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Parallel Translation on Kähler Manifolds*

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Abstract In this paper, the author establishs a real-valued function on Kähler manifolds by holomorphic sectional curvature under parallel translation. The author proves if such functions are equal for two simply-connected, complete Kähler manifolds, then they are holomorphically isometric.

Keywords Kähler manifold, Holomorphic sectional curvature, Parallel translation
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1 Introduction

A classic theorem in [3] shows that a Riemann manifold is a Riemannian locally symmetric space if and only if the sectional curvature is invariant under all parallel translations. A Hermitian symmetric space is of course a Riemannian symmetric space of even dimension, which implies the holomorphic sectional curvature of a Hermitian symmetric space is invariant under all parallel translations. But what will happen for general Kähler manifolds? The object of this paper is to characterize complete simply connected Kähler manifolds by their holomorphic sectional curvature and its behaviour under parallel translations.

Consider two complete simply connected Kähler manifolds and fix a point on each. Any holomorphic isomorphism of the holomorphic tangent space at one of the points onto the holomorphic tangent space at the other induces, through parallel translation, a correspondence between broken geodesics emanating from the one and broken geodesics emanating from the other. We asserts that if the holomorphic sectional curvature parallel translates in the same way along corresponding singly broken geodesics, then the two manifolds are holomorphically isometric.

We now state our main theorem here. Let d be the complex dimension of the Kähler manifolds we consider. Z will be the space of all triples (a, b, Q) where $a \in \mathbb{C}^d, b \in \mathbb{C}^d$ and Q is any complex 1-dimensional subspace of $\mathbb{C}^d$. For each complete complex d-dimensional Kähler manifold M, $m \in M$, and unitary frame $e_1, \cdots, e_d$ at m, we define a real-valued function L on Z as follows. Let $(a, b, Q) \in Z$ with $a = (a^1, \cdots, a^d), b = (b^1, \cdots, b^d)$. Let α be the geodesic segment of length $|a|$ with $\alpha(0) = m, \alpha'(0) = a^i e_i$. Let n be the final point of α. Let $f_1, \cdots, f_d$ be the unitary frame at n obtained by parallel translating the $e_1, \cdots, e_d$ along α. Let β be the geodesic segment of length $|b|$ with $\beta(0) = n, \beta'(0) = b^i f_i$. Let P_0 be the holomorphic section at m into which Q is carried by the holomorphic isomorphism which carries δ_i into e_i (where

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δ_i is the ith unit point in $\mathbb{C}^d$). Let P be the parallel translate of P_0 along $\beta\alpha$. We define $L(a, b, Q) = K(P)$, where $K(P)$ is the holomorphic sectional curvature of P.

Theorem 1.1 *Let M and $'M$ be complete simply connected complex d-dimensional Kähler manifolds, m a point of M, $'m$ a point of $'M$, $e_1, \cdots, e_d$ a unitary frame at m and $'e_1, \cdots, 'e_d$ a unitary frame at $'m$. Let L and $'L$ be the corresponding function defined as above on the corresponding spaces. If $L =' L$, then M and $'M$ are holomorphically isometric under a holomorphic isometry which carries m into $'m$ and whose differential carries each e_i into $'e_i$. This holomorphic isometry is trivially unique.*

2 Connections on the Bundle of Bases

Let M be a complex manifold of complex dimension d. Let $\{z^1, \cdots, z^d\}$ be a set of local complex coordinates, with $z^\alpha = x^\alpha + \mathrm{i}x^{d+\alpha}$, so that $\{x^1, \cdots, x^d, x^{d+1}, \cdots, x^{2d}\}$ are local real coordinates. Let $T_R M$ denote the real tangent bundle of M, it is a real bundle of rank $2d$ equipped with a complex structure J. Let $T^{1,0}M$ denote the holomorphic tangent bundle of M. As well known, the bundles $T^{1,0}M$ and $T_R M$ are isomorphic. For example, we can choose an explicit isomorphism, the bundle map $^\circ : T^{1,0}M \to T_R M$, given by

$$v^\circ = v + \overline{v}, \quad \forall v \in T^{1,0}M.$$

It is easily known that $^\circ$ is a real isomorphism preserving J. The inverse $_\circ : T_R M \to T^{1,0}M$ is given by

$$u_\circ = \frac{1}{2}(u - \mathrm{i}Ju), \quad \forall u \in T_R M.$$

Furthermore, if $v = v^\alpha \frac{\partial}{\partial z^\alpha} \in T^{1,0}M$ locally, then setting $v^\alpha = u^\alpha + \mathrm{i}u^{\alpha+d}$, $v^\circ = u^a \frac{\partial}{\partial x^a}$. Conversely, if $u = u^a \frac{\partial}{\partial x^a}$, then $u_\circ = (u^\alpha + \mathrm{i}u^{\alpha+d}) \frac{\partial}{\partial z^\alpha}$.

Let W be any linear space over $\mathbb{C}$; then we have a natural complex manifold structure on W. For any such W, we have a natural linear holomorphic isomorphism of W_w onto W (for each $w \in W$) that we shall denote by α^w. It is defined as follows. Let $e_1, \cdots, e_n$ be any base of W and $z^1, \cdots, z^n$ be its dual base; then $\alpha^w\big(a^i \frac{\partial}{\partial z^i}(w)\big) = a^i e_i$. It is easily checked it is well-defined. If W and W' are complex linear spaces, f is a linear map of $W \to W'$, and $w \in W$, then it is obvious that $\alpha^{f(w)} \circ df \circ (\alpha^w)^{-1} = f$, where df is the tangent map of f at w.

Let M be a Kähler manifold of complex d-dimension. For any $m \in M$, M_m will denote either $T^{1,0}_m M$ or $T^R_m M$ depending on the actual situation. If $(e_1, \cdots, e_d)$ is a unitary base of M_m, then it is easily checked that $(e_1^\circ, \cdots, e_d^\circ, Je_1^\circ, \cdots, Je_d^\circ)$ is an orthogonal base of M_m and conversely.

Let $\pi : T^{1,0}M \to M$ denote the holomorphic tangent bundle of M. If $\{(U_\alpha, z^i_\alpha) : \alpha \in I\}$ is a local coordinate system on M, we write $\{(U_\alpha, \psi_\alpha); \alpha \in I\}$ be the locally trivialized structure of the bundle $\pi : T^{1,0}M \to M$, where $\psi_\alpha : U_\alpha \times \mathbb{C}^d \to \pi^{-1}(U_\alpha)$ are holomorphic isomorphisms. For $1 \le a \le d$, define

$$S^\alpha_a(p) = \psi_\alpha(p, \delta_a), \quad \forall p \in U_\alpha.$$

Then $S^\alpha = (S^\alpha_1, \cdots, S^\alpha_d)$ is a local field of bases of $T^{1,0}M$ on U_α. For each point $p \in U_\alpha$, we use $B(p)$ to denote the set of all bases of complex vector space $\pi^{-1}(p)$; then there is a 1-1 correspondence between the complex general linear group $GL(d; \mathbb{C})$ and $B(p)$. In fact, for any $A \in GL(d; \mathbb{C})$, the corresponding base is

$$f(p) = (f_1(p), \cdots, f_d(p)) = (S^\alpha_1(p), \cdots, S^\alpha_d(p)) \cdot A = S^\alpha(p) \cdot A,$$

that is

$$f_a(p) = A_a^b \psi_\alpha(p, \delta_b) = \psi_{\alpha,p}(A_a^b \delta_b),$$

where $\psi_{\alpha,p} = \psi_\alpha(p, \cdot) : \mathbb{C}^d \to \pi^{-1}(p)$ is a holomorphic isomorphism. Let $B(M) = \bigcup_{p \in M} B(p)$ and define $\widetilde{\pi} : B(M) \to M$ such that $\widetilde{\pi}(B(p)) = \{p\}$ for any $p \in M$. It is clear that we can make $(B(M), \widetilde{\pi})$ a holomorphic principal bundle on M naturally. In fact, for any $\alpha \in I$, define $\varphi_\alpha : U_\alpha \times GL(d;\mathbb{C}) \to \widetilde{\pi}^{-1}(U_\alpha)$ by

$$\varphi_\alpha(p, A) = S^\alpha(p) \cdot A, \quad \forall (p, A) \in U_\alpha \times GL(d;\mathbb{C}).$$

Then we can define a complex differential structure on $B(M)$ such that the above φ_α becomes a holomorphic isomorphism. In particular, $\varphi_\alpha^{-1} : \widetilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times GL(d;\mathbb{C})$ supplies a local coordinate system for $B(M)$, which is denoted by $(\widetilde{\pi}^{-1}(U_\alpha); z_\alpha^i, A_a^b)$. It is clear that bundles $\widetilde{\pi} : B(M) \to M$ and $\pi : T^{1,0}M \to M$ share the same family of transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(d;\mathbb{C})\}$, where $g_{\alpha\beta}(p) = \psi_{\alpha,p}^{-1} \circ \psi_{\beta,p} = \varphi_{\alpha,p}^{-1} \circ \varphi_{\beta,p}$.

