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A Note on Randomly Weighted Sums of Dependent Subexponential Random Variables*

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Abstract The author obtains that the asymptotic relations

$$\mathbb{P}\Big(\sum_{i=1}^{n}\theta_i X_i > x\Big) \sim \mathbb{P}\Big(\max_{1\le m\le n}\sum_{i=1}^{m}\theta_i X_i > x\Big) \sim \mathbb{P}\Big(\max_{1\le i\le n}\theta_i X_i > x\Big) \sim \sum_{i=1}^{n}\mathbb{P}(\theta_i X_i > x)$$

hold as $x \to \infty$, where the random weights $\theta_1, \cdots, \theta_n$ are bounded away both from 0 and from ∞ with no dependency assumptions, independent of the primary random variables $X_1, \cdots, X_n$ which have a certain kind of dependence structure and follow non-identically subexponential distributions. In particular, the asymptotic relations remain true when $X_1, \cdots, X_n$ jointly follow a pairwise Sarmanov distribution.

Keywords Randomly weighted sums, Subexponential distributions, Ruin probabilities, Insurance and financial risks
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1 Introduction

Throughout, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim \frac{a(x)}{b(x)} = 1$; we write $a(x) = O(b(x))$ if $\limsup \frac{a(x)}{b(x)} < \infty$; we write $a(x) \asymp b(x)$ if both $a(x) = O(b(x))$ and $b(x) = O(a(x))$ hold; and we write $a(x) = o(b(x))$ if $\lim \frac{a(x)}{b(x)} = 0$. For any distribution F, we denote its (right) tail by $\overline{F}(x) = 1 - F(x) = F(x, \infty)$, $x \in (-\infty, \infty)$.

Throughout, let $n \ge 1$ be a fixed integer and let $X_1, \cdots, X_n$ be n real-valued random variables (r.v.s), called primary r.v.s, and let $\theta_1 \cdots, \theta_n$ be n positive r.v.s, called random weights, independent of the primary r.v.s.

In this note, we will continue to investigate the tail behavior of the randomly weighted sums S_n^θ and the maximum M_n^θ, which are defined by

$$S_n^\theta = \sum_{i=1}^{n}\theta_i X_i, \quad M_n^\theta = \max_{1\le m\le n} S_m^\theta. \tag{1.1}$$

The randomly weighted sums and their maximum in (1.1) play important roles in many applied probability fields such as financial insurance, risk theory, queueing theory and so on. A well-known example in risk theory interprets the weights as discount factors and the primary

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r.v.s as the net losses of an insurance company to analyze the ruin probability (see [11]). In economics, the primary r.v.s can be interpreted as net incomes of an investment and the weights as random return rates (see [12]). Hence, the randomly weighted sums and their maximum have been well studied in the literature. One of the famous results is described below, which was obtained by [11].

Theorem 1.1 *Let $X_1, \cdots, X_n$ be independent r.v.s with a common subexponential distribution F (see Definition 2.1 below) and let the random weights $\theta_1, \cdots, \theta_n$ be bounded away both from 0 and from ∞, that is, there are two positive constants a and b such that*

$$\mathbb{P}(a \le \theta_i \le b) = 1 \quad \text{for } i = 1, 2, \cdots, n. \tag{1.2}$$

Then it holds that

$$\mathbb{P}(S_n^\theta > x) \sim \mathbb{P}(M_n^\theta > x) \sim \mathbb{P}\Big(\max_{1\le i\le n} \theta_i X_i > x\Big) \sim \sum_{i=1}^n \mathbb{P}(\theta_i X_i > x). \tag{1.3}$$

However, the assumption of independence among the primary r.v.s is far unrealistic, and hence, many scholars have proved that the relations (1.3) still hold in which the primary r.v.s have a kind of dependence structure and belong to some subclasses of the subexponential distribution class, see [2, 6, 10, 13, 16], etc. It is worth noting that the above results usually need an additional condition that the primary r.v.s have negligible left tail with respect to the right tail, that is,

$$F(-x) = o(\overline{F}(x)). \tag{1.4}$$

Recently, Cheng and Cheng [3] proved that the asymptotic relations (1.3) still hold when the primary r.v.s are dependent and subexponential under an extra condition that

$$\lim \sup_{c\in[a,b]} \frac{F\Big(-\dfrac{h(x)}{c}\Big)}{\overline{F}\Big(\dfrac{x}{c}\Big)} = 0 \tag{1.5}$$

holds for some positive function h.

