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Sufficient Conditions for Amalgamated 3-Manifolds to be ∂-Irreducible*

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Abstract In this paper, the authors give some sufficient conditions for an amalgamated 3-manifold along a compact connected surface F with boundary to be ∂-irreducible in terms of distances between some kinds of vertex subsets of the curve complex and the arc complex of F.

Keywords ∂-Irreducibility, Irreducibility, Amalgamated 3-manifold, Curve complex, Arc complex
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1 Introduction

Let M_1 and M_2 be two compact connected orientable 3-manifolds with boundary, $F_i \subset \partial M_i$ be a compact connected surface, $i = 1, 2$, and $h : F_1 \to F_2$ be a homeomorphism. We call the 3-manifold $M = M_1 \cup_F M_2$, obtained by gluing M_1 and M_2 together via h, an amalgamated 3-manifold of M_1 and M_2 along F, where $F = F_1 = F_2$ in M.

Clearly, if M_1 and M_2 are compression bodies and $\partial_+ M_1 = F = \partial_+ M_2$, then $M_1 \cup_F M_2$ is a Heegaard splitting for M. It is well known that any compact connected orientable 3-manifold admits a Heegaard splitting, and any closed orientable 3-manifold can be obtained by Dehn surgery on a link in S^3. So Heegaard splittings and Dehn fillings can be viewed as typical ways to construct 3-manifolds by means of amalgamations. The amalgamation of two 3-manifolds, as well as the amalgamation of two Heegaard splittings, have been studied extensively in recent 30 years.

One of the interesting questions might be: Under what conditions, the amalgamated 3-manifolds are ∂-irreducible?

Przytycki's theorem (see [17]) (1983) on the incompressibility of one relator 3-manifolds can be regarded as a first approach to the question. Jaco [10] then generalized Przytycki's result to the well-known Handle Addition Theorem in 1984. Hence after, several generalizations on adding 2-handles to 3-manifolds have been made, see, for example, [2, 13, 18], etc.

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If M admits an amalgamation $H_1 \cup_F H_2$ along a connected surface F with boundary, where both H_1 and H_2 are handlebodies, then $H_1 \cup_F H_2$ is called an H'-splitting for M. It is shown in [3] that each compact connected orientable 3-manifold with boundary admits an H'-splitting. In [14], a necessary and sufficient condition for an amalgamation of two handlebodies to be a handlebody is given.

The curve complex on a surface was first defined by Harvey [5] in late 1970s, and the concept of the Heegaard distances of Heegaard splittings was introduced to study 3-manifolds by Hempel [8] in 2001. Since then, much significant progress on the study on amalgamated 3-manifolds (as well as on amalgamated Heegaard splittings) via the distance has been made, refer to, for example, [11, 16, 19], etc.

For an amalgamation $M_1 \cup_F M_2$ of 3-manifolds M_1 and M_2 along a common boundary component F of M_1 and M_2, Li [15] introduced a kind of distance $d(\mathcal{U}_1, \mathcal{U}_2)$ between some two subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of vertices in $\mathcal{C}(F)$, and proved that there is a number K depending on M_1 and M_2, such that if $d(\mathcal{U}_1, \mathcal{U}_2) > K$, then M is irreducible and ∂-irreducible.

In this paper, we give some sufficient conditions for an amalgamated 3-manifold along a compact connected surface with boundary to be ∂-irreducible and irreducible in terms of distances between some kinds of vertex subsets of the curve complex and the arc complex, see Theorems 3.1–3.2 in Section 3.

The paper is organized as follows. In Section 2, we briefly introduce some definitions and preliminaries. The statements of the main results and their proofs are given in Section 3.

2 Preliminaries

Throughout this paper, all 3-manifolds and surfaces are compact and orientable. For a submanifold X of Y, we denote the interior of X by $\mathrm{int}(X)$, the closure of X by $\overline{X}$, the number of connected components of X by $|X|$, the closed regular neighborhood of X by $\eta(X)$. The concepts and terminologies which are not defined in the paper are all standard, referring to, for example, [6–9].

Let F be a connected surface with boundary. A simple arc γ properly embedded in F is inessential in F if γ cuts off a disk from F; otherwise, γ is essential in F. A simple close curve (s.c.c. for short) α in F is inessential in F if α bounds a disk in F; otherwise, α is essential in F.

Let M be a connected 3-manifold. A disk D properly embedded in M is inessential in M if D cuts off a 3-ball from M; otherwise, D is essential in M. A 2-Sphere S embedded in M is inessential in M if S bounds a 3-ball in M; otherwise, S is essential in M. M is reducible if M contains an essential 2-sphere. M is irreducible if M is not reducible.

Let M be a 3-manifold, F be a surface either in ∂M or properly embedded in M. If one of the following conditions is satisfied:

(1) F is an inessential 2-sphere in M, or

(2) F is a disk in ∂M, or F is an inessential disk in M, or

(3) there is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D is essential in F,

we say F is compressible in M. In case (3), the disk D is called a compression disk of F in M. F is incompressible if it is not compressible in M. We say M is ∂-reducible if ∂M is compressible in M. Otherwise, M is ∂-irreducible.