Let $F(M)$ be the real submanifold of $B(M)$ consisting of all $(m, e_1, \cdots, e_d)$ which the $\{e_i\}$ is a unitary base of M_m. Then $F(M)$ is a holomorphic subbundle of $B(M)$.

Both the structure group and the fiber of $B(M)$ are $G = GL(d;\mathbb{C})$, all non-singular d by d matrices with complex matrix elements. The Lie algebra $\mathfrak{L}$ of G is all of the left invariant vector fields on G, which is isomorphic to $\mathfrak{gl}(d;\mathbb{C})$, all $d \times d$ complex matrices. There is a natural isomorphism of the Lie algebra $\mathfrak{L}$ of G onto a Lie algebra $\widetilde{\mathfrak{L}}$ of vertical vector fields on $B(M)$ which will be defined below. In fact, let $A \in \mathfrak{L}$, we will assign a vertical vector field W on $B(M)$ to A. For any $b \in B(M)$, consider any strip map $\varphi : U \times G \to B(M)$ such that $b \in \varphi(U \times G)$. If $\varphi_m(f) = b$ where $\varphi_m = \varphi(m, \cdot)$, we define $W(b) = d\varphi_m A(f)$. It can be checked this definition is independent of the strip map φ; then it is well-defined. The map $A \to W$ is the isomorphism from $\mathfrak{L}$ to $\widetilde{\mathfrak{L}}$, which we denote by λ. For a natural base for L, $V_1^1, \cdots, V_d^d$ such that $V_i^j(e) = \frac{\partial}{\partial A_i^j}(e)$, we define vector fields E_i^j on $B(M)$ by $E_i^j = \lambda V_i^j$.

Let D be the Hermite connection on M. Let $p \in M$ be any fixed point, $\sigma_0 \in \widetilde{\pi}^{-1}(p)$; then $\sigma_0 = (\sigma_1, \cdots, \sigma_d)$ is a base of $\pi^{-1}(p) = T_p^{1,0}M$. Let $\gamma : [0, b] \to M$ be a smooth curve on M with $\gamma(0) = p$. It is well known there exists a unique family of vector fields $\sigma_a(u), 0 \le u \le b, 1 \le a \le d$, parallel along γ, with $\sigma_a(0) = \sigma_a$. Thus $\sigma(u) = (\sigma_1(u), \cdots, \sigma_d(u))$ is a field of base parallel along γ with $\sigma(0) = \sigma_0$, $\sigma(u) \in B(M)$ for $0 \le u \le b$. $\sigma(u)$ is called the horizontal lift of $\gamma(u)$ on $B(M)$ through σ_0, and $\sigma'(0)$ is called the horizontal lift of $\gamma'(0) \in T_p^{1,0}M$ at $\sigma_0 \in \widetilde{\pi}^{-1}(p)$. Such vectors are called horizontal vectors. We denote by H_{σ_0} the set of all horizontal vectors at σ_0, which is a subspace of $B(M)_{\sigma_0}$ and will be called a horizontal subspace at σ_0. We also call the distribution H the induced connection on $B(M)$ by D.

A holomorphic vector $t \in B(M)_{\sigma_0}$ is said to be vertical if $d\widetilde{\pi} t = 0$. The linear space of vertical vectors at σ_0 is called vertical subspace at σ_0 and denoted by V_{σ_0}. It is clear that

$$B(M)_{\sigma_0} = H_{\sigma_0} \oplus V_{\sigma_0}.$$

Under local coordinates (z_α^i, A_a^b) as above, let

$$\omega_a^b = (A^{-1})_c^b (dA_a^c + A_a^d \Gamma_{di}^c dz^i),$$

where $\{\Gamma_{ai}^b\}$ are the Christoffel symbols of the Hermite connection on M. It can be checked directly that $\{\omega_a^b\}$ are well-defined on the whole $B(M)$. Thus we can call $\omega = (\omega_a^b)$, a $d \times d$ matrix of 1-form elements, the 1-form of the connection H.

The curvature form is defined by $\Omega = d\omega - \omega \wedge \omega$, which is a $d \times d$ matrix of $(1,1)$-form elements. We now define certain $(1,0)$-vector fields $E_1, \cdots, E_d$ on $B(M)$ as follows. If $b = (m, e_1, \cdots, e_d) \in B(M)$, then $E_i(b)$ is the unique element of H_b that projects to e_i under $d\widetilde{\pi}$. We also introduce certain $(1,0)$-forms $\omega^1, \cdots, \omega^d$ on $B(M)$, which is independent on H, by if $t \in B(M)_b$, then $\omega^i(t) =$ the ith coefficient of $d\widetilde{\pi}t$ when $d\widetilde{\pi}t$ is expressed linearly in terms of the base $e_1, \cdots, e_d$. So $d\widetilde{\pi}t = \omega^i(t)e_i$. It is obvious that the ω^i and ω_j^k at b are a dual base of the $E_i(b)$ and $E_j^k(b)$.

Under the above local coordinate system, $\omega^i = (A^{-1})^i_j dz^j$, $E_i = A_i^j\big(\frac{\partial}{\partial z^j} - A_a^l \Gamma_{lj}^b \frac{\partial}{\partial A_a^b}\big)$, $E_i^j = A_i^k \frac{\partial}{\partial A_j^k}$. Notice that ω^i and E_i^j are holomorphic 1-form and vector fields on $B(M)$ respectively, while E_i and ω_a^b fail to be holomorphic.

It is well known or can be checked directly that the Cartan structural equations for Kähler manifolds are

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j.$$

In terms of vector fields E_i and E_j^k, the above structural equations can be expressed by the formula

$$[E_i^j, E_k] = \delta_k^j E_i, \quad [E_i, E_j] = -\sum_{k,l} \Omega_k^l(E_i, E_j)E_k^l = 0, \quad [E_i, \overline{E}_j] = -\sum_{k,l} \Omega_k^l(E_i \overline{E}_j)E_k^l.$$

A property of the Ω_i^j that will be useful later is

$$\Omega_i^j = \frac{1}{2} K_{ik\bar{l}}^j \omega^k \wedge \overline{\omega}^l \tag{2.1}$$

for certain C^∞ functions $K_{ik\bar{l}}^j$ on $B(M)$, which can be checked directly under local coordinate system.

3 The Complex Exponential Mappings

When M is looked as a Riemann manifold with the induced Riemann metric, the exponential map exp and Exp have been defined which can be found in [1]. For each $m \in M$, we define $\exp_m^C : M_m \to M$ by $\exp_m^C = \exp_m \cdot^\circ$ as complex exponential map on M. More precisely, if $p \in M_m$, σ_p is the unique geodesic with $\sigma_p(0) = m$ and whose holomorphic tangent vector at m is p, then $\exp_m^C p = \sigma_p(1)$. For each $b = (m, e_1, \cdots, e_d) \in F(M)$, we define $\mathrm{Exp}_b^C : M_m \to B(M)$ as follows. If $\widetilde{\sigma}_p$ is the unique horizontal curve (the holomorphic tangent vectors are horizontal) in $B(M)$ with $\widetilde{\sigma}_p(0) = b$ and $\widetilde{\sigma}_p$ lying over σ_p, i.e., $\widetilde{\pi} \circ \widetilde{\sigma}_p = \sigma_p$, then $\mathrm{Exp}_b^C p = \widetilde{\sigma}_p(1)$. Obviously, Exp_b^C carries rays through the origin in M_m into the corresponding horizontal curves through b.

