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Nonlocal Symmetries of the Camassa-Holm Type Equations*

Lu ZHAO[1] Changzheng QU[1]

Abstract A class of nonlocal symmetries of the Camassa-Holm type equations with bi-Hamiltonian structures, including the Camassa-Holm equation, the modified Camassa-Holm equation, Novikov equation and Degasperis-Procesi equation, is studied. The nonlocal symmetries are derived by looking for the kernels of the recursion operators and their inverse operators of these equations. To find the kernels of the recursion operators, the authors adapt the known factorization results for the recursion operators of the KdV, modified KdV, Sawada-Kotera and Kaup-Kupershmidt hierarchies, and the explicit Liouville correspondences between the KdV and Camassa-Holm hierarchies, the modified KdV and modified Camassa-Holm hierarchies, the Novikov and Sawada-Kotera hierarchies, as well as the Degasperis-Procesi and Kaup-Kupershmidt hierarchies.

Keywords Nonlocal symmetry, Recursion operator, Camassa-Holm equation, Modified Camassa-Holm equation, Novikov equation, Degasperis-Procesi equation, Liouville correspondence
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1 Introduction

A remarkable property to integrable systems is the existence of an infinite number of generalized symmetries (also Lie-Bäcklund symmetres) (see [30, 33–34]), which is relevant to the existence of an infinite number of conservation laws due to the Noether theorem (see [34]). Such property is related closely to integrable properties of the integrable systems such as the Lax-pair and bi-Hamiltonian structure etc. A simple and effective method to obtain an infinite number of symmetries of integrable equations is to look for their recursion operators. In a number of papers, this property has been employed to classify integrable equations of certain forms so as to obtain a large classes of new integrable equations.

Beyond the generalized symmetries, integrable equations also admit various nonlocal symmetries, which are an extension of the local Lie symmetry and generalized symmetry, they usually depend upon the integral of solutions and eigenfunction functions of the Lax-pair. For instance, the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0 \tag{1.1}$$

Manuscript received October 30, 2017.
[1]Center for Nonlinear Studies and Department of Mathematics, Ningbo University, Ningbo 315211, Zhejiang, China. E-mail: quchangzheng@nbu.edu.cn 18892614926@163.com
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admits the nonlocal symmetry $\sigma = (\Psi^2)_x$, where Ψ is the eigenfunction of the isospectral problem of the KdV equation. This symmetry can be used to reduce the KdV equation to a finite-dimensional dynamical system. The recursion operators of integrable equations can also be used to obtain nonlocal symmetries of the KdV equation (see [16–18, 27–29, 43]).

Some non-integrable nonlinear evolution equations such as the nonlinear diffusion equations also admit nonlocal symmetries. The nonlocal symmetries were used by Bluman et al. [2–3] to linearize nonlinear diffusion equations, and by Akhatov et al. [1] to perform classification of symmetries and obtain exact solutions of nonlinear diffusion equations. Some systems of nonlinear diffusion equations were proved to admit nonlocal symmetries (see [38]), which can be used to linearize systems of nonlinear diffusion equations (see [2]). A geometric formulation of nonlocal symmetries was formulated by Vinogradov and Krasilshchik [45]. There are a number of papers to consider other different kinds of nonlocal symmetries of nonlinear evolution equations, see [19, 21–22, 35, 37–38, 40] and the references therein.

The integrable Camassa-Holm (CH for short) type equations (see [4–5, 7, 10, 12, 15, 32, 36]) have attracted much attention in recent years because of their several remarkable properties. Those equations are closely related to the well-known classical integrable systems. For example, the CH equation and modified CH (mCH for short) equation can be transformed to first equation respectively in negative hierarchies of the KdV and mKdV equations via Liouville transformations (see [20, 26, 31]). The nonlocal symmetries of the CH equation depending on the eigenfunctions of the isospectral problem have been discussed in [39, 41]. It seems there are no such nonlocal symmetries for the mCH equation, Novikov equation and Degasperis-Procesi (DP for short) equations (see [44]). In addition, these equations admit only the trivial Lie point symmetries, translation for time t and space variable x and a dilation for time t and dependent variable. As a class of typical integrable equations, they should possess rich symmetry groups. Therefore, it is of great interest to explore other symmetries of the CH-type equations.

