2\gamma+12}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_6 &\le \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^6(I)}^2\|u_t^k\|_{L^3(I)}\|u_{xx}^k\|_{L^2(I)} \le \Phi_K^{\gamma+6}(t)\|u_{xt}^k\|_{L^2(I)}\\
&\le C_\varepsilon\Phi_K^{2\gamma+12}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_7 &\le \|\rho^k\|_{L^\infty(I)}^{\frac12}\|u_t^{k-1}\|_{L^6(I)}\|u_x^k\|_{L^2(I)}\|\sqrt{\rho^k}u_t^k\|_{L^3(I)} \le C\Phi_K^{\frac94}(t)\|u_{xt}^{k-1}\|_{L^2(I)}\|u_{xt}^k\|_{L^2(I)}^{\frac12}\\
&\le C_{\frac12}\Phi_K^{\frac92}(t)\|u_{xt}^k\|_{L^2(I)} + \frac12\|u_{xt}^{k-1}\|_{L^2(I)}^2 \le C_{\frac12,\eta}\Phi_K^9(t) + \eta\|u_{xt}^k\|_{L^2(I)}^2 + \frac12\|u_{xt}^{k-1}\|_{L^2(I)}^2,\\
\mathrm{I}_8 &\le \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^6(I)}\|u_t^k\|_{L^3(I)}\|f_x\|_{L^2(I)} \le C_\varepsilon\Phi_K^4(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_9 &\le \|\rho^k\|_{L^\infty(I)}\|u^{k-1}\|_{L^6}\|u_{xt}^k\|_{L^3(I)}\|f\|_{L^2(I)} \le C\Phi_K^2(t)\|u_{xt}^k\|_{L^2(I)} \le C_\varepsilon\Phi_K^4(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_{10} &\le \|\rho^k\|_{L^6(I)}\|f_t\|_{L^2(I)}\|u_t^k\|_{L^3(I)} \le C_\varepsilon\Phi_K^2(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2,\\
\mathrm{I}_{11} &\le \|[(\rho^k)^\alpha]_t\|_{L^2(I)}\|u_x^k\|_{L^\infty(I)}\|u_{xt}^k\|_{L^2(I)} \le C\Phi_K^3(t)\|u_{xt}^k\|_{L^2(I)} \le C_\varepsilon\Phi_K^6(t) + \varepsilon\|u_{xt}^k\|_{L^2}^2,\\
\mathrm{I}_{12} &\le \|[(\rho^k)^\alpha]_t\|_{L^2(I)}\|u_x^k\|_{L^\infty(I)}^{p-1}\|u_{xt}^k\|_{L^2(I)} \le \Phi_K^{p+1}(t)\|u_{xt}^k\|_{L^2(I)} \le C_\varepsilon\Phi_K^{2(p+1)}(t) + \varepsilon\|u_{xt}^k\|_{L^2(I)}^2.
\end{aligned}
$$

Taking $\varepsilon, \eta > 0$ small enough, substituting these estimates into (2.23), and integrating over $(\tau, t) \subset (0, t)$, we can obtain

$$
\begin{aligned}
&\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}^2 + \int_\tau^t \|u_{xt}^k\|_{L^2(I)}^2(s)\mathrm{d}s + \int_\tau^t\int_0^1 |u_x^k|^{p-2}|u_{xt}^k|^2\mathrm{d}x\mathrm{d}s\\
&\le \|\sqrt{\rho^k}u_t^k(\tau)\|_{L^2(I)}^2 + C\int_\tau^t \Phi_K^{2p(\gamma+3)}(s)\mathrm{d}s + \frac12\int_\tau^t \|u_{xt}^{k-1}\|_{L^2(I)}^2\mathrm{d}s. \qquad (2.24)
\end{aligned}
$$

Applying (2.2) and compatibility condition (1.4), letting $\tau \to 0$, we have

$$\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}^2 \le C\Big(1+\int_\tau^t \Phi_K^{2p(\gamma+3)}(s)\mathrm{d}s\Big).$$

So we obtain

$$\begin{aligned}&\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}^2+\int_\tau^t \|u_{xt}^k\|_{L^2(I)}^2(s)\mathrm{d}s+\int_\tau^t\int_0^1 |u_x^k|^{p-2}(u_{xt}^k)^2\mathrm{d}x\mathrm{d}s\\ &\le C\Big(1+\int_\tau^t \Phi_K^{2p(\gamma+3)}(s)\mathrm{d}s\Big)\end{aligned} \tag{2.25}$$

for $1 \le k \le K$. We complete the proof of lemma.

