Chin. Ann. Math. Ser. B
41(3), 2020, 465–478
DOI: 10.1007/s11401-020-0211-z

Chinese Annals of Mathematics, Series B

Construction of Lagrangian Submanifolds in Complex Hyperquadric*

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Abstract In this paper, the authors present a method to construct the minimal and H-minimal Lagrangian submanifolds in complex hyperquadric Q_n from submanifolds with special properties in odd-dimensional spheres. The authors also provide some detailed examples.

Keywords Lagrangian submanifold, Minimal, H-minimal
2000 MR Subject Classification 53C40, 53C42, 53D12.

1 Introduction

Let (N, J, g, ω) be a Kähler manifold with complex dimension n, where J is the complex structure, g is the Riemann metric and ω is the Kähler form. An immersion $f : \Sigma \to N$ from a real n-dimensional manifold Σ into N is called Lagrangian if $f^*\omega = 0$. A vector field V along a Lagrangian immersion f is called Hamiltonian variation (see [19]) if the associated 1-form $\alpha_V := (V \rfloor \omega)_\Sigma$ is exact on Σ. A smooth family $\{f_t\}$ of immersions from Σ into N is called Hamiltonian deformation if its derivative is Hamiltonian, and a Lagrangian immersion f is called Hamiltonian minimal (or H-minimal for short) if it satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathrm{Vol}(f_t(\Sigma)) = 0$ for all Hamiltonian deformation. The Euler-Lagrange equation of H-minimal Lagrangian submanifolds is $\delta\alpha_H = 0$, where H is the mean curvature vector field of f and δ is the co-differential operator on Σ with respect to the induced metric. In particular, minimal Lagrangian submanifolds are trivially H-minimal.

In the past few decades, many geometricians constructed minimal or H-minimal Lagrangian submanifolds in the complex space forms. Anciaux and Castro [1] constructed examples of H-minimal Lagrangian immersions in $\mathbb{C}^n$ by using curves in two-dimensional space forms and Legendrian immersions in odd-dimensional spheres. Castro, Li and Urbano [2] used the Legendrian immersions in odd-dimensional spheres and anti-de Sitter spaces to construct minimal

Manuscript received June 22, 2016. Revised August 19, 2018.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11331002, 11471299, 11871445).

and H-minimal Lagrangian submanifolds in $\mathbb{CP}^n$ and $\mathbb{CH}^n$. Castro and Urbano [3] gave new examples of minimal Lagrangian tori in $\mathbb{CP}^2$, and in [4] they constructed unstable H-minimal Lagrangian tori in $\mathbb{C}^2$. Chen and Garay [6] classified H-minimal Lagrangian submanifolds with constant curvature in $\mathbb{CP}^3$ with positive nullity. Helen and Romon [8–9] studied a general construction of H-minimal Lagrangian surfaces in $\mathbb{C}^2$ and $\mathbb{CP}^2$ from the point of view of completely integrable systems. Ma and her cooperators [11–12, 14] studied the Lagrangian tori in $\mathbb{CP}^2$ from different viewpoints. Mironov [15–18] constructed some examples of H-minimal and minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{CP}^n$ for higher dimensional cases. Li, Ma and Wei [10] constructed a class of compact minimal Lagrangian submanifolds in complex hyperquadrics by studying Gauss maps of compact rotational hypersurfaces in the unit sphere. Ma and Ohnita [13] determined completely the Hamiltonian stability of all compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as the images of the Gauss map of homogeneous isoparametric hypersurfaces in the unit spheres, by harmonic analysis on homogeneous spaces and fibrations on homogeneous isoparametric hypersurfaces. In this paper, we will construct minimal and H-minimal Lagrangian submanifolds in the complex hyperquadric $Q_n = \{[Z] \in \mathbb{CP}^{n+1} \mid (Z, \overline{Z}) = 0\}$, which is a complex submanifold of the complex projective space $\mathbb{CP}^{n+1}$.

Let $\mathbb{C}^n$ be the complex Euclidean space endowed with the standard Hermitian inner product $(z, w) = \sum_{j=1}^{n} z_j \overline{w}_j$ for $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$. The real part of $(\ , \ )$ determines a metric $\langle \ , \ \rangle$ on $\mathbb{C}^n$, i.e., $\langle \ , \ \rangle = \mathrm{Re}(\ , \ )$. The Liouville 1-form on $\mathbb{C}^n$ is given by $\Omega = -\frac{\mathrm{i}}{2}((dz, z) - (z, dz))$, and the Kähler form of $\mathbb{C}^n$ is $\omega_{\mathbb{C}^n} = \frac{d\Omega}{2}$. Let $\mathbf{S}^{2n+1}(1)$ be the $(2n+1)$-dimensional unit sphere in $\mathbb{C}^{n+1}$, an immersion ϕ from n-dimensional Σ into $\mathbb{S}^{2n+1}(1) \subset \mathbb{C}^n$ is called Legendrian if $\phi^*\Omega = 0$. It is easy to check that an isometric immersion $f : \Sigma \to \mathbf{S}^{2n+3}(1)$, $p \mapsto Z(p)$ satisfies $f^*\Omega = 0$ and $Z \cdot Z = 0$ gives a Lagrangian immersion $F = \pi \circ f$ from Σ into Q_n, where $\pi : \mathbf{S}^{2n+1}(1) \to \mathbb{CP}^n$, $Z \mapsto [Z]$ is the Hopf fibration of $\mathbf{S}^{2n+1}(1)$ on the complex projective space $\mathbb{CP}^n$. Basing on these fundamental facts, we can construct minimal and H-minimal Lagrangian submanifolds in the complex hyperquadric.

Our main theorems are as follows.