The goal of this paper is to investigate the nonlocal symmetries of the CH-type equations including CH equation, the mCH equation, Novikov equation and DP equation. This work is motivated by three observations. First, in light of the factorization of the recursion operators and their inverse operators, some nonlocal symmetries of the KdV equation, mKdV equation, Sawada-Kotera equation (see [42]) and Kaup-Kupershmidt equation (see [25]) can be derived by seeking for the kernels of the recursion operators and their inverse operators, see the references [18, 27–29] and therein. Second, recent studies [23–24] show that the CH equation (see [4, 11]), mCH equation (see [12–13, 36]), Novikov equation (see [20, 32]) and DP equation (see [9–10]) can be mapped into respectively the first ones in the KdV, mKdV, Sawada-Kotera and Kaup-Kupershmidt hierarchies (see [23–24, 26, 31]). It turns out that the recursion operators or their dual recursion operator of the CH, mCH, Novikov and DP hierarchies are interconnected with the recursion operators of the KdV, mKdV, Sawada-Kotera and Kaup-Kupershmidt hierarchies. Using the factorization of recursion operators of the KdV, mKdV, SK and KK hierarchies and their relationships with the CH, mCH, Novikov and DP hierarchies. Consequently, we obtain nonlocal symmetries of these equations.

The outline of this paper is as follows. In Section 2, we construct nonlocal symmetries of the

CH equation by using the results in [26, 31] and the factorization of the recursion operator of the KdV equation. The nonlocal symmetries for the modified CH equation are found by employing the results in [23] and the factorization of the recursion operator of the mKdV equation (see [28]). The nonlocal symmetries for the Novikov and DP equations will be discussed respectively in Sections 4–5 by using the results in [24] and [18, 27].

2 Nonlocal Symmetries of the CH Equation

The CH equation

$$m_t = 2u_x m + u m_x, \quad m = u - u_{xx} \tag{2.1}$$

can be expressed as the bi-Hamiltonian system

$$m_t = \mathcal{K}\,\frac{\delta H_1[u]}{\delta m} = \mathcal{J}\,\frac{\delta H_2[u]}{\delta m}, \tag{2.2}$$

with the Hamiltonian functionals

$$\begin{aligned} H_1[u] &= \frac{1}{2}\int (u^2 + u_x^2)\mathrm{d}x, \\ H_2[u] &= \frac{1}{2}\int u(u^2 + u_x^2)\mathrm{d}x \end{aligned} \tag{2.3}$$

and the Hamiltonian operators

$$\mathcal{K} = -(mD_x + D_x m), \quad \mathcal{J} = -D_x(1 - D_x^2), \tag{2.4}$$

which leads to the the recursion operator of CH equation

$$\widetilde{\mathcal{R}}_1 = \mathcal{K}\mathcal{J}^{-1}. \tag{2.5}$$

It is well-known that the CH equation is closely related to the KdV equation (see [11, 26, 31])

$$v_t + v_{yyy} + 6vv_y = 0. \tag{2.6}$$

This fact is reflected in two aspects. On the one hand, the bi-Hamiltonian structures of the CH equation can be obtained from the KdV equation via the tri-Hamiltonian duality approach (see [12–13, 16]). On the other hand, the CH equation can be transformed to the first equation in the negative flow of the KdV hierarchy (see [6]). The recursion operator of the KdV equation reads (see [33–34])

$$\mathcal{R}_1 = D_y^2 + 2v + v_y D_y^{-1}, \tag{2.7}$$

which permits the factorization (see [28])

$$\mathcal{R}_1 = \psi^{-2} D_y \psi^2 D_y \psi^2 D_y \psi^{-2} D_y^{-1}, \tag{2.8}$$

where ψ is the eigenfunction of the Schrödinger operator with the potential $v(y)$ and zero eigenvalue

$$\psi_{yy} + v\psi = 0. \tag{2.9}$$

It is easy to find the inverse of $\mathcal{R}$, given by (see [28])

$$\mathcal{R}_1^{-1} = D_y\psi^2 D_y^{-1}\psi^{-2}D_y^{-1}\psi^{-2}D_y^{-1}\psi^2. \tag{2.10}$$

The relationship between the recursion operators of the KdV and CH equations is established in the following lemma (see [11, 26, 31]).

