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The Coarse ℓ^p-Novikov Conjecture and Banach Spaces with Property (H)*

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Abstract In this paper, for $1 < p < \infty$, the authors show that the coarse ℓ^p-Novikov conjecture holds for metric spaces with bounded geometry which are coarsely embeddable into a Banach space with Kasparov-Yu's Property (H).

Keywords The coarse ℓ^p-Novikov conjecture, Banach spaces with Property (H), Coarse geometry, K-Theory

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1 Introduction

The coarse Novikov conjecture (cf. [4, 6, 15]) is one of the most important problems in noncommutative geometry. It implies the Gromov-Lawson-Rosenberg conjecture for non-existence of positive scalar curvature on aspherical manifolds, and the Gromov's zero-in-the-spectrum conjecture on noncompact complete Riemannian manifolds. Let X be a discrete metric space, e.g. a discrete net of a Riemannian manifold or a finitely generated group with the word length metric. The space X is said to have bounded geometry if for any $r > 0$ there is $N > 0$ such that any ball of radius r in X contains at most N elements. The coarse Novikov conjecture for X states that the higher index map

$$\mu : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*(X))$$

is injective, where on the left hand side, $K_*(P_d(X))$ is the K-homology group of the Rips complex $P_d(X)$ of X at scale $d > 0$, while on the right hand side, $K_*(C^*(X))$ is the K-theory group of the Roe C^*-algebra $C^*(X)$ of X.

In [17], Yu introduced a localization C^*-algebra $C_L^*(P_d(X))$ to establish the following com-

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mutative diagram

$$\begin{array}{ccc} & & \lim\limits_{d\to\infty} K_*(C_L^*(P_d(X))) \\ & \overset{\mu_L}{\underset{\cong}{\nearrow}} & \Big\downarrow e_* \\ \lim\limits_{d\to\infty} K_*(P_d(X)) & \xrightarrow[\mu]{} & K_*(C^*(X)). \end{array}$$

He showed that the local index maps μ_L is an isomorphism. It follows that the coarse Novikov conjecture for X is equivalent to the statement that the evaluation map at the K-theory level:

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))) \to \lim_{d\to\infty} K_*(C^*(P_d(X))) \cong K_*(C^*(X))$$

is injective.

In recent years, operator algebras on L^p-spaces have been studied quite extensively (cf. [2–3, 8–10]). One might naturally tend to formulate and study an L^p-version of the coarse Baum-Connes conjecture or an L^p-coarse Novikov conjecture. In fact, this direction of generalization has a very strong motivation, namely, the powerful Dirac-dual Dirac method in Kasparov's KK-theory for the Baum-Connes conjecture (cf. [5]) or the Strong Novikov conjecture breaks down in the case of discrete groups with Kazhdan's Property (T). As opposite of amenability, groups with Kazhdan's Property (T) do not admit a proper affine isometric action on a Hilbert spaces. And it is well known that most of Gromov hyperbolic groups have Kazhdan's Property (T). However, Yu [19] proved that any hyperbolic group admits a proper affine isometric action on an ℓ^p-space for some large $p > 2$. This remarkable discovery suggests that one might go beyond Hilbert spaces to consider higher index problems on general L^p-spaces. However, difficulty on L^p-spaces is much more tremendous than one might expect. For example, so far, there are no reasonable generalization of the Roe algebra on L^p-spaces. Only the Roe algebra on ℓ^p for $1 \le p < \infty$ is defined (cf. [2]), so that the ℓ^p-version of the coarse Baum-Connes conjecture and the coarse ℓ^p-Novikov conjecture have been properly formulated. In [3], Chung and Nowak showed that expanders are counterexamples to the coarse ℓ^p-Baum-Connes conjectures. In [14], Shan and Wang proved the coarse geometric ℓ^p-Novikov conjecture for metric spaces with bounded geometry which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature. In [20], Zhang and Zhou proved that K-theory for ℓ^p-Roe algebra are independent of $1 < p < \infty$ for spaces with finite asymptotic dimension.

Let X and Y be two metric spaces. Recall that a map $f : X \to Y$ is said to be a coarse embedding if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}_+$ such that:

(1) $\rho_1(d(x, y)) \le d(f(x), f(y)) \le \rho_2(d(x, y))$;

(2) $\lim\limits_{r\to\infty} \rho_i(r) = \infty$ for $i = 1, 2$.

Kasparov and Yu [7] introduced a geometric condition to Banach spaces, called Property (H), and proved the strong Novikov conjecture for groups coarsely embeddable into a Banach space with Property (H).

Definition 1.1 (cf. [7]) *A real Banach space V is said to have Property* (H) *if there exists an increasing sequence of finite dimensional subspaces $\{V_n\}_{n\in\mathbb{N}}$ of V, and an increasing sequence of finite dimensional subspaces $\{W_n\}_{n\in\mathbb{N}}$ of a real Hilbert space such that*

(i) $\bigcup\limits_{n\in N} V_n$ *is dense in V,*

(ii) *there exists a uniformly continuous map $\psi : S\big(\bigcup\limits_{n\in N} V_n\big) \to S\big(\bigcup\limits_{n\in N} W_n\big)$ such that the restriction of ψ to $S(V_n)$ is a homeomorphism onto $S(W_n)$ for each $n \in N$, where $S(\cdot)$ denotes the unit sphere of a subspace of a Banach space.*

For example, the Banach space ℓ^p has Property (H) for any $p \geq 1$. In [1], Chen, Wang and Yu prove the coarse Novikov conjecture for discrete metric spaces with bounded geometry which are coarsely embeddable into Banach spaces with Property (H).

The main purpose of this paper is to prove the following result, which generalizes the above result to the ℓ^p setting.

Theorem 1.1 *Let X be a discrete metric space with bounded geometry, and let $1 < p < \infty$. If X admits a coarse embedding into a Banach space with Property* (H), *then the coarse geometric ℓ^p-Novikov conjecture holds for X, i.e., the index map*

$$e_* : \lim_{d\to\infty} K_*(B^p_L(P_d(X))) \to K_*(B^p(X))$$

is injective.

This paper is organized as follows. In Section 2, we recall the formulation of the coarse ℓ^p-Novikov conjecture, and the idea of ℓ^p-localization. In Section 3, we define the twisted ℓ^p-Roe algebra and its ℓ^p-localization counterpart for a metric space which admits a coarse embedding into a Banach space with Property (H). We construct uniformly almost flat Bott generators to establish a Bott map β from the K-theory of the ℓ^p-Roe algebra to the K-theory of the twisted ℓ^p-Roe algebra, and a Bott map β_L between the K-theory of the corresponding localization algebras. In Section 4, we discuss various ideals of the twisted algebras and show that the evaluation map from the twisted ℓ^p-localization algebra to the twisted ℓ^p-Roe algebra induces an isomorphism at the K-theory level. In Section 5, we complete the proof of the main result of this paper. To do so, we construct one more Bott map β_L^∞ and use it, together with the results in previous sections, to show the injectivity of β_L, which implies Theorem 1.1.

2 The Coarse ℓ^p-Novikov Conjecture

In this section, we shall recall the concepts of the ℓ^p-Roe algebras (cf. [3, 12]), Yu's ℓ^p-localization algebras (cf. [3, 17]) and the coarse geometric ℓ^p-Novikov conjecture.

For $r > 0$, an r-net in X is a discrete subset $Y \subset X$ such that for any $y_1, y_2 \in Y$, $d(y_1, y_2) \geq r$ and for any $x \in X$ there is a $y \in Y$ such that $d(x, y) < r$. A general metric space X is called to have bounded geometry if X has an r-net Y for some $r > 0$ such that Y has bounded geometry.

Throughout the paper, $p > 1$. And $\mathcal{K}_p = \mathcal{K}(\ell^p)$, the set of all compact operators over ℓ^p.

Definition 2.1 (cf. [3, 12]) *Let X be a proper metric space (a metric space is called proper if every closed ball is compact), and fix a countable dense subset $Z \subset X$. Let T be a bounded operator on $\ell^p(Z, \ell^p)$, and write $T = (T(x,y))_{x,y\in Z}$ so that each $T(x,y)$ is a bounded operator on ℓ^p. T is said to be locally compact if*

(i) *each $T(x,y)$ is a compact operator on ℓ^p;*

(ii) *for every bounded subset $B \subset X$, the set*

$$\{(x,y) \in (B \times B) \cap (Z \times Z) \mid T(x,y) \neq 0\}$$

is finite.

The propagation of T is defined to be

$$prop(T) = \inf\{S > 0 \mid T(x,y) = 0 \text{ for all } x, y \in Z \text{ with } d(x,y) > S\}.$$

The algebraic Roe algebra of X, denoted by $\mathbb{C}^p[X]$, is the subalgebra of $B(\ell^p(Z,\ell^p))$ consisting of all finite propagation, locally compact operators. The ℓ^p-Roe algebra of X, denoted by $B^p(X)$, is the closure of $\mathbb{C}^p[X]$ in $B(\ell^p(Z,\ell^p))$. $B^p(X)$ does not depend on the choice of Z. See [20] *for a proof.*

Definition 2.2 (cf. [3, 17]) *Let X be a proper metric space, and let $\mathbb{C}^p[X]$ be its algebraic Roe algebra. Let $\mathbb{C}^p_L[X]$ be the algebra of bounded, uniformly norm-continuous functions $g : [0,\infty) \to \mathbb{C}^p(X)$ such that $prop(g(t)) \to 0$ as $t \to \infty$. Equip $\mathbb{C}^p_L[X]$ with the norm*

$$\|g\| := \sup_{t\in[0,\infty)} \|g(t)\|_{B^p(X)}.$$

The completion of $\mathbb{C}^p_L[X]$ under this norm, denoted by $B^p_L(X)$, is the ℓ^p-localization algebra of X.