Let M be a 3-manifold, and F be a connected surface either lying in ∂M or properly embedded in M. Suppose that F is neither a disk nor a 2-sphere. It follows from Dehn Lemma that F is incompressible in M if and only if the homomorphism $\pi_1(F) \to \pi_1(M)$ induced by the inclusion is injective. A 3-manifold is called a Haken manifold if it is irreducible and contains a 2-sided incompressible surface.

Let M be a connected 3-manifold with boundary, F be a surface properly embedded in M. If F cuts off a 3-manifold X which is homeomorphic to $F \times I$, we say that F is boundary parallel in M. F is essential in M if F is incompressible and not boundary parallel in M.

Lemma 2.1 *Let D be an essential disk in M, and Δ be a disk in M such that $\Delta \cap D = \alpha$ is an arc in $\partial\Delta$, $\Delta \cap \partial M = \beta$ is an arc in $\partial\Delta$, and $\alpha \cap \beta = \partial\alpha = \partial\beta$, $\alpha \cup \beta = \partial\Delta$. α cuts D into two sub-disks D' and D''. Set $D_1 = D' \cup \Delta$ and $D_2 = D'' \cup \Delta$. Then at least one of D_1 and D_2 is essential in M.*

The operation in Lemma 2.1, from D to D_1 and D_2, is called a ∂-compression of D along Δ. Refer to [6] for a proof of Lemma 2.1.

Definition 2.1 *Let M_i be a connected 3-manifold with boundary, S_i be a boundary component of M_i, $F_i \subset S_i$ be a connected sub-surface of S_i, $i = 1, 2$, and $h : F_1 \to F_2$ be a homeomorphism. The 3-manifold $M = M_1 \cup_h M_2$ obtained by gluing M_1 and M_2 via h is called an amalgamation of M_1 and M_2. Denote by F the surface $F_1 = F_2$ in M, and call F a splitting surface of M. We usually denote M by $M_1 \cup_F M_2$, and call M an amalgamated 3-manifold along F.*

In particular, when F is a disk, $M_1 \cup_F M_2$ is called a boundary connected sum of M_1 and M_2, and is denoted by $M_1 \#_\partial M_2$; when both M_1 and M_2 are compression bodies, and $F = \partial_+ M_1 = \partial_+ M_2$, $M_1 \cup_F M_2$ is called a Heegaard splitting for M, and F is called a Heegaard surface in M; when both M_1 and M_2 are handlebodies, $M_1 \cup_F M_2$ is called an H'-splitting for M, and F (possibly non-closed) is called an H'-surface in M.

It is a well-known fact that any compact connected 3-manifold admits a Heegaard splitting, and it is shown in [3] that any compact connected 3-manifold with boundary admits an H'-splitting.

Let $V \cup_S W$ be a Heegaard splitting for M. $V \cup_S W$ is reducible (weakly reducible, resp.) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 = \partial D_2$ ($\partial D_1 \cap \partial D_2 = \emptyset$, resp.). Otherwise, $V \cup_S W$ is irreducible (strongly irreducible, resp.).

It is a theorem of Haken (see Haken's Lemma [4]) that any Heegaard splitting of a reducible 3-manifold is reducible, and a theorem of Casson-Gordon [2] that if $V \cup_S W$ is a weakly reducible Heegaard splitting for M, then either $V \cup_S W$ is reducible, or M is Haken.

The following proposition is a well-known fact, refer to [12] for a proof.

Proposition 2.1 *Let $M = M_1 \cup_F M_2$ be an amalgamation of two 3-manifolds M_1 and M_2*

along F. Suppose that F is incompressible in both M_1 and M_2.

(1) *Then M is irreducible if and only if both M_1 and M_2 are irreducible.*

(2) *F is a closed surface. Then M is ∂-irreducible if and only if both M_1 and M_2 are ∂-irreducible.*

Let M be a connected 3-manifold with boundary, S be a boundary component of M, and L be a simple closed curve in S. If there exists an essential disk D in M with $|L \cap \partial D| = 1$, L is called a longitude of M, and $(L, \partial D)$ is called a longitude-meridian pair of M.

Proposition 2.2 *Let $(L, \partial D)$ be a longitude-meridian pair on a boundary component S of 3-manifold M.*

(1) *If S is a torus, then $M = T\#M'$, where T is a solid torus with $\partial T = S$, and $(L, \partial D)$ is a longitude-meridian pair of T.*

(2) *If $g(S) \geq 2$, there exists a separating disk E properly embedded in M such that E cuts M into a solid torus T' with the longitude-meridian pair $(L, \partial D)$ and a 3-manifold M'', and $M = T'\#_\partial M''$.*

Proof (1) S is a torus. Push L slightly to L' in $\text{int}(M)$ by isotopy such that $|L' \cap D| = 1$. Let $N = \eta(S \cup D \cup L')$ be a closed regular neighborhood of $S \cup D \cup L'$ in M. $\partial N = S \cup S^*$, where S^* is a 2-sphere which cuts M into N and a 3-manifold M^*. Denote by T (M', resp.) the 3-manifold obtained by filling in a 3-ball to N (M^*, resp.) along the 2-sphere component S^*. Then T is a solid torus with $\partial T = S$, $(L, \partial D)$ is a longitude-meridian pair of T, and $M = T\#M'$.