Lemma 2.1 (see [26, 31]) *The recursion operators of the KdV and CH equations are related by*

$$\widetilde{\mathcal{R}}_1 = m\mathcal{R}_1^{-1}m^{-1}. \tag{2.11}$$

The proof of Lemma 2.1 relies on the Liouville transformation

$$y = \int_{-\infty}^{x} m^{\frac{1}{2}}(\xi)\mathrm{d}\xi, \quad v(y) = \frac{1}{m}\Big(\frac{1}{4} - m^{\frac{1}{4}}(m^{-\frac{1}{4}})_{xx}\Big). \tag{2.12}$$

This allows us to obtain

$$D_x = m^{\frac{1}{2}}D_y. \tag{2.13}$$

Using (2.10) and (2.12)–(2.13), we obtain the recursion operator of the CH equation (2.1) given by

$$\widetilde{\mathcal{R}}_1 = m^{\frac{1}{2}}D_x\psi^2 D_x^{-1}m^{\frac{1}{2}}\psi^{-2}D_x^{-1}m^{\frac{1}{2}}\psi^{-2}D_x^{-1}\psi^2 m^{-\frac{1}{2}}. \tag{2.14}$$

Clearly, its inverse operator is

$$\widetilde{\mathcal{R}}_1^{-1} = m^{\frac{1}{2}}\psi^{-2}D_x\psi^2 m^{-\frac{1}{2}}D_x\psi^2 m^{-\frac{1}{2}}D_x\psi^{-2}D_x^{-1}m^{-\frac{1}{2}}, \tag{2.15}$$

where $\psi(t,x)$ satisfies

$$\psi_{xx} - \frac{1}{2}\frac{m_x}{m}\psi_x + \Big(\frac{1}{4} - m^{\frac{1}{4}}(m^{-\frac{1}{4}})_{xx}\Big)\psi = 0.$$

By looking for the kernels of $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_1^{-1}$, we obtain the nonlocal symmetries of the CH equation (2.1) governed by the following theorem.

Theorem 2.1 *The CH equation* (2.1) *admits the following nonlocal symmetries*:

$$\begin{aligned}
K_0 &= m^{\frac{1}{2}}D_x\psi^2,\\
K_1 &= m^{\frac{1}{2}}D_x\psi^2 D_x^{-1}m^{\frac{1}{2}}\psi^{-2},\\
K_2 &= m^{\frac{1}{2}}D_x\psi^2 D_x^{-1}m^{\frac{1}{2}}\psi^{-2}D_x^{-1}m^{\frac{1}{2}}\psi^{-2},\\
J_0 &= m^{\frac{1}{2}}\psi^{-2}D_x\psi^2 m^{-\frac{1}{2}}D_x\psi^2 m^{-\frac{1}{2}}D_x\psi^{-2}.
\end{aligned}$$

3 Nonlocal Symmetries of the mCH Equation

The mCH equation reads

$$m_t + ((u^2 - u_x^2)m)_x = 0, \qquad m = u - u_{xx}, \tag{3.1}$$

which can be written in bi-Hamiltonian form (see[36])

$$m_t = \mathcal{K}_1 \frac{\delta H_3[u]}{\delta m} = \mathcal{J}_1 \frac{\delta H_4[u]}{\delta m}, \tag{3.2}$$

where $\mathcal{K}_1$ and $\mathcal{J}_1$, given by

$$\mathcal{K}_1 = -D_x m D_x^{-1} m D_x, \quad \mathcal{J}_1 = -D_x(1 - D_x^2) \tag{3.3}$$