It is clear that $\widetilde{\pi} \circ \mathrm{Exp}_b^C = \exp_{\widetilde{\pi}b}^C$ and (1) $\exp_m^C$ and Exp_b^C are holomorphic; (2) $\exp_m^C up = \sigma_p(u)$ and $\mathrm{Exp}_b^C up = \widetilde{\sigma}_p(u)$ for all real u; (3) the holomorphic tangent maps $d\exp_m^C$ and $d\,\mathrm{Exp}_b^C$ are non-singular at O.

We carry the $\omega^i, \omega_i^j, \Omega_i^j$ back, via Exp_b^C to forms $\theta^i|_b, \theta_i^j|_b, \Theta_i^j|_b$ on M_m, i.e., $\theta^i|_b = \omega^i \circ d\,\mathrm{Exp}_b^C, \theta_i^j|_b = \omega_i^j \circ d\,\mathrm{Exp}_b^C, \Theta_i^j|_b = \Omega_i^j \circ d\,\mathrm{Exp}_b^C$.

In this section, the point $b = (m, e_1, \cdots, e_d)$ will be kept fixed, so for the remainder of the section, we sometimes drop it. We fix the following notation for this section. We let $z^1, \cdots, z^d$ denote the dual base of $e_1, \cdots, e_d$; thus the z^i are linear functions on M_m and a holomorphic coordinate system of M_m considered as a complex manifold. We let $z = (\Sigma |z^i|^2)^{\frac{1}{2}}$.

Then if $\mathrm{Exp}_b^C p = (n, f_1, \cdots, f_d)$, we shall sometimes write $m(p)$ for n and $e_i(p)$ for f_i, i.e., $\mathrm{Exp}_b^C p = (m(p), e_1(p), \cdots, e_d(p))$. So $m(0) = m, e_i(0) = e_i$.

If γ is any ray from O to p in M_m and σ is the corresponding geodesic from m to $n = \exp_m^C p$, i.e., $\sigma = \exp_m^C \circ \gamma$, then we shall call σ the natural geodesic from m to $\exp_m^C p$. Note that the $e_i(p)$ are the parallel translates of the e_i along the natural geodesic from m to $\exp_m^C p$ for $\mathrm{Exp}_b^C \circ \gamma$ is a horizontal curve lying over σ from $(m, e_1, \cdots, e_d)$ to $(m(p), e_1(p), \cdots, e_d(p))$.

Proposition 3.1 (1) *At* $O \in M_m$, $d \exp_m^C = \alpha^0$ *where* α^0 *is the natural map of* $(M_m)_O \to M_m$.

(2) *If* t *is the holomorphic tangent vector to the ray* $\gamma : \gamma(\lambda) = \lambda c^i e_i$ *at any point on the ray, then* $\theta_i^j(t) = 0$ *and* $\theta^i(t) = c^i$.

(3) *If* t *is a holomorphic tangent vector to* M_m *at* p, *then* $d \exp_m^C t = \theta^i|_b(t) e_i(p)$.

Proof (1) If $t \in (M_m)_O$, then $t = c^i \frac{\partial}{\partial z^i}$ for some complex numbers c^i. Then clearly t is the holomorphic tangent vector of the ray $\gamma : \gamma(v) = v c^i e_i$. The mapping $\exp_m^C$ carries this ray into the geodesic whose holomorphic tangent vector at m is $c^i e_i$. Thus it carries the holomorphic tangent vector to that ray, which is $c^i \frac{\partial}{\partial z^i}$ into the holomorphic tangent vector to that geodesic, which is $c^i e_i$, i.e., $d \exp_m^C = \alpha^0$.

(2) Since Exp_b^C carries γ into a horizontal curve; hence $d\,\mathrm{Exp}_b^C t$ is horizontal. Then $\theta_i^j(t) = \omega_i^j(d\,\mathrm{Exp}_b^C t) = 0$. Since $\mathrm{Exp}_b^C \circ \gamma$ is horizontal and lies over a geodesic, we know that $\omega^i(d\,\mathrm{Exp}_b^C t)$ is constant when t varies through the various holomorphic tangent vectors to γ, i.e., $\theta^i(t)$ is constant on these t. So it suffices to prove, for t the holomorphic tangent vector to this ray at O, that $\theta^i(t) = c^i$. We have known that $d \exp_m^C t = c^i e_i$. From this and the fact that $\widetilde{\pi} \circ \mathrm{Exp}_b^C = \exp_m^C$, it follows that $d\,\mathrm{Exp}_b^C t = c^j E_j(b)$. Hence

$$\theta^i(t) = \omega^i(d\,\mathrm{Exp}_b^C t) = \omega^i(c^j E_j) = c^i.$$

(3) It can be deduced from the following:

$$\begin{aligned} \theta^i|_b(t) &= \omega^i(d\,\mathrm{Exp}_b^C t) \\ &= i\text{th coefficient of } d\widetilde{\pi} \circ d\,\mathrm{Exp}_b^C t \text{ with respect to the } e_i(p) \\ &= i\text{th coefficient of } d \exp_m^C t \text{ with respect to the } e_i(p). \end{aligned}$$

The formula in (3) of the above proposition shows that

$$\langle d \exp_m^C s, d \exp_m^C t \rangle = \Sigma \theta^i(s) \overline{\theta^i}(t), \tag{3.1}$$

$$\| d \exp_m^C s \|^2 = \Sigma |\theta^i(s)|^2 \tag{3.2}$$

for any $s, t \in (M_m)_p$.

The Cartan structural equation, when carried back to M_m under Exp_b^C, becomes

$$d\theta^i = \theta^j \wedge \theta_j^i, \quad d\theta_i^j = \theta_i^k \wedge \theta_k^j + \Theta_i^j.$$

Let ρ be a map of the unit square $[0,1] \times [0,1]$ in $R^2 \cong C$ into $B(M)$ which can be extended to a C^∞ map of some neighborhood of the square into $B(M)$. For each $v \in [0,1]$, let ρ^v be the curve $\rho^v(u) = \rho(u, v)$. Let $\rho^i = \omega^i \circ d\rho, \rho_i^j = \omega_i^j \circ d\rho, P_i^j = \Omega_i^j \circ d\rho$. Let U and V be the vector fields of partial differentiation with respect to the first and second coordinate axes in R^2.

Lemma 3.1 *If each ρ^v is horizontal and lies over a geodesic, then*
(1) $U\rho^i(V) = V\rho^i(U) + \rho^k(U)\rho^i_k(V)$;
(2) $U\rho^j_i(V) = P^j_i(U,V)$;
(3) $U^2\rho^i(V) = \rho^k(U)P^i_k(U,V)$.

Proof We know $[U,V]=0$, because the ρ^v are horizontal we have $\rho^j_i(U)=0$, and because $\widetilde{\pi}\circ\rho^v$ is a geodesic we have $U\rho^i(U)=0$. In each of the following steps we use these facts.

The first structural equation gives

$$U\rho^i(V) - V\rho^i(U) = \rho^k(U)\rho^i_k(V),$$

proving (1). Applying U to (1) gives

$$\begin{aligned} U^2\rho^i(V) &= UV\rho^i(U) + \rho^k(U)U\rho^i_k(V) \\ &= [U,V]\rho^i(U) + \rho^k(U)U\rho^i_k(V) \\ &= \rho^k(U)U\rho^i_k(V). \end{aligned}$$

The second structural equation gives

$$U\rho^j_i(V) = P^j_i(U,V).$$

This is (2) and combined with the previous formula gives (3), so the lemma is proved.

Lemma 3.2 *Let γ be the ray through O in M_m defined by $\gamma(t) = t(c^1e_1+\cdots+c^de_d)$, where the c^i are any complex numbers with $\Sigma|c^i|^2=1$, and let W be the field of holomorphic tangent vectors to γ. Let A be any constant holomorphic tangent vector field on M_m, i.e., $A = a^i\frac{\partial}{\partial z^i}$, where the a^i are complex numbers. Then*
(1) *at O, we have $\theta^i(uA) = \theta^i(uA^\circ) = 0, W^\circ\theta^i(uA) = W^\circ\theta^i(uA^\circ) = a^i$;*
(2) $W^\circ\theta^j_i(uA^\circ) = \Theta^j_i(W^\circ, uA^\circ)$;
(3) $(W^\circ)^2\theta^i(uA^\circ) = c^k\Theta^i_k(W^\circ, uA^\circ)$.