However, both (1.4) and (1.5) are superfluous in Theorem 1.1, which exclude many primary r.v.s (e.g. r.v.s with symmetrical distributions) that satisfy Theorem 1.1.

In this note, we will relax the condition (1.5) to that

$$\mathbb{P}(X_i > x, X_j \le -h(x)) = o(\mathbb{P}(X_i > x)), \quad 1 \le i, j \le n \tag{1.6}$$

holds for some positive function h, and will obtain that the asymptotic relations (1.3) still hold when $X_1, \cdots, X_n$ are dependent r.v.s with non-identically subexponential distributions. We remark that, when the primary r.v.s are independent, then they automatically satisfy the extra conditions (1.6), and hence, our result covers Theorem 1.1. Furthermore, if the primary r.v.s have a pairwise Sarmanov distribution (see Section 2 below), they automatically satisfy the condition (1.6) also.

The rest of this note consists of two sections. In Section 2, after introducing some preliminaries on heavy-tailed distribution subclasses and some dependence structures among r.v.s, we will present the main results of this note. In Section 3, we will prove the main results.

2 Preliminaries and Main Results

2.1 Some heavy-tailed distribution subclasses

To model the dangerous claim sizes in the insurance industry, most practitioners select the claim-size distribution from the heavy-tailed distribution class. By definition, an r.v. X or its distribution F is said to be (right) heavy-tailed if $\int_0^\infty \mathrm{e}^{\varepsilon y} F(\mathrm{d}y) = \infty$ holds for all $\varepsilon > 0$. To this end, we now introduce some important subclasses of heavy-tailed distribution class, one of which is the subexponential distribution class.

Definition 2.1 *A distribution F supported on $[0, \infty)$ is said to be subexponential, denoted by $F \in \mathcal{S}$, if it is unbounded above (that is, $\overline{F}(x) > 0$ for all $x \geq 0$) and the relation*

$$\overline{F^{*n}}(x) \sim n\overline{F}(x)$$

holds for some (or equivalently for all) $n = 2, 3, \cdots$, where F^{*n} denotes the n-fold convolution of F with itself. Furthermore, a distribution F supported on $(-\infty, \infty)$ is still said to be subexponential, if F^+ does so, where $F^+(x) = F(x)\mathbf{1}(x \geq 0)$ for $x \in (-\infty, \infty)$ and $1(A)$ is the indicator function of the set A.

It is well-known that if a distribution F supported on $[0, +\infty)$ or $(-\infty, +\infty)$ is subexponential, then it is long-tailed.

Definition 2.2 *A distribution F supported on $[0, +\infty)$ or $(-\infty, \infty)$ is said to be long-tailed, denoted by $F \in \mathcal{L}$, if F is unbounded above and $\overline{F}(x + t) \sim \overline{F}(x)$ holds for any $t \in (-\infty, \infty)$.*

The long-tailed distribution class has the following important properties: If $F \in \mathcal{L}$, then the function class

$$\mathcal{H}(F) = \Big\{h \text{ on } [0, \infty) : h(x) \uparrow \infty,\ \frac{h(x)}{x} \downarrow 0 \text{ and } \overline{F}(x - h(x)) \sim \overline{F}(x)\Big\}$$

is not empty, and if $h \in \mathcal{H}(F)$, then $ch \in \mathcal{H}(F)$ for any $c > 0$, for instance see [4]. It is clear that if $h_1 \in \mathcal{H}(F)$ and $h_1 \geq h_2 \uparrow \infty$, $\frac{h_2(x)}{x} \downarrow 0$, then $h_2 \in \mathcal{H}(F)$. For more details on the classes $\mathcal{S}$ and $\mathcal{L}$, the reader can refer to [5] and so on.

2.2 Dependence structures

Inspired by [7, 14–15], Cheng [3] introduced a new dependence structure as follows.