(2) $g(S) \geq 2$. Let $T' = \eta(D \cup L)$ be a closed regular neighborhood of $D \cup L$ in M. Then T' is a solid torus. Denote $\overline{M - T'}$ by M''. Then $T' \cap M'' = E$ is a separating disk properly embedded in M, $(L, \partial D)$ is a longitude-meridian pair of T', and $M = T'\#_\partial M''$.

For an annulus $A = S^1 \times I$, $J = S^1 \times \frac{1}{2}$ is called a core curve of A. In the following, we collect some facts on an amalgamation of two 3-manifolds along an annulus.

Proposition 2.3 *Let $M = M_1 \cup_A T$ be an amalgamated 3-manifold of M_1 and T along an annulus A, where T is a solid torus, and the core curve of A is a longitude of T. Then $M \cong M_1$.*

Proof Note that when the core curve of A is a longitude of T, $T \overset{h}{\cong} A \times I$ with $h(A) = A \times 0$, the conclusion follows directly.

Let M_1 and M_2 be 3-manifolds with boundary. Suppose that M_1 has a boundary component S_1 with a longitude-meridian pair $(L, \partial D) \subset S_1$, and A_1 is a regular neighborhood of L in S_1. Let $A_2 \subset \partial M_2$ be an annulus, and $M = M_1 \cup_A M_2$ be an amalgamation of M_1 and M_2 via a homeomorphism $h: A_1 \to A_2$. If S_1 is a torus, then by Proposition 2.2(1), $M_1 = T\#M_1'$, where T is a solid torus with $\partial T = S_1$, and $(L, \partial D)$ is a longitude-meridian pair of T, thus $M = M_1 \cup_A M_2 = (M_1'\#T) \cup_A M_2 = M_1'\#(T \cup_A M_2)$. By Proposition 2.3, $T \cup_A M_2 \cong M_2$, so $M \cong M_1'\#M_2$. In particular, if both M_1' and M_2 are ∂-irreducible, it follows from Proposition

2.1(2) that M is ∂-irreducible. If $g(S_1) \geq 2$, then by Proposition 2.2(2), $M_1 = T'\#_\partial M_1''$, where T' is a solid torus with the longitude-meridian pair $(L, \partial D)$. Thus $M = M_1 \cup_A M_2 = (M_1''\#_\partial T') \cup_A M_2 = M_1''\#_\partial(T' \cup_A M_2) \cong M_1''\#_\partial M_2$. In particular, ∂M is compressible.

The following theorem is the well-known Jaco's Handle Addition Theorem.

Theorem 2.1 (Handle Addition Theorem) *Let M be an irreducible and ∂-reducible 3-manifold, and J be a simple closed curve on ∂M. Suppose that $\partial M - J$ is incompressible in M. Let M_J be the 3-manifold obtained by attaching a 2-handle to M along J. Then either*

(1) *M_J is ∂-irreducible, or*

(2) *M_J is a 3-ball, where M is a solid torus, and J is a longitude for M.*

Remark 2.1 Theorem 2.1 in case that M is a handlebody was first proved by Przytycki [17] in 1983 by an algebraic approach, then it was generalized to the Handle Addition Theorem by Jaco [10] in 1984. Some generalizations have been made hence later, see, for example, [2, 13, 18].

For a compact connected orientable surface S, we use $g = g(S)$, $b = b(S)$ to denote the genus of S, the number of boundary components of S, respectively, and denote S by $S_{g,b}$. We call $S_{0,b}$ a planar surface when $b > 0$, $S_{g,0}$ a closed surface of genus g, and simply denote it by S_g. For a compact connected sub-surface F of S_k with $b(F) > 0$, if each boundary component of F is essential in S, then it is not hard to see that $g(F) \leq k - 1$ and $b(F) \leq 2(k - g(F))$. Thus, an annulus is the only sub-surface on a torus.

The following two propositions are a direct consequence of Proposition 2.2 and Theorem 2.1 (see [12]).

