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On Existence of the Even L_p Gaussian Minkowski Problem for $p > n$*

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Abstract This paper concerns the even L_p Gaussian Minkowski problem in n-dimensional Euclidean space $\mathbb{R}^n$. The existence of the solution to the even L_p Guassian Minkowski problem for $p > n$ is obtained.

Keywords Convex body, Existence, L_p Gaussian surface area measure, The even L_p Gaussian Minkowski problem
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1 Introduction

In n-dimensional Euclidean space $\mathbb{R}^n$, a compact convex set with non-empty interior is called a convex body. Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$ and $\mathcal{K}^n_o$ denote the set of all convex bodies in $\mathbb{R}^n$ containing the origin in their interiors. The set of all origin-symmetric convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n_e$. Clearly, $\mathcal{K}^n_e \subset \mathcal{K}^n_o \subset \mathcal{K}^n$.

As one of main parts of the Brunn-Minkowski theory, a Minkowski problem characterizes a geometric measure generated by convex bodies: Given a non-zero finite Borel measure on unit sphere in $\mathbb{R}^n$, what are the necessary and sufficient conditions such that the given measure is a geometric measure generated by a convex body? Furthermore, if such a convex body exists, is it unique? These two problems are called existence and uniqueness of the solution to Minkowski problem. There is a long history for study of Minkowski problem which greatly promotes developments of the Brunn-Minkowski theory and fully non-linear partial differential equations (see [38, 41]). We will review some Minkowski problems shortly.

In 1990s, Lutwak [30] introduced the L_p surface area measure $S_p(K,\cdot)$ of convex body $K \in \mathcal{K}^n_o$ by the variational formula of the n-dimensional volume (Lebesgue measure) V_n for L_p Minkowski combination as follows.

For $p \in \mathbb{R} \setminus \{0\}$,

$$\lim_{t\to 0^+} \frac{V_n(K +_p t \cdot L) - V_n(K)}{t} = \frac{1}{p} \int_{S^{n-1}} h^p_L(u) \mathrm{d}S_p(K,u), \tag{1.1}$$

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where $K +_p t \cdot L$ is the L_p Minkowski combination of $K, L \in \mathcal{K}_o^n$ (see the details in (2.2)), and h_L is the support function of L (see the details in (2.1)). Note that the case for $p = 0$ can be defined by similar way. It is called cone-volume measure which has intuitive geometric significance and is the only one among all L_p surface area measure that is $\mathrm{SL}(n)$ invariant.

When $p = 1$, the L_1 surface area measure $S_1(K, \cdot)$ is the well-known classical surface area measure S_K, that is, $S_1(K, \cdot) = S_K$. The Minkowski problem associated with the classical surface area measure is called the classical Minkowski problem: What are the necessary and sufficient conditions such that a given non-zero finite Borel measure on unit sphere is the classical surface area measure of a convex body? Many important works for the existence and uniqueness of this problem are due to Minkowski [35–36], Alexandrov [1–2], Fenchel-Jessen [15] and others.

In [30], Lutwak not only introduced the L_p surface area measure but also studied associated Minkowski problem called L_p Minkowski problem as follows.

***L_p* Minkowski problem** For a fixed p and a given non-zero finite Borel measure μ on S^{n-1}, what are necessary and sufficient conditions in order that there exists a convex body K in $\mathbb{R}^n$ such that its L_p surface area measure $S_p(K, \cdot)$ is equal to μ, that is,

$$S_p(K, \cdot) = \mu?$$

The volume normalized form of this problem is called the normalized L_p Minkowski problem.

What are necessary and sufficient conditions in order that there exists a convex body K in $\mathbb{R}^n$ such that

$$\frac{S_p(K, \cdot)}{V(K)} = \mu?$$

Note that, when $p \neq n$, the L_p Minkowski problem and its normalized version are equivalent. This is due to positive homogeneity of degree $(n - p)$ of the L_p surface area measure $S_p(K, \cdot)$.