Theorem 1.1 *Let $f : \Sigma_1 \to \boldsymbol{S}^{2q+3}(1) \subset \mathbb{C}^{q+2}$, $p \mapsto Z(p)$ be an isometric immersion from q-dimensional manifold Σ_1 into $\boldsymbol{S}^{2q+3}(1)$, which satisfies $f^*\Omega = 0$ and $(Z, \overline{Z}) = 0$. Let $\phi : \Sigma_2 \to \boldsymbol{S}^{2m-1}(1) \subset \mathbb{C}^m$, $p \mapsto w(p)$ be a Legendrian immersion from Σ_2 into $\boldsymbol{S}^{2m-1}(r)$, which satisfies $(w, \overline{w}) = 0$. Define the new map as follows*:

$$\widetilde{F} : \Sigma_1 \times \Sigma_2 \times T^1 \to \boldsymbol{S}^{2n+3}(1), \quad (p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(Z, \mathrm{e}^{\mathrm{i}t} w).$$

Then $F = \pi \circ \widetilde{F}$ is a Lagrangian immersion from $\Sigma_1 \times \Sigma_2 \times T^1$ into Q_n, where $n = q + m$. Moreover, we have

(1) *F is minimal if and only if* $\pi \circ f : \Sigma_1 \to Q_q$ *is minimal and*

$$\widehat{H}^C_\phi = \frac{1 - nr^2}{r^2(1 + r^2)}\, w,$$

where $\widehat{H}^C_\phi$ *is the complex mean curvature vector of* ϕ *and* w *is position vector of* Σ_2 *in* $\mathbb{C}^m$;

(2) *F is* H*-minimal if and only if*

$$\delta\alpha_{\widetilde{H}} + \delta\alpha_{\widehat{H}} = 0,$$

where $\widetilde{H}$, $\widehat{H}$ *are the mean complex mean curvatures of* $\pi \circ f$ *and* $\psi : \Sigma_2 \times T^1 \to \boldsymbol{S}^{2m-1}(r) \subset \mathbb{C}^m$, $(w, \mathrm{e}^{\mathrm{i}t}) \mapsto \mathrm{e}^{\mathrm{i}t} w$, *respectively.*

By using Theorem 1.1 and Proposition 4.1, we have the following theorem.

Theorem 1.2 *Let* $f : \Sigma_1 \to \boldsymbol{S}^{2q+3}(1) \subset \mathbb{C}^{q+2}$, $p \mapsto Z(p)$ *be an isometric immersion from* q*-dimensional manifold* Σ_1 *into* $\boldsymbol{S}^{2q+3}(1)$, *which satisfies* $f^*\Omega = 0$ *and* $(Z, \overline{Z}) = 0$. *Let* m *be an even number and* $q + m = n$, *and define the map* $F : \Sigma_1 \times \boldsymbol{S}^{m-1} \times T^1 \to Q_n$ *by*

$$(p, x, \mathrm{e}^{\mathrm{i}t}) \mapsto \frac{1}{2}[\sqrt{2}Z, \mathrm{e}^{\mathrm{i}t}(x + \mathtt{i}\mathcal{J}x)],$$

where $\mathcal{J}$ *is defined in Section* 4. *Then, we have*

(1) *if* $q = m - 1$ *and* $\pi \circ f : \Sigma_1 \to Q_q$ *is minimal, then* F *is a minimal Lagrangian immersion*;

(2) *if* $\pi \circ f : \Sigma_1 \to Q_q$ *is H-minimal, then* F *is an* H*-minimal Lagrangian immersion.*

Remark 1.1 It is known that (see [1]) the minimal Lagrangian submanifolds and Lagrangian submanifolds with parallel mean curvature vector are automatically H-minimal. The explicit examples provided in Section 4 are Lagrangian submanifolds with parallel mean curvature vector.

Throughout this paper we will agree on the following ranges of indices:

$$\begin{aligned} &0 \le A,\, B,\, C,\, \cdots \le n; \quad && 1 \le \alpha,\, \beta,\, \gamma,\, \cdots \le n;\\ &1 \le j,\, k,\, l,\, \cdots \le q; && q{+}1 \le \lambda,\, \mu,\, \nu,\, \cdots \le n, \end{aligned}$$

and we also agree on conventions of the conjugate like $\overline{\omega}_{A\overline{B}} = \omega_{\overline{A}B}$, $\overline{f}^{\alpha}_{i} = f^{\overline{\alpha}}_{i}$, etc.

2 Preliminaries

2.1 Basic formulae of submanifolds in Kähler manifold

Let Σ be a smooth Riemannian manifold with real dimension q. Locally, we choose an orthonormal frame field $\{e_j\}$ of Σ, with the dual $\{\theta^j\}$. Then the first Cartan's structure equation is given by

$$d\theta^j = -\theta^j_k \wedge \theta^k, \quad \theta^j_k + \theta^k_j = 0, \tag{2.1}$$

where θ_k^j are the connection 1-forms with respect to θ^j. Let (N, J, g, ω) be a Kähler manifold with complex dimension n. Locally, we choose a unitary frame field $\{\varepsilon_\alpha\}$ of (1,0)-type of N, with the dual $\{\varphi_\alpha\}$. Then the structure equation is given by

$$d\varphi_\alpha = -\varphi_{\beta\overline{\alpha}} \wedge \varphi_\beta, \quad \varphi_{\alpha\overline{\beta}} + \varphi_{\overline{\beta}\alpha} = 0, \tag{2.2}$$

where $\varphi_{\beta\overline{\alpha}}$ are the connection 1-forms with respect to φ_α.

Let $F : \Sigma \to N$ be an isometric immersion. Set

$$F^*\varphi_\alpha = F_j^\alpha\, \theta^j. \tag{2.3}$$

Taking the exterior derivative on both side of (2.3), using (2.1)–(2.3), we obtain

$$(dF_j^\alpha - F_k^\alpha \theta_j^k + \varphi_{\beta\overline{\alpha}} F_j^\beta) \wedge \theta^j = 0. \tag{2.4}$$

Define the covariant derivative of F_j^α by

$$DF_j^\alpha := dF_j^\alpha - F_k^\alpha \theta_j^k + \varphi_{\beta\overline{\alpha}} F_j^\beta = F_{jk}^\alpha \theta^k. \tag{2.5}$$

Then, we have $F_{jk}^\alpha = F_{kj}^\alpha$ by using (2.4). The tensor field $\sum\limits_{j,k,\alpha} F_{jk}^\alpha\ \theta^j \otimes \theta^k \otimes \varepsilon_\alpha$ is a smooth section of the bundle $T^*\Sigma \otimes T^*\Sigma \otimes T^{(1,0)}N$, which is called the complex second fundamental form of F. For the relations between the real second fundamental form and complex second fundamental form, one can refer to [7] for details. By taking the trace, we call $H^{\mathbb{C}} = \sum\limits_{j,\alpha} F_{jj}^\alpha\ \varepsilon_\alpha$ the complex mean curvature vector field of F. It is known that F is minimal if and only if $H^{\mathbb{C}} = 0$.