Proposition 2.4 *Let M_1 and M_2 be irreducible 3-manifolds, $M = M_1 \cup_A M_2$ be an amalgamation of M_1 and M_2 along an annulus A. Suppose $\partial M_i - A$ is incompressible in M_i, $i = 1, 2$. Then M is ∂-irreducible if and only if either the core curve J is not a longitude of M_i for $i = 1, 2$, or J is a longitude for M_i and M_j has incompressible boundary for $\{i, j\} = \{1, 2\}$.*

Proposition 2.5 *Let $M = M_1 \cup_A M_2$ be an amalgamation of irreducible 3-manifolds M_1 and M_2 along an annulus A. Then M is reducible if and only if the core curve J bounds a disk in M_i and there exists an essential planar surface P in M_j whose boundary curves are all parallel to J on ∂M_i for $\{i, j\} = \{1, 2\}$.*

From now on, we only consider the amalgamated 3-manifold $M = M_1 \cup_F M_2$ along F, where F is a compact connected sub-surface of a boundary component S_i of M_i, $g(S_i) \geq 2$, $i = 1, 2$, and $\chi(F) < 0$ (i.e., F is neither a disk nor an annulus).

For an essential simple closed curve or arc γ on $S = S_{g,b}$, the isotopic class of γ is denoted by $\widehat{\gamma}$. If γ is parallel to a component of ∂S, we say that γ is peripheral in S. Otherwise, γ is non-peripheral in S.

Definition 2.2 (1) *Let $S = S_{g,b}$. The curve complex of S, denoted by $\mathcal{C}(S)$, is the complex* *whose vertices are the isotopy classes of essential non-peripheral simple closed curves in S, and $k+1$ pairwise distinct vertices determine a k-simplex if they are represented by pairwise disjoint curves on S. For any two vertices α and β in $\mathcal{C}(S)$, an edge path (from α to β) is a sequence $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ of vertices in $\mathcal{C}(S)$, such that α_{i-1} and α_i span a 1-simplex in $\mathcal{C}(S)$ for $1 \le i \le n$. n is called the length of the edge path. The distance of α and β is the smallest integer $n \ge 0$ such that there is an edge path from α to β of length n, and is denoted by $d(\alpha, \beta)$.*

(2) *For $S = S_{g,b}$ with $b \ge 1$, the arc complex $\mathcal{A}(S)$ is defined in a similar way : Vertices are the isotopy classes of essential arcs in S. A collection of $k+1$ pairwise distinct vertices span a k-simplex if they are represented by pairwise disjoint arcs on S. The distance in $\mathcal{A}(S)$ between two vertices is the minimal possible number of edges in an edge path between them.*

(3) *For two vertex subsets $V_1, V_2 \subset \mathcal{C}(S)$ or $V_1, V_2 \subset \mathcal{A}(S)$, the distance of V_1 and V_2 is defined to be*

$$d(V_1, V_2) = \min\{d(\alpha, \beta) \mid \alpha \in V_1, \beta \in V_2\}.$$

Remark 2.2 (1) Let $W_1 \cup_S W_2$ be a Heegaard splitting, set

$$V_i = \{\widehat{\alpha} \in \mathcal{C}(S) \mid \alpha \text{ bounds an essential disk in } W_i\}, \quad i = 1, 2.$$

The $D(S) = d(V_1, V_2)$ is called the distance of the Heegaard splitting $W_1 \cup_S W_2$.

The curve complex $\mathcal{C}(S)$ of a closed surface S was first defined by Harvey [5] in late 1970s, and the Heegaard distance $D(S)$ was introduced by Hempel [8] in 2001. It is clear that $V \cup_S W$ is reducible if and only if $D(S) = 0$, $V \cup_S W$ is weakly reducible if and only if $D(S) \le 1$.

(2) If F is an annulus or a pair of pants, $\mathcal{C}(F) = \emptyset$. If F is a torus, or a once-punctured torus, or a fourth-punctured 2-sphere, $\mathcal{C}(F)$ consists only vertices (there is no 1-simplex in $\mathcal{C}(F)$).

Let M be a compact connected 3-manifold, S be a boundary component of M with $g(S) \ge 2$, F is a compact connected sub-surface of S with $\partial F \ne \emptyset$ and $\chi(F) < 0$, and each component of ∂F is essential in S. For an essential s.c.c J on S, we assume that J is in a position that J intersects ∂F transversely and $J \cap \partial F$ is minimal among the curves in $\widehat{J}$.

Definition 2.3 *Denote the following vertex subset of $\mathcal{A}(F)$,*

$$\{\widehat{\gamma} \in \mathcal{A}(F) \mid \gamma \textit{ is a component of } F \cap \partial D, \textit{ where } D \textit{ is an essential disk in } M\}$$

by $\mathcal{A}_D(F; M)$, and following vertex subset of $\mathcal{C}(F)$,

$$\{\widehat{J} \in \mathcal{C}(F) \mid \exists \textit{ an essential planar surface } P \subset M, \partial P \cap \partial F = \emptyset, \textit{ and } J \textit{ is a component of } (\partial P) \cap F\}$$

by $\mathcal{C}_P(F; M)$.

Note that for a $\widehat{J} \in \mathcal{C}_P(F; M)$, J is a boundary component of an essential planar surface P in M with $\partial P \cap \partial F = \emptyset$ and $J \subset F$, P may have some other boundary components lying in $\overline{\partial M - F}$; $\mathcal{C}_D(F; M)$ denotes the collection of vertices $\widehat{J}$ in $\mathcal{C}(F)$ with $J \subset F$ and J bounding an essential disk in M.