Applying Lemmas 2.2–2.4, we can deduce that

$$\Phi_K(t) \le C\exp\Big(\int_0^t \Phi_K^{2p(\gamma+3)}(s)\mathrm{d}s\Big). \tag{2.26}$$

Thanks to this integral inequality, we can easily show that there exists a time $T^* \in (0,T)$ depending only on initial value and parameters of C such that

$$\sup_{0\le t\le T^*} \Phi_K(t) \le C. \tag{2.27}$$

So, we have the estimate as follows:

$$\begin{aligned}&\sup_{0\le t\le T^*}(\|\rho^k(t)\|_{H^1(I)}+\|(\rho^k)^\alpha(t)\|_{H^1(I)}+\|u^k(t)\|_{W_0^{1,p}(I)\bigcap H^2(I)}+\|\sqrt{\rho^k}u_t^k(t)\|_{L^2(I)}\\ &+\|\rho_t^k(t)\|_{L^2(I)}+\|(\rho^k)_t^\alpha(t)\|_{L^2(I)})+\int_0^{T^*}\|u_{xt}^k\|_{L^2(I)}^2\mathrm{d}s+\int_0^{T^*}\int_0^1|u_x^k|^{p-2}|u_{xt}^k|^2\mathrm{d}x\mathrm{d}s\\ &\le C.\end{aligned} \tag{2.28}$$

Remark 2.1 The C in (2.28) is independent of the lower bound of density δ, so it also have estimate (2.28) for vacuum state.

2.2 Convergence of approximate solutions

In this section, we will show the approximate solution (ρ^k, u^k) to be strong convergence. Set

$$\overline{\rho}^{k+1}=\rho^{k+1}-\rho^k, \quad \overline{u}^{k+1}=u^{k+1}-u^k.$$

Then $(\overline{\rho}^{k+1},\overline{u}^{k+1})$ satisfies the following system:

$$\overline{\rho}_t^{k+1}+(\overline{\rho}^{k+1}u^k)_x+(\rho^k\overline{u}^k)_x=0, \tag{2.29}$$

$$\begin{aligned}&\rho^{k+1}\overline{u}_t^{k+1}+\rho^{k+1}u^k\overline{u}_x^{k+1}+P_x(\rho^{k+1})-P_x(\rho^k)-[\mu(\rho^{k+1})(u_x^{k+1}+|u_x^{k+1}|^{p-2}u_x^{k+1})]_x\\ &+[\mu(\rho^k)(u_x^k+|u_x^k|^{p-2}u_x^k)]_x=-\overline{\rho}^{k+1}u_t^k-\overline{\rho}^{k+1}u^ku_x^k-\rho^k\overline{u}^ku_x^k+\overline{\rho}^{k+1}f\end{aligned} \tag{2.30}$$

with the initial boundary conditions

$$(\overline{\rho}^{k+1},\overline{u}^{k+1})|_{t=0}=(\rho_0,u_0), \quad x\in[0,1]; \quad \overline{u}^{k+1}|_{x=0}=\overline{u}^{k+1}|_{x=1}=0, \quad t\in[0,T]. \tag{2.31}$$

Multiplying (2.30) by $\overline{u}^{k+1}$, and integrating it over $(0,\ 1)$ with respect to x, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_0^1 (\overline{u}_x^{k+1})^2\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_0^1 (\overline{u}_x^{k+1})^2\mathrm{d}x + (p-1)\int_0^1 \mu(\rho^{k+1})|\theta u_x^{k+1} \\
&\quad + (1-\theta)u_x^k|^{p-2}(\overline{u}_x^{k+1})^2\mathrm{d}x \\
&\le B^k(t)\|\overline{\rho}^{k+1}\|_{L^2(I)}^2 + C\|\sqrt{\rho^k}\overline{u}^k\|_{L^2(I)}^2, \qquad (2.32)
\end{aligned}$$

where $B^k(t) = C(1+\|u_{xt}^k\|_{L^2(I)}^2)$.

Multiplying (2.29) by $\overline{\rho}^{k+1}$, and integrating it over $(0,1)$ with respect to x, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\rho}^{k+1}\|_{L^2(I)}^2 &\le C\|\overline{\rho}^{k+1}\|_{L^2(I)}^2\|u_x^k\|_{L^\infty(I)} + \|\rho^k\|_{H^1}\|\overline{u}_x^k\|_{L^2(I)}\|\overline{\rho}^{k+1}\|_{L^2(I)} \\
&\le C\|\overline{\rho}^{k+1}\|_{L^2(I)}^2\|u_{xx}^k\|_{L^2(I)} + C_\eta\|\rho^k\|_{H^1(I)}^2\|\overline{\rho}^{k+1}\|_{L^2(I)}^2 + \eta\|\overline{u}_x^k\|_{L^2(I)}^2 \\
&\le D_\eta^k(t)\|\overline{\rho}^{k+1}\|_{L^2(I)}^2 + \eta\|\overline{u}_x^k\|_{L^2(I)}^2, \qquad (2.33)
\end{aligned}$$

where $D_\eta^k(t) = C\|u_{xx}^k\|_{L^2(I)} + C_\eta\|\rho^k\|_{H^1(I)}^2$ for all $t \le T^*$ and $k \ge 1$. And duo to the estimate (2.28), we have

$$\int_0^t B^k(s)\mathrm{d}s \le C(1+t), \quad \int_0^t D_\eta^k(s)\mathrm{d}s \le C + C_\eta t.$$