Let H be the real mean curvature vector field of F. Through direct calculations, we obtain

$$\alpha_H := (H \rfloor \omega)_\Sigma = H_j\ \theta^j, \quad H_j = \frac{\mathtt{i}}{2}(F_{kk}^\alpha F_j^{\overline{\alpha}} - F_{kk}^{\overline{\alpha}} F_j^\alpha). \tag{2.6}$$

Therefore, the co-differential of α_H is given by

$$\delta\alpha_H = -\sum_j H_{jj}, \tag{2.7}$$

where $H_{jk}\theta^k := dH_j - H_k\theta_j^k$ is the covariant derivative of H_j.

2.2 Lagrangian submanifolds in complex hyperquadric Q_n

The complex projective space $\mathbb{CP}^{n+1}$ is the set of all 1-dimensional complex line through the origin in $\mathbb{C}^{n+2}$, or equivalently, $\mathbb{CP}^{n+1} \cong U(n+2)/U(1) \times U(n+1)$. We always view $Q_n = \{[Z] \in \mathbb{CP}^{n+1} \mid Z \cdot Z = 0\}$ as a complex submanifold in $\mathbb{CP}^{n+1}$.

Let $Z_0, Z_1, \cdots, Z_n, Z_{n+1}$ be a moving frame of $\mathbb{C}^{n+2}$. We set

$$dZ_A = \sum_{C=1}^{n+2} \omega_{A\overline{B}}\, Z_B, \tag{2.8}$$

where $\omega_{A\overline{B}} = (dZ_A, Z_B)$ are the Maurer-Cartan forms of $U(n+2)$. They are skew-Hermitian, i.e.,

$$\omega_{A\overline{B}} + \omega_{\overline{B}A} = 0. \tag{2.9}$$

Taking the exterior derivative of (2.8), we obtain the Maurer-Cartan equation of $U(n+2)$:

$$d\omega_{A\overline{B}} = \sum_{C=1}^{n+2} \omega_{A\overline{C}} \wedge \omega_{C\overline{B}}, \tag{2.10}$$

which plays an important role in our later calculations. The quadratic form

$$ds^2_{FS} = \sum_{\alpha=1}^{n+1} \omega_{0\overline{\alpha}}\omega_{\overline{0}\alpha}$$

defines a Kählerian metric on $\mathbb{CP}^{n+1}$, so-called the Fubini-Study metric, and the Kähler form of ds^2_{FS} is

$$\omega_{FS} = \frac{\mathtt{i}}{2}\sum_{\alpha=1}^{n+1} \omega_{0\overline{\alpha}} \wedge \omega_{\overline{0}\alpha}.$$

To study the geometry of the complex hyperquadric Q_n, locally, we choose a moving frame $Z_0, \cdots, Z_n, Z_{n+1} = \overline{Z}_0$ associated to Q_n. Noticing that $Z_0 \cdot Z_0 = 0$, we have

$$\omega_{0\,\overline{n+1}} = (dZ_0, \overline{Z}_0) = 0. \tag{2.11}$$

So, the metric induced from the Fubini-Study metric on Q_n is given by

$$ds^2_{Q_n} = \sum_{\alpha=1}^{n} \omega_{0\overline{\alpha}}\omega_{\overline{0}\alpha},$$

and the Kähler form is

$$\omega_{Q_n} = \frac{\mathtt{i}}{2}\sum_{\alpha=1}^{n} \omega_{0\overline{\alpha}} \wedge \omega_{\overline{0}\alpha}.$$

Set $\varphi_\alpha := \omega_{0\overline{\alpha}}$, $\alpha = 1, \cdots, n$. Then $\{\varphi_\alpha\}$ is a unitary frame field on Q_n of $(1,0)$-type. Therefore, by the Maurer-Cartan equation (2.10) and (2.11), we obtain the structure equation

$$d\varphi_\alpha = -\sum_{\beta=1}^{n} \varphi_{\beta\overline{\alpha}} \wedge \varphi_\beta, \quad \varphi_{\beta\overline{\alpha}} = \omega_{\beta\overline{\alpha}} - \omega_{0\overline{0}}\delta_{\alpha\beta}, \quad \varphi_{\beta\overline{\alpha}} + \varphi_{\overline{\alpha}\beta} = 0, \tag{2.12}$$

where $\varphi_{\beta\overline{\alpha}}$ are the connection 1-forms with respect to φ_α.

Let Σ be a smooth manifold with real dimension n, and let F be an immersion from Σ into Q_n. Let $U \subset \Sigma$ be an open set. We say $Z : U \to U(n+2)$, $p \mapsto (Z_0, Z_1, \cdots, Z_n, Z_{n+1} = \overline{Z}_0)(p)$ is a moving frame along F if $F(p) = [Z_0(p)]$ for all $p \in U$. For the moving frame along a Lagrangian immersion, we have the following proposition.

Proposition 2.1 *Let (Σ, ds^2_Σ) be an n-dimensional Riemannian manifold with the metric $ds^2_\Sigma = \sum\limits_{i=1}^{n}(\theta^i)^2$. Let F be a Lagrangian isometric immersion from Σ into Q_n. Then, for every point $p \in \Sigma$, there is a small neighborhood around p and a moving frame Z along F such that*

$$\omega_{0\overline{0}} = \omega_{0\,\overline{n+1}} = 0, \quad \omega_{0\overline{i}} = \theta^i, \quad 1 \le i \le n, \tag{2.13}$$

where $\omega_{A\overline{B}}$ are the pull-back of the Maurer-Cartan forms of $U(n+2)$ via Z^*.

Proof It is similar to the proof of [7, Proposition 2.2] by the fact that the complex structure of Q_n inherits from $\mathbb{CP}^{n+1}$. This completes the proof.

Let $F : \Sigma \to Q_n$ be a Lagrangian isometric immersion, and θ^α be an orthonormal frame field on Σ. By Proposition 2.1, there exists a moving frame $Z_0, Z_1, \cdots, Z_n, Z_{n+1} = \overline{Z}_0$ along F such that

$$\varphi_\alpha = \omega_{0\overline{\alpha}} = \theta^\alpha. \tag{2.14}$$

For later use, we set

$$\omega_{\alpha\overline{\beta}} = \Lambda_{\alpha\overline{\beta},\gamma}\,\theta^\gamma, \quad \omega_{0\overline{0}} = \Lambda_{0\overline{0},\gamma}\,\theta^\gamma \tag{2.15}$$

and

$$\theta^\alpha_\beta = \Gamma^\alpha_{\gamma\beta}\,\theta^\gamma, \tag{2.16}$$

the connection 1-forms with respect to θ^α.