Definition 2.3 *For any fixed integer $n \geq 1$, real-valued r.v.s $X_1, \cdots, X_n$ are called conditionally linearly wide dependent (CLWD for short) on an interval $\Lambda \subset (0, \infty)$, if there exist positive constants $x_0 = x_0(n)$ and $K = K(n)$ such that for all $x, y > x_0$ and $2 \leq m \leq n$,*

$$\mathbb{P}\Big(\sum_{i=1}^{m-1} c_i X_i^+ > x \Big| c_m X_m = y\Big) \leq K\mathbb{P}\Big(\sum_{i=1}^{m-1} c_i X_i^+ > x\Big) \tag{2.1}$$

holds uniformly for all $(c_1, c_2, \cdots, c_m) \in \Lambda^m$, where $X_i^+ = X_i 1(X_i \geq 0)$, $1 \leq i \leq n$.

Notice that when y is not a possible value of $c_m X_m$, namely $\mathbb{P}(c_m X_m \in \Delta) = 0$ for some open interval Δ containing y, the conditional probability in (2.1) is simply understood as unconditional probability.

The CLWD dependence structure allows a wide choice of r.v.s. For the sake of illustration, we introduce the following pairwise Sarmanov distribution.

Definition 2.4 *Random variables $X_1, \cdots, X_n$ jointly follow a pairwise Samarnov distribution, if it has the form*

$$\mathbb{P}\Big(\bigcap_{i=1}^{n}(X_i \in \mathrm{d}x_i)\Big) = \Big(1 + \sum_{1\le k<j\le n} \omega_{kj}\phi_k(x_k)\phi_j(x_j)\Big)\prod_{i=1}^{n} F_i(\mathrm{d}x_i), \tag{2.2}$$

where $F_1, \cdots, F_n$ are corresponding marginal distributions, $\phi_1(\cdot), \cdots, \phi_n(\cdot)$ are kernel functions, and $\omega_{kj}, 1 \le k < j \le n$, are real numbers such that

$$E[\phi_i(X_i)] = 0, \quad 1 \le i \le n, \tag{2.3}$$

and

$$1 + \sum_{1\le k<j\le n} \omega_{kj}\phi_k(x_k)\phi_j(x_j) \ge 0, \quad (x_1, \cdots, x_n) \in (-\infty, \infty)^n.$$

Note that if all ω_{kj} are 0, then $X_1, \cdots, X_n$ are independent. It is well-known that the kernels ϕ_i and the real numbers ω_{kj} offer us flexibility in constructing desired dependence structures. For more details on multivariate Sarmanov distributions, one can refer to [8–9] among others. There are many choices for the kernels $\phi_i(x)$. A special choice is to take $\phi_i(x) = 1 - F_i(x) - F_i(x-)$, $i = 1, \cdots, n$, which leads to the well-known pairwise Farlie-Gumbel-Morgenstern (FGM for short) distribution (see [1]).

[3, Example 2.1] proved that, if r.v.s $X_1, \cdots, X_n$ jointly follow a pairwise Sarmanov distribution of the form (2.2)–(2.3) which satisfy

$$|\phi_i(x)| \le 1, \quad 1 \le i \le n \text{ for all } x \in (-\infty, \infty), \tag{2.4}$$

and

$$\sum_{1\le k<j\le n} |\omega_{kj}| < 1, \tag{2.5}$$

then they are CLWD on any interval $\Lambda \subset (0, \infty)$.

2.3 Main results

Now, we are ready to state our main results as follows.

Theorem 2.1 *Let $X_1, \cdots, X_n$ be real-valued r.v.s with distributions $F_1, \cdots, F_n$ and be CLWD on the interval $[a, b]$ for some real numbers $0 < a < b < \infty$, and let $\theta_1, \cdots, \theta_n$ be r.v.s which satisfy* (1.2)*, independent of the primary r.v.s $X_1, \cdots, X_n$. Assume that $F_i \in \mathcal{L}$ and $\overline{F}_i(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{S}$ and all $i = 1, 2, \cdots, n$. If there exists a positive function $h \in \bigcap_{i=1}^{n} \mathcal{H}(F_i)$ such that* (1.6) *holds, then the relations in* (1.3) *hold.*

Remark 2.1 Obviously, if r.v.s $X_1, \cdots, X_n$ are independent, then they satisfy (1.6) automatically. Hence, Theorem 2.1 covers Theorem 1.1.