are the Hamiltonian operators, and

$$H_3[u] = \int (u^2 + u_x^2)\mathrm{d}x, \qquad H_4[u] = \frac{1}{4}\int (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4)\mathrm{d}x \tag{3.4}$$

are the Hamiltonian functionals, which gives the recursion operator of the mCH equation

$$\widetilde{\mathcal{R}}_2 = \mathcal{K}_1 \mathcal{J}_1^{-1} \tag{3.5}$$

of the mCH equation (see [36]). As for the CH case, the mCH equation is related to the mKdV equation (see [23, 36])

$$Q_\tau + Q_{yyy} + 6Q^2 Q_y = 0. \tag{3.6}$$

Indeed, the mCH equation can be obtained from the mKdV equation via the tri-Hamiltonian duality approach (see [36]). Recently, it was shown in [23] that the mCH equation is mapped into the first equation in the negative flow of the mKdV hierarchy via the Liouville transformation

$$Q(\tau, y) = \frac{1}{2m(t,x)}, \quad y = \int^x m(t,\xi)\mathrm{d}\xi, \quad \tau = t. \tag{3.7}$$

The mKdV equation is a bi-Hamiltonian equation, which can be written as

$$Q_\tau = \overline{\mathcal{K}}_1 \frac{\delta \overline{H}_3[Q]}{\delta Q} = \overline{\mathcal{J}}_1 \frac{\delta \overline{H}_4[Q]}{\delta Q}, \tag{3.8}$$

where

$$\begin{aligned} \overline{\mathcal{K}}_1 &= -\frac{1}{4}D_y^3 - D_y Q D_y^{-1} Q D_y, \\ \overline{\mathcal{J}}_1 &= -D_y, \\ \overline{H}_3[Q] &= 2\int Q^2 \mathrm{d}y, \\ \overline{H}_4[Q] &= \frac{1}{2}\int (Q^4 - Q_y^2)\mathrm{d}y. \end{aligned} \tag{3.9}$$

The recursion operator of the mKdV equation is then given by

$$\mathcal{R}_2 = \overline{\mathcal{K}}_1 \overline{\mathcal{J}}_1^{-1} = D_y^2 + Q^2 + Q_y D_y^{-1} Q. \tag{3.10}$$

The following lemma gives the relationship between $\widetilde{\mathcal{R}}_2$ and $\mathcal{R}_2$.

Lemma 3.1 (see [23]) *Let $\mathcal{K}_1$ and $\mathcal{J}_1$ be the two compatible Hamiltonian operators* (3.3) *of the mCH equation* (3.1), *and $\overline{\mathcal{K}}_1$ and $\overline{\mathcal{J}}_1$ the two compatible Hamiltonian operators* (3.9) *of the mKdV equation* (3.6). *Then there holds the identity*

$$\widetilde{\mathcal{R}}_2 = -4\Big(1 - \frac{Q_y}{4Q^3}D_y + \frac{1}{4Q^2}D_y^2\Big)\mathcal{R}_2^{-1}\Big(1 - \frac{Q_y}{4Q^3}D_y + \frac{1}{4Q^2}D_y^2\Big)^{-1}. \tag{3.11}$$

Note that the operator can be expressed as

$$1 - \frac{Q_y}{4Q^3}D_y + \frac{1}{4Q^2}D_y^2 = \frac{1}{4}(hQ)^{-1}D_y h^2 Q^{-1} D_y h^{-1}, \tag{3.12}$$

where Q is the solution of (3.6) and $h(t,y)$ satisfies the equation

$$h_{yy} - \frac{Q_y}{Q}h_y + 4Q^2 h = 0. \tag{3.13}$$

Using (3.11)–(3.12) and the following factorization for $\mathcal{R}_2$ (see [27])

$$\mathcal{R}_2 = D_y g^2 D_y g^{-4} Q^{-1} D_y g^2 D_y^{-1} Q,$$

where g satisfies the Schrödinger equation (2.9) with v repalced by Q, we get the following result.