The evaluation homomorphism e from $B^p_L(X)$ to $B^p(X)$ is defined by $e(g) = g(0)$ for $g \in B^p_L(X)$.

Definition 2.3 *Let X be a discrete metric space with bounded geometry. For each $d \geq 0$, the Rips complex $P_d(X)$ at scale d is defined to be the simplicial polyhedron in which the set of vertices is X, and a finite subset $\{x_0, x_1, \cdots, x_q\} \subset X$ spans a simplex if and only if $d(x_i, x_j) \leq d$ for all $0 \leq i, j \leq q$.*

Endow $P_d(X)$ with the spherical metric. Recall that on each path connected component of $P_d(X)$, the spherical metric is the maximal metric whose restriction to each simplex $\big\{\sum_{i=0}^{q} t_i x_i \mid t_i \geq 0, \sum_{i=0}^{q} t_i = 1\big\}$ is the metric obtained by identifying the simplex with S^q_+ via the map

$$\sum_{i=0}^{q} t_i x_i \mapsto \Big(\frac{t_0}{\sqrt{\sum_{i=0}^{q} t_i^2}}, \frac{t_1}{\sqrt{\sum_{i=0}^{q} t_i^2}}, \cdots, \frac{t_q}{\sqrt{\sum_{i=0}^{q} t_i^2}}\Big),$$

where $S_+^q := \big\{(s_0, s_1, \cdots, s_q) \in \mathbb{R}^{q+1} \mid s_i \geq 0, \sum_{i=0}^{q} s_i^2 = 1\big\}$ is endowed with the standard Riemannian metric. If y_0, y_1 belong to two different connected components Y_0, Y_1 of $P_d(X)$, respectively, we define

$$d(y_0, y_1) = \min\{d(y_0, x_0) + d_X(x_0, x_1) + d(x_1, y_1) \mid x_0 \in X \cap Y_0, x_1 \in X \cap Y_1\}.$$

In [17], Yu proved that the local index map from K-homology to K-theory of localization algebra is an isomorphism for finite-dimensional simplicial complexes. In [11], Qiao and Roe generalized this isomorphism to general locally compact metric spaces. Therefore for $1 < p < \infty$, considering the analogs of ℓ^p-Roe algebra and ℓ^p-localization algebra, we define the following assembly map which is equivalent to the original map when $p = 2$. The following conjecture is called the coarse ℓ^p-Novikov conjecture.

Conjecture If X is a discrete metric space with bounded geometry, then the index map

$$e_* : \lim_{d\to\infty} K_*(B_L^p(P_d(X))) \to \lim_{d\to\infty} K_*(B^p(P_d(X))) \cong K_*(B^p(X))$$

is injective.

3 Twisted ℓ^p-Roe Algebras and Twisted ℓ^p-Localization Algebras

In this section, we shall define the twisted ℓ^p-Roe algebra and its localization counterpart for a metric space of bounded geometry which admits a coarse embedding into a Banach space with Property (H). We shall then construct a Bott map β from the K-theory of the ℓ^p-Roe algebra to the K-theory of the twisted ℓ^p-Roe algebra, and a Bott map β_L between the K-theory of the corresponding ℓ^p-localization algebras, to build the following commutative diagram for each $d \geq 0$,

$$\begin{array}{ccc} K_*(B_L^p(P_d(X))) & \xrightarrow{\beta_L} & K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))) \\ \downarrow{\scriptstyle e_*} & & \downarrow{\scriptstyle e_*^{\mathcal{A}}} \\ K_*(B^p(P_d(X))) & \xrightarrow{\beta} & K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))), \end{array}$$

where $e_*^{\mathcal{A}}$ is the homomorphism induced by the evaluation map from the twisted ℓ^p-localization algebra to the twisted ℓ^p-Roe algebra. The diagram plays a central role in the proof of Theorem 1.1.

3.1 The twisted ℓ^p-Roe algebra

Let X be a discrete metric space with bounded geometry which admits a coarse embedding $f : X \to V$ into a real Banach space V with Property (H). There exists an increasing sequence of finite dimensional subspaces V_n of V, and an increasing sequence of finite dimensional subspaces W_n of a real Hilbert space such that

(i) $\bigcup_{n\in N} V_n$ is dense in V,

(ii) there exists a uniformly continuous map $\psi : S\big(\bigcup_{n\in N} V_n\big) \to S\big(\bigcup_{n\in N} W_n\big)$ such that the restriction of ψ to $S(V_n)$ is a homeomorphism onto $S(W_n)$ for each $n \in N$.

By a slight modification, we may assume without loss of generality that $f(X) \subset \bigcup_{n\in N} V_n$ and the subspaces V_n and W_n are all even dimensional. For each $d \geq 0$, the coarse embedding $f : X \to \bigcup_{n\in N} V_n$ can be extended a coarse embedding $f : P_d(X) \to \bigcup_{n\in N} V_n$ as follows: For any point

$$z = \sum_{x\in X} c_x x \in P_d(X), \quad c_x \geq 0, \quad \sum_{x\in X} c_x = 1,$$

where all but finitely many coefficients c_x are zero, we define

$$f(z) = \sum_{x\in X} c_x f(x) \in \bigcup_{n\in N} V_n.$$

For each $n \in N$, let Cliff(W_n) be the complex Clifford algebra of W_n with respect to the relation $w^2 = \|w\|^2$ for all $w \in W_n$. Let

$$\mathcal{A}_n := C_0(V_n, \mathrm{Cliff}(W_n))$$

be the Banach algebra of all bounded continuous functions from V_n to Cliff(W_n) which vanish at infinity.

Let $\mathcal{H}_n := L^2(V_n, \mathrm{Cliff}(W_n))$, the set of all L^2 sections of Cliff(W_n), which is a Hilbert space. $\mathcal{A}_n$ acts on $\mathcal{H}_n$ by pointwise multiplication. For $a \in \mathcal{A}_n$ and $h \in \mathcal{H}_n$. Define $a_{\max} := \max\{\|a(x)\| \mid x \in V_n\}$. Then $\|a \cdot h\| \leq a_{\max}\|h\|$ and $\mathcal{A}_n \subset B(\mathcal{H}_n)$. For $n \in \mathbb{N}$, define $\mathcal{H}_{n,p}^{(m)} := \mathcal{H}_n \oplus_p \cdots \oplus_p \mathcal{H}_n$, the ℓ^p-direct sum of m copies of $\mathcal{H}_n$. The ℓ^p-norm of $\mathcal{H}_{n,p}^{(m)}$ is defined as

$$\|(h_1, \cdots, h_m)\|_p := \Big(\sum_{i=1}^{m} \|h_i\|^p\Big)^{\frac{1}{p}}$$

for $h_1, \cdots, h_m \in \mathcal{H}_n$.

Let $\mathcal{M}_m(\mathcal{A}_n)$ be the set of $m \times m$ matrices with entries in $\mathcal{A}_n$. Then elements of $\mathcal{M}_m(\mathcal{A}_n)$ act on $\mathcal{H}_{n,p}^{(m)}$ by matrix multiplication. For $M = (M_{i,j})_{i,j\in\{1,\cdots,m\}} \in \mathcal{M}_m(\mathcal{A}_n)$ and $h^{(m)} \in \mathcal{H}_{n,p}^{(m)}$,

$$\|M \cdot h^{(m)}\| \leq \max_{i,j\in\{1,\cdots,m\}}\{(M_{i,j})_{\max}\} \cdot \|h^{(m)}\|.$$

Hence $\mathcal{M}_m(\mathcal{A}_n) \subset B(\mathcal{H}_{n,p}^{(m)})$. Let $r_{m,m+1} : \mathcal{H}_{n,p}^{(m+1)} \to \mathcal{H}_{n,p}^{(m)}$ be the projection map defined by

$$r_{m,m+1}(h_1, \cdots, h_m, h_{m+1}) := (h_1, \cdots, h_m)$$

for $(h_1, \cdots, h_m, h_{m+1}) \in \mathcal{H}_{n,p}^{(m+1)}$. Define $r_{m,m+1}^* : \mathcal{M}_m(\mathcal{A}_n) \to \mathcal{M}_{m+1}(\mathcal{A}_n)$ by

$$(r_{m,m+1}^*(T))(v) := i_{m,m+1}(T(r_{m,m+1}(v)))$$

for all $T \in \mathcal{M}_m(\mathcal{A}_n)$ and $v \in \mathcal{H}_{n,p}^{(m+1)}$, where $i_{m,m+1} : \mathcal{H}_{n,p}^{(m)} \to \mathcal{H}_{n,p}^{(m+1)}$ is the canonical inclusion defined by $(h_1, \cdots, h_m) \mapsto (h_1, \cdots, h_m, 0)$. This is equivalent to embed $\mathcal{M}_m(\mathcal{A}_n)$

into $\mathcal{M}_{m+1}(\mathcal{A}_n)$ by placing matrices at the top left corner and inserting 0 at the right column and the bottom line. And $\|r^*_{m,m+1}(M)\| \le \|M\|$ for all $M \in \mathcal{M}_m(\mathcal{A}_n)$.