It is not hard to see that the L_p Minkowski problem is the generalization of the classical Minkowski problem ($p = 1$). What's more, the L_p Minkowski problem has other two special cases: The centro-affine Minkowski problem ($p = -n$) and the logarithmic Minkowski problem ($p = 0$), see [6, 9, 13, 26, 39–40, 42, 52–53]. So far, there are many results for the existence, uniqueness, regularity and continuity of the (normalized) L_p Minkowski problem. For more references, one can see [11, 18, 22, 25, 28, 30–31, 33, 44, 54–56]. As an important application, the solutions to the L_p Minkowski problem are powerful tools for discovering a kind of new affine isoperimetric inequalities of which some new (sharp) affine L_p Sobolev inequalities are important parts, see [12, 19–21, 32, 46, 48]. Besides, the solutions to this problem have close relation to some important flows, see [3–4, 39–40].

As "dual" case of the L_p Minkowski problem, the dual Minkowski problem was posed by Huang-Lutwak-Yang-Zhang [23] and is a characterization problem for dual curvature measure defined by the variational formula of the dual volume (see [29]) for L_1 Minkowski combination. Note that the dual volume is a generalization of volume. Recently, the dual Minkowski problem and its generalization have rapid developments for the existence, uniqueness, regularity and continuity, see [5, 7, 10, 16, 34, 43, 45, 47, 49–51].

It is well-known that the volume (Lebesgue measure) is an important notion in $\mathbb{R}^n$ and has some beautiful properties such as translation invariance, homogeneity and so on. In Brunn-Minkowski theory, the Gaussian probability measure γ_n also has hot attention and is defined by

$$\gamma_n(E) = \frac{1}{(\sqrt{2\pi})^n} \int_E \mathrm{e}^{-\frac{|x|^2}{2}} \mathrm{d}x,$$

where E is a subset of $\mathbb{R}^n$ and $|x|$ is the absolute value of $x \in E$. $\gamma_n(E)$ is called the Gaussian volume of E. It is a nature problem to study the Brunn-Minkowski theory for the Gaussian volume γ_n. Since the Gaussian volume γ_n does not have translation invariance and homogeneity, there are more difficulties to study the corresponding Brunn-Minkowski theory. The Brunn-Minkowski inequality and the Minkowski inequality for the Gaussian volume γ_n are studied in [8, 14, 17, 37].

Recently, Huang, Xi and Zhao [24] defined the Gaussian surface area measure $S_{\gamma_n,K}$ of convex body $K \in \mathcal{K}_o^n$ by the variational formula of the Gaussian volume γ_n as follows.

For $K, L \in \mathcal{K}_o^n$,

$$\lim_{t \to 0^+} \frac{\gamma_n(K + tL) - \gamma_n(K)}{t} = \int_{S^{n-1}} h_L(u) \mathrm{d}S_{\gamma_n,K}(u). \tag{1.2}$$

What's more, they posed the corresponding Minkowski problem called Gaussian Minkowski problem.

The Gaussian Minkowski problem Given a finite Borel measure μ on the unit sphere S^{n-1}, what are the necessary and sufficient conditions on μ so that there exists a convex body K such that

$$S_{\gamma_n,K} = \mu?$$

If K exists, is it unique?

The Gaussian volume normalized form of this problem is called the normalized Gaussian Minkowski problem. When μ is even, this problem is called the even Gaussian Minkowski problem. In [24], Huang, Xi and Zhao studied the even (normalized) Gaussian Minkowski problem and obtained some results for the existence and uniqueness of this problem.

By the variational formula of the Gaussian volume γ_n for L_p Minkowski combination, Liu [27] defined the L_p Gaussian surface area measure $S_{p,\gamma_n}(K, \cdot)$ of convex body $K \in \mathcal{K}_o^n$.

For $K, L \in \mathcal{K}_o^n$ and $p \neq 0$,

$$\lim_{t \to 0} \frac{\gamma_n(K +_p t \cdot L) - \gamma_n(K)}{t} = \frac{1}{p} \int_{S^{n-1}} h_L^p(u) \mathrm{d}S_{p,\gamma_n}(K, u). \tag{1.3}$$

In particular, when $p = 1$, it is the Gaussian surface area measure, that is, $S_{1,\gamma_n}(K, \cdot) = S_{\gamma_n,K}$. The corresponding Minkowski problem is called the L_p Gaussian Minkowski problem as follows.