Proposition 3.1 *The recursion operator $\widetilde{\mathcal{R}}_2$ and its inverse of the mCH equation has the following factorizations*:

$$\begin{aligned}\widetilde{\mathcal{R}}_2 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (Qh)^{-1} D_y g^{-2} D_y^{-1} Q g^4 D_y^{-1} g^{-2} \\ &\quad \cdot D_y^{-1} h D_y^{-1} Q h^{-2} D_y^{-1} Q h, \\ \widetilde{\mathcal{R}}_2^{-1} &= (Qh)^{-1} D_y h^2 Q^{-1} D_y h^{-1} D_y g^2 D_y g^{-4} Q^{-1} D_y g^2 \\ &\quad \cdot D_y^{-1} Q h D_y^{-1} Q h^{-2} D_y^{-1} Q h.\end{aligned} \tag{3.14}$$

By looking for the kernels of $\widetilde{\mathcal{R}}_2$ and $\widetilde{\mathcal{R}}_2^{-1}$, we are able to obtain nonlocal symmetries of the mCH equation.

Theorem 3.1 *The mCH equation* (3.1) *possesses the following nonlocal symmetries:*

$$\begin{aligned}\overline{J}_0 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y h^{-1} D_y g^2 D_y Q^{-1} g^{-4} D_y g^2, \\ \overline{J}_1 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y h^{-1} D_y g^2 D_y Q^{-1} g^{-4} D_y g^2 D_y^{-1}(Qh), \\ \overline{J}_2 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y h^{-1} D_y g^2 D_y Q^{-1} g^{-4} D_y g^2 D_y^{-1} Q h D_y^{-1}(Qh^{-2}), \\ \overline{K}_0 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (hQ)^{-1} D_y g^{-2}, \\ \overline{K}_1 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (hQ)^{-1} D_y g^{-2} D_y^{-1}(Qg^4), \\ \overline{K}_2 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (hQ)^{-1} D_y g^{-2} D_y^{-1}(Qg^4) D_y^{-1} g^{-2}, \\ \overline{K}_3 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (hQ)^{-1} D_y g^{-2} D_y^{-1}(Qg^4) D_y^{-1} g^{-2} D_y^{-1} h, \\ \overline{K}_4 &= (Qh)^{-1}D_y h^2 Q^{-1} D_y (hQ)^{-1} D_y g^{-2} D_y^{-1}(Qg^4) D_y^{-1} g^{-2} D_y^{-1} h D_y^{-1}(Qh^{-2}).\end{aligned} \tag{3.15}$$

4 Nonlocal Symmetries of the Novikov Equation

The Novikov equation with cubic nonlinearities

$$m_t = 3uu_x m + u^2 m_x, \quad m = u - u_{xx} \tag{4.1}$$

arises from the symmetry classification of a class of nonlinear evolution equations involving both cubic and quadratic nonlinearities (see [32]). The Lax pair formulation with 3×3 isospectral problem and bi-Hamiltonian structure were established in [9], it can be written in bi-Hamiltonian form (see [20])

$$m_t = \mathcal{K}_2 \frac{\delta H_5[u]}{\delta m} = \mathcal{J}_2 \frac{\delta H_6[u]}{\delta m}, \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{K}_2 &= \frac{1}{2} m^{\frac{1}{3}} D_x m^{\frac{2}{3}} (4D_x - D_x^3)^{-1} m^{\frac{2}{3}} D_x m^{\frac{1}{3}}, \\
\mathcal{J}_2 &= (1 - D_x^2) m^{-1} D_x m^{-1} (1 - D_x^2), \\
H_5[u] &= 9 \int (u^2 + u_x^2) \mathrm{d}x, \\
H_6[u] &= \frac{1}{6} \int um\partial_x^{-1} m (1 - \partial_x^2)^{-1} (u^2 m_x + 3uu_x m) \mathrm{d}x \\
&= \frac{1}{6} \int (u^4 m^2 - u_t m_t) \mathrm{d}x.
\end{aligned} \tag{4.3}$$