Let $\mathcal{M}_\infty(\mathcal{A}_n)$ be the inductive limit of $\{\mathcal{M}_m(\mathcal{A}_n)\}_{m=1}^\infty$. Define $\mathcal{H}^\infty_{n,p}$ to be the ℓ^p-direct sum of infinite copies of $\mathcal{H}_n$ with the ℓ^p-norm

$$\|\{h_i\}_{i=1}^\infty\|_p := \Big(\sum_{i=1}^\infty \|h_i\|^p\Big)^{\frac{1}{p}}$$

for $\{h_i\}_{i=1}^\infty \in \mathcal{H}^\infty_{n,p}$. Then $\mathcal{H}^\infty_{n,p} \cong \ell^p(\mathbb{N}, \mathcal{H}_n)$ and all $\mathcal{M}_m(\mathcal{A}_n)$ can be considered as subalgebras of $B(\mathcal{H}^\infty_{n,p})$.

Denote by $\mathcal{A}_n \otimes_{alg} \mathcal{K}_p$ the algebraic tensor product of $\mathcal{A}_n$ and $\mathcal{K}_p$. Naturally $\mathcal{A}_n \otimes_{alg} \mathcal{K}_p$ acts on $\mathcal{H}^\infty_{n,p}$ and $\mathcal{A}_n \otimes_{alg} \mathcal{K}_p \subset B(\mathcal{H}^\infty_{n,p})$. Let $\mathcal{A}_n \otimes_p \mathcal{K}_p := \overline{\mathcal{A}_n \otimes_{alg} \mathcal{K}_p}^{B(\mathcal{H}^\infty_{n,p})}$. It follows that $\mathcal{A}_n \otimes_p \mathcal{K}_p \cong \mathcal{M}_\infty(\mathcal{A}_n)$.

For a function $h \in \mathcal{A}_n \otimes_p \mathcal{K}_p \cong C_0(V_n, \mathrm{Cliff}(W_n) \otimes_p \mathcal{K}_p)$, the support of h is defined to be

$$\mathrm{Supp}(h) = \overline{\{v \in V_n \mid h(v) \neq 0\}} \subset V_n.$$

Let $\prod\limits_{n=1}^\infty \mathcal{A}_n \otimes_p \mathcal{K}_p$ be the Banach algebra direct product, i.e., the algebra of all bounded sequences $(h_1, \cdots, h_n, \cdots)$ with $h_n \in \mathcal{A}_n \otimes_p \mathcal{K}_p$ for all $n \in \mathbb{N}$, and let $\bigoplus\limits_{n=1}^\infty \mathcal{A}_n \otimes_p \mathcal{K}_p$ be the ideal of $\prod\limits_{n=1}^\infty \mathcal{A}_n \otimes_p \mathcal{K}_p$ generated by those sequences $(h_1, \cdots, h_n, \cdots)$ such that $\lim\limits_{n\to\infty} \|h_n\| = 0$. Denote the quotient algebra as

$$\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}) := \frac{\prod\limits_{n=1}^\infty \mathcal{A}_n \otimes_p \mathcal{K}_p}{\bigoplus\limits_{n=1}^\infty \mathcal{A}_n \otimes_p \mathcal{K}_p}.$$

Take a countable dense subset $Z_d \subset P_d(X)$ for each $d \ge 0$ in such a way that $Z_d \subset Z_{d'}$ whenever $d < d'$.

Denote by $\#A$ the number of elements in a set A.

Definition 3.1 *For each $d \ge 0$, define*

$$\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

to be the set of all bounded functions

$$T : Z_d \times Z_d \to \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$$

such that

(1) *for any bounded subset $B \subset X$, the set*

$$\{(x, y) \in B \times B \cap Z \times Z \mid T(x, y) \neq 0\}$$

is finite;

(2) *there exists* $L > 0$ *such that*

$$\#\{y \in Z \mid T(x,y) \neq 0\} < L, \quad \#\{y \in Z \mid T(y,x) \neq 0\} < L$$

for all $x \in Z$;

(3) *there exists* $R \geq 0$ *such that* $T(x,y) = 0$ *whenever* $d(x,y) > R$ *for* $x, y \in Z$; (*The least such* R *is called the propagation of* T.)

(4) *there exists* $r > 0$ *such that, for all* $x, y \in Z_d$, $T(x,y)$ *is of the form*

$$T(x,y) = [(h_1, \cdots, h_n, \cdots)] \in \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$$

where $h_n \in \mathcal{A}_n \otimes_p \mathcal{K}_p \cong C_0(V_n, \mathrm{Cliff}(W_n) \otimes_p \mathcal{K}_p)$ *such that*

$$\mathrm{Supp}(h_n) \subset \mathrm{Ball}_{V_n}(f(x), r)$$

for $n \in \mathbb{N}$ *large enough such that* $f(x) \in V_n$.

The algebraic structure for $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ is defined by regarding elements T as $Z_d \times Z_d$-matrices. Let

$$E = \Big\{ \sum_{x \in Z_d} a_x[x] \,\Big|\, a_x \in \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}), \sum_{x\in Z_d} \|a_x\|^p \text{ converges} \Big\}.$$

Then E is a L^p-X-module over $\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$:

$$\Big(\sum_{x\in Z_d} a_x[x]\Big)a = \sum_{x\in Z_d} a_x a[x]$$

for any $a \in \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$. The algebra $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ acts on E by the formula

$$T\Big(\sum_{x\in Z_d} a_x[x]\Big) = \sum_{x\in Z_d} \Big(\sum_{y\in Z_d} T(x,y)a_y\Big)[x]$$

for $T \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ and $\sum_{x\in Z_d} a_x[x] \in E$. Note that T is a module homomorphism. Let $B(E)$ be the Banach algebra of all module homomorphisms from E to E.

Definition 3.2 *The twisted* ℓ^p*-Roe algebra* $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ *is defined to be the norm completion of* $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ *in* $B(E)$.

Definition 3.3 *For each* $d \geq 0$, *define*

$$\mathbb{C}^p_L[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

to be the algebra of all bounded, uniformly norm-continuous functions

$$g : [0,\infty) \to \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

such that

(1) *for any bounded subset $B \subset X$, the set*

$$\{(x,y) \in B \times B \cap Z \times Z \mid (g(t))(x,y) \neq 0\}$$

is finite for any $t \in [0,\infty)$;

(2) *there exists $L > 0$ such that*

$$\#\{y \in Z \mid (g(t))(x,y) \neq 0\} < L, \quad \#\{y \in Z \mid (g(t))(y,x) \neq 0\} < L$$

for any $t \in [0,\infty)$ and for all $x \in Z$;

(3) *there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} R(t) = 0$ such that $(g(t))(x,y) = 0$ whenever $d(x,y) > R(t)$ for $x, y \in Z$;*

(4) *there exists $r > 0$ such that, for all $t \in [0,\infty)$ and $x, y \in Z_d$, if $(g(t))(x,y)$ is of the form*

$$(g(t))(x,y) = [(h_1, \cdots, h_n, \cdots)] \in \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$$

with $h_n \in \mathcal{A}_n \otimes_p \mathcal{K}_p \cong C_0(V_n, \mathrm{Cliff}(W_n) \otimes_p \mathcal{K}_p)$, then

$$\mathrm{Supp}(h_n) \subset \mathrm{Ball}_{V_n}(f(x), r)$$

for all $n \in \mathbb{N}$ large enough such that $f(x) \in V_n$.

Definition 3.4 *The twisted ℓ^p-localization algebra $B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ is defined to be the norm completion of $\mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ with respect to the norm $\|g\|_\infty = \sup_{t\in[0,\infty)} \|g(t)\|$, where the norm $\|g(t)\|$ is the norm in $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$.*

The evaluation homomorphism $\mathrm{e}^{\mathcal{A}}$ from $B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ to $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ defined by $\mathrm{e}^{\mathcal{A}}(g) = g(0)$ induces a homomorphism on K-theory:

$$\mathrm{e}_*^{\mathcal{A}} : \lim_{d\to\infty} K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))) \to \lim_{d\to\infty} K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

3.2 Uniformly almost flat Bott generators

In this subsection, we shall recall some basic facts of uniformly almost flat bundles.

For any $n \in \mathbb{N}$, $x \in V_n$ and $r > 0$, define a function

$$f_{x,r}^{(n)} : V_n \to W_n \subset \mathrm{Cliff}(W_n)$$

by the formula

$$f_{x,r}^{(n)}(v) = \phi_r(\|v - x\|)\psi\Big(\frac{v - x}{\|v - x\|}\Big)$$

where $\psi : S\big(\bigcup_{n\in N} V_n\big) \to S\big(\bigcup_{n\in N} W_n\big)$ is the uniformly continuous function as in the definition of Property (H) for V, and the function

$$\phi_r : [0,\infty) \to [0,\infty)$$

is defined by

$$\phi_r(t) = \begin{cases} 0, & \text{if } 0 \le t \le \dfrac{r}{2}; \\ \dfrac{2t}{r} - 1, & \text{if } \dfrac{r}{2} \le t \le r; \\ 1, & \text{if } t \ge r. \end{cases}$$

The following result describes a certain "uniform almost flatness" of the functions $\{f_{x,r}^{(n)}\}_{n\in\mathbb{N},x\in V_n,r>0}$.

Lemma 3.1 *For any $R > 0$, $\varepsilon > 0$, there exists $r > 0$ large enough such that the family of functions $\{f_{x,r}^{(n)}\}_{n\in\mathbb{N},x\in V_n,r>0}$ constructed above satisfy the following condition*:

$$\|x - y\| < R \Rightarrow \sup_{v\in V_n} \|f_{x,r}^{(n)}(v) - f_{y,r}^{(n)}(v)\| < \varepsilon$$

for all $x, y \in V_n$, $n \in \mathbb{N}$.