Notice that $F^\alpha_\beta = \delta^\alpha_\beta$. By using (2.14), we obtain the complex second fundamental form $F^\alpha_{\beta\gamma}$ of F, that is

$$F^\alpha_{\beta\gamma} = -\Gamma^\alpha_{\gamma\beta} + \Lambda_{\beta\overline{\alpha},\gamma} - \delta_{\alpha\beta}\Lambda_{0\overline{0},\gamma} \tag{2.17}$$

by (2.5), (2.12) and (2.15)–(2.16). So, by using (2.6), we obtain

$$\alpha_H = \sum_{\beta=1}^{n} H_\beta\,\theta^\beta, \quad H_\beta = \sum_{\gamma=1}^{n} \frac{\mathtt{i}}{2}(F^\beta_{\gamma\gamma} - F^{\overline{\beta}}_{\gamma\gamma}). \tag{2.18}$$

2.3 Spherical Lagrangian submanifolds in $\mathbb{C}^m$

It is known that the spherical Lagrangian submanifolds are closely related to Legendrian submanifolds. We want to study the relationship of complex second fundamental form and complex mean curvature between the spherical Lagrangian submanifolds and Legendrian submanifolds.

Chen [5] proved that a spherical Lagrangian submanifold in $\mathbb{C}^m$ must take the form

$$\psi : \Sigma_2 \times T^1 \to \mathbb{C}^m, \quad (p, \mathrm{e}^{\mathtt{i}t}) \mapsto z = \mathrm{e}^{\mathtt{i}t} w(p),$$

where

$$\phi : \Sigma_2 \to \mathbf{S}^{2m-1}(r), \quad p \mapsto w(p)$$

is a Legendrian immersion.

Noticing that ϕ is Legendrian, locally, one can choose an orthonormal frame field $e_{q+1}, \cdots, e_{n-1}, e_n = \frac{w}{r}$ such that

$$dw = \sum_{\lambda=q+1}^{n-1} \theta^\lambda\, e_\lambda, \quad ds^2_{\Sigma_2} = \sum_{\lambda=q+1}^{n-1} (\theta^\lambda)^2.$$

By using Legendrian condition, one can check that e_λ is also a unitary frame field, i.e., $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$. Set

$$de_\lambda = \omega_{\lambda\overline{\mu}}\, e_\mu, \quad \omega_{\lambda\overline{\mu}} = (de_\lambda, e_\mu).$$

The fact $(e_\lambda, e_\mu) = \delta_{\lambda\mu}$ implies

$$\omega_{\lambda\overline{\mu}} + \omega_{\overline{\mu}\lambda} = 0. \tag{2.19}$$

Obviously, we have

$$(dw, e_\lambda) = \theta^\lambda, \quad \omega_{\lambda\overline{n}} = -\omega_{\overline{n}\lambda} = -\overline{\Big(\frac{1}{r}dw, e_\lambda\Big)} = -\frac{1}{r}\theta^\lambda. \tag{2.20}$$

Denote by θ^λ_μ the connection 1-forms with respect to θ^λ. Set

$$\theta^\lambda_\mu = \Gamma^\lambda_{\nu\mu}\, \theta^\nu, \quad \phi^*\omega_{\lambda\overline{\mu}} = \Lambda_{\lambda\overline{\mu},\nu}\, \theta^\nu.$$

The complex second fundamental form of ϕ is given by

$$\phi^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} + \Lambda_{\mu\overline{\lambda},\nu} \quad \text{for } p+1 \le \lambda < n, \tag{2.21}$$

$$\phi^n_{\lambda\mu} = -\frac{1}{r}\delta_{\lambda\mu} \quad \text{for } p+1 \le \lambda < n \tag{2.22}$$

by (2.5), (2.20) and the fact that $\phi^\lambda_\mu = \delta_{\lambda\mu}$ for $p+1 \le \mu < n$.

For the Lagrangian immersion ψ, we have

$$dz = \sum_{\lambda=p+1}^{n-1} \mathrm{e}^{\mathtt{i}t}\theta^\lambda\, e_\lambda + \mathtt{i}r\mathrm{e}^{\mathtt{i}t}dt\, e_n,$$

which gives

$$ds^2_{\Sigma_2\times T^1} = \sum_{\lambda=q+1}^{n} (\widehat{\theta}^\lambda)^2,$$

where $\widehat{\theta}^\lambda = \theta^\lambda$ and $\widehat{\theta}^n = rdt$. The connection 1-forms $\widehat{\theta}^\lambda_\mu$ w.r.t. $\widehat{\theta}^\lambda$ are given by

$$\widehat{\theta}^\lambda_\mu = \theta^\lambda_\mu, \quad \widehat{\theta}^n_\lambda = 0 \tag{2.23}$$

for $p+1 \le \lambda < n$. Set

$$\widehat{\theta}^\lambda_\mu = \widehat{\Gamma}^\lambda_{\nu\mu}\,\widehat{\theta}^\nu, \quad \psi^*\omega_{\lambda\overline{\mu}} = \widehat{\Lambda}_{\lambda\overline{\mu},\nu}\,\widehat{\theta}^\nu.$$

The complex second fundamental form of ϕ is given by

$$\psi^\lambda_{\mu\nu} = -\mathrm{e}^{\mathtt{i}t}(\widehat{\Gamma}^\lambda_{\nu\mu} - \widehat{\Lambda}_{\mu\overline{\lambda},\nu}) \quad \text{for } p+1 \le \lambda,\mu,\nu < n, \tag{2.24}$$

$$\psi^\lambda_{n\mu} = \psi^\lambda_{\mu n} = \frac{\mathtt{i}\mathrm{e}^{\mathtt{i}t}}{r}\delta^\lambda_\mu, \quad \psi^\lambda_{nn} = 0 \quad \text{for } p+1 \le \lambda,\mu < n, \tag{2.25}$$

$$\psi^n_{\lambda\mu} = -\frac{\mathrm{e}^{\mathtt{i}t}}{r}\delta_{\lambda\mu}, \tag{2.26}$$

by (2.5), (2.20) and the fact that $\psi^\lambda_\mu = \mathrm{e}^{\mathtt{i}t}\delta_{\lambda\mu}$ for $p+1 \le \lambda < n$, $\psi^n_\mu = \mathtt{i}\mathrm{e}^{\mathtt{i}t}\delta_{n\mu}$.