This gives the recursion operator of the Novikov equation

$$\widetilde{\mathcal{R}}_3 = \mathcal{K}_2 \mathcal{J}_2^{-1}. \tag{4.4}$$

It was shown in [20, 24] that the Novikov hierarchy is related to the Sawada-Kotera hierarchy

$$Q_\tau = \overline{\mathcal{K}}_2 \frac{\delta \overline{H}_5[Q]}{\delta Q}, \quad \overline{\mathcal{J}_2 \mathcal{K}}_2[Q] = \frac{\delta \overline{H}_6[Q]}{\delta Q}, \tag{4.5}$$

where

$$\begin{aligned}
\overline{\mathcal{K}}_2 &= -(D_y^3 + 2QD_y + 2D_y Q), \\
\overline{\mathcal{J}}_2 &= 2D_y^3 + 2D_y^2 Q D_y^{-1} + 2D_y^{-1} Q D_y^2 + Q^2 D_y^{-1} + D_y^{-1} Q^2, \\
\overline{H}_5 &= \frac{1}{6} \int (Q^3 - 3Q_y^2) \mathrm{d}y,
\end{aligned} \tag{4.6}$$

whose corresponding integrable hierarchy is then generated by the recursion operator

$$\mathcal{R}_3 = \overline{\mathcal{K}}_2 \overline{\mathcal{J}}_2. \tag{4.7}$$

In fact, there is relationship between the recursion operator for the Novikov equation and the dual recursion operator of the Sawada-Kotera equation (see [24]). It was addressed in [20] that the Novikov equation (4.1) is related to the first equation in the negative Sawada-Kotera hierarchy, this fact was verified recently in [24].

Lemma 4.1 (see [24]) *Under the Liouville transformation*

$$y=\int^x m^{\frac{2}{3}}(t,\xi)\mathrm{d}\xi,\quad \tau=t,\quad Q(\tau,\xi)=\frac{4}{9}m^{-\frac{10}{3}}m_x^2-\frac{1}{3}m^{-\frac{7}{3}}m_{xx}-m^{-\frac{4}{3}},\tag{4.8}$$

the relation

$$\widetilde{\mathcal{R}}_3=\mathcal{K}_2\mathcal{J}_2^{-1}=mD_y(\overline{\mathcal{J}}_2\overline{\mathcal{K}}_2)^{-1}D_y^{-1}\tag{4.9}$$

holds.

It follows from the above lemma that the recursion operator $\widehat{\mathcal{R}}$ of the Novikov equation permits the following factorization

$$\begin{aligned}\widetilde{\mathcal{R}}_3&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^2D_xg\\&\quad\cdot D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}gm^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}gm^{-\frac{1}{3}},\\\widetilde{\mathcal{R}}_3^{-1}&=m^{\frac{1}{3}}g^{-1}D_xg^2m^{-\frac{2}{3}}D_xg^{-1}m^{-\frac{2}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-1}\\&\quad\cdot D_x^{-1}g^{-2}D_xg^2m^{-\frac{2}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-2}D_x^{-1}m^{-\frac{1}{3}}.\end{aligned}\tag{4.10}$$

Thus we arrive at the following result for the Novikov equation.

Theorem 4.1 *The Novikov equation* (4.1) *possesses the following nonlocal symmetries*:

$$\begin{aligned}\widetilde{K}_0&=m^{\frac{1}{3}}D_xg^2=2m^{\frac{1}{3}}gg_x,\\\widetilde{K}_1&=m^{\frac{1}{3}}D_xg^2D_x^{-1}(g^{-2}m^{\frac{2}{3}}),\\\widetilde{K}_2&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}(g^{-2}m^{\frac{2}{3}}),\\\widetilde{K}_3&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}(gm^{\frac{2}{3}}),\\\widetilde{K}_4&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^2D_xgD_x^{-1}(g^{-2}m^{\frac{2}{3}}),\\\widetilde{K}_5&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^2D_xgD_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}(gm^{\frac{2}{3}}),\\\widetilde{K}_6&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^2D_xgD_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}gm^{\frac{2}{3}}D_x^{-1}(gm^{\frac{2}{3}}),\\\widetilde{K}_7&=m^{\frac{1}{3}}D_xg^2D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^{-2}m^{\frac{2}{3}}D_x^{-1}g^2D_xgD_x^{-1}g^{-2}m^{\frac{2}{3}}\\&\quad\cdot D_x^{-1}gm^{\frac{2}{3}}D_x^{-1}gm^{\frac{2}{3}}D_x^{-1}(g^{-2}m^{\frac{2}{3}}),\\\widetilde{J}_0&=g^{-1}m^{\frac{1}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-1}m^{-\frac{2}{3}}D_xg^{-1}m^{-\frac{2}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-1},\\\widetilde{J}_1&=g^{-1}m^{\frac{1}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-1}m^{-\frac{2}{3}}D_xg^{-2}m^{-\frac{2}{3}}D_xg^2m^{-\frac{2}{3}}\\&\quad\cdot D_xg^{-1}D_x^{-1}g^{-2}D_xg^2m^{-\frac{2}{3}}D_xg^2m^{-\frac{2}{3}}D_xg^{-2}.\end{aligned}\tag{4.11}$$

5 Nonlocal Symmetries of the DP Equation

The DP equation

$$n_t=3v_xn+vn_x,\quad n=v-v_{xx}\tag{5.1}$$

can be obtained from the governing equation for shallow-water waves (see [8]), which can also be written in bi-Hamiltonian form (see [9])

$$n_t=\mathcal{L}\,\frac{\delta H_7[u]}{\delta m}=\mathcal{D}\,\frac{\delta H_8[u]}{\delta m},\tag{5.2}$$

where

$$
\begin{aligned}
&\mathcal{L} = n^{\frac{2}{3}} D_x\, n^{\frac{1}{3}}\, (D_x - D_x^3)^{-1}\, n^{\frac{1}{3}}\, D_x\, n^{\frac{2}{3}},\\
&\mathcal{D} = D_x(1 - D_x^2)(4 - D_x^2),\\
&H_7[u] = \frac{9}{2}\int n\mathrm{d}x,\\
&H_8[u] = \frac{1}{6}\int u^3\mathrm{d}x.
\end{aligned}
\tag{5.3}
$$

This provides the recursion operator of the DP equation (see [9])

$$\widetilde{\mathcal{R}}_4 = \mathcal{L}\mathcal{D}^{-1}. \tag{5.4}$$

The relationship between DP equation and KK equation was indicated in [9], and it was proved in [24] that DP equation (5.1) is related to the first one in the negative flow of KK equation (see [9, 24])

$$P_\tau + P_{yyyyy} + 20QQ_{yyy} + 50Q_yQ_{yy} + 80Q^2Q_y = 0 \tag{5.5}$$

via the Liouville transformation

$$P(\tau, y) = n^{-\frac{1}{2}}\Big(D_x^2 - \frac{1}{4}\Big)n^{-\frac{1}{6}}, \quad y = \int^x n^{\frac{1}{3}}(t,\xi)\mathrm{d}\xi, \quad \tau = t. \tag{5.6}$$

It was known (see [14]) that equation (5.5) is a generalized bi-Hamiltonian system:

$$P_\tau = \widetilde{\mathcal{L}}\frac{\delta \overline{H}_7[P]}{\delta P}, \quad \widetilde{\mathcal{D}}\widetilde{\mathcal{L}}[P] = \frac{\delta \overline{H}_8[P]}{\delta P}, \tag{5.7}$$

where

$$
\begin{aligned}
\widetilde{\mathcal{L}} &= -(D_y^3 + 2PD_y + 2D_yP),\\
\widetilde{\mathcal{D}} &= D_y^3 + 6(PD_y + D_yP) + 4(D_y^2PD_y^{-1} + D_y^{-1}PD_y^2) + 32(P^2D_y^{-1} + D_y^{-1}P^2),\\
\overline{H}_7 &= \frac{1}{6}\int\Big(\frac{8}{3}P^3 - \frac{1}{2}P_y^2\Big)\mathrm{d}y.
\end{aligned}
\tag{5.8}
$$