3 Main Results

The following theorem gives a sufficient condition for an amalgamated 3-manifold along a surface F with non-empty boundary to be ∂-irreducible.

Theorem 3.1 *Let $M = M_1 \cup_F M_2$ be an amalgamation of 3-manifolds M_1 and M_2 along F, where F is lying in a component S_i of ∂M_i with $g(S_i) \geq 2$, $i = 1, 2$, and F is neither an ith-punctured 2-sphere ($i \leq 4$), nor a once-punctured torus. Suppose that the following conditions are satisfied*:

(i) $\partial M_i - F$ *is incompressible in* M_i, $i = 1, 2$;

(ii) $d(\mathcal{A}_D(F; M_1), \mathcal{A}_D(F; M_2)) > 0$;

(iii) $d(\mathcal{C}_D(F; M_i), \mathcal{C}_P(F; M_j)) > 1$ *for* $\{i, j\} = \{1, 2\}$.

Then M is ∂-irreducible.

Proof Assume that M is ∂-reducible. Let D be a compression disk of ∂M in M, such that D is in general position with F. If $D \cap F = \emptyset$, then D is a properly embedded disk in M_i with $\partial D \in \partial M_i - F$, $i = 1$ or 2, then D is a compression disk of $\partial M_i - F$ in M_i, contradicting to the assumption (i). Therefore, $D \cap F \neq \emptyset$. D can be viewed as a $2n$-polygon whose edges lie in $\overline{\partial M_1 - F}$ and $\overline{\partial M_2 - F}$ alternatively. Set $c(D) = (2n, |D \cap F|)$, call $c(D)$ the complexity of D. We compare the complexities in lexicographical order which is $(a, b) < (c, d)$ if and only if $a < c$ or $a = c$ and $b < d$. Choose a compression disk of ∂M, still denoted by D, such that D is in general position with F, and D has the least complexity among all such compression disks of ∂M up to isotopy.

Claim 1 Each arc component of $D \cap F$ is essential on F.

Otherwise, there exists an arc component α of $D \cap F$, so α cuts out of a disk E from F and $\text{int}(E)$ contains no arc component of $D \cap F$. If $\text{int}(E)$ contains circle components of $D \cap F$, choose a circle component σ of $D \cap F$ such that σ is innermost in E, i.e., σ bounds a disk E' in $\text{int}(E)$ with $\text{int}(E') \cap D = \emptyset$. σ bounds a disk D_0 in $\text{int}(D)$. Push the disk $\overline{(D - D_0)} \cup E'$ slightly in M by isotopy, we get a disk D^* with $\partial D^* = \partial D$, and $|D^* \cap F| < |D \cap F|$ (therefore $c(D^*) < c(D)$), see Figure 1 below, contradicting to the minimality of $c(D)$. Thus $\text{int}(E) \cap D = \emptyset$.

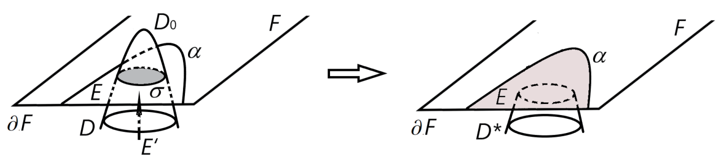

Figure 1 D to D^*

α cuts D into two sub-disks D_1 and D_2. Set $D' = D_1 \cup E$, $D'' = D_2 \cup E$. Then by Lemma 2.1, at least one of D' and D'' is an essential disk of M, and after an isotopy around E, $\max\{|D' \cap F|, |D'' \cap F|\} \leq |D \cap F| - 1$, again contradicting to the minimality of $c(D)$. Hence, Claim 1 holds.

Claim 2 Each component of $D \cap \overline{(\partial M_i - F)} = \partial D \cap \overline{(\partial M_i - F)}$ is essential on $\overline{(\partial M_i - F)}$, $i = 1, 2$.

Otherwise, there is an edge β of ∂D, which is a component of $\partial D \cap \overline{(\partial M_i - F)}$, such that $\partial\beta$ bounds an arc s in ∂F and $s \cup \beta$ bounds a disk $E \subset \overline{(\partial M_i - F)}$ on ∂M with $\mathrm{int}(E) \cap \partial D = \emptyset$. Since $E \subset \partial M$, $D \cap E$ has no circle component. Thus $\mathrm{int}(E) \cap D = \emptyset$. Push $D \cup E$ slightly by isotopy in M to get a properly embedded disk D' in M, then D' is isotopic to D in M, but D' is a $(2n-2)$-polygon, again contradicting to the minimality of $c(D)$. In fact, if $\partial\beta$ is the boundary components of an arc component γ of $D \cap F$, then $D' \cap F = (D \cap F - \{\gamma\}) \cup \{\gamma'\}$, where γ' is a circle component, see Figure 2(1) below; if the two arc components γ_1 and γ_2 of $D \cap F$ are incident to the two points of $\partial\beta$, then γ_1 and γ_2 will merge to a single component γ' of $D' \cap F$, $D' \cap F = (D \cap F - \{\gamma_1, \gamma_2\}) \cup \{\gamma'\}$, see Figure 2(2) below.