The L_p Gaussian Minkowski problem For fixed p and a given non-zero finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ in order that there exists a convex body $K \in \mathcal{K}_o^n$ such that

$$S_{p,\gamma_n}(K, \cdot) = \mu?$$

If f is the density of the given measure μ, then the corresponding Monge-Ampère type equation on S^{n-1} is as follows.

For $u \in S^{n-1}$,

$$\frac{1}{(\sqrt{2\pi})^n} \mathrm{e}^{-\frac{|\nabla h(u)|^2+h^2(u)}{2}} h^{1-p}(u) \det(\nabla^2 h(u) + h(u) I) = f(u),$$

where $h : S^{n-1} \to (0, +\infty)$ is the function to be found, $\nabla h, \nabla^2 h$ are the gradient vector and the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix. When μ or f is even, this problem is called the even L_p Gaussian Minkowski problem.

The existence of the solution to the normalized L_p Guassian Minkowski problem is obtained by Liu [27] as follows.

Theorem A (see [27]) *For $p > 0$, let μ be a non-zero finite Borel measure on S^{n-1} and be not concentrated in any closed hemisphere. Then there exist a $K \in \mathcal{K}_o^n$ and a positive constant λ such that*

$$\frac{\lambda}{p} S_{p,\gamma_n}(K, \cdot) = \mu.$$

Due to the lack of homogeneity of L_p Gaussian surface area measure $S_{p,\gamma_n}(K, \cdot)$, it is difficult that the coefficient $\frac{\lambda}{p}$ is eliminated. Next, the even L_p Gaussian Minkowski problem is considered as follows.

The even L_p Gaussian Minkowski problem For fixed p and a given non-zero even finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ in order that there exists a convex body $K \in \mathcal{K}_e^n$ such that

$$S_{p,\gamma_n}(K, \cdot) = \mu?$$

For even case, Liu [27] eliminated the coefficient $\frac{\lambda}{p}$ and obtained the following result.

Theorem B (see [27]) *For $p \geq 1$, let μ be a non-zero even finite Borel measure on S^{n-1} and be not concentrated in any closed hemisphere with $|\mu| < \sqrt{\frac{2}{\pi}} r^{-p} a \mathrm{e}^{-\frac{a^2}{2}}$, the r and a are chosen such that $\gamma_n(rB) = \gamma_n(P) = \frac{1}{2}$, symmetry strip $P = \{x \in \mathbb{R}^n : |x_1| \leq a\}$. Then there exists a unique $K \in \mathcal{K}_e^n$ with $\gamma_n(K) > \frac{1}{2}$ such that*

$$S_{p,\gamma_n}(K, \cdot) = \mu.$$

In this paper, when $p > n$, we find that the existence of the solution to the even L_p Guassian Minkowski problem can do not need the condition $|\mu| < \sqrt{\frac{2}{\pi}} r^{-p} a \mathrm{e}^{-\frac{a^2}{2}}$ in Theorem B, and we obtain the following result.

Theorem 1.1 *Let $p > n$ and μ be a non-zero even finite Borel measure not concentrated in any closed hemisphere of S^{n-1}. Then, there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$S_{p,\gamma_n}(K_0, \cdot) = \mu.$$

2 Preliminaries

In this section, we list some notations and recall some basic facts about convex bodies. For vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the standard inner product in $\mathbb{R}^n$. The boundary of the Euclidean unit ball $B_n = \{x \in \mathbb{R}^n : \sqrt{x \cdot x} \leq 1\}$ is denoted by S^{n-1} called unit sphere. The n-dimensional volume (Lebesgue measure) of B_n is denoted by ω_n. We write ∂K and int K for the boundary and the set of all interiors of convex body K in $\mathbb{R}^n$, respectively. Let $\partial' K$ denote the subset of ∂K with unique outer unit normal. According to the context of this paper, $|\cdot|$ can denote different meanings: The absolute value, the standard Euclidean norm on $\mathbb{R}^n$ and the total mass of a finite measure. Let $C(S^{n-1})$ denote the set of continuous functions defined on S^{n-1}, and let $C^+(S^{n-1})$ denote the set of strictly positive functions in $C(S^{n-1})$.