From Theorem 2.1 and [3, Example 2.1], we obtain the following corollary directly.

Corollary 2.1 *Suppose that $X_1, \cdots, X_n$ are real-valued r.v.s with distributions $F_1, \cdots, F_n$ and jointly follow a pairwise Sarmanov distribution of the form* (2.2) *which satisfy* (2.3)*–*(2.5)*. Assume that $F_i \in \mathcal{L}$ and $\overline{F}_i(x) \asymp \overline{F}(x)$ for some $F \in \mathcal{S}$ and all $i = 1, 2, \cdots, n$. Let $\theta_1, \cdots, \theta_n$ be r.v.s satisfying* (1.2) *for some real numbers $0 < a < b < \infty$, independent of the primary r.v.s $X_1, \cdots, X_n$. Then the relations in* (1.3) *hold.*

Now, using Theorem 2.1, we will continue to investigate the asymptotic behavior for the finite-time ruin probability in a discrete-time risk model with both insurance and financial risks, which was discussed by many authors, see, e.g. [3] and the references therein.

Consider the following discrete-time risk model: For any integer $i \geq 1$, the net insurance loss within period i, which is equal to the total claim amount minus the total premium income, is denoted by a real-valued r.v. X_i with a distribution F_i. Suppose that the insurer invests his/her wealth in a discrete-time financial market consisting of a risk-free bond with a stochastic interest rate $I_i > 0$ and a risky stock with a stochastic return rate $R_i > -1$ during period i, where $\{I_i,\ i = 1, 2, \cdots;\ R_i,\ i = 1, 2, \cdots\}$ are independent of $\{X_i,\ i = 1, 2, \cdots\}$. Suppose that, in the beginning of every period, the insurer invests a constant proportion $\pi_i \in [0, 1)$ of his/her current wealth in the stock and keeps the rest in the bond.

Denote by U_i the insurer's wealth at time i, and by $U_0 = x > 0$ the deterministic initial capital. Clearly, we have

$$\begin{aligned} U_i &= ((1-\pi_i)(1+I_i) + \pi_i(1+R_i))U_{i-1} - X_i \\ &= Y_i^{-1} U_{i-1} - X_i, \quad i = 1, 2, \cdots, \end{aligned} \tag{2.6}$$

where

$$Y_i = \frac{1}{(1-\pi_i)(1+I_i) + \pi_i(1+R_i)}, \quad i = 1, 2, \cdots.$$

As usual, the probability of ruin by time n is defined as

$$\psi(x, n) = P\Big(\min_{0 \leq i \leq n} U_i < 0 \Big| U_0 = x\Big). \tag{2.7}$$

Theorem 2.2 *In the above-mentioned risk model, suppose that the net losses* $X_1, \cdots, X_n$ *are CLWD on the interval* $(0, 1]$ *and there exists some constant* $0 < c < \infty$ *such that*

$$\mathbb{P}(0 \leq I_i \leq c) = 1, \quad \mathbb{P}(-1 < R_i \leq c) = 1, \quad 1 \leq i \leq n. \tag{2.8}$$

Assume that $F_i \in \mathcal{L}$ *and* $\overline{F}_i(x) \asymp \overline{F}(x)$ *for some* $F \in \mathcal{S}$ *and all* $i = 1, 2, \cdots, n$*. If there exists a positive function* $h \in \bigcap_{i=1}^{n} \mathcal{H}(F_i)$ *such that* (1.6) *holds, then*

$$\psi(x, n) \sim \sum_{i=1}^{n} P\Big(X_i \prod_{j=1}^{i} Y_j > x\Big). \tag{2.9}$$

From Theorem 2.2 and [3, Example 2.1], we obtain the following corollary directly.