Proof We apply the previous lemma to the 2-cube $\rho = \mathrm{Exp}^C_b\circ p$, where p is the mapping of the unit square into M_m defined by

$$p(u,v) = u\sum_i(c^i + a^iv)e_i.$$

Trivial computations show that $dpU = W^\circ, dpV = uA^\circ$; the definition of Exp^C_b makes each ρ^v a horizontal curve lying over a geodesic. Hence (2) and (3) of Lemma 3.1 imply (2) and (3) of this lemma. To prove (1) we note, following through the definition of ρ, $\rho^i(U)(0,v) = c^i + a^iv$, hence $V\rho^i(U) = a^i$. Then (1) follows from (1) of Lemma 3.1.

The above lemma shows immediately that a flat Kähler manifold is locally holomorphically isometric to C^d. In fact, if the curvature is 0 (the vanishing of holomorphic sectional curvature can imply the vanishing of sectional curvature for the induced Riemann metric), then it shows along any ray out from the origin in M_m that $\theta^i(zA)$ is a linear function, hence $\theta^i(A)$ is a constant and then by (1) it follows that $\theta^i(A) = a^i$. Then by (3.2), $\|d\exp^C_m A\|^2 = \Sigma|\theta^i(A)|^2 = \Sigma|a^i|^2 = \|A\|^2$. So the differential of $\exp^C_m$ is a locally holomorphic isometry. If M is complete and simply connected, this locally holomorphic isometry will be a holomorphic isometry of M_m onto M. We will show below the case of arbitrary holomorphic curvature.

The set of conjugate points of m in M_m is the set of all $p\in M_m$ such that $d\exp^C_m$ is singular at p, i.e., there exists $t\neq 0$ in $(M_m)_p$ with $d\exp^C_m t = 0$. Using (3.2), one sees that $p\in M_m$ is conjugate to m if and only if there is a $t \neq 0$ in $(M_m)_p$ with all $\theta^i(t) = 0$. The conjugate locus, or set of first conjugate points of m in M_m is the set of all those conjugate points p of m in M_m such that no points between O and p in the ray from O to p is a conjugate point. The set of conjugate points and the conjugate locus of m in M are the images under $\exp^C_m$ of these sets in M_m. Since Lemma 3.2 shows that the $\theta^i$are determined by the Θ^j_i, we get the conjugate points are determined by the curvature. If S is the set of non-conjugate points of m in M_m and we put a new Kähler metric on S by $\|t\|^2 = \Sigma|\theta^i(t)|^2$, then $\exp^C_m$ becomes a locally holomorphic isometry of S onto its image in M.

4 The Significance of Θ^j_i

Let M and $'M$ be complete complex d-dimensional Kähler manifolds; m and $'m$ will be fixed points of M and $'M$ respectively; $e_1, \cdots, e_d$ will be a fixed unitary base of M_m and $'e_1, \cdots,' e_d$ be a fixed unitary base of $'M_m$. In general, if Q is any object associated with M, then $'Q$ will be the corresponding object associated with $'M$. However, when the corresponding object associated with $'M$ has several pre-primes in its symbol, we usually drop most of them, allowing one or more pre-primes to indicate that the rest of them are properly there. Again we drop the subscript b which remains fixed.

From now on, R will be a fixed linear transformation of $M_m \to' M_m$ carrying $e_i \to' e_i$; thus R is a holomorphic isometry of M_m onto $'M_m$.

Theorem 4.1 *If $\Theta^j_i =' \Theta^j_i \circ dR$ and $p \in M,' p \in' M$ are not conjugate points of m and $'m$ respectively, then*

(1) $\theta^i =' \theta^i \circ dR$;

(2) *R carries the set of conjugate points and the conjugate locus of M_m onto the set of conjugate points and the conjugate locus of $'M_m$;*

(3) *there exists a neighborhood P of p and a neighborhood O of $\exp^C_m p$ such that $\exp^C_m$ is a holomorphic isomorphism of P onto O; there exist similar $'P,' O$ for $'M$ and $'p$. For any such $P, O,' P,' O$ for which $RP =' P$, the mapping $'\exp^C_m \circ R \circ (\exp^C_m)^{-1}$ is a holomorphic isometry of O onto $'O$.*

Proof (1) It is clear that $\theta^i = \omega^i \circ d\,\mathrm{Exp}^C_b$ is a holomorphic 1-form on M_m. Using Lemma 3.2, we see that $\theta^i(uA) = \theta^i(uA^\circ)$ has a second derivative along any ray of a certain expression involving the Θ^j_i. Since the same is true for $'\theta^i(uA)$; then (1) holds.

(2) It follows from (1) and the characterization of the conjugate points in terms of the θ^i.

(3) Since p is not a conjugate point of m, (2) shows that $'p$ will not be a conjugate point of $'m$ and the inverse function theorem implies the existence of such $P, O,' P,' O$. Let S be the set of non-conjugate points of m in M_m, and put on S the Kähler metric in which, for $t \in S_s, \|t\|^2 = \Sigma|\theta^i(t)|^2$; it is a Kähler metric since it is a pulling back metric. Then (3.2) shows that $(\exp^C_m)^{-1}$ is a holomorphic isometry of O onto P. Similarly we make $'S$, the non-conjugate points of $'m$ in $'M$, into a Kähler manifolds by defining $\|t\|^2 = \Sigma|'\theta^i(t)|^2$ for $t \in' S_s$, and have that $'\exp^C_m$ is a holomorphic isometry of $'P$ onto $'O$.

Because we are assuming $\Theta^j_i =' \Theta^j_i \circ dR$, (1) implies $\theta^i =' \theta^i \circ dR$, thus R is a holomorphic isometry of S onto $'S$. Hence if $RP =' P$ we see that the indicated mapping is a holomorphic isometry.

For each $s \in M_n$ at each $n \in M$, we define a linear map $T_s : M_n \to M_n$ as follows. If $f_1, \cdots, f_d$ is any base of M_n and $\widetilde{s}$ is any (1,0)-vector at $c = (n, f_1, \cdots, f_d)$ which lies over s, i.e., $d\widetilde{\pi}\widetilde{s} = s$, then

$$T_s f_i = -\Omega_i^j(\widetilde{s}, \overline{\widetilde{s}}) f_j.$$

Noticing that (2.1), it is easily checked that this definition is independent of the particular choices of $f_1, \cdots, f_d, \widetilde{s}$ with the above properties. The holomorphic section curvature $K(P)$ of the holomorphic section spanned by s is

$$K(P) = \langle T_s s, s\rangle / \|s\|^4, \tag{4.1}$$

where $\|s\|^2 = \langle s, s\rangle$.

We now define a function L_m on the holomorphic sections of M_m, $L_m(Q)$, for Q a holomorphic section of M_m at $q \in M_m$, will be the holomorphic sectional curvature of a holomorphic section P of M at $\exp_m^C q$. P is obtained by first translating Q to a holomorphic section Q_0 at O, carrying Q_0 to a holomorphic section P_0 at m by $d\exp_m^C$; and then translating P_0 parallel to itself along the natural geodesic from m to $\exp_m^C q$.

For any manifold M, we denote by M^2 a topological space whose points are all (m, P) where $m \in M$ and P is any holomorphic section at m. We define the topology of M^2 in terms of the topology on $B(M)$. We have a natural mapping, that we denote by α, of $B(M) \to M^2$: $\alpha(n, f_1, \cdots, f_d) = (n,$ the complex 1-dimensional subspace of M_n spanned by $f_1)$. We define the topology on M^2 to be the finest one in which this mapping is continuous, i.e., a set V in M^2 is open if and only if $\alpha^{-1}(V)$ is open in $B(M)$. It is easily seen that α is an open mapping.