A convex body $K \in \mathcal{K}_o^n$ is uniquely determined by its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

It is not hard to see that support functions are positively homogeneous of degree one and subadditive. For $K \in \mathcal{K}_o^n$, its support function h_K is continuous and strictly positive on the unit sphere S^{n-1}. The support hyperplane H_K of $K \in \mathcal{K}_o^n$ with respect to outer unit normal $v \in S^{n-1}$ is defined by

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}.$$

Clearly, $H_K(v) \cap K \subseteq \partial K$ for all $v \in S^{n-1}$.

The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of convex body $K \in \mathcal{K}_o^n$ is another important function for $K \in \mathcal{K}_o^n$, and it is given by

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Note that the radial function ρ_K of $K \in \mathcal{K}_o^n$ is positively homogeneous of degree -1, and it is continuous and strictly positive on the unit sphere S^{n-1}. For each $u \in S^{n-1}$, $\rho_K(u)u \in \partial K$.

The set $\mathcal{K}_o^n$ can be endowed with Hausdorff metric and radial metric which mean the distance between two convex bodies. The Hausdorff metric of $K, L \in \mathcal{K}_o^n$ is defined by

$$||h_K - h_L|| = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

The radial metric of $K, L \in \mathcal{K}_o^n$ is defined by

$$||\rho_K - \rho_L|| = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

It is an important fact that the two metrics are mutually equivalent, then, for $K, K_i \in \mathcal{K}_o^n$,

$$h_{K_i} \to h_K \text{ uniformly if and only if } \rho_{K_i} \to \rho_K \text{ uniformly.}$$

If $||h_{K_i} - h_K|| \to 0$ or $||\rho_{K_i} - \rho_K|| \to 0$ as $i \to +\infty$, we call the sequence $\{K_i\}$ converges to K.

The polar body K^* of $K \in \mathcal{K}_o^n$ is given by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It is clear that $K^* \in \mathcal{K}_o^n$ and $K = (K^*)^*$. There exists an important fact on $\mathbb{R}^n \setminus \{0\}$ between K and its polar body K^*:

$$h_K = \frac{1}{\rho_{K^*}} \quad \text{and} \quad \rho_K = \frac{1}{h_{K^*}}.$$

Then, for $K, K_i \in \mathcal{K}_o^n$, we can obtain the following result:

$$K_i \to K \quad \text{if and only if} \quad K_i^* \to K^*.$$

For $f \in C^+(S^{n-1})$, the Wullf shape $[f]$ of f is defined by

$$[f] = \{x \in \mathbb{R}^n : x \cdot u \le f(u) \text{ for all } u \in S^{n-1}\}.$$

It is not hard to see that $[f]$ is a convex body in $\mathbb{R}^n$ and $h_{[f]} \le f$. In addition, $[h_K] = K$ for all $K \in \mathcal{K}_o^n$.

By the concept of Wullf shape, the L_p Minkowski combination can be defined for all $p \in \mathbb{R}$. When $p \ne 0$, for $K, L \in \mathcal{K}_o^n$ and $s, t \in \mathbb{R}$ satisfying that $sh_K^p + th_L^p$ is strictly positive on S^{n-1}, the L_p Minkowski combination $s \cdot K +_p t \cdot L$ is defined by

$$s \cdot K +_p t \cdot L = [(sh_K^p + th_L^p)^{\frac{1}{p}}]. \tag{2.2}$$

When $p = 0$, the L_p Minkowski combination $s \cdot K +_0 t \cdot L$ is defined by

$$s \cdot K +_0 t \cdot L = [h_K^s h_L^t].$$

By the variational formula (1.3) of the Gaussian volume γ_n for L_p Minkowski combination, the integral expression of L_p Gaussian surface area measure is obtained in [27].

Suppose $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$. For each Borel set $\eta \subseteq S^{n-1}$, L_p Gaussian surface area $S_{p,\gamma_n}(K, \cdot)$ of K is defined by

$$S_{p,\gamma_n}(K, \eta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\nu_K^{-1}(\eta)} (x \cdot \nu_K(x))^{1-p} \mathrm{e}^{-\frac{|x|^2}{2}} \mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.3}$$

Here $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

By the definition (1.2) of Gaussian surface area measure $S_{\gamma_n,K}$, L_p Gaussian surface area measure $S_{p,\gamma_n}(K, \cdot)$ can be rewritten as follows:

$$S_{p,\gamma_n}(K, \eta) = \int_\eta h_K^{1-p}(u) \mathrm{d}S_{\gamma_n,K}(u). \tag{2.4}$$

When $p = 1$, we have $S_{\gamma_n,K} = S_{1,\gamma_n}(K, \cdot)$.