In summary, the complex second fundamental forms of ϕ, ψ are given by

$$\sum_{p+1\le\mu,\nu<n,\ \lambda} \phi^\lambda_{\mu\nu}\theta^\mu \otimes \theta^\nu \otimes e_\lambda, \quad \sum_{\lambda,\mu,\nu} \psi^\lambda_{\mu\nu}\widehat{\theta}^\mu \otimes \widehat{\theta}^\nu \otimes e_\lambda,$$

respectively.

Proposition 2.2 *Notations as above, the complex second fundamental forms of ϕ, ψ have the relation*

$$\psi^\lambda_{\mu\nu} = \mathrm{e}^{\mathtt{i}t}\,\phi^\lambda_{\mu\nu}$$

for $p+1 \le \mu,\nu < n$, $p+1 \le \lambda \le n$.

Proof It follows from (2.21)–(2.24), (2.26) and the fact $\widetilde{\theta}^\lambda = \theta^\lambda$ for $p+1 \le \lambda \le n$. This completes the proof.

Remark 2.1 We will use the unitary frame field $\mathrm{e}^{\mathtt{i}t}e_\lambda, \mathtt{i}\mathrm{e}^{\mathtt{i}t}e_n$ instead of e_λ, e_n in Section 3.

3 Proof of Main Theorem

Let Σ_1 be a q-dimensional smooth Riemannian manifold, and let $f : \Sigma_1 \to \mathbf{S}^{2q+3}(1) \subset \mathbb{C}^{n+2}$, $p \mapsto \widetilde{Z}_0(p)$ be an isometric immersion with $f^*\Omega = 0$ and $(\widetilde{Z}_0, \overline{\widetilde{Z}_0}) = 0$. Then $\widetilde{f} = \pi \circ f : \Sigma_1 \to Q_q \subset \mathbb{CP}^{q+1}$, $p \mapsto [\widetilde{Z}_0]$ is a Lagrangian immersion. Since $f^*\Omega = 0$, it is easy to check that

$$\widetilde{\omega}_{0\overline{0}} = (d\widetilde{Z}_0, \widetilde{Z}_0) = 0. \tag{3.1}$$

By Proposition 2.1, we can choose pairwise Hermitian orthogonal local frame field $\widetilde{Z}_1, \cdots, \widetilde{Z}_q$ s.t. $\widetilde{Z}_0, \widetilde{Z}_1, \cdots, \widetilde{Z}_q, \widetilde{Z}_{q+1} = \overline{\widetilde{Z}_0}$ is a moving frame along $\widetilde{f}$, satisfying

$$\widetilde{\omega}_{0\overline{j}} = (d\widetilde{Z}_0, \widetilde{Z}_j) = \widetilde{\theta}^j \tag{3.2}$$

are real 1-forms. As before, set

$$\widetilde{\omega}_{j\overline{k}} = (d\widetilde{Z}_j, \widetilde{Z}_k) = \widetilde{\Lambda}_{j\overline{k},l}\,\widetilde{\theta}^l, \tag{3.3}$$

and denote by $\widetilde{\theta}^j_k = \widetilde{\Gamma}^j_{lk}\,\widetilde{\theta}^l$ the connection 1-forms with respect to $\widetilde{\theta}^j$. By using (2.17) and (3.1), we obtain the complex second fundamental form of $\widetilde{f}$, that is

$$\widetilde{f}^j_{kl} = -\widetilde{\Gamma}^j_{lk} + \widetilde{\Lambda}_{k\bar{j},l}. \tag{3.4}$$

Let $\psi : \Sigma_2 \times T^1 \to \mathbf{S}^{2m-1}(r) \subset \mathbb{C}^m$, $p \mapsto z(p)$ be a Lagrangian isometric immersion with $(z, \overline{z}) = 0$. Define a new map by

$$\widetilde{F} : \Sigma_1 \times \Sigma_2 \times T^1 \to \mathbf{S}^{2n+3}(1), \quad (p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}(\widetilde{Z}_0, z),$$

where $q + m = n$. In the following, we will study the map

$$F = \pi \circ \widetilde{F} : \Sigma_1 \times \Sigma_2 \times T^1 \to Q_n \subset \mathbb{CP}^{n+1}, \quad (p_1, p_2) \mapsto \frac{1}{\sqrt{1+r^2}}[\widetilde{Z}_0, z].$$

Choose the moving frame $Z_0, Z_1, \cdots, Z_n, Z_{n+1}$ along f as follows:

$$\begin{aligned}
Z_0 &= \frac{1}{\sqrt{1+r^2}}(\widetilde{Z}_0, z),\\
Z_j &= (\widetilde{Z}_j, 0), \quad 1 \le j \le q,\\
Z_\lambda &= (0, e_\lambda), \quad q+1 \le \lambda < n,\\
Z_n &= \frac{1}{\sqrt{1+r^2}}(-\mathtt{i}r\widetilde{Z}_0, e_n),\\
Z_{n+1} &= \overline{Z}_0,
\end{aligned}$$

where $\widetilde{Z}_0$, $\widetilde{Z}_j$ and e_λ, e_n (here e_λ is the one in Subsection 2.3 multiply by $\mathrm{e}^{\mathtt{i}t}$, e_n is the one in Subsection 2.3 multiply by $\mathtt{i}\mathrm{e}^{\mathtt{i}t}$) are as they were in the context of f and ψ, respectively. Notice that $\omega_{A\overline{B}} = (dZ_A, Z_B)$. By using (2.20) and (3.1), through direct calculations, we obtain

$$\omega_{0\bar{j}} = \frac{1}{\sqrt{1+r^2}}\widetilde{\omega}_{0\bar{j}} = \frac{1}{\sqrt{1+r^2}}\widetilde{\theta}^j =: \theta^j, \tag{3.5}$$

$$\omega_{0\bar{\lambda}} = \frac{1}{\sqrt{1+r^2}}\widehat{\theta}^\lambda =: \theta^\lambda, \quad q+1 \le \lambda < n, \tag{3.6}$$

$$\omega_{0\bar{n}} = \frac{1}{1+r^2}(dz, e_n) = \frac{1}{1+r^2}\widehat{\theta}^n := \theta^n. \tag{3.7}$$

Similarly, we also have

$$\omega_{0\bar{0}} = \mathtt{i}r\theta^n, \quad \omega_{j\bar{k}} = \widetilde{\omega}_{j\bar{k}}, \quad \omega_{j\bar{\lambda}} = 0, \quad \omega_{j\bar{n}} = -\mathtt{i}r\theta^j, \quad \omega_{\lambda\bar{\mu}} = \widehat{\omega}_{\lambda\bar{\mu}}, \tag{3.8}$$

$$\omega_{j\bar{n}} = -\mathtt{i}r\theta^j, \quad \omega_{\lambda\bar{\mu}} = \widehat{\omega}_{\lambda\bar{\mu}}, \quad \omega_{\lambda\bar{n}} = \frac{\mathtt{i}}{r}\theta^\lambda, \quad \omega_{n\bar{n}} = \frac{\mathtt{i}}{r}\theta^n, \tag{3.9}$$

where $q+1 \le \lambda,\ \mu < n$.

