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# Equivariant Chern Classes of Orientable Toric Origami Manifolds*

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(*Dedicated to Professor Mikiya Masuda on his 70th birthday*)

**Abstract** A toric origami manifold, introduced by Cannas da Silva, Guillemin and Pires, is a generalization of a toric symplectic manifold. For a toric symplectic manifold, its equivariant Chern classes can be described in terms of the corresponding Delzant polytope and the stabilization of its tangent bundle splits as a direct sum of complex line bundles. But in general a toric origami manifold is not simply connected, so the algebraic topology of a toric origami manifold is more difficult than a toric symplectic manifold. In this paper they give an explicit formula of the equivariant Chern classes of an oriented toric origami manifold in terms of the corresponding origami template. Furthermore, they prove the stabilization of the tangent bundle of an oriented toric origami manifold also splits as a direct sum of complex line bundles.

**Keywords** Equivariant Chern classes, Toric origami manifolds, Unitary structures, Spin structures
**2000 MR Subject Classification** 55R40, 55N91, 53D05, 57S12

## 1 Introduction

Toric symplectic manifolds are classified by Delzant and by his famous result we know that toric symplectic manifolds are nonsingular projective toric varieties. Equivariant Chern classes of a nonsingular projective toric variety can be described explicitly in terms of the corresponding Delzant polytope and the stabilization of the tangent bundle of a toric symplectic manifold splits as a direct sum of complex line bundles (see [3, 6–7]). In this paper we generalize these results to orientable toric origami manifolds introduced by Cannas da Silva, Guillemin and Pires in [4].

A toric origami manifold is a generalization of a toric symplectic manifold from symplectic geometric point of view. It is well-known that a toric symplectic manifold is orientable and simply connected. However, a toric origami manifold may not be simply connected and even not be orientable (see [11, 16]). In [4], the authors generalized Delzant's result to toric origami manifolds and showed that there is a bijective correspondence between toric origami manifolds

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and origami templates. Inspired by Yu's result in [15] and Masuda's work in [12], we express the equivariant Chern classes of an oriented toric origami manifold in terms of origami templates (Theorem 4.1). Moreover we show that the stabilization of the tangent bundle of an oriented toric origami manifold splits as a direct sum of complex line bundles (Theorem 5.1).

This paper is organized as follows. In Section 2 we recall some necessary definitions and properties of toric origami manifolds. In Section 3 we check that the unitary structure given in [5] is $T$-invariant for an oriented toric origami manifold. In Section 4 we give an explicit formula for equivariant Chern classes of oriented toric origami manifolds. In Section 5 we show that the stabilization of the tangent bundle of an oriented toric origami manifold splits as a direct sum of complex line bundles. In Section 6 we apply the results in Section 4 to determining when an oriented toric origami manifold with fixed points is a spin manifold. In Section 7 which is an appendix, we show that $H_T^{\mathrm{even}}(M)$ is torsion free as a $\mathbb{Z}$-module for an oriented toric origami manifold $M$ by applying the arguments in [2].

## 2 Toric Origami Manifolds

### 2.1 Origami manifolds

**Definition 2.1** *A folded symplectic form on a* $2n$ *dimensional manifold* $M$ *is a closed* $2$*-form* $\omega$ *satisfying the following two conditions*:

(1) $\omega^n$ *vanishes transversally on a submanifold* $i : Z \hookrightarrow M$;

(2) $i^*\omega$ *has maximal rank, i.e.,* $(i^*\omega)^{n-1}$ *does not vanish.*

*We call* $(M, \omega)$ *a folded symplectic manifold and the submanifold* $Z$ *is called the folding hypersurface or fold.*

When $M$ is oriented, $M \setminus Z$ is the disjoint union of $M^+ = \{x \in M \mid \omega^n(x) > 0\}$ and $M^- = \{x \in M \mid \omega^n(x) < 0\}$. In particular, $Z$ is a coörientable hypersurface in $M$ and $Z$ is also an oriented hypersurface. Therefore the normal bundle $N(Z)$ of $Z$ in $M$ is trivial.

**Definition 2.2** *An origami manifold is a folded symplectic manifold* $(M, \omega)$ *whose null foliation is fibrating with oriented circle fibers,* $\pi$, *over a compact base* $B$. *The form* $\omega$ *is called an origami form and the null foliation, i.e., the vertical bundle of* $\pi$ *is called the null fibration.*

For an oriented origami manifold $(M, \omega)$, let $E \to Z$ be the kernel of the bundle map induced by $\omega$ from $TM_{|Z}$ to $T^*M_{|Z}$ and $F := E \cap TZ$. Since $F$ is the subbundle of $TZ$ along the circle fiber, we have an oriented non-vanishing section $v$ of $F$. The normal line bundle of $Z$ in $M$ is trivial, so $E$ is a direct sum of two trivial line bundles. Namely, $E \cong N(Z) \oplus F$. Moreover if there is a compact connected abelian Lie group $T \cong (S^1)^d$ acts on $M$ preserving $\omega$, then $E \cong N(Z) \oplus F$ as $T$-invariant vector bundles. By the proof of [4, Theorem 3.2], the following theorem shows that we can extend $E$ to an equivariant neighborhood $\mathcal{U}$ of $Z$.