Corollary 2.2 *In the above-mentioned risk model, suppose that there exists a constant* $0 < c < \infty$ *such that* (2.8) *holds. Suppose that the net losses* $X_1, \cdots, X_n$ *jointly follow a pairwise Sarmanov distribution of the form* (2.2) *which satisfy* (2.3)–(2.5)*. Assume that* $F_i \in \mathcal{L}$ *and* $\overline{F}_i(x) \asymp \overline{F}(x)$ *for some* $F \in \mathcal{S}$ *and all* $i = 1, 2, \cdots, n$*. Then the relation* (2.9) *holds.*

Remark 2.2 Note that in Corollaries 2.1–2.2, condition (1.6) does not appear since it is automatically satisfied.

3 Proofs of the Main Results

To prove Theorem 2.1, we will prepare a proposition on the uniformly asymptotic behavior of weighted sums of heavy-tailed increments, which extends [11, Proposition 5.1] from the independent case to the dependent one.

Proposition 3.1 *Let $X_1,\cdots,X_n$ be real-valued r.v.s with distributions $F_1,\cdots,F_n$ and be CLWD on the interval $[a,b]$. Assume that $F_i\in\mathcal{L}$ and $\overline{F}_i(x)\asymp\overline{F}(x)$ for some $F\in\mathcal{S}$ and all $i=1,2,\cdots,n$. If there exists some positive function $h\in\bigcap\limits_{i=1}^{n}\mathcal{H}(F_i)$ such that (1.6) holds for all $1\le i,j\le n$, then the relation*

$$\mathbb{P}\Big(\sum_{i=1}^{n}c_iX_i>x\Big)\sim\sum_{i=1}^{n}\mathbb{P}(c_iX_i>x) \tag{3.1}$$

holds uniformly for all $(c_1,c_2,\cdots,c_n)\in[a,b]^n$, where the uniformity is understood as

$$\limsup\sup_{(c_1,c_2,\cdots,c_n)\in[a,b]^n}\left|\frac{\mathbb{P}\Big(\sum\limits_{i=1}^{n}c_iX_i>x\Big)}{\sum\limits_{i=1}^{n}\mathbb{P}(c_iX_i>x)}-1\right|=0.$$

To prove the proposition, we need two lemmas which give the uniform asymptotic upper and lower bounds of weighted sums of heavy-tailed increments respectively.

Following the proof of [3, Lemma 3.2] with some obvious changes, we obtain the following lemma.

Lemma 3.1 *Let $X_1,\cdots,X_n$ be real-valued r.v.s with distributions $F_1,\cdots,F_n$ and be CLWD on the interval $[a,b]$. If $F_i\in\mathcal{L}$ and $\overline{F}_i(x)\asymp\overline{F}(x)$ for some $F\in\mathcal{S}$ and all $i=1,2,\cdots,n$, then we have*

$$\limsup\sup_{(c_1,\cdots,c_n)\in[a,b]^n}\frac{\mathbb{P}\Big(\sum\limits_{i=1}^{n}c_iX_i>x\Big)}{\sum\limits_{i=1}^{n}\mathbb{P}(c_iX_i>x)}\le 1. \tag{3.2}$$

The next lemma deals with asymptotic lower bound for weighted sums.

Lemma 3.2 *Let $X_1,\cdots,X_n$ be real-valued r.v.s with distributions $F_1,\cdots,F_n$ and be CLWD on the interval $[a,b]$. Assume that $F_i\in\mathcal{L}$ and $\overline{F}_i(x)\asymp\overline{F}(x)$ for some $F\in\mathcal{S}$ and all $i=1,2,\cdots,n$. If there exists a positive function $h\in\bigcap\limits_{i=1}^{n}\mathcal{H}(F_i)$ such that (1.6) holds, then we have*

$$\liminf\inf_{(c_1,\cdots,c_n)\in[a,b]^n}\frac{\mathbb{P}\Big(\sum\limits_{i=1}^{n}c_iX_i>x\Big)}{\sum\limits_{i=1}^{n}\mathbb{P}(c_iX_i>x)}\ge 1. \tag{3.3}$$

Proof We use the induction method. When $n=1$, the conclusion is obvious. Suppose that (3.3) holds for any fixed positive integer $n=m-1$, we aim to show that it also holds for $n=m$, where $2\le m\le n$. For notational convenience, we write $S_m^c=\sum\limits_{i=1}^{m}c_iX_i$ from now on.