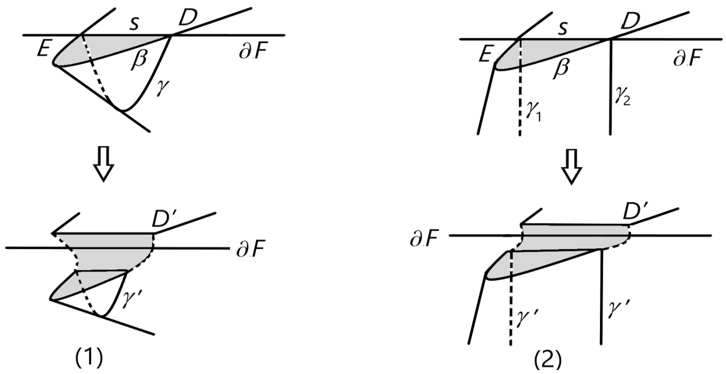

Figure 2 D to D'

Claims 1 and 2 imply that ∂D intersects ∂F essentially. In the following, we divide it into two cases to discuss.

Case 1 $D \cap F$ consists of arc components.

Set $A = D \cap F = \{\beta_1, \cdots, \beta_m\}$. Then A cuts D into $m+1$ disks $\Delta_0, \Delta_1, \cdots, \Delta_m$, each Δ_i is a polygon with even number of edges which is properly embedded in M_1 or M_2, $0 \le i \le m$. By Claims 1 and 2, $\partial\Delta_i \cap \partial F$ has the smallest possible intersection number, $0 \le i \le m$.

Claim 3 For each i, say $\Delta_i \in M_j$ ($j = 1$ or 2), Δ_i is an essential disk in M_j, $0 \le i \le m$.

Otherwise, $\partial\Delta_i$ bounds a disk E in ∂M_j. A similar argument to the proof of Claim 1 implies this can not happen.

Now let β be an arc component of $D \cap F$ which is outermost on D, i.e., β cuts out of a sub-disk, say, $\Delta_1 \subset M_1$, from D with $\mathrm{int}(\Delta_1) \cap F = \emptyset$. Without loss of generality, we assume that Δ_2 is the polygon such that Δ_1 and Δ_2 have the edge β in common. Clearly, $\Delta_2 \subset M_2$. By Claim 3, both Δ_1 and Δ_2 are essential disks in M_1 and M_2, respectively. By a slight isotopy of Δ_1 to Δ_1' in M_1, $\partial\Delta_1' \cap \partial\Delta_2 = \emptyset$. This contradicts to the assumption (ii). So Case 1 can not happen.

Case 2 $D \cap F = C \cup A$, where C ($\neq \emptyset$) consists of all the circle components of $D \cap F$, and A consists of all the arc components of $D \cap F$ (possibly, $A = \emptyset$).

Set $P_1 = D \cap M_1$, $P_2 = D \cap M_2$.

Claim 4 P_i is incompressible in M_i, $i = 1, 2$. In particular, each disk in P_i is essential in M_i, $i = 1, 2$.

Assume that it is not the case, say, P_1 is inessential in M_1. If some disk component Σ of P_1 is boundary parallel in M_1, then $\partial\Sigma$ bounds a disk E in ∂M_1, and $\Sigma \cup E$ bounds a 3-ball X in M_1. We may push Σ to E crossing X and push further slightly in a neighborhood of X by isotopy to get a new compression disk D^* of ∂M in M with $\partial D^* = \partial D$, but $c(D^*) < c(D)$, contradicting to the minimality of $c(D)$. Thus, each disk component of P_1 is essential in M_1. If P_1 is compressible in M_1, then there exists a compression disk Ω of P_1 in M_1. Ω is lying in a non-disk component of P_1. $\partial\Omega$ bounds a disk Δ^* in D. Set $D^{**} = \overline{(D - \Delta^*)} \cup \Omega$. Then D^{**} is a compression disk of ∂M in M with $\partial D^{**} = \partial D$, and it is clear that $c(D^{**}) < c(D)$, again contradicting to the minimality of $c(D)$. Thus P_1 is incompressible in M_1. This finishes the proof of Claim 4.

A circle component γ in C is called nested if the interior of the disk bounded by γ in D contains a non-empty subset of C; otherwise, it is the innermost.

Subcase 2.1 $D \cap F$ has at least a nested component.