**Theorem 2.1** (see [4–5]) *Let $(M, \omega, T)$ be an oriented origami manifold with a torus $T$ action. Then there exists an equivariant neighborhood of $Z$ with a $T$-equivariant diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ such that*

$$\varphi^* \omega = p^* i^* \omega + d(t^2 p^* \alpha),$$

*where $T$ acts trivially on $(-\varepsilon, \varepsilon)$, $p : Z \times (-\varepsilon, \varepsilon) \to Z$ is the projection onto the first factor, $t$ is the real coordinate on the interval $(-\varepsilon, \varepsilon)$ and $\alpha$ is a $T$-invariant $S^1$-connection on $Z$ for a chosen principal $S^1$ action, $S^1 \hookrightarrow Z \xrightarrow{\pi} B$. We call such $\mathcal{U}$ is an equivariant Moser model.*

### 2.2 Toric origami manifolds

**Definition 2.3** *A toric origami manifold $(M, \omega, T, \mu)$, abbreviated as $M$, is a compact connected origami manifold $(M, \omega)$ equipped with an effective Hamiltonian action of a torus $T$ with $\dim T = \frac{1}{2} \dim M$ and with a choice of a corresponding moment map $\mu$.*

**Remark 2.1** When the folding hypersurface $Z = \emptyset$, $(M, \omega, T, \mu)$ is a toric symplectic manifold, so toric origami manifolds are generalizations of toric symplectic manifolds.

Let $\mathcal{D}_n$ denote the set of all Delzant polytopes in $\mathbb{R}^n$ (w.r.t. a given lattice), $\mathcal{F}_n$ — the set of all their facets and $G$ a connected graph (loops and multiple edges are allowed) with the vertex set $V$ and the edge set $E$.

**Definition 2.4** *An $n$-dimensional origami template consists of a connected graph $G$, called the template graph, and a pair of maps $\Psi_V \colon V \to \mathcal{D}_n$ and $\Psi_E \colon E \to \mathcal{F}_n$ such that:*

1. *If $e \in E$ is an edge of $G$ with endpoints $v_1, v_2 \in V$, then $\Psi_E(e)$ is a facet of both polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes coincide near $\Psi_E(e)$ (this means that there exists an open neighborhood $U$ of $\Psi_E(e)$ in $\mathbb{R}^n$ such that $U \cap \Psi_V(v_1) = U \cap \Psi_V(v_2)$);*
2. *if $e_1, e_2 \in E$ are two edges of $G$ adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi_V(v)$.*

*The facets of the form $\Psi_E(e)$ for $e \in E$ are called the fold facets of the origami template.*

**Definition 2.5** *An origami template $(G, \Psi_V, \Psi_E)$ is called orientable if the template graph $G$ is 2-colorable.*

**Theorem 2.2** (see [4, Theorem 3.13]) *Toric origami manifolds are classified by origami templates up to equivariant symplectomorphism preserving the moment maps. More specifically, at the level of symplectomorphism classes (on the left hand side), there is a one-to-one correpondence*

$$\{2n\text{-}diml\ toric\ origami\ manifolds\} \to \{n\text{-}diml\ origami\ templates\}$$

$$(M^{2n}, \omega, T^n, \mu) \mapsto \mu(M).$$

*Moreover, oriented toric origami manifolds correspond to oriented origami templates.*

## 3 $T$-Invariant Unitary Structure on Toric Origami Manifolds

By [5, Theorem 2] there exists a complex structure $J$ on $TM \oplus \underline{\mathbb{R}}^2$ for an oriented folded symplectic manifold $(M, \omega)$. The following is the main idea of the proof in [5].

$\omega$ is symplectic on $M \setminus \mathcal{U}$, so there is a complex structure $J_0$ compatible with $\omega$ on $TM|_{M\setminus\mathcal{U}}$. Since $J_0$ gives an opposite orientation on $M^+$ and $M^-$, $J_0$ can not extend to $TM$. However the authors in [5] constructed a deformation of $J_0 \oplus J_{sta}$ on $(TM \oplus \underline{\mathbb{R}}^2)|_{\mathcal{U}}$, where $J_{sta}$ is the standard complex structure on $\mathbb{R}^2$, such that there is a complex structure $J$ on $TM \oplus \underline{\mathbb{R}}^2$ which extends the complex structure

$$\begin{cases} J_0 \oplus J_{sta} & \text{on } M^+ \setminus \mathcal{U}, \\ J_0 \oplus -J_{sta} & \text{on } M^- \setminus \mathcal{U}. \end{cases}$$

Hence $M$ is a unitary manifold.