Let $l(x)=\frac{b}{a\wedge 1}h(x)$, where $a\wedge 1=\min(a,1)$. Clearly, $h\in\bigcap\limits_{i=1}^{n}\mathcal{H}(F_i)$ implies $l\in\bigcap\limits_{i=1}^{n}\mathcal{H}(F_i)$.

For any fixed $\varepsilon>0$, from [3, Lemma 3.1] and the induction hypothesis, there exists a constant $x_1>0$ such that

$$\mathbb{P}(c_iX_i>x+(m-1)l(x))\ge(1-\varepsilon)\mathbb{P}(c_iX_i>x),\quad 1\le i\le m \tag{3.4}$$

and

$$\mathbb{P}(S^c_{m-1} > x) \geq (1-\varepsilon) \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x) \tag{3.5}$$

hold for all $x > x_1$ and for all $(c_1, c_2, \cdots, c_m) \in [a,b]^m$. For the sake of convenience, we can pick the above constant x_1 sufficiently large so that $x \geq (m-2)l(x) > x_0$ holds for all $x > x_1$, where x_0 is determined in Definition 2.3. Hence, for all $x > x_1$, we have the following decomposition:

$$\begin{aligned}\mathbb{P}(S^c_m > x) \geq\ & \mathbb{P}(S^c_{m-1} > x + l(x), c_m X_m > -l(x)) \\ & + \mathbb{P}(S^c_{m-1} \in [-(m-1)l(x), (m-1)l(x)], c_m X_m > x + (m-1)l(x)) \\ =:\ & \mathrm{J}_1(x) + \mathrm{J}_2(x).\end{aligned} \tag{3.6}$$

First we estimate $\mathrm{J}_1(x)$. We use the following further decomposition:

$$\begin{aligned}\mathrm{J}_1(x) &= \mathbb{P}(S^c_{m-1} > x + l(x)) - \mathbb{P}(S^c_{m-1} > x + l(x), c_m X_m \leq -l(x)) \\ &\geq \mathbb{P}(S^c_{m-1} > x + l(x)) - \mathbb{P}\Big(c_m X_m \leq -l(x), \bigcup_{i=1}^{m-1} \{c_i X_i > x\}\Big) \\ &\quad - \mathbb{P}\Big(S^c_{m-1} > x, \bigcap_{i=1}^{m-1} \{c_i X_i \leq x\}\Big) \\ &\geq \mathbb{P}(S^c_{m-1} > x + l(x)) - \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x, c_m X_m \leq -l(x)) \\ &\quad - \mathbb{P}\Big(\sum_{i=1}^{m-1} c_i X_i^+ > x, \bigcap_{i=1}^{m-1} \{c_i X_i^+ \leq x\}\Big) \\ &=: \mathrm{J}_{11}(x) - \mathrm{J}_{12}(x) - \mathrm{J}_{13}(x).\end{aligned} \tag{3.7}$$

From now on, we will fix the weights $(c_1, c_2, \cdots, c_m) \in [a,b]^m$. For $\mathrm{J}_{11}(x)$, it follows from (3.4) and (3.5) that

$$\mathrm{J}_{11}(x) \geq (1-2\varepsilon) \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x) \tag{3.8}$$

holds for all $x > x_1$.

To estimate $\mathrm{J}_{12}(x)$, note that condition (1.6) implies that there exists a constant $x_2 > 0$ such that $\mathbb{P}(X_i > x, X_j \leq -h(x)) \leq \varepsilon \mathbb{P}(X_i > x)$ holds for all $x > x_2$ and $1 \leq i, j \leq m$, which yields that

$$\mathbb{P}(c_i X_i > x, c_j X_j \leq -l(x)) \leq \varepsilon \mathbb{P}(c_i X_i > x) \tag{3.9}$$

holds for all $x > b x_2$ and $1 \leq i, j \leq m$. In fact, if $c_i > 1$, then it follows from $l(x) \geq l\big(\frac{x}{c_i}\big)$ that

$$\begin{aligned}\mathbb{P}(c_i X_i > x, c_j X_j \leq -l(x)) &\leq \mathbb{P}\Big(X_i > \frac{x}{c_i}, X_j \leq -\frac{1}{b} l\Big(\frac{x}{c_i}\Big)\Big) \\ &\leq \mathbb{P}\Big(X_i > \frac{x}{c_i}, X_j \leq -h\Big(\frac{x}{c_i}\Big)\Big);\end{aligned}$$