By the definition of L_p Gaussian surface area measure (2.3), the following result is obtained.

For $K \in \mathcal{K}_o^n$ and $p \in \mathbb{R}$, $S_{p,\gamma_n}(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure S_K, and

$$\mathrm{d}S_{p,\gamma_n}(K, u) = \frac{1}{(\sqrt{2\pi})^n} \mathrm{e}^{-\frac{|\nabla h_K(u)|^2 + h_K^2(u)}{2}} h_K^{1-p}(u) \mathrm{d}S_K(u), \quad u \in S^{n-1}.$$

Besides, if K is C_+^2, then

$$\mathrm{d}S_{p,\gamma_n}(K,u)=\frac{1}{(\sqrt{2\pi})^n}\mathrm{e}^{-\frac{|\nabla h_K(u)|^2+h_K^2(u)}{2}}h_K^{1-p}(u)\det(\nabla^2 h_K(u)+h_K(u)I)\mathrm{d}u,\quad u\in S^{n-1}.$$

Hence, L_p Gaussian surface area measure is also absolutely continuous with respect to the spherical Lebesgue measure.

3 Existence

In this section, the existence of the solution to the even L_p Gaussian Minkowski problem is studied. By variational method, the first step for solving the existence of even L_p Gaussian Minkowski problem is properly to convert this problem to an optimization problem as follows.

Optimization problem For $p\neq 0$ and a given non-zero even finite Borel measure μ on S^{n-1}, does there exist a convex body in $\mathcal{K}_e^n$ that attains the supremum,

$$\sup\{\Gamma_p(K):K\in\mathcal{K}_e^n\}? \tag{3.1}$$

Here the functional $\Gamma_p:\mathcal{K}_e^n\to\mathbb{R}$ is given by

$$\Gamma_p(K)=-\frac{1}{p}\int_{S^{n-1}}h_K^p\mathrm{d}\mu+\gamma_n(K),\quad K\in\mathcal{K}_e^n.$$

Since the set of support functions of convex bodies in $\mathcal{K}_e^n$ is a subset of $C_e^+(S^{n-1})$, the functional Γ_p can be extended to a functional on $C_e^+(S^{n-1})$, $\Gamma_p:C_e^+(S^{n-1})\to\mathbb{R}$, defined by

$$\Gamma_p(h)=-\frac{1}{p}\int_{S^{n-1}}h^p\mathrm{d}\mu+\gamma_n([h]),\quad h\in C_e^+(S^{n-1}).$$

Then, $\Gamma_p(h_K)=\Gamma_p(K)$ for all $K\in\mathcal{K}_e^n$. Together with the definition of Wulff shape, we obtain the following interesting lemma.

Lemma 3.1 Let $p\neq 0$.

(1) If $K_0\in\mathcal{K}_e^n$ satisfies

$$\Gamma_p(K_0)=\sup\{\Gamma_p(K):K\in\mathcal{K}_e^n\},$$

then

$$\Gamma_p(h_{K_0})=\sup\{\Gamma_p(h):h\in C_e^+(S^{n-1})\}.$$

(2) If $h_0\in C_e^+(S^{n-1})$ satisfies

$$\Gamma_p(h_0)=\sup\{\Gamma_p(h):h\in C_e^+(S^{n-1})\},$$

then

$$\Gamma_p([h_0])=\sup\{\Gamma_p(K):K\in\mathcal{K}_e^n\}.$$

Proof By $h_{[h]} \leq h$ for all $h \in C_e^+(S^{n-1})$, we have

$$\Gamma_p(h_{[h]}) \geq \Gamma_p(h).$$

If

$$\Gamma_p(K_0) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\},$$

then, for each $h \in C_e^+(S^{n-1})$,

$$\Gamma_p(h_{K_0}) = \Gamma_p(K_0) \geq \Gamma_p([h]) = \Gamma_p(h_{[h]}) \geq \Gamma_p(h).$$