Let $\gamma \in C$ be a nested component such that the interior of the disk bounded by γ in D contains no nested circle in C. Denote the disk bounded by γ in D by Δ, and the subset of circles in C which lie in Δ by C'. $C' \neq \emptyset$, say $C' = \{c_1, \cdots, c_k\}$, each c_i bounds a disk σ_i in D with $\mathrm{int}(\sigma_i) \cap F = \emptyset$, $1 \le i \le k$. Set $P_\gamma = \Delta - \bigcup\limits_{i=1}^{k} \sigma_i$, say $P_\gamma \subset M_1$, thus $\sigma_i \subset M_2$, $1 \le i \le k$.

Claim 5 P_γ is essential in M_1.

Otherwise, by Claim 4, P_γ is incompressible in M_1. So P_γ is boundary parallel in M_1. Thus P_γ is separating in M_1 which cuts M_1 into two pieces M_1' and M_1'', say, $M_1' = P_\gamma \times I$, and $P_\gamma = P_\gamma \times 0$, $P_\gamma' = \overline{\partial M_1' - P_\gamma}$. If $P_\gamma' \subset F$, then we may push P_γ to P_γ' by isotopy in M_1, then a little bit further in M_2, to get a new compression disk D' of ∂M in M with $\partial D' = \partial D$, but $|D' \cap F| < |D \cap F|$ (hence $c(D') < c(D)$), contradicting to the minimality of $c(D)$. Otherwise, P_γ' contains some components of ∂F. Note that P_γ' is a planar surface homeomorphic to P_γ. Let δ be a component of ∂F lying in P_γ', then δ is separating in P_γ' and $\overline{P_\gamma' - \delta}$ has two planar surface components Q and Q'. Let Q be the planar surface such that $\mathrm{int}(Q)$ contains no boundary component of F and γ be not a boundary component of Q. Clearly, all the boundary components of Q other than δ bound disks $E_1, \cdots, E_l$ in M_2. Set $D'' = Q \bigcup\limits_{j=1}^{l} E_j$. It is clear that δ is essential on ∂M. We perform an isotopy on D'' by pushing $\mathrm{int}(D'')$ to $\mathrm{int}(M)$, then the disk D_δ after this isotopy is a compression disk of ∂M with $|D_\delta \cap F| < |D \cap F|$ (hence $c(D_\delta) < c(D)$), again contradicting to the minimality of $c(D)$.

Thus, P_γ is essential in M_1. By assumption (i), no component of P_γ is parallel to a component of F on F. But the disks in M_2 bounded by ∂P_γ can be moved in M_2 to be disjoint from ∂P_γ by isotopy, contradicting to the assumption (iii).

Subcase 2.2 $D \cap F$ has no nested component. In the case, each c in C is the innermost in D.

First consider the case of $A = \emptyset$. Say $\partial D \subset \partial M_1 - F$. Then $P = D \cap M_1$ is a connected planar surface in M_1 with one boundary component ($= \partial D$) lying in $\partial M_1 - F$ and all the others lying in F, and $D \cap M_2$ is a non-empty set which consists of pairwise disjoint essential disks in M_2. If no boundary component of P is parallel to a boundary component of F on F, then P is essential in M_1. This contradicts to the assumption (iii) and the conclusion holds in this case.

Now assume P is not essential in M_1. By Claim 4, P is incompressible in M_1. So P is boundary parallel in M_1. Denote $P \cap (\partial M_1 - F)$ by δ and $P \cap F$ by $\delta_1, \delta_2, \cdots, \delta_k$. Then each δ_i bounds an essential disk in M_2. δ_i is not parallel to a component of F on F, for $i = 1, 2, \cdots, k$. Otherwise $\partial M_2 - F_2$ is compressible in M_2. This contradicts to the assumption (i). Suppose P is parallel to a subsurface P' of ∂M_1 and $\partial P' = \{\delta, \delta_1, \delta_2, \cdots, \delta_k\}$. Since $\{\delta_1, \delta_2, \cdots, \delta_k\}$ lie in F and δ lies in $\partial M_1 - F$, there is at least one component of ∂F, say δ', lying in P'. Then δ' cuts a planar surface from P' which contains $\delta_{i_1}, \delta_{i_2}, \cdots, \delta_{i_m}$. This implies that δ' bounds an essential disk in M_2 and $\partial M_2 - F_2$ is compressible in M_2. This contradicts to the assumption (i). So P is essential in M_1 and the conclusion holds in this case.

If $A \neq \emptyset$, for an outermost arc $\beta \in A$, β cuts out of a disk Δ' from D without any other component in $A - \{\beta\}$ lying in Δ'. If Δ' contains a component of C, all the components $\delta_1, \cdots, \delta_k$ of C lying in Δ' are non-nested. So each δ_i bounds a disk σ_i' in D with $\text{int}(\sigma_i') \cap F = \emptyset$, $1 \leq i \leq k$. Set $P' = \Delta' - \bigcup\limits_{i=1}^{k} \sigma_i'$. As before, each δ_i is essential in F, so we get a contradiction to the assumption (iii). Thus Δ' contains no component of C.