Proof For given $R > 0$, $\varepsilon > 0$, take r large enough, say, $r > \frac{10R}{\varepsilon}$. Then the result follows from the uniform continuity of ψ from Property (H) of V and elementary calculations.

Note that $f_{x,r}^{(n)} \in C_b(V_n, \mathrm{Cliff}(W_n))$, the Banach algebra of all bounded continuous functions from V_n to $\mathrm{Cliff}(W_n)$, such that

$$(f_{x,r}^{(n)})^2 - 1 \in \mathcal{A}_n = C_0(V_n, \mathrm{Cliff}(W_n)).$$

The invertible element

$$[f_{x,r}^{(n)}] \in C_b(V_n, \mathrm{Cliff}(W_n))/C_0(V_n, \mathrm{Cliff}(W_n))$$

defines an element in $K_1(C_b(V_n, \mathrm{Cliff}(W_n))/C_0(V_n, \mathrm{Cliff}(W_n)))$. With the help of the index map

$$\partial : K_1(C_b(V_n, \mathrm{Cliff}(W_n))/C_0(V_n, \mathrm{Cliff}(W_n))) \to K_0(C_0(V_n, \mathrm{Cliff}(W_n))),$$

we obtain an element

$$\partial([f_{x,r}^{(n)}]) \in K_0(C_0(V_n, \mathrm{Cliff}(W_n))) \cong \mathbb{Z},$$

where recall that we assume that all the subspaces V_n are of even dimensional. It follows from the construction of $f_{x,r}^{(n)}$ that, for every $x \in V_n$ and $r > 0$, the element $\partial([f_{x,r}^{(n)}])$ is nothing but the Bott generator of $K_0(C_0(V_n, \mathrm{Cliff}(W_n)))$.

The element $\partial([f_{x,r}^{(n)}])$ can be expressed explicitly as follows. Let

$$\begin{aligned} W_{x,r} &= \begin{pmatrix} 1 & f_{x,r}^{(n)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{x,r}^{(n)} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{x,r}^{(n)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ b_{x,r}^{(n)} &= W_{x,r} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_{x,r}^{-1}, \\ b_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then both $b_{x,r}^{(0)}$ and b_0 are idempotents in $\mathcal{M}_2(\mathcal{A}_n^+)$, where $\mathcal{A}_n^+$ is the Banach algebra unitization of $\mathcal{A}_n$. It is easy to check that

$$b_{x,r}^{(n)} - b_0 \in C_c(V_n, \mathrm{Cliff}(W_n)) \otimes \mathcal{M}_2(\mathbb{C}),$$

the algebra of 2×2 matrices of compactly supported continuous functions, with

$$\mathrm{Supp}(b_{x,r}^{(n)} - b_0) \subset \mathrm{Ball}_{V_n}(f(x), r) := \{v \in V_n \mid \|v - x\| \le r\},$$

where for a matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of functions on V_n we define the support of a by

$$\mathrm{Supp}(a) = \bigcup_{i,j=1}^{2} \mathrm{Supp}(a_{i,j}).$$

Now we have the explicit expression

$$\partial([f_{x,r}^{(n)}]) = [f_{x,r}^{(n)}] - [b_0] \in K_0(\mathcal{A}_n).$$

Lemma 3.2 (Uniform almost flatness of the Bott generators (cf. [13])) *The family of idempotents* $\{b_{x,r}^{(n)}\}_{n\in\mathbb{N}, x\in V_n, r>0}$ *in* $\mathcal{M}_2(\mathcal{A}_n^+) = C_0(V_n, \mathrm{Cliff}(W_n))^+ \otimes \mathcal{M}_2(\mathbb{C})$ *constructed above are uniformly almost flat in the following sense: For any* $R > 0$ *and* $\varepsilon > 0$, *there exists* $r > 0$ *large enough such that, for any* $n \in \mathbb{N}$ *and any* $x, y \in V_n$, *we have*

$$\|x - y\| < R \Rightarrow \sup_{v\in V_n} \|b_{x,r}^{(n)}(v) - b_{y,r}^{(n)}(v)\|_{\mathrm{Cliff}(W_n)\otimes\mathcal{M}_2(\mathbb{C})} < \varepsilon.$$

Proof Straightforward from Lemma 3.1.

It would be convenient to introduce the following notion.

Definition 3.5 *For* $R > 0$, $\varepsilon > 0$, $r > 0$, *a family of idempotents* $\{b_x^{(n)}\}_{n\in\mathbb{N}, x\in V_n}$ *in*

$$\mathcal{M}_2(\mathcal{A}_n^+) = C_0(V_n, \mathrm{Cliff}(W_n))^+ \otimes \mathcal{M}_2(\mathbb{C}),$$

$n \in \mathbb{N}$, *is said to be* $(R, \varepsilon; r)$*-flat if*

(1) *for any* $x, y \in V_n$, $n \in \mathbb{N}$ *with* $\|x - y\| < R$, *we have*

$$\sup_{v\in V_n} \|b_x^{(n)}(v) - b_y^{(n)}(v)\|_{\mathrm{Cliff}(W_n)\otimes\mathcal{M}_2(\mathbb{C})} < \varepsilon;$$

(2) $b_x^{(n)} - b_0 \in C_c(V_n, \mathrm{Cliff}(W_n)) \otimes \mathcal{M}_2(\mathbb{C})$ *and*

$$\mathrm{Supp}(b_x^{(n)} - b_0) \subset \mathrm{Ball}_{V_n}(f(x), r) := \{v \in V_n \mid \|v - x\| \le r\}.$$

3.3 Construction of the Bott map β

In this subsection, we shall use the uniformly almost flat Bott generators for $K_0(\mathcal{A}_n)$ constructed in the above subsection to construct a Bott map

$$\beta : K_*(B^p(P_d(X))) \to K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

Definition 3.6 *For each $d \geq 0$, let Z_d be a countable dense subset of $P_d(X)$. Define $\mathbb{C}^p[P_d(X)]$ to be the set of all bounded functions*

$$T : Z_d \times Z_d \to \mathcal{K}_p$$

such that

(1) *for any bounded subset $B \subset P_d(X)$, the set*

$$\{(x, y) \in B \times B \cap Z_d \times Z_d \mid T(x, y) \neq 0\}$$

is finite;

(2) *there exists $L > 0$ such that*

$$\#\{y \in Z_d \mid T(x, y) \neq 0\} < L, \quad \#\{y \in Z_d \mid T(y, x) \neq 0\} < L$$

for all $x \in Z_d$;

(3) *there exists $R \geq 0$ such that $T(x, y) = 0$ whenever $d(x, y) > R$ for $x, y \in Z_d$.*

The product structure on $\mathbb{C}^p[P_d(X)]$ is defined by

$$(T_1T_2)(x, y) = \sum_{z \in Z_d} T_1(x, z)T_2(z, y).$$

The algebra $\mathbb{C}^p[P_d(X)]$ acts on $\ell^p(Z_d, \ell^p)$. The operator norm completion of $\mathbb{C}^p[P_d(X)]$ with respect to this action is isomorphic to $B^p(P_d(X))$ when X has bounded geometry.

Note that $B^p(P_d(X))$ is stable in the sense that

$$B^p(P_d(X)) \cong B^p(P_d(X)) \otimes \mathcal{M}_k(\mathbb{C})$$

for all natural number k. Indeed, since $\mathcal{M}_k(\mathbb{C})$ is finite dimensional, the ℓ^p-tensor product is equivalent to the canonical tensor product. Notice that $B^p(P_d(X)) \otimes_p \mathcal{M}_k(\mathbb{C})$ has a canonical faithful representation on $\ell^p(Z, \ell^p) \otimes_p \mathbb{C}^k \cong \ell^p(Z, \ell^p \otimes_p \mathbb{C}^k) \cong \ell^p(Z, \ell^p)$. Since $\mathcal{K}(\ell^p) \otimes \mathcal{M}_k(\mathbb{C}) \cong \mathcal{K}(\ell^p \otimes \mathbb{C}^k) \cong \mathcal{K}(\ell^p)$, one has that $B^p(P_d(X)) \cong B^p(P_d(X)) \otimes_p \mathcal{M}_k(\mathbb{C})$ by definition. Thus, any element in $K_0(B^p(P_d(X)))$ can be expressed as a difference of the K_0-classes of two idempotents in $B^p(P_d(X))$. To define the Bott map

$$\beta : K_*(B^p(P_d(X))) \to K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n \in \mathbb{N}}))),$$

it suffices to specify the value $\beta([P])$ in $K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n \in \mathbb{N}})))$ for any idempotent $P \in B^p(P_d(X))$.

Let $P \in B^p(P_d(X))$ be an idempotent. For any $0 < \varepsilon_1 < \frac{1}{100}$, take an element $Q \in \mathbb{C}^p[P_d(X)]$ such that

$$\|P - Q\| < \frac{\varepsilon_1}{4}\|P\|.$$

Then $\|Q - Q^2\| < \varepsilon_1$, and there is $R_{\varepsilon_1} > 0$ such that $Q(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}$ for $x, y \in Z_d$.