Together with $h_{K_0} \in C_e^+(S^{n-1})$, we obtain

$$\Gamma_p(h_{K_0}) = \sup\{\Gamma_p(h) : h \in C_e^+(S^{n-1})\}.$$

If

$$\Gamma_p(h_0) = \sup\{\Gamma_p(h) : h \in C_e^+(S^{n-1})\},$$

then, for each $K \in \mathcal{K}_e^n$,

$$\Gamma_p([h_0]) = \Gamma_p(h_{[h_0]}) \geq \Gamma_p(h_0) \geq \Gamma_p(h_K) = \Gamma_p(K).$$

Thus, by $[h_0] \in \mathcal{K}_e^n$, we have

$$\Gamma_p([h_0]) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\}.$$

To obtain the existence of L_p Gaussian Minkowski problem, the following variational formula for Gaussian volume γ_n is needed.

Lemma 3.2 (see [27]) *Let $K \in \mathcal{K}_e^n$, $p \neq 0$ and $f \in C_e(S^{n-1})$. Suppose $\delta > 0$ is a sufficiently small constant such that, for all $u \in S^{n-1}$ and $t \in (-\delta, \delta)$,*

$$h_t^p(u) = h_K^p(u) + tf(u) > 0.$$

Then,

$$\lim_{t \to 0} \frac{\gamma_n([h_t]) - \gamma_n(K)}{t} = \frac{1}{p} \int_{S^{n-1}} f(u) \mathrm{d}S_{p,\gamma_n}(K, u). \tag{3.2}$$

The following result shows that a solution to optimization problem (3.1) leads to a solution to the even L_p Guassian Minkowski problem.

Lemma 3.1 *Let $p \neq 0$ and μ be a finite even Borel measure not concentrated in any closed hemisphere of S^{n-1}. If there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\Gamma_p(K_0) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\},$$

then

$$\mu = S_{p,\gamma_n}(K_0, \cdot).$$

Proof Since

$$\Gamma_p(K_0) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\},$$

then, by Lemma 3.1, we have

$$\Gamma_p(h_{K_0}) = \sup\{\Gamma_p(h) : h \in C_e^+(S^{n-1})\}. \tag{3.3}$$

For each $f \in C_e(S^{n-1})$, define $h_t \in C_e^+(S^{n-1})$ by

$$h_t^p(u) = h_{K_0}^p(u) + tf(u), \quad t \in (-\delta, \delta)$$

for all $u \in S^{n-1}$, where $\delta > 0$ is a sufficiently small constant. Thus, from (3.3),

$$\Gamma_p(h_0) = \Gamma_p(h_{K_0}) \geq \Gamma_p(h_t), \quad t \in (-\delta, \delta).$$

Together with Lemma 3.2, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{dt}}\Gamma_p(h_t)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{dt}}\Big(-\frac{1}{p}\int_{S^{n-1}} h_t^p(u)\mathrm{d}\mu(u) + \gamma_n([h_t])\Big)\Big|_{t=0} \\
&= -\frac{1}{p}\int_{S^{n-1}} f(u)\mathrm{d}\mu(u) + \frac{1}{p}\int_{S^{n-1}} f(u)\mathrm{d}S_{p,\gamma_n}(K_0, u).
\end{aligned}$$

Since this holds for any $f \in C_e(S^{n-1})$, it follows that

$$\mu = S_{p,\gamma_n}(K_0, \cdot).$$

To obtain a solution to optimization problem (3.1), we need the following lemmas.

Lemma 3.4 *Let $p > n$. Then $\Gamma_p(rB_n) > 0$ for sufficiently small $r > 0$.*