Notice that θ^j, θ^λ, θ^n are real and linearly independent on $\Sigma_1 \times \Sigma_2 \times T^1$. Therefore F is an immersion and the induced metric on $\Sigma_1 \times \Sigma_2 \times T^1$ is given by

$$ds^2 = F^* ds^2_{Q_n} = \sum_\alpha (\theta^\alpha)^2, \tag{3.10}$$

where

$$\theta^j = \frac{1}{\sqrt{1+r^2}}\widetilde{\theta}^j, \quad \theta^\lambda = \frac{1}{\sqrt{1+r^2}}\widehat{\theta}^\lambda, \quad \theta^n = \frac{1}{1+r^2}\widehat{\theta}^n.$$

Choose the orthonormal frame field θ^α on $\Sigma_1\times\Sigma_2\times T^1$, then

$$F^\alpha_\beta = \delta_{\alpha\beta}. \tag{3.11}$$

Then the pull back of the Kähler form is given by

$$F^*\omega_{Q_n} = \frac{\mathtt{i}}{2}\sum_{\alpha=1}^{n}\theta^\alpha\wedge\theta^\alpha = 0,$$

which implies that F is a Lagrangian immersion.

Lemma 3.1 (see [7]) *Let*

$$d\widetilde{s}^2 = \sum_{\alpha=1}^{n}(\widetilde{\theta}^\alpha)^2, \quad ds^2 = \sum_{\alpha=1}^{n}(\theta^\alpha)^2 = \sum_{\alpha=1}^{n}(a_\alpha\widetilde{\theta}^\alpha)^2$$

be two Riemannian metrics, where a_α are positive constants. Set

$$\widetilde{\theta}^\alpha_\beta = \widetilde{\Gamma}^\alpha_{\gamma\beta}\,\widetilde{\theta}^\gamma, \quad \theta^\alpha_\beta = \Gamma^\alpha_{\gamma\beta}\,\theta^\gamma.$$

Then

(1) *if $a_1 = \cdots = a_n = a$, we have*

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{a}\widetilde{\Gamma}^\alpha_{\beta\gamma}; \tag{3.12}$$

(2) *if $a_1 = \cdots = a_{n-1} = a$, $a_n = a^2$, we have*

$$\Gamma^\lambda_{\mu\mu} = \frac{1}{a}\widetilde{\Gamma}^\lambda_{\mu\mu}, \quad \Gamma^\lambda_{nn} = \frac{1}{a}\widetilde{\Gamma}^\lambda_{nn}, \quad \Gamma^n_{\mu\mu} = \frac{1}{a^2}\widetilde{\Gamma}^n_{\mu\mu}, \quad \Gamma^n_{nn} = \frac{1}{a^2}\widetilde{\Gamma}^n_{nn}, \tag{3.13}$$

where $1\le\lambda,\mu\le n-1$.

Notice that ds^2 is a product metric. By using (3.12)–(3.13), we obtain

$$\Gamma^j_{kl} = \sqrt{1+r^2}\,\widetilde{\Gamma}^j_{kl}, \quad \Gamma^\lambda_{\alpha j} = \Gamma^j_{\alpha\lambda} = 0, \quad \Gamma^\lambda_{\mu\mu} = \sqrt{1+r^2}\,\widehat{\Gamma}^\lambda_{\mu\mu}, \tag{3.14}$$

$$\Gamma^\lambda_{nn} = \sqrt{1+r^2}\,\widehat{\Gamma}^\lambda_{nn}, \quad \Gamma^n_{\mu\mu} = (1+r^2)\widehat{\Gamma}^n_{\mu\mu}, \quad \Gamma^n_{nn} = (1+r^2)\widehat{\Gamma}^n_{nn}, \tag{3.15}$$

where $q+1\le\lambda,\mu<n$. On the other hand, by using (3.8)–(3.9), we also have

$$\Lambda_{0\overline{0},\alpha} = \mathtt{i}r\delta_{n\alpha}, \quad \Lambda_{j\overline{k},l} = \sqrt{1+r^2}\,\widetilde{\Lambda}_{j\overline{k},l}, \quad \Lambda_{\lambda\overline{j},\alpha} = \Lambda_{j\overline{\lambda},\alpha} = 0, \quad \Lambda_{j\overline{n},j} = -\mathtt{i}r, \tag{3.16}$$

$$\Lambda_{\lambda\overline{\mu},\nu} = \sqrt{1+r^2}\,\widehat{\Lambda}_{\lambda\overline{\mu},\nu}, \quad \Lambda_{\lambda\overline{n},\lambda} = \widehat{\Lambda}_{\lambda\overline{n},\lambda} = \frac{\mathtt{i}}{r}, \quad \Lambda_{n\overline{\lambda},n} = 0, \quad \Lambda_{n\overline{n},n} = \widehat{\Lambda}_{n\overline{n},n} = \frac{\mathtt{i}}{r}, \tag{3.17}$$

where $q+1\le\lambda<n$.