Note that for an oriented toric origami manifold $(M, \omega, T, \mu)$, $T$ acts trivially on the fiber of $E \oplus \underline{\mathbb{R}}^2$, so the proof of [5, Theorem 2] also holds for equivariant settings. In the following we give the details of the proof which is from [5].

**Proposition 3.1** (see Equivariant version of [5, Theorem 2]) *Suppose $(M, \omega, T, \mu)$ is an oriented toric origami manifold. Then there is a $T$-invariant complex structure on $TM \oplus \underline{\mathbb{R}}^2$, where $\underline{\mathbb{R}}^2$ is the trivial bundle of rank 2 over $M$.*

**Proof** Let $Z$ be the folding hypersurface of $M$ and $\mathcal{U}$ be an equivariant Moser model of $Z$ as in Theorem 2.1. Let $E \to Z$ be the kernel of the bundle map induced by $\omega$ from $TM_{|Z}$ to $T^*M_{|Z}$, $F = E \cap TZ$ and $N(Z)$ be the normal bundle of $Z$ in $M$ as in Subsection 2.1. By Theorem 2.1 we can extend $E, F$ and $N(Z)$ to $\mathcal{U}$. Let $E^\perp$ be the kerner of $\alpha$ and $p^*E^\perp$ be the pullback of $E$ on $Z \times (-\varepsilon, \varepsilon)$, where $p : Z \times (-\varepsilon, \varepsilon) \to Z$ is the projection onto the first factor. Then the tangent bundle of $Z \times (-\varepsilon, \varepsilon)$ is isomorphic to $p^*E^\perp \oplus E = p^*E^\perp \oplus F \oplus N(Z)$. Let $\frac{\partial}{\partial t}$ be the canonical section of $N(Z)$ and $v$ be a section of $F$ such that $\alpha(v) \equiv 1$, where $\alpha$ is a $T$-invariant $S^1$-connection on $Z$ for a chosen principal $S^1$ action, $S^1 \hookrightarrow Z \xrightarrow{\pi} B$. Then for any tangent vector $v_1 \in p^*E^\perp$ at any point $(x, t_0) \in Z \times (-\varepsilon, \varepsilon) \setminus Z$, we have

$$\varphi^*\omega\Big(\frac{\partial}{\partial t}, v_1\Big) = 2t\alpha(v_1) = 0,$$

$$\varphi^*\omega(v, v_1) = t^2 d\alpha(v, v_1) = t^2(i_v d\alpha)(v_1) = t^2\alpha(v_1) = 0$$

by Cartan's identity $\mathcal{L}_v\alpha = \iota_v d\alpha + d(\iota_v\alpha)$ and the $S^1$ invariance of $\alpha$. Therefore $p^*E^\perp$ be the symplectic orthogonal complement of $E$ on $Z \times (-\varepsilon, \varepsilon) \setminus Z$.

We also denote $p^*E$ and $p^*E^\perp$ by $E$ and $E^\perp$, respectively, by abusing notations. Since $\varphi^*\omega$ is symplectic on $E^\perp$, there exists a $T$-invariant complex structure $J_{E^\perp}$ on $E^\perp$. Due to $\varphi^*\omega$ is a $T$-invariant symplectic form on $E|_{\mathcal{U}\setminus Z}$, there exists a $T$-invariant complex structure $J_E$ on $E|_{\mathcal{U}\setminus Z}$ such that on $t \leq -\frac{\varepsilon}{2}$,

$$J_E\eta = -v, \quad J_E v = \eta$$

and on $t \geq \frac{\varepsilon}{2}$,

$$J_E\eta = v, \quad J_E v = -\eta,$$

where $v$ and $\eta$ are $T$-invariant section of $E$.