Combining (2.32) with (2.33), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\|_{L^2(I)}^2 + \|\overline{\rho}^{k+1}(t)\|_{L^2(I)}^2) + \int_0^1 \overline{u}_x^{k+1}\mathrm{d}x \\
&\le E_\eta(t)\|\overline{\rho}^{k+1}(t)\|_{L^2(I)}^2 + C\|\sqrt{\rho^k}\overline{u}^k\|_{L^2(I)}^2 + \eta\|\overline{u}_x^k\|_{L^2(I)}^2, \qquad (2.34)
\end{aligned}$$

where $E_\eta(t)$ depends only on $B^k(t)$ and $D_\eta^k(t)$, and we have

$$\int_0^t E_\eta^k(s)\mathrm{d}s \le C + C_\eta t$$

for all $t \le T^*,\ k \ge 1$. Then integrating (2.34) over $(0,t) \subset (0,T_1)$ and using Gronwall's inequality, we obtain that

$$\begin{aligned}
&(\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\|_{L^2(I)}^2 + \|\overline{\rho}^{k+1}(t)\|_{L^2(I)}^2) + \int_0^1 \overline{u}_x^{k+1}\mathrm{d}x \\
&\le C\exp(C_\eta t)\int_0^t (\|\sqrt{\rho^k}\overline{u}^k(s)\|_{L^2(I)}^2 + \|\overline{u}^k(s)\|_{L^2(I)}^2)\mathrm{d}s.
\end{aligned}$$

Using recursive relation and Gronwall's inequality, we deduce that

$$\sum_{k=1}^K \Big[\sup_{0\le t\le T^*}(\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\|_{L^2(I)}^2 + \|\overline{\rho}^{k+1}(t)\|_{L^2(I)}^2) + \int_0^{T_1}\|\overline{u}^{k+1}(t)\|_{L^2(I)}^2\mathrm{d}t\Big] \le C. \qquad (2.35)$$

Considering (2.1), we can obtain that

$$\rho^k(t,x) \ge \delta\exp\Big\{-\int_0^T \|u_x^{k-1}(x,s)\|_{L^\infty(I)}\mathrm{d}s\Big\} > 0.$$

So we have $\rho^{k+1} \geq \delta C^{-1}$.

Combining (2.35), we deduce that (ρ^k, u^k) converges to (ρ, u) in the following sense:

$$\begin{cases} u^k \to u & \text{in } L^\infty(0,T^*;L^2(I)) \cap L^2(0,T^*;H_0^1(I)), \\ \rho^k \to \rho & \text{in } L^\infty(0,T^*;L^2(I)) \end{cases} \tag{2.36}$$

as $k \to \infty$.

By virtue of the lower semi-continuity of various norms, we deduce that (ρ, u) satisfies the following uniform estimate:

$$\begin{aligned} &\sup_{0\leq t\leq T^*} (\|\rho(t)\|_{H^1(I)} + \|\rho^\alpha(t)\|_{H^1(I)} + \|u(t)\|_{W_0^{1,p}\cap H^2(I)} \\ &+ \|\sqrt{\rho}u_t(t)\|_{L^2(I)} + \|\rho_t(t)\|_{L^2(I)}) + \int_0^{T^*} \|u_{xt}(s)\|^2_{L^2(I)} \mathrm{d}s \leq C. \end{aligned}$$

3 Proof of Theorem 1.1

In Subsection 2.1, we obtain the high order regularity estimations of solution. Next we will prove the existence of strong solution. We will finish the proof by two steps of $k \to \infty$ and $\delta \to 0^+$.