Proof of Theorem 1.1 According to the identities (2.17) and (3.4), together with (3.14) and (3.16), we obtain

$$F^j_{kk} = -\Gamma^j_{kk} + \Lambda_{k\bar{j},k} = \sqrt{1+r^2}\,(-\widetilde{\Gamma}^j_{kk} + \widetilde{\Lambda}_{k\bar{j},k}) = \sqrt{1+r^2}\,\widetilde{f}^j_{kk}. \tag{3.18}$$

Similarly, we obtain

$$F^j_{\lambda\lambda} = F^j_{nn} = F^\lambda_{jj} = 0, \quad F^\lambda_{\mu\mu} = \sqrt{1+r^2}\,\psi^\lambda_{\mu\mu}, \quad F^\lambda_{nn} = \sqrt{1+r^2}\,\psi^\lambda_{nn}, \tag{3.19}$$

$$F^n_{jj} = -\mathtt{i}r, \quad F^n_{\lambda\lambda} = (1+r^2)\psi^n_{\lambda\lambda} - \mathtt{i}r, \quad F^n_{nn} = (1+r^2)\psi^n_{nn} - 2\mathtt{i}r, \tag{3.20}$$

where $q+1 \le \lambda, \mu < n$. Therefore, f is minimal if and only if

$$\sum_{k=1}^{q} \widetilde{f}^j_{kk} = \sum_{\mu=q+1}^{n} \psi^\lambda_{\mu\mu} = 0, \qquad \sum_{\mu=q+1}^{n} \psi^n_{\mu\mu} = \frac{\mathtt{i}(n+1)r}{1+r^2},$$

where $1 \le j \le q$ and $q+1 \le \lambda < n$. Thus, the first statement in theorem follows from Proposition 2.2.

Denote the associated 1-forms $\alpha_{\widetilde{H}}$, $\alpha_{\widehat{H}}$, α_H of the Lagrangian immersions $\widetilde{f}$, ψ, F respectively by

$$\alpha_{\widetilde{H}} = \sum_{j=1}^{q} \widetilde{H}_j\, \widetilde{\theta}^j, \quad \alpha_{\widehat{H}} = \sum_{\lambda=q+1}^{n} \widehat{H}_\lambda\, \widehat{\theta}^\lambda, \quad \alpha_H = \sum_{\beta=1}^{n} H_\beta\, \theta^\beta.$$

By using (2.18) and (3.18)–(3.20), we obtain

$$H_j = \sqrt{1+r^2}\, \widetilde{H}_j, \quad H_\lambda = \sqrt{1+r^2}\, \widehat{H}_\lambda, \quad H_n = (1+r^2)\, \widehat{H}_n + (n+1)r, \tag{3.21}$$

where $q+1 \le \lambda < n$. Recall the definition of the covariant derivative DH_β of H_β, i.e.,

$$DH_\beta := H_{\beta\gamma}\theta^\gamma = dH_\beta - H_\gamma\theta^\gamma_\beta.$$

Then, by using Lemma 3.1 and (3.21), through direct calculations, we have

$$H_{jj} = (1+r^2)\widetilde{H}_{jj}, \quad H_{nn} = (1+r^2)\widehat{H}_{nn} + r^2(1+r^2)\widehat{H}_{n;n}, \tag{3.22}$$

$$H_{\lambda\lambda} = (1+r^2)\widehat{H}_{\lambda\lambda} - (1+r^2)[r^2\widehat{H}_n + (n+1)r]\widehat{\Gamma}^n_{\lambda\lambda}, \tag{3.23}$$

where $\widehat{H}_n = -\mathrm{Im}((\widehat{H}^C, e_n))$ is a smooth function on $\Sigma_2 \times T^1$ and $\widehat{H}_{n;n} = \langle \mathrm{grad}\,\widehat{H}, e_n\rangle$. Therefore, the co-differential of α_H is

$$\delta\alpha_H = (1+r^2)(\delta\alpha_{\widetilde{H}} + \delta\alpha_{\widehat{H}}) + (1+r^2)[(r^2\widehat{H}_n + (n+1)r)\widehat{\Gamma}^n_{\lambda\lambda} - r^2\widehat{H}_{n;n}], \tag{3.24}$$

where $\delta\alpha_{\widetilde{H}}$, $\delta\alpha_{\widehat{H}}$ are the co-differential of $\alpha_{\widetilde{H}}$, $\alpha_{\widehat{H}}$ with respect to metrics induced by $\widetilde{f}$, ψ on Σ_1, $\Sigma_2 \times T^1$, respectively. Then, (3.24) gives that F is H-minimal if and only if

$$(\delta\alpha_{\widetilde{H}} + \delta\alpha_{\widehat{H}}) = r^2\widehat{H}_{n;n}, \tag{3.25}$$

by the fact $\theta^n = \frac{rdt}{1+r^2}$ and hence $\theta^n_\lambda = \widehat{\Gamma}^n_{\mu\lambda}\theta^\mu = 0$. On the other hand, from Proposition 2.2 and (2.22), we know $\widehat{H}_n = -\frac{1}{r}$ which is independent of t, i.e., $\widehat{H}_{n;n} = 0$. This completes the proof of Theorem 1.1.

4 Examples

Let $m = 2\kappa$ be an even number, and let $\mathbb{R}^m$ be the m-dimensional Euclidean space. Define a linear transformation $\mathcal{J}$ by

$$\mathcal{J} : \mathbb{R}^m \to \mathbb{R}^m, \quad \mathcal{J}x = (-x^{\kappa+1}, \cdots, -x^m, x^1, \cdots, x^\kappa),$$

where $x = (x^1, \cdots, x^\kappa, x^{\kappa+1}, \cdots, x^m) \in \mathbb{R}^m$. It has the properties $\mathcal{J}^2 = -\mathrm{id}$ and

$$\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle x, y \rangle \tag{4.1}$$

for $x, y \in \mathbb{R}^m$. Consider the unit sphere $\mathbf{S}^{m-1}(1) = \{x \in \mathbb{R}^m \mid |x|^2 = 1\}$ in $\mathbb{R}^m$. Locally, we choose an orthonormal frame field $\widehat{e}_1, \cdots, \widehat{e}_{m-1}, \widehat{e}_m = x$, so that the structure equations are given by

$$dx = \sum_{\lambda=1}^{m-1} \widehat{\theta}^\lambda \, \widehat{e}_\lambda, \quad d\widehat{\theta}^\lambda = -\widehat{\theta}_\mu^\lambda \wedge \widehat{\theta}^\mu, \tag{4.2}$$

where the 1-forms $\widehat{\theta}^\lambda$, $\widehat{\theta}_\mu^\lambda$ are real and $\widehat{\theta}_\mu^\lambda$ satisfies $\widehat{\theta}_\mu^\lambda + \widehat{\theta}_\lambda^\mu = 0$. Further, we have

$$d\widehat{e}_\lambda = \sum_{\mu=1}^{m-1} \widehat{\theta}_\lambda^\mu \, \widehat{e}_\mu, \quad 1 \le \lambda, \mu < m. \tag{4.3}$$