Then we can make a deformation on $E \oplus \underline{\mathbb{R}}^2$ as follows. For $0 \leq \theta \leq \pi$, consider the matrix

$$A_\theta := \begin{pmatrix} 0 & \cos\theta & 0 & \sin\theta \\ -\cos\theta & 0 & \sin\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \\ -\sin\theta & 0 & -\cos\theta & 0 \end{pmatrix}.$$

Then $A_\theta^2 = -\mathrm{Id}$, $A_\theta^{\mathrm{T}} = -A_\theta$, where $A_\theta^{\mathrm{T}}$ means the transpose of $A_\theta$, and $A_0 = -A_\pi$. Set $B_t = A_\theta$, where $\theta = \frac{\pi}{\varepsilon}t + \frac{\pi}{2}$. Then $B_t$ is a complex structure of $E \oplus \underline{\mathbb{R}}^2$ over $Z \times \big[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\big]$, if we identifies $\mathbb{R}^2$ with $\mathbb{C}$ and define the complex structure of $\underline{\mathbb{R}}^2$ over $M^+$ by $(x, y) \mapsto x + \mathrm{i}y$ and over $M^-$ by $(x, y) \mapsto x - \mathrm{i}y$, then $B_t$ is defined on all $\mathcal{U}$.

Since $T$ acts on the fiber of $E \oplus \underline{\mathbb{R}}^2$ trivially, $B_t$ is a $T$-invariant complex structure on $E \oplus \underline{\mathbb{R}}^2$. This implies that $B_t \oplus J_{E^\perp}$ is a $T$-invariant complex structure on $E^\perp \oplus E \oplus \underline{\mathbb{R}}^2$. Namely $B_t \oplus J_{E^\perp}$ is a $T$-invariant complex structure on $T\mathcal{U} \oplus \underline{\mathbb{R}}^2$.

Set $M_\varepsilon = M \setminus \big(Z \times \big[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\big]\big)$. Since there exists a $T$-invariant complex structure $J_\varepsilon$ on $TM_\varepsilon$ compatible with $\omega$ such that $J_\varepsilon = J_E \oplus J_{E^\perp}$ over $Z \times \big(-\frac{3}{4}\varepsilon, -\frac{1}{2}\varepsilon\big) \cup Z \times \big(\frac{1}{2}\varepsilon, \frac{3}{4}\varepsilon\big)$ by choosing a $T$-invariant metric on $M_\varepsilon$, $B_t \oplus J_{E^\perp}$ essentially defines a $T$-invariant complex structure on $TM \oplus \underline{\mathbb{R}}^2$.

## 4 Chern Classes of Oriented Toric Origami Manifolds

Let $(M, \omega, T, \mu)$ be an oriented toric origami manifold and $\mathcal{P}$ be the origami template corresponding to $(M, \omega, T, \mu)$. By [10, Lemma 5.1], the torus action on $M$ is locally standard, so the orbit space $M/T$ is a nice manifold with corners by [3, Proposition 7.4.13]. Here nice manifold with corners means every codimension-$k$ face of $M/T$ is contained in exactly $k$ facets.

A closed, connected, codimension-two submanifold of $M$ is called characteristic if it is a connected component of the set fixed pointwise by a certain circle subgroup of $T$. Since $M$ is compact, there are only finite many characteristic submanifolds. We denote them by $M_i$, where $i = 1, 2, \cdots, m$. Let $\pi : M \to M/T$ be the quotient map, then $F_i := \pi(M_i)$ is a facet of $M/T$. Since $M_i$ is a $T$-invariant unitary submanifold of $M$ of codimension 2, we have a Gysin homomorphism $H_T^*(M_i) \to H_T^{*+2}(M)$. Let $\tau_i \in H_T^2(M)$ is the image of $1 \in H_T^0(M_i)$. In this section, we express the equivariant Chern classes of $TM \oplus \underline{\mathbb{R}}^2$ in terms of $\tau_i$.

From now on for a $T$-space $X$, $X_T$ denotes the orbit space of $ET \times X$ by the diagonal action on $ET \times X$.

### 4.1 Chern classes of equivariant Moser model

The following lemma is well-known for toric symplectic manifolds, but for the reader's convenience we shall give a proof.

**Lemma 4.1** *Let $M$ be a toric symplectic manifold of dimension $2n$ and $M_i$ be a $T$-invariant divisor of $M$. Then*

$$\tau_i|_{M_T \setminus (M_i)_T} = 0$$

**Proof** $\tau_i$ is the image of 1 in the following composition map

$$H^0((M_i)_T) \xrightarrow{\phi} H^2(M_T, M_T \setminus (M_i)_T) \to H^2(M_T),$$

where $\phi$ is Thom isomorphism. Therefore by the following exact sequence

$$H^2(M_T, M_T \setminus (M_i)_T) \to H^2(M_T) \to H^2(M_T \setminus (M_i)_T)$$

we have

$$\tau_i|_{M_T \setminus (M_i)_T} = 0.$$

Let $Z$ be a connected folding hypersurface of an oriented toric origami manifold $(M, \omega, T, \mu)$. Let $M_{i_j}$ be the characteristic submanifolds of $M$ such that $M_{i_j} \cap Z \neq \emptyset$ where $1 \leq i_j \leq m$ and $1 \leq j \leq k$. We denote $\tau_{i_j}|_{\mathcal{U}_T}$ by $\xi_j$.