For M a Kähler manifold, the holomorphic section curvature K is a real valued function on M^2 and we want to show that K is continuous. To prove this, it is convenient to introduce the following function $\widetilde{K}$ on $B(M)$: If $c = (n, f_1, \cdots, f_d) \in B(M)$, then

$$\widetilde{K}(c) = \text{ holomorphic section curvature of the holomorphic section spanned by } f_1.$$

We prove $\widetilde{K} \in C^\infty$ as follows. It is clear $\langle T_s s, s\rangle = -\sum\limits_{i,j,k} \omega^i(\widetilde{s})\overline{\omega^j(\widetilde{s})}\Omega_i^k(\widetilde{s}, \overline{\widetilde{s}}) h_{k\bar{j}}$, where $h_{k\bar{j}} = \langle f_k, f_j\rangle$. Taking $\widetilde{s} = E_1(c)$, we find $\widetilde{K} = -\Omega_1^k(E_1, \overline{E_1}) h_{k\bar{1}} (h_{1\bar{1}})^{-2}$. Since the $\Omega_i^j, E_i, h_{i\bar{j}}$ are all C^∞ functions, this shows $\widetilde{K} \in C^\infty$. Continuity of K now follows from the fact that α is open and $K^{-1}(V) = \alpha\widetilde{K}^{-1}(V)$ for V any subset of R. Since we have showed $\widetilde{K} \in C^\infty$, it would follow in essentially the same way that $K \in C^\infty$ if we had introduced the complex structure on M^2 and proved α is holomorphic.

We now give again the definition of L_m, but in slightly different terms. We first define a map that we denote by f from M_m^2 to M^2. If Q is the holomorphic section at $q \in M_m$ spanned by $a^i \frac{\partial}{\partial z^i}(q)$, then $P = f(Q)$ is the holomorphic section at $\exp_m^C q$ spanned by $a^i e_i$. Then we define $L_m = K \circ f$.

Thus continuity of L_m will follow from continuity of f. So we briefly indicate a proof that f is continuous. We define a function F from $B(M_m)$ to $B(M)$ by $F(p, h_1, \cdots, h_d) = \mathrm{Exp}_c^C p$, where $c = (m, \alpha^p h_1, \cdots, \alpha^p h_d)$. One can prove F is holomorphic and clearly $\alpha \circ F = f \circ \alpha$; this and openness of α imply the continuity of f. And once more, if α is holomorphic, so is f.

Theorem 4.2 *$\Theta_i^j =' \Theta_i^j \circ dR$ if and only if $L_m =' L_m \circ dR$.*

Proof We first show that $L_m =' L_m \circ dR$ implies $\Theta_i^j =' \Theta_i^j \circ dR$. Notice that the holomorphic section curvature on a Kähler manifold determines the curvature tensor of the induced Riemann metric, the expression (4.21) of K_{ikl}^j in [2] shows that

$$K_{ikl}^j =' K_{ikl}^j \circ R. \tag{4.2}$$

Next we consider $\Theta_i^j(T^\circ, uA^\circ)$, where $A = a^i \frac{\partial}{\partial z^i}$, the a^i being any complex numbers, and T being the unit radial holomorphic vector field on M_m. Because $\widetilde{\pi} \circ \mathrm{Exp}_b^C = \exp_m^C$ and (3) in Proposition 3.1 shows that $d\exp_m^C uA = \theta^i(uA)e_i(p)$, we see that the horizontal part of $d\,\mathrm{Exp}_b^C uA = \theta^i(uA)E_i$. Now consider any fixed ray γ emanating from O in M_m, say $\gamma(u) = uc^i e_i$, and with $\Sigma|c^i|^2 = 1$. At points on this ray, we have $d\,\mathrm{Exp}_b^C T = c^i E_i$. Hence we have, at such points

$$\begin{aligned}\Theta_i^j(T^\circ, uA^\circ) &= \Omega_i^j(d\,\mathrm{Exp}_b^C T + \overline{d\,\mathrm{Exp}_b^C T}, d\,\mathrm{Exp}_b^C uA + \overline{d\,\mathrm{Exp}_b^C uA})\\ &= \Omega_i^j(d\,\mathrm{Exp}_b^C T, \overline{d\,\mathrm{Exp}_b^C uA}) + \Omega_i^j(\overline{d\,\mathrm{Exp}_b^C T}, d\,\mathrm{Exp}_b^C uA)\\ &= \Omega_i^j(c^\alpha E_\alpha, \overline{\theta^\beta(uA)E_\beta}) + \Omega_i^j(\overline{c}^\alpha \overline{E}_\alpha, \theta^\beta(uA)E_\beta)\\ &= c^\alpha \overline{\theta^\beta(uA)} K_{i\alpha\beta}^j - \overline{c}^\alpha \theta^\beta(uA) K_{i\beta\alpha}^j.\end{aligned}$$

Using (3) in Lemma 3.2, this shows that

$$(W^\circ)^2 \theta^j(uA) = c^i(c^\alpha \overline{\theta^\beta(uA)} K_{i\alpha\beta}^j - \overline{c}^\alpha \theta^\beta(uA) K_{i\beta\alpha}^j).$$

Notice that $K_{ikl}^j =' K_{ikl}^j \circ R$ and θ^β are holomorphic 1-forms, we conclude that

$$\theta^\beta(uA) =' \theta^\beta(uA) \circ R. \tag{4.3}$$

Now let $B = b^i \frac{\partial}{\partial z^i}$; then

$$\begin{aligned}\Theta_i^j(uA^\circ, uB^\circ) &= \Omega_i^j((d\,\mathrm{Exp}_b^C uA)^\circ, (d\,\mathrm{Exp}_b^C uB)^\circ)\\ &= \theta^\alpha(uA)\overline{\theta^\beta(uB)} K_{i\alpha\beta}^j - \overline{\theta^\alpha(uA)}\theta^\beta(uB) K_{i\beta\alpha}^j,\end{aligned}$$

and we have the corresponding formula for $'\Theta_i^j$. This plus (4.2)–(4.3) proves the desired conclusion $\Theta_i^j =' \Theta_i^j \circ dR$ at all points other than O. The desired relation then holds also at O by continuity.

For the proof of the other half of this theorem, we define $\widetilde{f} : T^{1,0}(M_m) \to T^{1,0}M$ by $\widetilde{f}\big(a^i \frac{\partial}{\partial z^i}(p)\big) = a^i e_i(p)$. It is obvious that if Q is the holomorphic section at $p \in M_m$ spanned by s, then $f(Q)$ is the holomorphic section at $\exp_m^C p$ spanned by $\widetilde{f}(s)$.

We first consider holomorphic sections at points p which are not conjugate points of m; then the corresponding points Rp of M_m are not conjugate points of $'m$. We shall prove first that $L_m =' L_m \circ f$ at such points, then use continuity to obtain this at other points.

We now assume $\Theta_i^j =' \Theta_i^j \circ dR$. We see immediately that $\theta^i =' \theta^i \circ dR$ by Theorem 4.1.

Now consider any fixed $p \in M_m$, not conjugate to m. We define a map β of $(M_m)_p$ onto itself by $\beta(t) = (d\exp_m^C)^{-1} \circ \widetilde{f}(t)$. Clearly, $d\exp_m^C \circ \beta = \widetilde{f}$ and then because $\widetilde{\pi} \circ \mathrm{Exp}_b^C = \exp_m^C$ it follows that $(d\,\mathrm{Exp}_b^C \circ \beta)(s)$ lies over $(d\exp_m^C \circ \beta)(s) = \widetilde{f}(s)$. By Proposition 3.1, we see that $\beta(t) = \theta^i(t)\frac{\partial}{\partial z^i}(p)$. Since $dR\big(\frac{\partial}{\partial z^i}\big) = \frac{\partial}{\partial' z^i}$ and $\theta^i =' \theta^i \circ dR$, it follows from this that $dR \circ \beta =' \beta \circ dR$.