For any $\varepsilon_2 > 0$, take by Lemma 3.2 a family of $(R_{\varepsilon_1}, \varepsilon_2; r)$-flat idempotents $\{b_v^{(n)}\}_{n\in\mathbb{N}, x\in V_n}$ in $\mathcal{M}_2(\mathcal{A}_n^+)$, $n \in \mathbb{N}$, for a sufficiently large $r > 0$, such that

$$\mathrm{Supp}(b_v^{(n)} - b_0) \subset \mathrm{Ball}_{V_n}(f(x), r).$$

Denote

$$\mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}) := \frac{\prod_{n=1}^{\infty} \mathcal{A}_n^+ \otimes \mathcal{M}_2(\mathbb{C}) \otimes_p \mathcal{K}_p}{\bigoplus_{n=1}^{\infty} \mathcal{A}_n^+ \otimes \mathcal{M}_2(\mathbb{C}) \otimes_p \mathcal{K}_p},$$
$$\mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}) := \frac{\prod_{n=1}^{\infty} \mathcal{A}_n \otimes \mathcal{M}_2(\mathbb{C}) \otimes_p \mathcal{K}_p}{\bigoplus_{n=1}^{\infty} \mathcal{A}_n \otimes \mathcal{M}_2(\mathbb{C}) \otimes_p \mathcal{K}_p}.$$

Define

$$\widetilde{Q}, \widetilde{Q}_0 : Z_d \times Z_d \to \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$$

by the formula

$$\widetilde{Q}(x, y) = [(b_{f(x)}^{(1)} \otimes_p Q(x, y), \cdots, b_{f(x)}^{(n)} \otimes_p Q(x, y), \cdots)],$$
$$\widetilde{Q}_0(x, y) = [(b_0 \otimes_p Q(x, y), \cdots, b_0 \otimes_p Q(x, y), \cdots)],$$

respectively, for all $(x, y) \in Z_d \times Z_d$, where $b_{f(x)}^{(n)}$ is well defined for n large enough such that $f(x) \in V_n$, and $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\widetilde{Q}, \widetilde{Q}_0 \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

and

$$\widetilde{Q} - \widetilde{Q}_0 \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})].$$

Since X has bounded geometry, by the uniform almost flatness of the Bott generators (Lemma 3.2), we can choose ε_1 and then ε_2 small enough to obtain $\widetilde{Q}$, $\widetilde{Q}_0$ as constructed above such that $\|\widetilde{Q}^2 - \widetilde{Q}\| < \frac{1}{5}$ and $\|\widetilde{Q}_0^2 - \widetilde{Q}_0\| < \frac{1}{5}$.

It follows that the spectrum of either $\widetilde{Q}$ or $\widetilde{Q}_0$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $\chi : S_0 \sqcup S_1 \to \mathbb{C}$ be a continuous function such that $\chi(S_0) = \{0\}$, $\chi(S_1) = \{1\}$. Define $\Theta = \chi(\widetilde{Q})$ and $\Theta_0 = \chi(\widetilde{Q}_0)$ by functional calculus. Then Θ and Θ_0 are idempotents in the Banach algebra

$$B^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$$

with

$$\Theta - \Theta_0 \in B^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})).$$

Note that $B^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ is a closed two-sided ideal of

$$B^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})).$$

At this point we need to recall the difference construction in K-theory of Banach algebras introduced by Kasparov-Yu [6]. Let J be a closed two-sided ideal of a Banach algebra B. Let $p, q \in B$ be idempotents such that $p - q \in J$. Then a difference element $D(p, q) \in K_0(J)$ associated to the pair p, q is defined as follows. Let

$$Z(p,q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

We have

$$(Z(p,q))^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

Define

$$D_0(p,q) = (Z(p,q))^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p,q).$$

Let

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $D_0(p,q) \in \mathcal{M}_4(J^+)$ and $D_0(p,q) \equiv p_1$ modulo $\mathcal{M}_4(J)$. We define the difference element

$$D(p,q) := [D_0(p,q)] - [p_1]$$

in $K_0(J)$.

Now, for any idempotent $P \in B^p(P_d(X))$ representing an element $[P]$ in $K_0(B^p(P_d(X)))$, we define

$$\beta([P]) = D(\Theta, \Theta_0) \in K_0(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

The correspondence $[P] \to \beta([P])$ extends to a homomorphism, the Bott map,

$$\beta : K_0(B^p(P_d(X))) \to K_0(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

By suspension, we similarly define the Bott map for K_1,

$$\beta : K_1(B^p(P_d(X))) \to K_1(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

3.4 The Bott map β_L for ℓ^p-localization algebras

In this subsection, we shall construct a Bott map β_L for the K-theory of ℓ^p-localization algebras:

$$\beta_L : K_*(B_L^p(P_d(X))) \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

Definition 3.7 *For each $d \geq 0$, let Z_d be a countable dense subset of $P_d(X)$. Define $\mathbb{C}_L^p[P_d(X)]$ to be the algebra of all bounded, uniformly norm-continuous functions*

$$g : [0, \infty) \to \mathbb{C}^p[P_d(X)]$$

such that

(1) *for any bounded subset $B \subset P_d(X)$, the set*

$$\{(x, y) \in B \times B \cap Z_d \times Z_d \mid (g(t))(x, y) \neq 0\}$$

is finite for all $t \in [0, \infty)$;

(2) *there exists $L > 0$ such that*

$$\#\{y \in Z_d \mid (g(t))(x, y) \neq 0\} < L, \quad \#\{y \in Z_d \mid (g(t))(y, x) \neq 0\} < L$$

for all $t \in [0, \infty)$ and $x \in Z_d$;

(3) *there exists a bounded function $R : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} R(t) = 0$ such that $(g(t))(x, y) = 0$ whenever $d(x, y) > R(t)$.*

The ℓ^p-localization algebra $B_L^p(P_d(X))$ is isomorphic to the norm completion of $\mathbb{C}_L^p[P_d(X)]$ under the norm

$$\|g\|_\infty = \sup_{t\in[0,\infty)} \|g(t)\|$$

when X has bounded geometry. Note that $B_L^p(P_d(X))$ is stable in the sense that

$$B_L^p(P_d(X)) \cong B_L^p(P_d(X)) \otimes \mathcal{M}_k(\mathbb{C})$$

for all natural number k. Hence, any element in $K_0(B_L^p(P_d(X)))$ can be expressed as a difference of the K_0-classes of two idempotents in $B_L^p(P_d(X))$. To define the Bott map

$$\beta_L : K_0(B_L^p(P_d(X))) \to K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))),$$

we need to specify the value $\beta_L([P])$ in

$$K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})))$$

for any idempotent $g \in B_L^p(P_d(X))$ representing an element $[g] \in K_0(B_L^p(P_d(X)))$.

Let $g \in B_L^p(P_d(X))$ be an idempotent. For any $0 < \varepsilon_1 < \frac{1}{100}$, take an element $h \in \mathbb{C}_L^p[P_d(X)]$ such that

$$\|g - h\|_\infty < \frac{\varepsilon_1}{4}\|g\|_\infty.$$

Then $\|h-h^2\| < \varepsilon_1$, and there is a bounded function $R_{\varepsilon_1}(t) > 0$ with $\lim_{t\to\infty} R_{\varepsilon_1}(t) = 0$ such that $(h(t))(x,y) = 0$ whenever $d(x,y) > R_{\varepsilon_1}(t)$ for every t and all $x, y \in Z_d$. Let $\widetilde{R}_{\varepsilon_1} = \sup_{t\in\mathbb{R}_+} R(t)$. For any $\varepsilon_2 > 0$, take by Lemma 3.2 a family of $(\widetilde{R}_{\varepsilon_1}, \varepsilon_2; r)$-flat idempotents $\{b_v^{(n)}\}_{n\in\mathbb{N}, x\in V_n}$ in $\mathcal{M}_2(\mathcal{A}_n^+)$ for a sufficiently large $r > 0$. Define

$$\widetilde{h}, \widetilde{h}_0 : [0,\infty) \to \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

by the formulas

$$(\widetilde{h}(t))(x,y) = [(b_{f(x)}^{(1)} \otimes_p (h(t))(x,y), \cdots, b_{f(x)}^{(n)} \otimes_p (h(t))(x,y), \cdots)],$$
$$(\widetilde{h}_0(t))(x,y) = [(b_0 \otimes_p (h(t))(x,y), \cdots, b_0 \otimes_p (h(t))(x,y), \cdots)]$$

for all $t \in [0,\infty)$ and $x, y \in Z_d$, where $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we have

$$\widetilde{h}, \widetilde{h}_0 \in \mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

and

$$\widetilde{h} - \widetilde{h}_0 \in \mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})].$$

Since X has bounded geometry, by the uniform almost flatness of the Bott generators, we can choose ε_1 and then ε_2 small enough to obtain $\widetilde{h}$, $\widetilde{h}_0$, as constructed above, such that $\|\widetilde{h}^2 - \widetilde{h}\|_\infty < \frac{1}{5}$ and $\|\widetilde{h}_0^2 - \widetilde{h}_0\| < \frac{1}{5}$. The spectrum of either $\widetilde{h}$ or $\widetilde{h_0}$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $\chi : S_0 \sqcup S_1 \to \mathbb{C}$ be a continuous function such that $\chi(S_0) = \{0\}$, $\chi(S_1) = \{1\}$. Let $\eta = \chi(\widetilde{h})$ and $\eta_0 = \chi(\widetilde{h}_0)$. Then η and η_0 are idempotents in

$$B_L^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$$

with

$$\eta - \eta_0 \in B_L^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})).$$

Thanks to the difference construction, we define

$$\beta_L([g]) = D(\eta, \eta_0) \in K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

The correspondence $[g] \to \beta_L([g])$ extends to a homomorphism, the Bott map,

$$\beta_L : K_0(B_L^p(P_d(X))) \to K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

The case for K_1 can be dealt with by suspension. Thus, we obtain the Bott map β_L,

$$\beta_L : K_*(B_L^p(P_d(X))) \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

It follows from the constructions of β and β_L, we have the following commuting diagram

$$\begin{array}{ccc} K_*(B_L^p(P_d(X))) & \xrightarrow{\beta_L} & K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))) \\ \downarrow e_* & & \downarrow e_*^{\mathcal{A}} \\ K_*(B^p(P_d(X))) & \xrightarrow{\beta} & K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))). \end{array}$$

4 Isomorphism for Twisted Algebras

In this section, we shall prove the following result.