Proof Combining the definitions of Γ_p and γ_n with polar coordinates, we obtain

$$\begin{aligned}
\Gamma_p(rB_n) &= -\frac{1}{p}\int_{S^{n-1}} r^p \mathrm{d}\mu + \gamma_n(rB_n) \\
&= -\frac{|\mu|}{p}r^p + \frac{1}{(\sqrt{2\pi})^n}\int_{rB_n} \mathrm{e}^{-\frac{|x|^2}{2}}\mathrm{d}x \\
&\geq -\frac{|\mu|}{p}r^p + \frac{\mathrm{e}^{-\frac{r^2}{2}}}{(\sqrt{2\pi})^n}V_n(rB_n) \\
&= -\frac{|\mu|}{p}r^p + \frac{\omega_n}{(\sqrt{2\pi})^n}\mathrm{e}^{-\frac{r^2}{2}}r^n \\
&= r^p\Big(\frac{\omega_n}{(\sqrt{2\pi})^n}\mathrm{e}^{-\frac{r^2}{2}}r^{n-p} - \frac{|\mu|}{p}\Big).
\end{aligned}$$

Since, by $p > n$,

$$\lim_{r\to 0^+} \mathrm{e}^{-\frac{r^2}{2}} = 1 \quad \text{and} \quad \lim_{r\to 0^+} r^{n-p} = +\infty,$$

then

$$\lim_{r\to 0^+} \mathrm{e}^{-\frac{r^2}{2}} r^{n-p} = +\infty.$$

Therefore, for sufficiently small $r > 0$, we have $\Gamma_p(rB_n) > 0$.

Lemma 3.5 *Suppose K is a compact convex set in $\mathbb{R}^n$. If $\gamma_n(K) > 0$, then K is a convex body in $\mathbb{R}^n$, that is, $K \in \mathcal{K}^n$.*

Proof By the definition of the Gaussian volume γ_n, we have

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K \mathrm{e}^{-\frac{|x|^2}{2}} \mathrm{d}x \le \frac{1}{(\sqrt{2\pi})^n} \int_K \mathrm{d}x = \frac{1}{(\sqrt{2\pi})^n} V_n(K).$$

Together with $\gamma_n(K) > 0$,

$$V_n(K) \ge (\sqrt{2\pi})^n \gamma_n(K) > 0.$$

Therefore, compact convex set K has nonempty interior in $\mathbb{R}^n$, that is, K is a convex body in $\mathbb{R}^n$.

The following lemma shows that there exists a solution to optimization problem (3.1).

Lemma 3.6 *Let $p > n$ and μ be a finite even Borel measure not concentrated in any closed hemisphere of S^{n-1}. Then, there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\Gamma_p(K_0) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\}.$$

Proof Let $\{K_i\} \subseteq \mathcal{K}_e^n$ be a maximizing sequence for Γ_p, that is,

$$\lim_{i\to+\infty} \Gamma_p(K_i) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\}.$$

By Lemma 3.4, we deduce that $\lim\limits_{i\to+\infty} \Gamma_p(K_i) > 0$.

Since $K_i \in \mathcal{K}_e^n$, ρ_{K_i} is continuous on S^{n-1}. By the fact that S^{n-1} is compact, we could choose a constant $R_i > 0$ and a unit vector $u_i \in S^{n-1}$ such that

$$R_i = \rho_{K_i}(u_i) = \max\{\rho_{K_i}(u) : u \in S^{n-1}\}.$$

Then, $R_i u_i \in K_i$ and $K_i \subseteq R_i B_n$. By the definition of support function,

$$h_{K_i}(v) \ge R_i |u_i \cdot v|$$

for all $v \in S^{n-1}$. Since μ is not concentrated in any closed hemisphere of S^{n-1}, there exists a constant $c_0 > 0$ such that

$$\int_{S^{n-1}} |u \cdot v|^p \mathrm{d}\mu(v) \ge c_0$$

for all $u \in S^{n-1}$. Together with $K_i \subseteq R_i B_n$, we have

$$\begin{aligned}\Gamma_p(K_i) &= -\frac{1}{p} \int_{S^{n-1}} h_{K_i}^p(v) \mathrm{d}\mu(v) + \gamma_n(K_i) \\ &\le -\frac{R_i^p}{p} \int_{S^{n-1}} |u_i \cdot v|^p \mathrm{d}\mu(v) + \gamma_n(R_i B_n) \\ &\le -\frac{c_0}{p} R_i^p + \gamma_n(R_i B_n).\end{aligned}$$