**Proposition 4.1** *Let $(M, \omega, T, \mu)$ be an oriented toric origami manifold with a connected folding hypersurface $Z$. Then*

$$c^T(\mathcal{U}) = (1 + \xi_1) \cdots (1 + \xi_k),$$

*where $\mathcal{U}$ is an equivariant Moser model as in Theorem* 2.1.

**Proof** Let $\overline{M}^+ = \{x \in M \mid \omega^n(x) \geq 0\}$ and $\overline{M}^- = \{x \in M \mid \omega^n(x) \leq 0\}$. $\partial\overline{M}^+ = \partial\overline{M}^- = \{x \in M \mid \omega^n(x) = 0\} = Z$ which is a connected manifold. Therefore, $\overline{M}^+$ and $\overline{M}^-$ are connected manifolds. Hence $M \setminus Z$ has only two connected components. By [4], the orbit space $M/T$ is the union of two Delzant polytopes in $\mathbb{R}^n$ along a neighborhood of a facet which corresponds to the orbit space $Z/T$. Therefore, each face of $M/T$ is acyclic.

Since each face of $M/T$ is acyclic, $H_T^*(M)$ is a torsion free $H^*(BT)$-module by [14, Lemma 2.1 and Theorem 8.3]. Hence we have

$$c^T(M) = \prod_{i=1}^{m}(1 + \tau_i) \tag{4.1}$$

by [12, Theorem 3.1], where $m$ and $\tau_i$ are defined at the beginning of this section. Since $c^T(\mathcal{U}) = c^T(M)|_{\mathcal{U}_T}$, we have

$$c^T(\mathcal{U}) = (1 + \xi_1) \cdots (1 + \xi_k)$$

by Lemma 4.1 and (4.1).

### 4.2 Chern classes of $TM \oplus \underline{\mathbb{R}}^2$

**Lemma 4.2** *Let $M$ be an oriented toric origami manifold with a folding hypersurface $Z$. Then $H_T^*(Z)$ vanishes in odd degrees.*

**Proof** Since $T$ is a connected Lie group, each connected component $Z_i, i = 1, 2, \cdots, h$, of the folding hypersurface $Z$ is $T$-invariant. Therefore, we only need to prove the statement for every connected component $Z_i$ of $Z$. By [4, Theorem 3.2 (b)], the null fibration of $Z_i$ is a subgroup of $T$. We denote this subgroup of $T$ by $T_i$ which is isomorphic to $S^1$. Hence $H_T^*(Z_i) = H_{T/T_i}^*(Z_i/T_i)$. Since $Z/T_i$ is a toric symplectic manifold with $T/T_i$ action, $H_T^*(Z_i)$ vanishes in odd degrees. Hence $H_T^*(Z) = \overset{h}{\underset{i=1}{\oplus}} H_T^*(Z_i)$ vanishes in odd degrees.

**Theorem 4.1** *Let $(M, \omega, T, \mu)$ be a toric origami manifold with a $T$-invariant unitary structure $J$. Then*

$$c^T(M) = (1 + \tau_1) \cdots (1 + \tau_m),$$

*where $m$ and $\tau_i$ are defined at the beginning of this section.*

**Proof** Let $Z$ be the folding hypesurface of $M$ and $Z = \bigsqcup_{i=1}^{h} Z_h$, where $Z_i, 1 \le i \le h$ are connected components of $Z$ and $\mathcal{U}_i$ are an equivariant Moser model of $Z_i$ as in Theorem 2.1. Set $M \setminus Z = \bigsqcup_{i=1}^{k} N_i$, where $N_i, 1 \le i \le k$ are connected components of $M \setminus Z$ and $N_i$ is an open toric symplectic manifold. Set $d = (1 + \tau_1) \cdots (1 + \tau_m)$. Then

$$d|_{(N_i)_T} = c^T(N_i) \tag{4.2}$$

by the proof of Lemma 4.1 and

$$d|_{\mathcal{U}_i} = c^T(\mathcal{U}_i) \tag{4.3}$$

by Proposition 4.1. Note that

$$\Big(\bigsqcup_{i=1}^{k} N_i\Big) \cap \Big(\bigsqcup_{j=1}^{h} \mathcal{U}_j\Big) = \bigsqcup_{i=1}^{k}\bigsqcup_{j=1}^{h} (N_i \cap \mathcal{U}_j)$$