Now let Q be any holomorphic section at p, spanned by s. So $L_m(Q)$ is the holomorphic curvature of the holomorphic section P at $\exp_m^C p$ spanned by $\widetilde{f}(s)$. Using this and (4.1) and the fact that $(d\,\mathrm{Exp}_b^C \circ \beta)(s)$ lies over $\widetilde{f}(s)$, we have

$$\begin{aligned}L_m(Q) = \sum_{i,j} &\omega^i((d\,\mathrm{Exp}_b^C \circ \beta)(s))\overline{\omega^j((d\,\mathrm{Exp}_b^C \circ \beta)(s))}\Omega_i^j((d\,\mathrm{Exp}_b^C \circ \beta)(s),\\ &\overline{(d\,\mathrm{Exp}_b^C \circ \beta)(s))}/\alpha^2(\widetilde{f}(s), \overline{\widetilde{f}(s)})\\ = \sum_{i,j} &\theta^i(\beta s)\overline{\theta^j(\beta s)}\Theta_i^j(\beta s, \overline{\beta s})/\Big[\sum_i (\theta^i(\beta s))^2 \sum_j (\overline{\theta^j(\beta s)})^2 - \sum_i |\theta^i(\beta s)|^2\Big].\end{aligned}$$

And the corresponding formula of course holds for $'L('p)$. From these two formulas and the facts: (1) $\theta^i ='\theta^i \circ dR$; (2) $\Theta_i^j =' \Theta_i^j \circ dR$; (3) $dR \circ \beta =' \beta \circ dR$, it follows trivially that $L_m(Q) =' L_m(dRQ)$, i.e., $L_m =' L_m \circ dR$. This is for any holomorphic section Q at any p not conjugate to m. Using the well known facts that along each ray in M_m the conjugate points are isolated, it follows, by continuity, that $L_m =' L_m \circ dR$ for holomorphic sections at all points, completing the theorem.

5 Proof of Theorem 1.1

We again let M and $'M$ be complete, complex d-dimensional Kähler manifolds with $m, e_1, \cdots, e_d$, $b = (m, e_1, \cdots, e_d), 'm, 'e_1, \cdots, 'e_d, 'b = ('m, 'e_1, \cdots, 'e_d)$ fixed as before, including that the e_i and $'e_i$ are unitary bases. We continue to use the pre-prime systematically as before and R will again be the fixed linear map of $M_m \to' M_m$ carrying $e_i \to' e_i$. However the fixed b and $'b$ of this section need not be the same as those previously held fixed. We shall apply the results of earlier sections, stated there with b and $'b$ fixed, to points other than the fixed b and $'b$ of this section. We define that I_c where $c = (n, f_1, \cdots, f_d)$ is any point of $B(M)$, is the linear transformation of $C^d \to M_n$ carrying δ_i into f_i. Let O be the origin in $\mathbb{C}^d$ and we define

$$\exp_O^C = \exp_m^C \circ I_b, \quad \mathrm{Exp}_O^C = \mathrm{Exp}_b^C \circ I_b.$$

If r is any point in $\mathbb{C}^d$, we define

$$\exp_r^C = \exp_{\exp_O^C r}^C \circ I_{\mathrm{Exp}_O^C r}, \quad \mathrm{Exp}_r^C = \mathrm{Exp}_{\mathrm{Exp}_O^C r}^C \circ I_{\mathrm{Exp}_O^C r}.$$

Thus the effect of $\exp_r^C$ is to map C^d into M by first mapping it into M_m, then parallel translating M_m along the geodesic into which $\exp_m^C$ carries the ray from O to $I_b r$, then spraying onto M via the geodesics at $\exp_O^C r$. Clearly, $\pi \circ \mathrm{Exp}_r^C = \exp_r^C$. For $(r, s) \in \mathbb{C}^d \times \mathbb{C}^d$, we define

$$m(r) = \exp_r^C O = \exp_O^C r, \quad m(r, s) = \exp_r^C s,$$
$$b(r) = \mathrm{Exp}_r^C O = \mathrm{Exp}_O^C r, \quad b(r, s) = \mathrm{Exp}_r^C s.$$

Clearly $m(r, 0) = m(r), b(r, 0) = b(r)$. If $b(r, s) = (n, f_1, \cdots, f_d)$, then clearly $n = m(r, s)$ and we define $e_i(r, s) = f_i$, i.e., $b(r, s) = (m(r, s), e_1(r, s), \cdots, e_d(r, s))$; and we similarly define $e_i(r)$ and have $b(r) = (m(r), e_1(r), \cdots, e_d(r))$. Thus $e_i(r, 0) = e_i(r)$.

We let $B(p, \delta)$ be the open ball of radius δ about the point p, for p in any metric space. We define a function Δ on $\mathbb{C}^d$, whose values are positive real numbers or ∞, by $\Delta(r) = \sup\{\delta \mid \exp_r^C$ maps $B(O, \delta)$ onto $B(\exp_r^C O, \delta)$ and $'\exp_r^C$ maps $B(O, \delta)$ onto $B('\exp_r^C O, \delta)$ such that both maps are holomorphic isomorphisms$\}$. Thus Δ is bounded below by a positive number on any compact subset of $\mathbb{C}^d$. We let F be the subset of $\mathbb{C}^d \times \mathbb{C}^d$ consisting of those (r, s) with any point of C^d and $|s| < \Delta(r)$. We also define, for $m \in M$,

$$\Delta(m) = \sup\{\delta \mid \exp_m^C \text{ is a holomorphic isomorphism of } B(O, \delta) \text{ onto an open subset of } M\}.$$

Define an equivalence relation $\sim$, on the points of F by $(r_1, s_1,) \sim (r_2, s_2)$ if and only if all three of the following hold: (1) $m(r_1, s_1) = m(r_2, s_2)$, (2) $'m(r_1, s_1) =' m(r_2, s_2)$, (3) $b(r_1, s_1)g = b(r_2, s_2)$ implies $'b(r_1, s_1)g =' b(r_2, s_2)$ (bg denotes the transform of $b \in F(M)$ by g, an element of the unitary group, under the action of the group on $F(M)$). Define X to be the set of equivalence classes of this equivalence relation. Let I denote the natural map of $F \to X : I(r, s) =$ equivalence class containing (r, s); and let I_r be the map of $B(O, \Delta(r))$

into X defined by $I_r(s) = I(r,s)$. We also define a map e, of X into M, by $e(x) = m(r,s)$, if $(r,s) \in x$. Clearly $\exp_r^C = e \circ I_r$; it is obvious that e maps X onto M and $'e$ maps X onto $'M$ because M and $'M$ are complete.

The following lemma is due to [1].

Lemma 5.1 *Let n and p be points in M, and $\alpha_1, \alpha_2, \beta$ be paths with α_1 going from n to p, β from p to p, and α_2 from p to n. Let c be any point of $F(M)$ lying over n, d be the parallel translation of c along α_1, g be the holonomy element generated by d and β, h be the holonomy element generated by $\alpha_2\alpha_1$ and c. Then the holonomy element generated by $\alpha_2\beta\alpha_1$ and c is hg.*

Our procedure from this point is to make X into a topological space, show e and $'e$ are local homeomorphisms, use this to put a Kähler metric on X for which e and $'e$ are locally holomorphic isometries, prove X complete, and deduce that e and $'e$ are covering mappings. Then if M and $'M$ are simply connected, we conclude that e and $'e$ are homeomorphisms, thus $'e \circ e^{-1}$ is a holomorphic isometry of M onto $'M$.

We now define the topology on X by the condition that each I_r shall be an open mapping of $B(O, \Delta(r))$ into X, i.e., the topology is generated by all sets of the form $I_r V$ where r is any point in $\mathbb{C}^d$ and V is any open subset of $B(O, \Delta(r))$. Define $P_r = I_r B(O, \Delta(r))$.

Since $\exp_r^C$ maps $B(O, \Delta(r))$ 1-1 onto $B(m(r), \Delta(r))$ and $\exp_r^C = e \circ I_r$, we have that e maps P_r 1-1 onto $B(m(r), \Delta(r))$ and $'e$ maps P_r 1-1 onto $B('m(r), \Delta(r))$. Furthermore, I_r maps $B(O, \Delta(r))$ 1-1 onto P_r.

Lemma 5.2 *e and $'e$ are continuous.*

Proof Let $e(x) = n$, and V be any neighborhood of n. Let $(r,s) \in x$. Then $(\exp_r^C)^{-1} V \cap B(O, \Delta(r))$ is open, hence $P = I_r((\exp_r^C)^{-1} V \cap B(O, \Delta(r)))$ is open in X. It suffices to show that $x \in P$ and $e(P) \subset V$. In fact, we have $\exp_r^C s = e(x) = n \in V$, showing $s \in (\exp_r^C)^{-1} V$, and $(r,s) \in x$ implies $s \in B(O, \Delta(r))$. Thus $s \in (\exp_r^C)^{-1} V \cap B(O, \Delta(r))$, hence $x = I_r s \in P$. We have $e(P) \subset V$ because $y \in P$ implies $y = I_r s_1$ for some $s_1 \in (\exp_r^C)^{-1} V \cap B(O, \Delta(r))$, thus $e(y) = \exp_r^C s_1 \in V$.