And if $c_i < 1$, then it follows from $l(x) \geq c_i l\big(\frac{x}{c_i}\big) \geq a l\big(\frac{x}{c_i}\big)$ that

$$\mathbb{P}(c_i X_i > x, c_j X_j \leq -l(x)) \leq \mathbb{P}\Big(X_i > \frac{x}{c_i}, X_j \leq -\frac{a}{b} l\Big(\frac{x}{c_i}\Big)\Big)$$

$$= \mathbb{P}\Big(X_i > \frac{x}{c_i}, X_j \le -h\Big(\frac{x}{c_i}\Big)\Big).$$

Hence, (3.9) yields that

$$\mathrm{J}_{12}(x) \le \varepsilon \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x) \tag{3.10}$$

holds for all $x > bx_2$. From [3, Proposition 3.2] and Lemma 3.1, there exists a positive constant x_3, independent of $c_1, c_2, \cdots, c_m$, such that

$$\mathbb{P}\Big(\max_{1\le k\le m-1} c_k X_k^+ > x\Big) \ge (1-\varepsilon) \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x)$$

and

$$\mathbb{P}\Big(\sum_{k=1}^{m-1} c_k X_k^+ > x\Big) \le (1+\varepsilon) \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x)$$

hold for all $x > x_3$, which yields that

$$\begin{aligned}\mathrm{J}_{13}(x) &= \mathbb{P}\Big(\sum_{k=1}^{m-1} c_k X_k^+ > x\Big) - \mathbb{P}\Big(\max_{1\le k\le m-1} \{c_k X_k^+ > x\}\Big) \\ &\le 2\varepsilon \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x) \end{aligned}\tag{3.11}$$

holds for all $x > x_3$.

Plugging (3.8), (3.10)–(3.11) into (3.7), it follows that

$$\mathrm{J}_1(x) \ge (1-5\varepsilon) \sum_{i=1}^{m-1} \mathbb{P}(c_i X_i > x) \tag{3.12}$$

holds for all $x > \max\{x_1, bx_2, x_3\}$.

Now we estimate $\mathrm{J}_2(x)$. We have the following decomposition:

$$\begin{aligned}\mathrm{J}_2(x) &\ge \mathbb{P}(c_m X_m > x + (m-1)l(x)) - \mathbb{P}(c_m X_m > x, S_{m-1}^c < -(m-1)l(x)) \\ &\quad - \mathbb{P}(c_m X_m > x, S_{m-1}^c > (m-1)l(x)) \\ &=: \mathrm{J}_{21}(x) - \mathrm{J}_{22}(x) - \mathrm{J}_{23}(x). \end{aligned}\tag{3.13}$$

For $\mathrm{J}_{21}(x)$, clearly, it follows from (3.4) that

$$\mathrm{J}_{21}(x) \ge (1-\varepsilon)\mathbb{P}(c_m X_m > x) \tag{3.14}$$

holds for all $x > x_1$.

For $\mathrm{J}_{22}(x)$, it is clear that

$$\begin{aligned}\mathrm{J}_{22}(x) &\le \mathbb{P}\Big(c_m X_m > x, \bigcup_{i=1}^{m-1} \{c_i X_i \le -l(x)\}\Big) \\ &\le \sum_{i=1}^{m-1} \mathbb{P}(c_m X_m > x, c_i X_i \le -l(x)) \end{aligned}$$

$$\leq (m-1)\varepsilon\mathbb{P}(c_mX_m > x) \tag{3.15}$$

holds for all $x > bx_2$, where the last step is due to (3.9).