Define an immersion ψ from $\mathbf{S}^{m-1}(1) \times T^1$ into $\mathbb{C}^m$ by

$$(x, \mathrm{e}^{\mathtt{i}t}) \mapsto z = \frac{\sqrt{2}}{2} \mathrm{e}^{\mathtt{i}t} (x + \mathtt{i}\mathcal{J}x). \tag{4.4}$$

Choosing the moving frame along ψ to be

$$e_\lambda = \frac{\sqrt{2}}{2} \mathrm{e}^{\mathtt{i}t} (e_\lambda + \mathtt{i}\mathcal{J}e_\lambda), \quad 1 \le \lambda < m, \quad e_m = \mathtt{i}z. \tag{4.5}$$

Notice that the differential operator d is commute with $\mathcal{J}$, by using (4.1) and (4.2), we have

$$\theta^\lambda := (dz, e_\lambda) = \widehat{\theta}^\lambda, \quad 1 \le \lambda < m, \quad \theta^m := (dz, e_m) = dt. \tag{4.6}$$

It is easy to check that ψ is Lagrangian by the fact that θ^λ and θ^m are real. Set $\omega_{\lambda\mu} = (de_\lambda, e_\mu)$. Through direct calculations, we have

$$\omega_{\lambda\overline{\lambda}} = \omega_{m\overline{m}} = \mathtt{i}\theta^m, \quad \omega_{\lambda\overline{m}} = \mathtt{i}\theta^\lambda, \quad 1 \le \lambda < m, \quad \omega_{\lambda\overline{\mu}} = \theta_\lambda^\mu, \quad 1 \le \lambda < \mu < m. \tag{4.7}$$

Denote the connection 1-forms with respect to θ^λ by θ_μ^λ, we have

$$\theta_\lambda^m = 0, \quad \theta_\mu^\lambda = \widehat{\theta}_\mu^\lambda, \quad 1 \le \lambda, \mu < m. \tag{4.8}$$

From (2.21) and (4.7)–(4.8), we obtain

$$\psi_{\mu\mu}^\lambda = 0, \quad \psi_{\mu\mu}^m = \mathtt{i}, \quad 1 \le \mu \le m. \tag{4.9}$$

Proposition 4.1 *The map* $\psi: \boldsymbol{S}^{m-1}(1)\times T^1 \to \mathbb{C}^m$ *given by* $(x, \mathrm{e}^{\mathtt{i}t}) \mapsto z = \frac{\sqrt{2}}{2}\mathrm{e}^{\mathtt{i}t}(x+\mathtt{i}\mathcal{J}x)$ *is an* H*-minimal Lagrangian immersion in* $\mathbb{C}^m$*, and its complex mean curvature* $\widehat{H}^{\mathbb{C}}$ *satisfies*

$$\widehat{H}^{\mathbb{C}} = -m\, z.$$

Proof From (2.6), we have $\widehat{H}_\lambda = 0$ for $1 \le \lambda < m$ and $\widehat{H}_m = -m$. By using (2.7) and (4.8), we obtain $\delta\alpha_{\widehat{H}} = 0$. So, ψ is H-minimal. This completes the proof.

Let $\mathbf{x} : \Sigma_1 \to \mathbf{S}^{q+1}(1) \hookrightarrow \mathbb{R}^{q+2}$, $p \mapsto \mathbf{x}(p)$ be a hypersurface in $\mathbf{S}^{q+1}(1)$, and let $\mathbf{n}$ be the unit normal vector field of Σ_1 in $\mathbf{S}^{q+1}(1)$. Define a map $f : \Sigma_1 \to \mathbf{S}^{2q+3}(1)$ by

$$p \mapsto \frac{\sqrt{2}}{2}(\mathbf{x} + \mathtt{i}\mathbf{n}).$$

By [20, Proposition 3.1], we know that Gauss map $\pi \circ f : \Sigma_1 \to Q_q$ is a Lagrangian immersion, where $\pi : \mathbf{S}^{2q+3}(1) \to \mathbb{CP}^{q+1}$ is the Hopf fibration. Furthermore, $\pi \circ f$ is minimal (see [20]) if $\mathbf{x}$ is an isoparametric immersion. Thus, by Theorem 1.2, the map

$$F : \Sigma_1 \times \mathbf{S}^{m-1}(1) \times T^1 \to Q_n, \quad (\mathbf{x}, x, \mathrm{e}^{\mathtt{i}t}) \mapsto \frac{1}{2}[(\mathbf{x} + \mathtt{i}\mathbf{n}, \mathrm{e}^{\mathtt{i}t}(x + \mathtt{i}\mathcal{J}x))]$$

is H-minimal. Particularly, let $\mathbf{x} : \mathbf{S}^{q_1}(r_1) \times \mathbf{S}^{q_2}(r_2) \to \mathbf{S}^{q+1}(1)$, $q_1 + q_2 = q$, $r_1^2 + r_2^2 = 1$ be the Clifford hypersurface in $\mathbf{S}^{q+1}(1)$. It is known that Clifford hypersurfaces are isoparametric hypersurface. Then, the map

$$\widetilde{f} : \mathbf{S}^{q_1}(r_1) \times \mathbf{S}^{q_2}(r_2) \to Q_q, \quad (p_1, p_2) \mapsto \frac{\sqrt{2}}{2}\Big[(p_1, p_2) + \mathtt{i}\Big(-\frac{r_2}{r_1}p_1, \frac{r_1}{r_2}p_2\Big)\Big]$$

is a minimal Lagrangian immersion. Therefore, by Theorem 1.2, the map $F : \mathbf{S}^{q_1}(r_1) \times \mathbf{S}^{q_2}(r_2) \times \mathbf{S}^{m-1}(1) \times T^1 \to Q_n$ given by

$$((p_1, p_2), x, \mathrm{e}^{\mathtt{i}t}) \mapsto \frac{1}{2}\Big[\Big((p_1, p_2) + \mathtt{i}\Big(-\frac{r_2}{r_1}p_1, \frac{r_1}{r_2}p_2\Big)\Big), \mathrm{e}^{\mathtt{i}t}(x + \mathtt{i}\mathcal{J}x)\Big]$$

is H-minimal, and it is minimal if $q_1 + q_2 = m - 1$. Here, $q_1 + q_2 + m = n$.

Acknowledgement The authors would like to thank the referee for various helpful comments and suggestions.

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