Theorem 4.1 *Let X be a discrete metric space with bounded geometry which admits a coarse embedding into a Banach space with Property* (H)*. The evaluation map*

$$e_*^{\mathcal{A}}: \lim_{d\to\infty} K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))) \to \lim_{d\to\infty} K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})))$$

is an isomorphism.

The proof proceeds by decomposing the twisted algebras into various smaller ideals or subalgebras and applying a Mayer-Vietoris sequence argument. Let X be a discrete metric space with bounded geometry which admits a coarse embedding $f: X \to V$ into a Banach space V with Property (H). To begin with, we shall discuss ideals of the twisted algebras supported on open subsets of V.

Definition 4.1 *Let $O \subset V$ be an open subset of V. For each $d \geq 0$, define*

$$\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]_O$$

to be the subalgebra of $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ generated by the elements

$$T \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

for which

$$\mathrm{Supp}((T(x,y))^{(n)}) \subset O \cap V_n$$

for all $n \in \mathbb{N}$, $x, y \in Z_d$, where we denote

$$T(x,y) = [((T(x,y))^{(1)}, \cdots, (T(x,y))^{(n)}, \cdots)] \in \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}).$$

Define

$$B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O$$

to be the norm closure of $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]_O$ in $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$. Similarly, define

$$\mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]_O$$

to be the subalgebra of $\mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$ generated by those functions

$$g: [0,\infty) \to \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

such that $g(t) \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]_O$ for all $t \in [0,\infty)$. Define

$$B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O$$

to be the completion of $\mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]_O$ in $B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$.

It is easy to see that

$$B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O$$

is an ideal of $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$, and $B^p_L(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O$ is an ideal of

$$B^p_L(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})).$$

There is also an evaluation map

$$\mathrm{e}^{\mathcal{A}} : B^p_L(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O \to B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O$$

given by $\mathrm{e}^{\mathcal{A}}(g) = g(0)$.

Definition 4.2 (cf. [1]) *Let $\Gamma \subset X$ and $r > 0$. An open subset $O \subset V$ is said to be (Γ, r)-separate if*

(1) $O = \bigsqcup_{\gamma\in\Gamma} O_\gamma$, *where $O_\gamma \subset V$ such that $O_\gamma \cap O_{\gamma'} = \emptyset$ whenever $\gamma \neq \gamma'$.*

(2) $O_\gamma \subset \mathrm{Ball}_{V_n}(f(x), r)$ *for all $\gamma \in \Gamma$.*

Lemma 4.1 Supp*ose $\Gamma \subset X$ and $r > 0$. Then for any (Γ, r)-separate open subset O of V, the evaluation homomorphism induced on K-theory*

$$e^{\mathcal{A}}_* : \lim_{d\to\infty} K_*(B^p_L(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O) \to \lim_{d\to\infty} K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O)$$

is an isomorphism.

We need some preparations before we can prove Lemma 4.1. For any $d \geq 0$, let $(Y_\gamma)_{\gamma\in\Gamma}$ be a family of closed subsets of $P_d(X)$ indexed by Γ such that

(1) $\gamma \in Y_\gamma$ for all $\gamma \in \Gamma$,

(2) the family $(Y_\gamma)_{\gamma\in\Gamma}$ is uniformly bounded in the sense that there exists $M > 0$ such that

$$\mathrm{diameter}(Y_\gamma) \leq M$$

for all $\gamma \in \Gamma$.

In particular, we will mainly consider the following three cases of $(Y_\gamma)_{\gamma\in\Gamma}$:

(1) $Y_\gamma = \mathrm{Ball}_{P_d(X)}(\gamma, S) := \{x \in P_d(X) \mid d(x, \gamma) \leq S\}$, with respect to a common bound $S > 0$ for all $\gamma \in \Gamma$;

(2) $Y_\gamma = \Delta_\gamma$, a simplex in $P_d(X)$ with $\gamma \in \Delta_\gamma$ for each $\gamma \in \Gamma$;

(3) $Y_\gamma = \{\gamma\}$ for each $\gamma \in \Gamma$.

Let $O = \bigsqcup_{\gamma\in\Gamma} O_\gamma$ be a (Γ, r)-separate open subset of V. For each $\gamma \in \Gamma$, let

$$(\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_\gamma}$$

be the subalgebra of $\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})$ generated by those equivalence classes $[(h_1, \cdots, h_n, \cdots)]$ such that the functions $h_n \in \mathcal{A}_n \otimes_p \mathcal{K}_p \cong C_0(V_n, \mathrm{Cliff}(W_n) \otimes_p \mathcal{K}_p)$ satisfy

$$\mathrm{Supp}(h_n) \subset O_\gamma \cap V_n$$

for all $n \in \mathbb{N}$. We define

$$A^*((Y_\gamma)_{\gamma\in\Gamma}) = \prod_{\gamma\in\Gamma}(B^p(Y_\gamma) \otimes (\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_\gamma})$$
$$= \Big\{(T_\gamma)_{\gamma\in\Gamma} \,|\, T_\gamma \in B^p(Y_\gamma) \otimes (\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_\gamma},\ \sup_{\gamma\in\Gamma}\|b_\gamma\| < \infty\Big\},$$

where $B^p(Y_\gamma)$ is the ℓ^p-Roe algebra of Y_γ. Similarly we define $A_L^*((Y_\gamma)_{\gamma\in\Gamma})$ to be the subalgebra of

$$\Big\{(b_\gamma)_{\gamma\in\Gamma} \,|\, b_\gamma \in B_L^p(Y_\gamma) \otimes (\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_\gamma},\ \sup_{\gamma\in\Gamma}\|b_\gamma\| < \infty\Big\}$$

generated by elements $(b_\gamma)_{\gamma\in\Gamma}$ such that

(1) the function

$$(b_\gamma)_{\gamma\in\Gamma} : [0,\infty) \to \prod_{\gamma\in\Gamma}(B^p(Y_\gamma) \otimes (\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_\gamma})$$

is uniformly norm-continuous in $t \in [0,\infty)$;

(2) there exists a bounded function $R(t) > 0$ on $[0,\infty)$ with $\lim\limits_{t\to\infty} R(t) = 0$ such that

$$(b_\gamma(t))(x,y) = 0$$

whenever $d(x,y) > R(t)$ for all $\gamma \in \Gamma$, $x,y \in Z_d$ and $t \in [0,\infty)$.

For any $S > 0$, let $\Delta_\gamma(S)$ be the simplex with vertices $\{x \in X \mid d(x,\gamma) \le S\}$ in $P_d(X)$ for $d > S$.

Lemma 4.2 *Let $O = \bigsqcup_{\gamma\in\Gamma} O_\gamma$ be a (Γ, r)-separate open subset of V. Then*

(1) $B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O \cong \lim\limits_{S\to\infty} A^*((\mathrm{Ball}_{P_d(X)}(\gamma,S))_{\gamma\in\Gamma})$;

(2) $B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O \cong \lim\limits_{S\to\infty} A_L^*((\mathrm{Ball}_{P_d(X)}(\gamma,S))_{\gamma\in\Gamma})$;

(3) $\lim\limits_{d\to\infty} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O \cong \lim\limits_{S\to\infty} A^*((\Delta_\gamma(S))_{\gamma\in\Gamma})$;

(4) $\lim\limits_{d\to\infty} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O \cong \lim\limits_{S\to\infty} A_L^*((\Delta_\gamma(S))_{\gamma\in\Gamma})$.

Proof Similar to the arguments in [18, Section 6].

Now we turn to recall the notion of strong Lipschitz homotopy (cf. [18]).

Let $(X_\gamma)_{\gamma\in\Gamma}$ and $(Y_\gamma)_{\gamma\in\Gamma}$ be two families of uniformly bounded closed subspaces of $P_d(X)$ for some $d \ge 0$ with $\gamma \in X_\gamma$, $\gamma \in Y_\gamma$ for every $\gamma \in \Gamma$. A map $g : \bigsqcup_{\gamma\in\Gamma} X_\gamma \to \bigsqcup_{\gamma\in\Gamma} Y_\gamma$ is said to be Lipschitz if

(1) $g(X_\gamma) \subset Y_\gamma$ for each $\gamma \in \Gamma$;

(2) there exists a constant c, independent of $\gamma \in \Gamma$, such that

$$d(g(x), g(y)) \le c d(x,y)$$

for all $x, y \in X_\gamma$, $\gamma \in \Gamma$.

Let g_1, g_2 be two Lipschitz maps from $\bigsqcup_{\gamma\in\Gamma} X_\gamma$ to $\bigsqcup_{\gamma\in\Gamma} Y_\gamma$. We say g_1 is strongly Lipschitz homotopy equivalent to g_2 if there exists a continuous map

$$F:[0,1]\times\Big(\bigsqcup_{\gamma\in\Gamma} X_\gamma\Big)\to\bigsqcup_{\gamma\in\Gamma} Y_\gamma$$

such that

(1) $F(0,x)=g_1(x)$, $F(1,x)=g_2(x)$ for all $x\in\bigsqcup_{\gamma\in\Gamma} X_\gamma$;

(2) there exists a constant c for which $d(F(t,x),F(t,y))\leq cd(x,y)$ for all $x,y\in X_\gamma$, $\gamma\in\Gamma$, and $t\in[0,1]$;

(3) F is equi-continuous in t, i.e., for any $\varepsilon>0$ there exists $\delta>0$ such that $d(F(t_1,x),F(t_2,x))<\varepsilon$ for all $x\in\bigsqcup_{\gamma\in\Gamma} X_\gamma$ if $|t_1-t_2|<\delta$.