Lemma 5.3 *If $L =' L$, then $L_{m(r)} =' L_{m(r)} \circ dR_r$ for all $r \in C^d$, where R_r is the linear map of $M_{m(r)}$ onto $'M_{m(r)}$ which carries $e_i(r)$ into $'e_i(r)$.*

Proof Let S be any holomorphic section at $q \in M_{m(r)}$, spanned by $a^i \frac{\partial}{\partial z^i}(q)$, where the z^i are the linear coordinates on $M_{m(r)}$ dual to the $e_i(r)$. Let $I_{b(r)} t = q$. Then by definition $L_{m(r)}(S) = K(P)$, where P is the holomorphic section at $m(r,t)$ spanned by $a^i e_i(r,t)$. Also by definition this equals $L(r,t,Q)$, where Q is the complex 1-dimensional subspace of C^d spanned by $(a^1, \cdots, a^d)$. If $L =' L$, this means $L_{m(r)}(S) =' L_{m(r)}('S)$, where $'S$ is the holomorphic section of $'M_{m(r)}$ spanned by $a^i \frac{\partial}{\partial' z^i}('q)$, the $'z^i$ being the linear coordinates on $'M_{m(r)}$ dual to the $'e_i(r)$, and $'I_{b(r)} t =' q$. Since $R_r q =' q$ and $dR_r\left(\frac{\partial}{\partial z^i}\right) = \left(\frac{\partial}{\partial' z^i}\right)('q)$, the statement that $L_{m(r)}(S) =' L_{m(r)}('S)$ says that $L_{m(r)}(S) =' L_{m(r)}(dR_r S)$, thus $L_{m(r)} =' L_{m(r)} \circ dR_r$.

By the discussion in the previous sections, we can conclude that

$$\theta^i|_{b(r)} =' \theta^i|_{'b(r)} \circ dR_r, \quad \theta_i^j|_{b(r)} =' \theta_i^j|_{b(r)} \circ dR_r, \quad \Theta_i^j|_{b(r)} =' \Theta_i^j|_{b(r)} \circ dR_r.$$

For each $r \in C^d$, we define a map S_r from $B(m(r), \Delta(r))$ onto $B('m(r), \Delta(r))$ by

$$S_r =' \exp_r^C \circ (\exp_r^C)^{-1}|_{B(m(r), \Delta(r))}.$$

Lemma 5.4 *For each $r \in \mathbb{C}^d$, S_r is a holomorphic isometry of $B(m(r), \Delta(r))$ onto $B('m(r), \Delta(r))$ and for $|s| < \Delta(r)$, we have $S_r(m(r,s)) =' m(r,s)$ and $dS_r e_i(r,s) =' e_i(r,s)$.*

Proof It is obvious that $S_r m(r,s) ='m(r,s)$. Since

$$'\exp_r^C \circ (\exp_r^C)^{-1} =' \exp_{m(r)}^C \circ' I_{b(r)} \circ I_{b(r)}^{-1} \circ (\exp_{m(r)}^C)^{-1} =' \exp_{m(r)}^C R_r \circ (\exp_{m(r)}^C)^{-1},$$

we can conclude that S_r is a holomorphic isometry by Theorem 4.1. For the remainder of the lemma, we first consider $s = 0$. Then $dS_r = d'\exp_{m(r)}^C \circ dR_r \circ d(\exp_{m(r)}^C)^{-1} =' \alpha^0 \circ dR_r \circ (\alpha^0)^{-1} = R_r$. For a general s, we apply (3) in Proposition 3.1 with $b(r)$ for b, $e_i(r)$ for e_i and $e_i(r,s)$ for $e_i(p)$. It shows, for $t \in M_{m(r,s)}$, that $d\exp_{m(r)}^C t = \theta^i|_{b(r)}(t)e_i(r,s)$. Similarly, for $'t \in' M_{m(r,s)}$, $d'\exp_{m(r)}^C('t) =' \theta^i|_{b(r)}('t)'e_i(r,s)$. Taking $'t = dR_r t$, this gives $d'\exp_{m(r)}^C \circ dR_r t = '\theta^i|_{b(r)}(dR_r t)'e_i(r,s) = \theta^i|_{b(r)}(t)'e_i(r,s)$. Thus at $m(r,s)$, dS_r carries $\theta^i|_{b(r)}(t)e_i(r,s)$ into $\theta^i|_{b(r)}(t)'e_i(r,s)$. Since $|s| < \Delta(r)$, $m(r,s)$ can not be conjugate to $m(r)$ along $\exp_r^C$ of the ray from O to s, so this shows that dS_r carries $e_i(r,s)$ into $'e_i(r,s)$.

From this point, we shall often write $(\exp_r^C)^{-1}$ for $(\exp_r^C|_{B(O,\Delta(r))})^{-1}$.

Lemma 5.5 *Let (r_1,s_1) and (r_2,s_2) be in F. If $(r_1,s_1) \sim (r_2,s_2)$, then there is a neighborhood O_1 of s_1 and a neighborhood O_2 of s_2, with $O_i \subset B(0,\Delta(r_i))$ and such that all the following hold*:

(1) $(\exp_{r_2}^C)^{-1} \circ \exp_{r_1}^C$ *is a holomorphic isomorphism, mapping O_1 onto O_2*;

(2) $('\exp_{r_2}^C)^{-1} \circ' \exp_{r_1}^C$ *is the same as* $(\exp_{r_2}^C)^{-1} \circ \exp_{r_1}^C$ *on* O_1;

(3) *if $p_1 \in O_1$ and $p_2 \in O_2$, then $\exp_{r_1}^C p_1 = \exp_{r_2}^C p_2$ implies $(r_1,p_1) \sim (r_2,p_2)$.*

Proof Let $n = \exp_{r_1}^C s_1 = \exp_{r_2}^C s_2$, $'n =' \exp_{r_1}^C s_1 =' \exp_{r_2}^C s_2$. Choose a positive real number ε such that $B(n,\varepsilon) \subset \exp_{r_1}^C B(O,\Delta(r_1)) \cap \exp_{r_2}^C B(O,\Delta(r_2))$, $B('n,\varepsilon) \subset' \exp_{r_1}^C B(O,\Delta(r_1)) \cap$ $'\exp_{r_2}^C B(O,\Delta(r_2))$, $\varepsilon < \Delta(n)$ and $\varepsilon < \Delta('n)$. Define

$$O_1 = (\exp_{r_1}^C)^{-1} B(n,\varepsilon), \quad O_2 = (\exp_{r_2}^C)^{-1} B(n,\varepsilon).$$

Then conclusion (1) above holds trivially.

Next we show $S_{r_1} = S_{r_2}$ on $B(n,\varepsilon)$. Since we know S_{r_1} and S_{r_2} are holomorphic isometries and both carry n into $'n$, it suffices to show both (a) $B('n,\varepsilon)$ is included in the component of $'n$ of $S_{r_1}(B(m(r_1),\Delta(r_1))) \cap S_{r_2}(B(m(r_2),\Delta(r_2)))$ and (b) $dS_{r_1} = dS_{r_2}$ at n. The choice of ε above makes it clear that $S_{r_i}(B(m(r_i),\Delta(r_i)))$ contains $B('n,\varepsilon)$, so (a) holds. To prove (b), it is sufficient to show that $dS_{r_1} e_i(r_2,s_2) =' e_i(r_2,s_2)$. By assumption that $(r_1,s_1) \sim (r_2,s_2)$, we know $b(r_2,s_2) = b(r_1,s_1)g$ implies $'b(r_2,s_2) =' b(r_1,s_1)g$. Let $g = (g_i^j)$ and these statements say

$$e_i(r_2,s_2) = g_i^j e_j(r_1,s_1), \quad 'e_i(r_2,s_2) = g_i^j\, 'e_j(r_1,s_1),$$

hence $dS_{r_1} e_i(r_2,s_2) = g_i^j dS_{r_1} e_j(r_1,s_1) = g_i^j\, 'e_j(r_1,s_1) =' e_2(r_2,s_2)$, proving (b) and thus showing that $S_{r_1} = S_{r_2}$ on $B(n,\varepsilon)$.