Finally, there is a constant $x_4 > bx_0$ such that $\mathbb{P}(bX_i^+ > l(x)) < \varepsilon$ holds for all $x > x_4$ and $i = 1, \cdots, m$. It follows that

$$\begin{aligned}
\mathrm{J}_{23}(x) &\leq \int_x^\infty \mathbb{P}(S_{m-1}^c > (m-1)l(x) \mid c_mX_m = y)\mathrm{d}\mathbb{P}(c_mX_m \leq y) \\
&\leq M\mathbb{P}\Big(\sum_{i=1}^{m-1} c_iX_i^+ > (m-1)l(x)\Big)\mathbb{P}(c_mX_m > x) \\
&\leq M\sum_{i=1}^{m-1}\mathbb{P}(bX_i^+ > l(x))\mathbb{P}(c_mX_m > x) \\
&\leq (m-1)M\varepsilon\mathbb{P}(c_mX_m > x)
\end{aligned} \tag{3.16}$$

holds for all $x > \max\{x_1, x_4\}$. Plugging (3.14)–(3.16) into (3.13), we obtain that

$$\mathrm{J}_2(x) \geq (1 - (1 + 2(m-1)M)\varepsilon)\mathbb{P}(c_mX_m > x) \tag{3.17}$$

holds for all $x > \max\{x_1, x_2, x_4\}$. Clearly, it follows from (3.6), (3.12) and (3.17) that

$$\mathbb{P}(S_m^c > x) > (1 - (2(m-1)M + 5)\varepsilon)\sum_{i=1}^m \mathbb{P}(c_iX_i > x)$$

holds for all $x > \max\{x_1, bx_2, x_3, x_4\}$. Thus (3.3) holds for $n = m$ due to the arbitrariness of ε. This ends the proof of Lemma 3.2.

Proof of Proposition 3.1 The conclusion follows from Lemmas 3.1–3.2 directly.

Now we are standing in a position to prove the main results.

Proof of Theorem 2.1 Note that for each $n \geq 1$, we have

$$S_n^\theta \leq M_n^\theta \leq \sum_{i=1}^n \theta_iX_i^+.$$

Thus by the condition on $\theta_i, 1 \leq i \leq n$ and using [3, Proposition 3.2] and Propositions 3.1, we will obtain the relations (1.3) immediately.

Proof of Corollary 2.1 For any $1 \leq i < j \leq n$, it is clear that (X_i, X_j) follows a bivariate Samarnov distribution, that is,

$$P(X_i \in \mathrm{d}x_i, X_j \in \mathrm{d}x_j)) = (1 + \omega_{ij}\phi_i(x_i)\phi_j(x_j))F_i(\mathrm{d}x_i)F_j(\mathrm{d}x_j).$$

Hence, (1.6) follows from (2.4) immediately, and Corollary 2.1 follows from Theorem 2.1 and [3, Example 2.1].

Proof of Theorem 2.2 Clearly, for any real numbers $0 < a < b < \infty$, $X_1, \cdots, X_n$ are CLWD on the interval $[a, b]$. In fact, since $X_1, \cdots, X_n$ are CLWD on the interval $(0, 1]$, for any integer $2 \leq m \leq n$, there exist positive numbers x_0 and K such that (2.1) holds uniformly for all $(c_1, c_2, \cdots, c_m) \in (0, 1]^m$ and $x, y > x_0$, which yields that the (2.1) holds uniformly for all $(c_1, c_2, \cdots, c_m) \in [a, b]^m$ and $x, y > bx_0$ since

$$\mathbb{P}\Big(\sum_{i=1}^{m-1} c_iX_i^+ > x \mid c_mX_m = y\Big) = \mathbb{P}\Big(\sum_{i=1}^{m-1} \frac{c_i}{b}X_i^+ > \frac{x}{b} \mid \frac{c_m}{b}X_m = \frac{y}{b}\Big) \leq K\mathbb{P}\Big(\sum_{i=1}^{m-1} c_iX_i^+ > x\Big).$$

The rest of the proof is similar to that of [3, Theorem 4.1], so we omit it.

Proof of Corollary 2.2 The proof is similar to that of Corollary 2.1, so we omit it.

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