We say the family $(X_\gamma)_{\gamma\in\Gamma}$ is strongly Lipschitz homotopy equivalent to the family $(Y_\gamma)_{\gamma\in\Gamma}$ if there exist Lipschitz maps $g_1:\bigsqcup_{\gamma\in\Gamma} X_\gamma\to\bigsqcup_{\gamma\in\Gamma} Y_\gamma$ and $g_2:\bigsqcup_{\gamma\in\Gamma} Y_\gamma\to\bigsqcup_{\gamma\in\Gamma} X_\gamma$ such that g_1g_2 and g_2g_1, respectively strongly Lipschitz homotopy equivalent to identity maps.

Define $A^*_{L,0}((Y_\gamma)_{\gamma\in\Gamma})$ to be the subalgebra of $A^*_L((Y_\gamma)_{\gamma\in\Gamma})$ consisting of elements $(b_\gamma)_{\gamma\in\Gamma}$ satisfying $b_\gamma(0)=0$ for all $\gamma\in\Gamma$.

Lemma 4.3 (cf. [18]) *If $(X_\gamma)_{\gamma\in\Gamma}$ is strongly Lipschitz homotopy equivalent to $(Y_\gamma)_{\gamma\in\Gamma}$, then $K_*(A^*_{L,0}((X_\gamma)_{\gamma\in\Gamma}))$ is isomorphic to $K_*(A^*_{L,0}((Y_\gamma)_{\gamma\in\Gamma}))$.*

Let e be the evaluation homomorphism from $A^*_L((X_\gamma)_{\gamma\in\Gamma})$ to $A^*((X_\gamma)_{\gamma\in\Gamma})$ given by

$$(b_\gamma)_{\gamma\in\Gamma}\mapsto(b_\gamma(0))_{\gamma\in\Gamma}.$$

Lemma 4.4 (cf. [18]) *Let $O=\bigsqcup_{\gamma\in\Gamma} O_\gamma$ be a (Γ,r)-separate open subset of V as above, and let $d\geq 0$. If $(\Delta_\gamma)_{\gamma\in\Gamma}$ is a family of simplices in $P_d(X)$ such that $\gamma\in\Delta_\gamma$ for all $\gamma\in\Gamma$, then the induces map*

$$e_*:K_*(A^*_L((\Delta_\gamma)_{\gamma\in\Gamma}))\to K_*(A^*((\Delta_\gamma)_{\gamma\in\Gamma}))$$

is an isomorphism.

Proof Note that $(\Delta_\gamma)_{\gamma\in\Gamma}$ is strongly Lipschitz homotopy equivalent to $(\gamma)_{\gamma\in\Gamma}$. By an argument of Eilenberg swindle, we have $K_*(A^*_{L,0}((\gamma)_{\gamma\in\Gamma}))=0$. Consequently, Lemma 4.4 follows from Lemma 4.3 and the six term exact sequence of Banach algebra K-theory.

Proof of Lemma 4.1 By Lemma 4.2, we have the following commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} B^p_L(P_d(X),\mathcal{Q}((\mathcal{A}_n\otimes_p\mathcal{K}_p)_{n\in\mathbb{N}}))_O & \xrightarrow{e^{\mathcal{A}}} & \lim\limits_{d\to\infty} B^p(P_d(X),\mathcal{Q}((\mathcal{A}_n\otimes_p\mathcal{K}_p)_{n\in\mathbb{N}}))_O\\
\downarrow\cong & & \downarrow\cong\\
\lim\limits_{S\to\infty} A^*_L((\Delta_\gamma(S))_{\gamma\in\Gamma}) & \xrightarrow{e} & \lim\limits_{S\to\infty} A^*((\Delta_\gamma(S))_{\gamma\in\Gamma}),
\end{array}$$

which induces the following commuting diagram at K-theory level

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O) & \xrightarrow{e_*^{\mathcal{A}}} & \lim\limits_{d\to\infty} K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_O) \\
\downarrow \cong & & \downarrow \cong \\
\lim\limits_{S\to\infty} K_*(A_L^*((\Delta_\gamma(S))_{\gamma\in\Gamma})) & \xrightarrow{e_*} & \lim\limits_{S\to\infty} K_*(A^*((\Delta_\gamma(S))_{\gamma\in\Gamma})).
\end{array}$$

Now Lemma 4.1 follows from Lemma 4.4 by using the above commuting diagram. This ends the proof.

Lemma 4.5 *Let N be a positive integer, and let $\Gamma_1, \cdots, \Gamma_N$ be N mutually disjoint subsets of X. For any $r > 0$, let*

$$O_{r,j} = \bigcup_{\gamma\in\Gamma_j} \mathrm{Ball}_V(f(\gamma), r)$$

for each $j \in \{1, 2, \cdots, N\}$. Then for any $r_0 > 0$ and $k \in \{1, 2, \cdots, N-1\}$, we have the following equalities:

(1)

$$\begin{aligned}
&\lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}} \\
&\quad + \lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup\limits_{j=1}^{k-1} O_{r,j}} \\
&= \lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup\limits_{j=1}^{k} O_{r,j}};
\end{aligned}$$

(2)

$$\begin{aligned}
&\lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}} \\
&\cap \lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup\limits_{j=1}^{k-1} O_{r,j}} \\
&= \lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}\cap\left(\bigcup\limits_{j=1}^{k-1} O_{r,j}\right)};
\end{aligned}$$

(3)

$$\begin{aligned}
&\lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}} \\
&\quad + \lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup\limits_{j=1}^{k-1} O_{r,j}} \\
&= \lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup\limits_{j=1}^{k} O_{r,j}};
\end{aligned}$$

(4)

$$\lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}}$$

$$\cap \lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{\bigcup_{j=1}^{k-1} O_{r,j}}$$
$$= \lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r,k}\cap\left(\bigcup_{j=1}^{k-1} O_{r,j}\right)}.$$

Proof Similar to the proof of [18, Lemma 6.3].

We are now ready to give a proof to Theorem 4.1.

Proof of Theorem 4.1 For any $r > 0$, define

$$O_r = \bigcup_{x\in X} \mathrm{Ball}_V(f(x), r),$$

where $f : X \to V$ is the coarse embedding, and

$$\mathrm{Ball}_V(f(x), r) := \{v \in V \mid \|v - f(x)\| \le r\}.$$

By definition, we have

$$B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})) \cong \lim_{r\to\infty} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r},$$
$$B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})) \cong \lim_{r\to\infty} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r}.$$

For any $d \ge 0$, if $r < r'$, then

$$B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r} \subset B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r'}},$$
$$B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r} \subset B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_{r'}}.$$

Consequently, it suffices to show that, for each $r_0 > 0$, the evaluation map

$$e_*^{\mathcal{A}} : \lim_{d\to\infty} K_*\Big(\lim_{r<r_0, r\to r_0} B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r}\Big)$$
$$\to \lim_{d\to\infty} K_*\Big(\lim_{r<r_0, r\to r_0} B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))_{O_r}\Big)$$

is an isomorphism.

Let $r_0 > 0$. Since X has bounded geometry, there exists $\overline{N} > 0$ such that $\#\mathrm{Ball}_X(x, r_0) < \overline{N}$ for all $x \in X$. It follows that there exists an integer N such that

(1) $X = \bigsqcup_{j=1}^{N} \Gamma_j$ for some subspaces $\Gamma_j \subset X$ with $\Gamma_j \cap \Gamma_{j'} = \emptyset$ whenever $\gamma \neq \gamma'$;

(2) for each $j \in \{1, 2, \cdots, N\}$, and for any distinct $\gamma, \gamma' \in \Gamma_j$, we have

$$\|f(\gamma) - f(\gamma_j)\| > 2r$$

in V.

For any $0 < r < r_0$ and each $j \in \{1, 2, \cdots, N\}$, let

$$O_{r,j} = \bigcup_{x\in\Gamma_j} \mathrm{Ball}_V(f(x), r).$$

Then $O_r = \bigcup_{j=1}^{N} O_{r,j}$, and each $O_{r,j}$ or

$$O_{r,j} \cap \Big(\bigcup_{i=1}^{j-1} O_{r,i}\Big)$$

are (Γ_j, r)-separate for any $j \in \{1, 2, \cdots, N\}$. Now, Theorem 4.1 follows from Lemma 4.5, together with a Mayer-Vietoris sequence argument. This ends the proof.