and $N_i \cap \mathcal{U}_j$ is either an emptyset or $T$-equivariant homotopic to $Z_j$ for some $1 \le j \le h$. Therefore $H_T^{\rm odd}\big(\big(\bigsqcup_{i=1}^{k} N_i\big) \cap \big(\bigsqcup_{j=1}^{h} \mathcal{U}_j\big)\big) = 0$ by Lemma 4.2. Consider the Mayer-Vietoris exact sequence in cohomology for the triple $\big(M, \bigsqcup_{i=1}^{k} N_i, \bigsqcup_{j=1}^{h} \mathcal{U}_j\big)$, then we have the following exact sequence:

$$0 \to H_T^{2j}(M) \to H_T^{2j}\Big(\bigsqcup_{i=1}^{k} N_i\Big) \oplus H_T^{2j}\Big(\bigsqcup_{i=1}^{h} \mathcal{U}_i\Big) \to H_T^{2j}\Big(\Big(\bigsqcup_{i=1}^{k} N_i\Big) \cap \Big(\bigsqcup_{j=1}^{h} \mathcal{U}_j\Big)\Big).$$

Hence the map

$$H_T^{2j}(M) \to H_T^{2j}\Big(\bigsqcup_{i=1}^{k} N_i\Big) \oplus H_T^{2j}\Big(\bigsqcup_{i=1}^{h} \mathcal{U}_i\Big). \tag{4.4}$$

is injective. Note that $c^T(M)|_{N_i} = c^T(N_i)$ and $c^T(M)|_{\mathcal{U}_i} = c^T(\mathcal{U}_i)$, so we obtain that

$$c^T(M) = d = (1 + \tau_1)\cdots(1 + \tau_m)$$

by (4.2)–(4.4).

## 5 Splitting of $TM \oplus \underline{\mathbb{R}}^2$

It is well-known that for a $2n$-dimensional toric symplectic manifold $M$, topologically $TM \oplus \underline{\mathbb{R}}^2$ splits as a direct sum of complex line bundles (see [6, Theorem 8.1.6]). Namely we have

$$TM \oplus \underline{\mathbb{C}}^{m-n} \cong L_1 \oplus L_2 \oplus \cdots \oplus L_m,$$

where $m$ is the number of facets of the Delzant polytope associated to $M$ and $L_i$ is the complex line bundle corresponding to the $i$-th facet of the Delzant polytope. In this section, we generalize this result to oriented toric origami manifolds.

**Lemma 5.1** *Let $M$ be an oriented toric origami manifold, then the equivariant cohomology group $H_T^{\text{even}}(M)$ is torsion free as a $\mathbb{Z}$-module.*

**Proof** See the appendix.

**Proposition 5.1** *Let $M$ be a $2n$-dimensional oriented toric origami manifold and $E, F$ be $T$-equivariant complex vector bundles over $M$ with the same rank. If the rank of $E$ and $F$ is greater than $n$ and $c^T(E) = c^T(F)$, then $E \cong F$ as complex vector bundles.*

**Proof** We denote the $h$-skeleton of $M_T$ by $(M_T)^h$. Hence for each $h \geq 0$, $(M_T)^h$ is a finite CW-complex. By [8, Lemma 2.34], $H_i((M_T)^{2n}) = H_i(M_T)$ for $0 \leq i \leq 2n-1$, so $H^{\text{even}}((M_T)^{2n})$ is torsion free by the universal coefficient theorem and Lemma 5.1.

By the Atiyah-Hirzebruch spectral sequence in [1], we have

$$E_2^{m,-m} = H^m((M_T)^{2n}, K^{-m}(*)).$$

Since

$$K^{-m}(*) = \begin{cases} 0, & \text{if } m = 2k+1, \\ \mathbb{Z}, & \text{if } m = 2k, \end{cases}$$

we have

$$E_2^{m,-m} = \begin{cases} 0, & \text{if } m = 2k+1, \\ H^{2k}((M_T)^{2n}; \mathbb{Z}), & \text{if } m = 2k. \end{cases}$$

Therefore,

$$\{E_2^{m,-m} = H^m((M_T)^{2n}, K^{-m}(*))\}$$

are free abelian groups in the second page of the Atiyah-Hirzebruch spectral sequence. Since the order of the differential of Atiyah-Hirzebruch spectral sequence is finite,

$$\{E_\infty^{m,-m}\}$$

are also free abelian groups. Therefore, the complex K-group $K((M_T)^{2n})$ is a free abelian groups by the trivial extension.

Since $c(E_T)=c(F_T)$, we have $c(E_T|_{(M_T)^{2n}})=c(F_T|_{(M_T)^{2n}})$ which implies that $ch(E_T|_{(M_T)^{2n}}) = ch(F_T|_{(M_T)^{2n}})$. By the freeness of $K((M_T)^{2n})$, we have

$$[E_T|_{(M_T)^{2n}}] = [F_T|_{(M_T)^{2n}}] \quad \text{in } K((M_T)^{2n}).$$

Since the map $i : M \to M_T$ is homotopy equivalent to a CW map $f : M \to (M_T)^{2n}$, we have

$$[E] = [f^*(E_T|_{(M_T)^{2n}})] = [f^*(F_T|_{(M_T)^{2n}})] = [F] \quad \text{in } K(M).$$

By the condition that the rank of $E$ and $F$ are greater than $n$, we have

$$E \cong F$$

as complex vector bundles.