Let Δ'' be the component of the surface obtained from cutting D open along A with $\Delta' \cap \Delta'' = \beta$. If Δ'' contains a component of C, we can similarly have a contradiction to the assumption (iii). Thus Δ'' contains no component of C. Say, $\Delta' \subset M_1$ and $\Delta'' \subset M_2$. As in Case 1, this will derive to a contradiction to the assumption (ii).

Remark 3.1 (1) The condition (iii) in Theorem 3.1 implies that there exists an edge path of length at least 2 in $\mathcal{C}(F)$ which rule out the possibilities that F is an ith-punctured 2-sphere ($i \leq 4$), or a once-punctured torus.

(2) The condition (iii) in Theorem 3.1 can be replaced by the following stronger condition:

(iii)′ $d(\mathcal{C}_P(F; M_1), \mathcal{C}_P(F; M_2)) > 1$.

(3) In the main theorem (Theorem 1.2) in [15] (as well as in [1]), as one of the conditions, the condition of $d(\mathcal{U}_1, \mathcal{U}_2) > K$ is required to guarantee the incompressibility of the boundary of the amalgamated 3-manifolds along closed boundary components, where K is a constant depending only on the factor manifolds and the genus of the amalgamating surfaces. The condition (iii) in Theorem 3.1 ($d(\mathcal{C}_D(F; M_i), \mathcal{C}_P(F; M_j)) > 1$) is unified to guarantee the incompressibility of the boundary of the amalgamated 3-manifold along subsurfaces with boundary.

The followings are a direct consequence of Theorem 3.1.

Corollary 3.1 *Let $M = H_1 \cup_F H_2$ be an H'-splitting of 3-manifold M with boundary, where $g(\partial H_i) \geq 2$, $i = 1, 2$, and $\chi(F) < 0$. Suppose that the following conditions are satisfied:*

(i) *$\partial H_i - \partial F$ is incompressible in H_i, $i = 1, 2$;*

(ii) *$d(\mathcal{A}_D(F; H_1), \mathcal{A}_D(F; H_2)) > 0$.*

Then M is ∂-irreducible.

Proof As in the proof of Theorem 3.1, if M is ∂-reducible, let D be a compression disk of ∂M in M with minimal complexity. Then $D \cap F$ consists of only arc components. Claims 1–3 in the proof of Theorem 3.1 will derive contradictions in all possibilities.

Remark 3.2 Let M be a 3-manifold with boundary, and S be a component of ∂M. Let $\mathcal{J}$ be a collection of pairwise disjoint simple closed curves on S. $\mathcal{J}$ is called disk-busting if $\bigcup_{J\in\mathcal{J}} J$ intersects each simple closed curve which bounds a disk in M nontrivially. In particular, if $\mathcal{J}$ contains a single simple closed curve J, such a J is called a disk-busting curve. It is clear that the condition (i) in Corollary 3.1 is equivalent to that ∂F is disk-busting in both handlebodies H_1 and H_2, and can be replaced by a stronger condition: One component of ∂F is disk-busting in H_1 and one component of ∂F is disk-busting in H_2.

In the next theorem, we give a sufficient condition for an amalgamated 3-manifold along a surface F with boundary to be irreducible.

Theorem 3.2 *Let $M = M_1 \cup_F M_2$ be an amalgamation of irreducible 3-manifolds M_1 and M_2 along F, where F is lying in a component S_i of ∂M_i with $g(S_i) \geq 2$, $i = 1, 2$, and F is neither an ith-punctured 2-sphere ($i \leq 4$), nor a once-punctured torus. Suppose the following conditions are satisfied:*

(i) *Each boundary component of F does not bound a disk in M_1 or M_2;*

(ii) *$d(\mathcal{C}_D(F; M_i), \mathcal{C}_P(F; M_j)) > 1$ for $\{i, j\} = \{1, 2\}$.*

Then M is irreducible.

Proof Otherwise, M is reducible. Let S be an essential 2-sphere in M which is in general position with F. Since both M_1 and M_2 are irreducible, so $S \cap F \neq \emptyset$, and $S \cap F$ consists of finitely many circles. Choose an essential 2-sphere in M, still denoted by S, such that $S \cap F$ is the minimal components among all such essential 2-spheres in M. For a component α of $S \cap F$ which is innermost on S, by the assumption (i), α is not boundary parallel in F, then as in the proof of Theorem 3.1, Claims 4–5 will derive a contradiction to the assumption condition (ii). This finishes the proof.

Corollary 3.2 *Let $M = M_1 \cup_F M_2$ be an amalgamation of irreducible 3-manifolds M_1 and M_2 along F, where F is lying in a component S_i of ∂M_i with $g(S_i) \geq 2$, $i = 1, 2$, and $\chi(F) < 0$. Suppose F is incompressible in both M_1 and M_2. Then M is irreducible.*

Proof Otherwise, let S be an essential 2-sphere in M so that S intersects F minimally. An innermost component of $S \cap F$ will bound a compression disk of F in M_1 or M_2, a contradiction to the assumption condition.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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