5 Proof of the Main Result

In this final section, we shall complete the proof of Theorem 1.1. To do so, we need to construct one more Bott map, β_L^∞. Let $Z_d \subset P_d(X)$ is the countable dense subset for each $d \geq 0$ as before. Let $\prod_{n=1}^{\infty} \mathcal{K}_p$ be the Banach algebra direct product of countably many copies of $\mathcal{K}_p$, and let $\bigoplus_{n=1}^{\infty} \mathcal{K}_p$ be the ideal of $\prod_{n=1}^{\infty} \mathcal{K}_p$ consists of those sequences $(k_1, \cdots, k_n, \cdots)$ such that $\lim_{n\to\infty} \|k_n\| = 0$. Denote the quotient algebra as

$$\mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}) := \frac{\prod_{n=1}^{\infty} \mathcal{K}_p}{\bigoplus_{n=1}^{\infty} \mathcal{K}_p}.$$

Definition 5.1 *For each $d \geq 0$, define*

$$\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$$

to be the set of all bounded functions

$$T : Z_d \times Z_d \to \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})$$

such that

(1) *for any bounded subset $B \subset P_d(X)$, the set*

$$\{(x, y) \in B \times B \cap Z_d \times Z_d \mid T(x, y) \neq 0\}$$

is finite;

(2) *there exists $L > 0$ such that*

$$\#\{y \in Z_d \mid T(x, y) \neq 0\} < L, \quad \#\{y \in Z_d \mid T(y, x) \neq 0\} < L$$

for all $x \in Z_d$;

(3) *there exists $R \geq 0$ such that $T(x, y) = 0$ whenever $d(x, y) > R$ for $x, y \in Z_d$.* (*The least such R is called the propagation of T.*)

The algebraic structure for $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ is defined by regarding elements T as $Z_d \times Z_d$-matrices. Let

$$E = \Big\{ \sum_{x\in Z_d} a_x[x] \,\Big|\, a_x \in \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}), \sum_{x\in Z_d} \|a_x\|^p \text{ converges} \Big\}.$$

Then E is a L^p-X-module over $\mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})$:

$$\Big(\sum_{x\in Z_d} a_x[x] \Big) a = \sum_{x\in Z_d} a_x a[x]$$

for any $a \in \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})$. The algebra $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ acts on E by the formula

$$T\Big(\sum_{x\in Z_d} a_x[x] \Big) = \sum_{x\in Z_d} \Big(\sum_{y\in Z_d} T(x,y) a_y \Big)[x]$$

for $T \in \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ and $\sum_{x\in Z_d} a_x[x] \in E$. Note that T is a module homomorphism.

Let $B(E)$ be the Banach algebra of all module homomorphisms from E to E.

Definition 5.2 *The Banach algebra $B^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))$ is defined to be the norm completion of $\mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ in $B(E)$.*

Definition 5.3 *For each $d \geq 0$, define*

$$\mathbb{C}^p_L[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$$

to be the algebra of all bounded, uniformly norm-continuous functions

$$g : [0,\infty) \to \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$$

such that

(1) *for any bounded subset $B \subset P_d(X)$, the set*

$$\{(x,y) \in B \times B \cap Z_d \times Z_d \mid (g(t))(x,y) \neq 0\}$$

is finite for all $t \in [0,\infty)$;

(2) *there exists $L > 0$ such that*

$$\#\{y \in Z_d \mid (g(t))(x,y) \neq 0\} < L, \quad \#\{y \in Z_d \mid (g(t))(y,x) \neq 0\} < L$$

for all $t \in [0,\infty)$ and $x \in Z_d$;

(3) *there exists a bounded function $R : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} R(t) = 0$ such that $(g(t))(x,y) = 0$ whenever $d(x,y) > R(t)$.*

Definition 5.4 *The ℓ^p-localization Banach algebra $B^p_L(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))$ is defined to be the norm completion of $\mathbb{C}^p_L[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ with respect to the norm*

$$\|g\|_\infty = \sup_{t\in[0,\infty)} \|g(t)\|.$$

We now define the Bott map

$$\beta_L^\infty : K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))) \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

Note that $B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))$ is also a stable Banach algebra. Let

$$g \in B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))$$

be an idempotent. For any $0 < \varepsilon_1 < \frac{1}{100}$, take an element $h \in \mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})]$ such that

$$\|g - h\|_\infty < \frac{\varepsilon_1}{4}\|g\|_\infty.$$

Then $\|h - h^2\| < \varepsilon_1$, and there is a bounded function $R_{\varepsilon_1}(t) > 0$ with $\lim_{t\to\infty} R_{\varepsilon_1}(t) = 0$ such that $(h(t))(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}(t)$ for every t and all $x, y \in Z_d$. Let $\widetilde{R}_{\varepsilon_1} = \sup_{t\in[0,\infty)} R(t)$. For any $\varepsilon_2 > 0$, take by Lemma 3.2 a family of $(\widetilde{R}_{\varepsilon_1}, \varepsilon_2; r)$-flat idempotents $\{b_v^{(n)}\}_{n\in\mathbb{N},x\in V_n}$ in $\mathcal{M}_2(\mathcal{A}_n^+)$ for a sufficiently large $r > 0$. Define

$$\widetilde{h}, \widetilde{h}_0 : [0, \infty) \to \mathbb{C}^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

by the formulas

$$(\widetilde{h}(t))(x, y) = [(b_{f(x)}^{(1)} \otimes_p k^{(1)}, \cdots, b_{f(x)}^{(n)} \otimes_p k^{(n)}, \cdots)],$$

$$(\widetilde{h}_0(t))(x, y) = [(b_0 \otimes_p k^{(1)}, \cdots, b_0 \otimes_p k^{(n)}, \cdots)]$$

for all $t \in [0, \infty)$ and $x, y \in Z_d$, where we denote

$$(h(t))(x, y) = [(k^{(1)}, \cdots, k^{(n)}, \cdots)] \in \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}),$$

and recall that $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we have

$$\widetilde{h}, \widetilde{h}_0 \in \mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})]$$

and

$$\widetilde{h} - \widetilde{h}_0 \in \mathbb{C}_L^p[P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})].$$

Since X has bounded geometry, by the uniform almost flatness of the Bott generators, we can choose ε_1 and then ε_2 small enough to obtain $\widetilde{h}$, $\widetilde{h}_0$, as constructed above, such that $\|\widetilde{h}^2 - \widetilde{h}\|_\infty < \frac{1}{5}$ and $\|\widetilde{h}_0^2 - \widetilde{h}_0\| < \frac{1}{5}$. The spectrum of either $\widetilde{h}$ or $\widetilde{h_0}$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $\chi : S_0 \sqcup S_1 \to \mathbb{C}$ be a continuous function such that $\chi(S_0) = \{0\}$, $\chi(S_1) = \{1\}$. Let $\eta = \chi(\widetilde{h})$ and $\eta_0 = \chi(\widetilde{h}_0)$. Then η and η_0 are idempotents in

$$B_L^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n^+) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$$

with

$$\eta - \eta_0 \in B_L^p(P_d(X), \mathcal{Q}((\mathcal{M}_2(\mathcal{A}_n) \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}})).$$

Thanks to the difference construction, we define

$$\beta_L^\infty([g]) = D(\eta, \eta_0) \in K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

The correspondence $[g] \to \beta_L^\infty([g])$ extends to a homomorphism, the Bott map,

$$\beta_L^\infty : K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))) \to K_0(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

The case for K_1 can be dealt with by suspension. Thus, we obtain the Bott map

$$\beta_L^\infty : K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))) \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))).$$

Lemma 5.1 *β_L^∞ is an isomorphism.*

Proof Since X has bounded geometry, and the K-theory of $B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))$ and $B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ has strong Lipschitz homotopy invariance, one can follow the argument in [17, Theorem 3.2] to reduce the problem to proving β_L^∞ is an isomorphism when $P_d(X)$ is 0-dimensional. In this case, it suffices to show $\beta_* : K_*(\mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})) \to K_*(\mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))$ is an isomorphism, which holds clearly by Bott periodicity theorem, see [1, 7]. This ends the proof.

Furthermore, there exists a natural homomorphism

$$\widetilde{\tau} : \prod_{n=1}^{\infty} K_*(B_L^p(P_d(X))) \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})))$$

which induces a homomorphism

$$\tau : \frac{\prod_{n=1}^{\infty} K_*(B_L^p(P_d(X)))}{\bigoplus_{n=1}^{\infty} K_*(B_L^p(P_d(X)))} \to K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})))$$

Lemma 5.2 *τ is an isomorphism.*

Proof Note that both group $\frac{\prod_{n=1}^{\infty} K_*(B_L^p(P_d(X)))}{\bigoplus_{n=1}^{\infty} K_*(B_L^p(P_d(X)))}$ and $K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}})))$ satisfies strong Lipschitz homotopy invariance and Mayer-Vietoris theorem by using a similar argument with [17, Theorem 3.2]. If $P_d(X)$ is a single point (equally speaking, if $P_d(X)$ is 0-dimensional), then τ is an isomorphism by definition. Then the general case follows from an argument of strong Lipschitz homotopy invariance and Mayer-Vietoris sequence as in [17, Theorem 3.2].

Finally, we are ready to complete the proof of the main result of this paper.

Proof of Theorem 1.1 Consider the commutative diagram

$$\begin{array}{ccc}
\dfrac{\prod_{n=1}^{\infty} K_*(B_L^p(P_d(X)))}{\bigoplus_{n=1}^{\infty} K_*(B_L^p(P_d(X)))} & \xrightarrow{\tau} & K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{K}_p)_{n\in\mathbb{N}}))) \\
\uparrow \zeta & & \downarrow \beta_L^{\infty} \\
K_*(B_L^p(P_d(X))) & \xrightarrow{\beta_L} & K_*(B_L^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))) \\
\downarrow e_* & & \downarrow e_*^{\mathcal{A}} \\
K_*(B^p(P_d(X))) & \xrightarrow{\beta} & K_*(B^p(P_d(X), \mathcal{Q}((\mathcal{A}_n \otimes_p \mathcal{K}_p)_{n\in\mathbb{N}}))),
\end{array}$$

where the map ζ is defined by the mapping $x \mapsto [(x, x, \cdots, x, \cdots)]$ via constant sequences. Clearly ζ is injective. Passing to inductive limit as $d \to \infty$, Theorem 1.1 follows from Theorem 4.1 and Lemmas 5.1–5.2. This completes the proof.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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