Let $J$ be a $T$-invariant unitary structure on $M$ given by Section 3. Let $L_i$ be a $T$-equivariant line bundle over $M$ such that $\tau_i = c_1^T(L_i)$ and this can be achieved by [9].

**Theorem 5.1** *Let $(M, \omega, T, \mu)$ be a toric origami manifold with the $T$-invariant unitary structure.*

(1) *If $m = n$, then $TM \oplus \underline{\mathbb{R}}^2$ is isomorphic to $L_1 \oplus \cdots \oplus L_m \oplus \underline{\mathbb{C}}$ as complex vector bundles.*

(2) *If $m > n$, then $TM \oplus \underline{\mathbb{R}}^2 \oplus \underline{\mathbb{C}}^{m-n-1}$ is isomorphic to $L_1 \oplus \cdots \oplus L_m$.*

**Proof** Since $c^T(L_i) = \tau_i$, we have

$$c^T(TM \oplus \underline{\mathbb{R}}^2) = c^T(L_1) \cdot c^T(L_2) \cdots c^T(L_m)$$

by Theorem 4.1. Therefore this theorem follows from Proposition 5.1.

**Remark 5.1** For an oriented toric origami manifold $M$ of dimension $2n$ which is not homeomorphic to $T^2$, the number of facets of $M/T$ is at least $n$, so $m \geq n$. When $M$ is homeomorphic to $T^2$, then $TM \oplus \underline{\mathbb{R}}^2$ is isomorphic to a trivial complex vector bundle.

**Example 5.1** Consider an origami template consisting of two copies of four gons in the following figure. The corresponding toric origami manifold $M$ is equivariant homeomorphic to $L(k; 1) \times S^1$, where $L(k; 1)$ is the lens space which is the orbit space $S^3/\mathbb{Z}_k$ of the unit sphere $S^3 \subset \mathbb{C}^2$ with the action of $\mathbb{Z}_k$ generated by the rotation $\alpha \cdot (z_1, z_2) = (\mathrm{e}^{\frac{2\pi \mathrm{i}}{k}} z_1, \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}} z_2)$. Then

$$TM \oplus \underline{\mathbb{R}}^2 \cong L_1 \oplus L_2 \oplus \underline{\mathbb{C}}.$$

## 6 Spin Toric Origami Manifolds

If $M$ is an oriented toric origami manifold with fixed points, we can give a necessary and sufficient condition for the existence of a spin structure on $M$. It is well-known that $M$ is spin if and only if $w_2(M) = 0$, where $w_2$ is the second Stiefel-Whitney class of $M$.

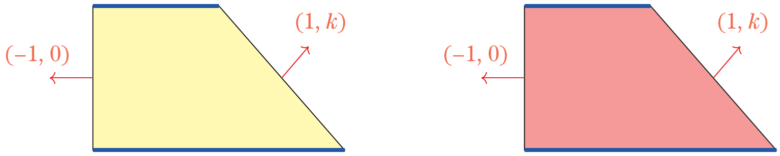


Figure 1 The origami template with two polygons

Consider the following commutative diagram

$$\begin{array}{ccc} ET\times_T M^\circ & \xrightarrow{i} & ET\times_T M \\ {\scriptstyle p^\circ}\downarrow & & \downarrow{\scriptstyle p} \\ M^\circ/T & \xrightarrow{\bar{i}} & M/T, \end{array}$$

where $i$ and $\bar{i}$ are the inclusions, $M^\circ$ is the $T$-free part of $M$, and $p^\circ$ and $p$ are the projection maps. Since the $T$-action on $M^\circ$ is free, $p^\circ : ET\times_T M^\circ \to M^\circ/T$ is a homotopy equivalence. Since $\bar{i} : M^\circ/T \to M/T$ is a homotopy equivalence, we have the surjection

$$i^* : H_T^*(M) \to H_T^*(M^\circ) \tag{6.1}$$

by the following commutative diagram

$$\begin{array}{ccc} H^*(ET\times_T M^\circ) & \xleftarrow{i^*} & H^*(ET\times_T M) \\ {\scriptstyle (p^\circ)^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ H^*(M^\circ/T) & \xleftarrow{(\bar{i})^*} & H^*(M/T). \end{array}$$

**Proposition 6.1** *$\tau_1, \tau_2, \cdots, \tau_m$ is a $\mathbb{Z}$-basis of $H_T^2(M;\mathbb{Z})$.*

This proposition follows from [15, Theorem 1.7], but in the following we give a direct proof of this proposition.