Suppose that $\{R_i\}$ is not a bounded sequence. Without loss of generality, we may assume that $\lim\limits_{i\to+\infty} R_i = +\infty$. By polar coordinates,

$$\gamma_n(R_iB_n) = \frac{1}{(\sqrt{2\pi})^n}\int_{R_iB_n} \mathrm{e}^{-\frac{|x|^2}{2}}\mathrm{d}x = \frac{n\omega_n}{(\sqrt{2\pi})^n}\int_0^{R_i} \mathrm{e}^{-\frac{t^2}{2}}t^{n-1}\mathrm{d}t.$$

Since the integral $\int_0^{+\infty} \mathrm{e}^{-\frac{t^2}{2}}t^{n-1}\mathrm{d}t$ is convergent, the sequence $\{\gamma_n(R_iB_n)\}$ is also convergent as $i \to +\infty$. Together with $\lim\limits_{i\to+\infty} R_i = +\infty$, we have

$$\Gamma_p(K_i) \le -\frac{c_0}{p}R_i^p + \gamma_n(R_iB_n) \to -\infty$$

as $i \to +\infty$. This is a contradiction to $\{K_i\}$ being a maximizing sequence. Therefore, $\{R_i\}$ is bounded, that is, the sequence $\{K_i\}$ is bounded.

By Blaschke selection theorem, without loss of generality, we may assume that $\{K_i\}$ converges to an origin-symmetric compact convex set K_0 in $\mathbb{R}^n$. From the continuity of the Gaussian volume with respect to the Hausdorff metric and the definition of Γ_p, we have

$$\gamma_n(K_0) = \lim_{i\to+\infty} \gamma_n(K_i) \ge \lim_{i\to+\infty} \Gamma_p(K_i) > 0.$$

Together with Lemma 3.5, we obtain $K_0 \in \mathcal{K}_e^n$. Since $\{K_i\}$ converges to $K_0 \in \mathcal{K}_e^n$ in the Hausdorff metric, then

$$\Gamma_p(K_0) = \lim_{i\to+\infty} \Gamma_p(K_i) = \sup\{\Gamma_p(K) : K \in \mathcal{K}_e^n\}.$$

By Lemmas 3.3 and 3.6, we obtain the existence of the solution to the even L_p Guassian Minkowski problem for $p > n$. Theorem 1.1 is rewritten as Theorem 3.1 as follows.

Theorem 3.1 *Let $p > n$ and μ be a non-zero even finite Borel measure on S^{n-1} not concentrated in any closed hemisphere of S^{n-1}. Then, there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$\mu = S_{p,\gamma_n}(K_0, \cdot).$$

Remark 3.1 In Theorem 3.1, we eliminate the condition $|\mu| < \sqrt{\frac{2}{\pi}}r^{-p}a\mathrm{e}^{\frac{-a^2}{2}}$ in Theorem B. But, the following problem deserves to be considered.

Is the condition $|\mu| < \sqrt{\frac{2}{\pi}}r^{-p}a\mathrm{e}^{\frac{-a^2}{2}}$ necessary for $p > n$? In other words, when $p > n$, does any $K \in \mathcal{K}_e^n$ satisfy the following inequality:

$$|S_{p,\gamma_n}(K,\cdot)| < \sqrt{\frac{2}{\pi}}r^{-p}a\mathrm{e}^{\frac{-a^2}{2}}?$$

If the answer is negative, then Theorem B does not cover Theorem 3.1 for $p > n$.

The following isoperimetric type inequality can be derived from Ehrhard inequality by Liu [27] and gives a negative answer of the problem in Remark 3.1.

Lemma 3.7 (see [27]) *For $p \ge 1$, let $K \in \mathcal{K}_e^n$ and symmetry strip $P = \{x \in \mathbb{R}^n : |x_1| \le a\}$ with $\gamma_n(K) = \gamma_n(P) = \frac{1}{2}$. Then,*

$$|S_{p,\gamma_n}(K,\cdot)| \ge \sqrt{\frac{2}{\pi}}r^{-p}a\mathrm{e}^{\frac{-a^2}{2}},$$

where r is chosen such that $\gamma_n(rB)=\frac{1}{2}$.

By Lemma 3.7, it is meaningful to eliminate the condition $|\mu|<\sqrt{\frac{2}{\pi}}r^{-p}ae^{\frac{-a^2}{2}}$ in this paper. Hence, when $p>n$, Theorem B does not cover Theorem 3.1.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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