For $k \to \infty$. Since (ρ^k, u^k) is a smooth solution of (2.1)–(2.2), it satisfies the following identities:

$$\int_0^1 \rho^k(x,t)\varphi(x,t)\mathrm{d}x - \int_0^t\int_0^1 (\rho^k\varphi_t + \rho^k u^k\varphi_x)(x,s)\mathrm{d}x\mathrm{d}s = \int_0^1 \rho_0\varphi(x,0)\mathrm{d}x, \tag{3.1}$$

where $\varphi \in C([0,T^*];H^1(I)), \varphi_t \in L^\infty([0,T^*];L^2(I))$, and

$$\begin{aligned} &\int_0^1 \rho^k(x,t)u^k(x,t)\phi(x,t)\mathrm{d}x - \int_0^t\int_0^1 (\rho^k u^k\phi_t + \rho^k(u^k)^2\phi_x \\ &- \mu(\rho^k)(u_x^k + |u_x^k|^{p-2}u_x^k)\phi_x + P(\rho^k)\phi_x)(x,s)\mathrm{d}x\mathrm{d}s = \int_0^1 \rho_0 u_0\phi(x,0)\mathrm{d}x, \end{aligned} \tag{3.2}$$

where $\varphi \in C([0,T^*];H^1(I)) \cap L^\infty([0,T^*];H^2(I)),\ \varphi_t \in L^2([0,T^*];H_0^1(I))$.

Let

$$\begin{aligned} \sum_{i=1}^3 I_i^k &= \int_0^1 (\rho^k - \rho)\varphi\mathrm{d}x - \int_0^t\int_0^1 [(\rho^k-\rho)\varphi_t + (\rho^k u^k - \rho u)\varphi_x](x,s)\mathrm{d}x\mathrm{d}s, \\ \sum_{i=1}^6 W_i^k &= \int_0^1 (\rho^k u^k - \rho u)\phi(x,t)\mathrm{d}x - \int_0^t\int_0^1 [(\rho^k u^k - \rho u)\phi_t + (\rho^k(u^k)^2 - \rho u^2)\phi_x \\ &\quad - (\mu(\rho^k)u_x^k - \mu(\rho)u_x)\phi_x - (\mu(\rho^k)|u_x^k|^{p-2}u_x^k - \mu(\rho)|u_x|^{p-2}u_x)\phi_x \\ &\quad + (P(\rho^k) - P(\rho))\phi_x](x,s)\mathrm{d}x\mathrm{d}s. \end{aligned}$$

In a similar way as the method of proof of existence in [25], we prove that, as $k \to \infty$,

$$\begin{cases} I_i^k \to 0, & i = 1,2,3, \\ W_i^k \to 0, & i = 1,2,3,4,5,6. \end{cases} \tag{3.3}$$

So (ρ, u) satisfies (3.1)–(3.2).

Secondly, for $\delta \to 0^+$. Let $\rho_0^\delta = \rho_0 + \delta \geq \delta > 0$ and $u_\delta^0 \in H_0^1 \cap H^2$ satisfies the compatibility condition (1.4). From the conclusion of Subsection 2.1, we obtain that there exists $T^* > 0$ such that the initial-boundary problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - [\tau(u_x)]_x + P_x = \rho f, \\ (\rho, u)|_{t=0} = (\rho_0^\delta, u_0^\delta), \quad u|_{x=0} = u|_{x=1} = 0, \end{cases}$$

exist a unique solution (ρ^δ, u^δ) which satisfies the uniform estimate as follows

$$\sup_{0 \leq t \leq T^*} (\|\rho^\delta(t)\|_{H^1(I)} + \|(\rho^\delta)^\alpha(t)\|_{H^1(I)} + \|u^\delta(t)\|_{W_0^{1,p}(I) \cap H^2(I)}$$
$$+ \|\sqrt{\rho^\delta} u_t^\delta(t)\|_{L^2(I)} + \|\rho_t^\delta(t)\|_{L^2(I)}) + \int_0^{T^*} \|u_{xt}^\delta(s)\|_{L^2(I)}^2 \mathrm{d}s \leq C,$$

where C is a positive constant and is independent of δ. So we have the following strong convergence:

$$\begin{cases} \rho^\delta \to \rho \quad \text{in } L^\infty(0, T^*; L^2(I)), \\ u^\delta \to u \quad \text{in } L^\infty(0, T^*; L^2(I)) \cap L^2(0, T^*; H_0^1(I)) \end{cases}$$

as $\delta \to 0^+$. (ρ, u) satisfies the following uniform estimate:

$$\sup_{0 \leq t \leq T^*} (\|\rho(t)\|_{H^1(I)} + \|\rho^\alpha(t)\|_{H^1(I)} + \|u(t)\|_{W_0^{1,p}(I) \cap H^2(I)}$$
$$+ \|\sqrt{\rho} u_t(t)\|_{L^2(I)} + \|\rho_t(t)\|_{L^2(I)}) + \int_0^{T^*} \|u_{xt}(s)\|_{L^2(I)}^2 \mathrm{d}s \leq C.$$

The existence of solution in Theorem 1.1 is proved. Furthermore, we can obtain the uniqueness of strong solution using the same method as in the Subsection 2.2.

Theorem 1.1 is proved.

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