The recursion operator of (5.5) is

$$\mathcal{R}_4 = \widetilde{\mathcal{L}}\widetilde{\mathcal{D}}. \tag{5.9}$$

The relationship between the recursion operator of the DP equation and the dual recursion operator of the KK equation was established in [24].

Lemma 5.1 (see [24]) *Under the Liouville transformation* (5.6), *there holds*

$$\widetilde{\mathcal{R}}_4 = \mathcal{L}\mathcal{D}^{-1} = nD_y(\widetilde{\mathcal{D}}\widetilde{\mathcal{L}})^{-1}(nD_y)^{-1}. \tag{5.10}$$

To obtain nonlocal symmetries of the Kaup-Kupershmidt equation (5.5), we apply the following factorization on the operator $\widetilde{\mathcal{D}}\widetilde{\mathcal{L}}$ (see [18])

$$\widetilde{\mathcal{D}}\widetilde{\mathcal{L}} = D_y^{-1}\widetilde{g}^{-4}D_y\widetilde{g}^2D_y\widetilde{g}^2D_y\widetilde{g}^2D_y\widetilde{g}^2D_y\widetilde{g}^4D_y^{-1}\widetilde{g}^{-2}D_y\widetilde{g}^2D_y\widetilde{g}^2D_y\widetilde{g}^{-2}, \tag{5.11}$$

where $\widetilde{g}$ satisfies (2.9) with v replaced by P. Using Lemma 5.1 and the Liouville transformation (5.6), we get the following result.

The recursion operator $\widetilde{\mathcal{R}}_4$ of the DP equation (5.1) and its inverse operator admit the following factorizations:

$$\begin{aligned} \widetilde{\mathcal{R}}_4 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^2 D_x \widetilde{g}^{-4} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} \\ &\quad \cdot D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^4 n^{-\frac{2}{3}}, \\ \widetilde{\mathcal{R}}_4^{-1} &= n^{\frac{2}{3}} \widetilde{g}^{-4} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^4 \\ &\quad \cdot D_x^{-1} \widetilde{g}^{-2} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^{-2} D_x^{-1} n^{-\frac{2}{3}}. \end{aligned} \tag{5.12}$$

We thus arrive at the following result.

Theorem 5.1 *The DP equation* (5.1) *admits the following nonlocal symmetries:*

$$\begin{aligned} \widehat{K}_0 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 = 2n^{\frac{2}{3}} \widetilde{g}\widetilde{g}_x, \\ \widehat{K}_1 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{K}_2 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{K}_3 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} (\widetilde{g}^{-4} n^{\frac{1}{3}}), \\ \widehat{K}_4 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^2 D_x \widetilde{g}^{-4} D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{K}_5 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^2 D_x \widetilde{g}^{-4} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{K}_6 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^2 D_x \widetilde{g}^{-4} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{K}_7 &= n^{\frac{2}{3}} D_x \widetilde{g}^2 D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^2 D_x \widetilde{g}^{-4} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} \\ &\quad \cdot D_x^{-1} \widetilde{g}^{-2} n^{\frac{1}{3}} D_x^{-1} (\widetilde{g}^{-2} n^{\frac{1}{3}}), \\ \widehat{J}_0 &= n^{\frac{2}{3}} \widetilde{g}^{-4} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^4, \\ \widehat{J}_1 &= n^{\frac{2}{3}} \widetilde{g}^{-4} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^4 D_x^{-1} \widetilde{g}^{-2} \\ &\quad \cdot D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^2 n^{-\frac{1}{3}} D_x \widetilde{g}^{-2}. \end{aligned} \tag{5.13}$$

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