Let us write $\exp_{r_1}^C$ for the mapping of O_1 into $B(n,\varepsilon)$, $\exp_{r_2}^C$ for the mapping of O_2 into $B(n,\varepsilon)$. Because $S_{r_1} = S_{r_2}$ on $B(n,\varepsilon)$, we have from the definition of S_r,

$$'\exp_{r_1}^C \circ (\exp_{r_1}^C)^{-1} =' \exp_{r_2}^C \circ (\exp_{r_2}^C)^{-1}.$$

Thus $('\exp_{r_2}^C)^{-1} \circ' \exp_{r_1}^C = (\exp_{r_2}^C)^{-1} \circ \exp_{r_1}^C$, proving (2).

Now we prove (3). Fix any $p_1 \in O_1$ and $p_2 \in O_2$ with $\exp_{r_1}^C p_1 = \exp_{r_2}^C p_2$. By (2), we know that $'\exp_{r_1}^C p_1 =' \exp_{r_2}^C p_2$. It remains to show $b(r_2,p_2) = b(r_1,p_1)h$ implies $'b(r_2,p_2) = 'b(r_1,p_1)h$. We know that $b(r_2,s_2) = b(r_1,s_1)g$ implies $'b(r_2,s_2) =' b(r_1,s_1)g$.

Let ρ_i be the geodesic into which $\exp_O^C$ carries the ray from O to r_i, σ_i be the geodesic into which $\exp_{r_i}^C$ carries the ray from O to s_i, let $\beta = \sigma_2\rho_2\rho_1^{-1}\sigma_1^{-1}$. Let α_2 be the unique geodesic in $B(n,\varepsilon)$ from n to $m(r_1,p_1)=m(r_2,p_2)$, $\alpha_2^{-1}=\alpha_1$. Let $\overline{\alpha}_1$ be the unique horizontal curve over α_1 which ends at $b(r_1,s_1)$ and let c be its initial point. Since the holonomy element by $\alpha_2\alpha_1$ and c is the identity, it follows from Lemma 5.1 that the holonomy element generated by $\alpha_2\beta\alpha_1$ and c is g; similarly, the holonomy element generated by ${}'\alpha_2'\beta'\alpha_1$ and ${}'c$ is g.

Let τ_i be the geodesic into which $\exp_{r_i}^C$ carries the ray from O to p_i, $\delta=\tau_2\rho_2\rho_1^{-1}\tau_1^{-1}$, $\gamma_i=\alpha_2\sigma_i\tau_i^{-1}$, and let k_i be the holonomy element generated by $b(r_i,p_i)$ and γ_i. Then the holonomy element generated by $b(r_1,p_1)$ and δ is clearly $k_1k_2^{-1}g$, i.e., $b(r_2,p_2)=b(r_1,p_1)k_1k_2^{-1}g$.

Since S_{r_i} carries $\tau_i\to{}'\tau_i$, $\sigma_i\to{}'\sigma_i$, $\alpha_i\to{}'\alpha_i$ and dS_{r_i} carries $e_i(r_j,p_j)\to{}'e_i(r_j,p_j)$, $e_i(r_j,s_j)\to{}'e_i(r_j,s_j)$, and $e_i(r_j)\to{}'e_i(r_j)$, it follows that the holonomy element generated by ${}'\gamma_i$ and ${}'b(r_i,p_i)$ is the same as that generated by γ_i and $b(r_i,p_i)$, thus is k_i. Then it follows that the holonomy element generated by ${}'\delta$ is $k_ik_2^{-1}g$, i.e., ${}'b(r_2,p_2)={}'b(r_1,p_1)k_1k_2^{-1}g$. This proves (3).

Lemma 5.6 *Each I_r is continuous.*

Proof It is sufficient to show, for each such finite intersection, that $I_r^{-1}(I_{r_1}P_1\cap\cdots\cap I_{r_k}P_k)$ is open in $B(O,\Delta(r))$, where all $r_i\in\mathbb{C}^d$ and $P_i\subset B(O,\Delta(r))$ are open. Let $x\in I_{r_1}P_1\cap\cdots\cap I_{r_k}P_k$, $x=I_rs=I_{r_j}s_j$. By the previous lemma, we can find neighborhoods $O_1,\cdots,O_k$ of s and neighborhoods $Q_1,\cdots,Q_k$ of $s_1,\cdots,s_k$ such that $O_i\subseteq B(O,\Delta(r)),Q_i\subseteq P_i$, $(\exp_{r_i}^C)^{-1}\circ\exp_r^C$ is a holomorphic isomorphism of O_i onto Q_i, and for $t_i\in O_i$ and $q_i\in Q_i$, we have $(r,t_i)\sim(r,q_i)$ if $\exp_r^C t_i=\exp_{r_i}^C q_i$.

Let $V=O_1\cap\cdots\cap O_k$ and we show $I_rV\subseteq I_{r_1}P_1\cap\cdots\cap I_{r_k}P_k$. Let $t\in V$ and we must show $I_rt\in I_{r_i}Q_i$ for each i. Let $q_i=(\exp_{r_i}^C)^{-1}\circ\exp_r^C t$. We have $(r,t)\sim(r,q_i)$, hence $I_rt=I_rq_i\in I_rQ_i$.

Furthermore, we have the following lemma.

Lemma 5.7 *For any r_1 and r_2 in C^d, the mappings I_{r_1} and I_{r_2} are holomorphic related, i.e., $(I_{r_2}^{-1}|_{P_{r_1}\cap P_{r_2}})\circ I_{r_1}$ is holomorphic.*

Proof Let $x\in P_{r_1}\cap P_{r_2}$ with $x=I_{r_1}s_1=I_{r_2}s_2$. Choose the neighborhoods O_1 and O_2 of s_1 and s_2 given by Lemma 5.5. Then on O_1, we have $(I_{r_2}^{-1}|_{P_{r_1}\cap P_{r_2}})\circ I_{r_1}=(\exp_{r_2}^C)^{-1}\circ\exp_{r_1}^C$. Since the latter is holomorphic, so is the former. This holds for every such x, so the lemma is proved.

This lemma shows that the mappings I_r induce a complex structure on X and we henceforth consider X as a complex manifold in this way. Since $\exp_r^C, {}'\exp_r^C$ and I_r are holomorphic maps on $B(O,\Delta(r))$, it follows that e and ${}'e$ are holomorphic maps of X into M and ${}'M$. We now define the Kähler structure on X by the condition that e and ${}'e$ shall be locally holomorphic isometries.

Definition 5.1 *If u and $v\in X_x$, we define $\langle u,v\rangle=\langle deu,dev\rangle=\langle d'eu,d'ev\rangle$.*

The second equality holds because the S_r are holomorphic isometries. It is clear that we now have made X into a Kähler manifold, and so that e and ${}'e$ are locally holomorphic isometries.

Lemma 5.8 *X is complete.*

Proof Let $x_0=I_OO$. For each ray ρ emanating from O in C^d, we find $I_\rho O$ is a geodesic in X; this follows from the facts that $e\circ I_\rho O=\exp_\rho^C O=\exp_O\circ\rho$ is a geodesic in M and the local holomorphic isometry of X with M. Since the rays ρ are infinitely extendable, we see that these geodesics are infinitely extendable. They are also all the geodesics emanating from x_0, hence X is complete.

The following theorem is obvious. In fact, it suffices to prove ϕ is 1-1, but it is a direct result of the corresponding theorem of Riemann manifolds if we look M and N as Riemann manifolds.

Theorem 5.1 *Let N and M be Kähler manifolds of complex dimension d with N complete and ϕ be a locally holomorphic isometry of N onto M. If M is simply connected, then ϕ is a globally holomorphic isometry.*

Now we can finish the proof of Theorem 1.1, our main theorem. In fact, by the above theorem, we see that e and $'e$ are homeomorphisms, thus $'e \circ e^{-1}$ is a homeomorphism of M onto $'M$. Because e and $'e$ are locally holomorphic isometries, $'e \circ e^{-1}$ is also a locally holomorphic isometry. It clearly carries m into $'m$ and its differential carries e_i into $'e_i$. Then the conclusion holds by the above theorem.

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