**Proof** Let $M_1, \cdots, M_m$ be all the characteristic submanifolds of $M$ and

$$X := M \setminus \bigcup_{i\neq j} M_i \cap M_j.$$

Set $M_i^\circ := M_i \cap X$, then $M_i^\circ \cap M_j^\circ = \emptyset$ for $i \neq j$. Since $M_i \cap M_j = \emptyset$ or it is a codimension 4 submanifold of $M$, $H_T^2(M) \xrightarrow{\iota^*} H_T^2(X)$ is an isomorphism, where $\iota : X \to M$ is the inclusion map. Hence it suffices to show that $\tau_1, \cdots, \tau_m$ is a $\mathbb{Z}$-basis of $H_T^2(X;\mathbb{Z})$, where $\tau_i$ means $\tau_i|_{X_T}$ by abusing notations.

Consider the following exact sequence

$$\begin{aligned} &\to \cdots \\ &\to H_T^1(X, M^\circ) \to H_T^1(X) \to H_T^1(M^\circ) \\ &\to H_T^2(X, M^\circ) \to H_T^2(X) \to H_T^2(M^\circ) \\ &\to \cdots . \end{aligned} \tag{6.2}$$

Since (6.1) is surjection, $H_T^*(X) \to H_T^*(M^\circ)$ is also surjection by the composition map

$$H_T^*(M) \to H_T^*(X) \to H_T^*(M^\circ).$$

Hence (6.2) splits into short exact sequence

$$0 \to H_T^2(X, M^\circ) \to H_T^2(X) \to H_T^2(M^\circ) \to 0.$$

Since $H_T^2(M^\circ) = H^2(M^\circ/T)$ and $M^\circ$ is homotopy equivalent to a wedge of circles, $H_T^2(M^\circ) = 0$. Therefore the restriction map

$$H_T^2(X, M^\circ) \to H_T^2(X)$$

is an isomorphism. Note that $M^\circ = X \setminus \bigcup_{i=1}^{m} M_i^\circ$ and $M_i^\circ \cap M_j^\circ = \emptyset$, so $H_T^2(X, M^\circ) \cong \bigoplus_{i=1}^{m} H_T^2(X, X \setminus M_i^\circ)$ by relative Mayer-Vietoris sequence (see [8, pp.204]). Hence

$$\bigoplus_{i=1}^{m} H_T^2(X, X \setminus M_i^\circ) \cong H_T^2(X)$$

by the restriction maps. Since $\tau_i$ is the image of 1 in the following composition map

$$H_T^0(M_i^\circ) \xrightarrow{\phi} H_T^2(X, X \setminus M_i^\circ) \to H_T^2(X),$$

where $\phi$ is Thom isomorphism, this implies that $\tau_1, \tau_2, \cdots, \tau_m$ is a $\mathbb{Z}$ basis of $H_T^2(X)$. This completes the proof of the proposition.

**Remark 6.1** The above argument holds for any coefficient.

The following proposition also holds for oriented toric origami manifold by the proof of [13, Proposition 2.2] and Proposition 6.1.

**Corollary 6.1** (see [13, Proposition 2.2]) *To each $i \in [m]$, there is a unique element $v_i \in H_2(BT)$ such that*

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT)$$

*where $\langle , \rangle$ is the natural pairing between cohomology and homology for any coefficients.*

**Corollary 6.2** *If $M$ is an oriented toric origami manifold with fixed point, then $w_2(M) = 0$ if and only if $\tau_1 + \cdots + \tau_m \in \mathrm{im}(\pi^*)$, where $\pi^* : H^*(BT; \mathbb{Z}_2) \to H_T^*(M; \mathbb{Z}_2)$ induced by the projection $\pi : M_T \to BT$.*

**Proof** Let $p \in M^T$, then we have a $T$-equivariant map $s : p \to M$, and $s^* : H_T^*(M) \to H^*(BT)$. Note that $\pi^* \circ s^* = \mathrm{id}$, so $\pi^* : H^*(BT) \to H_T^*(M)$ is injective. Hence we have the following short exact sequence

$$0 \to H^2(BT; \mathbb{Z}_2) \xrightarrow{\pi^*} H_T^2(M; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2). \tag{6.3}$$

Since $w_2(M) = w_2^T(M)|_M$, $w_2^T(M)|_M = 0$ if and only if $\tau_1 + \cdots + \tau_m \in \mathrm{im}(\pi^*)$ by (6.3).

**Example 6.1** Let $(M, \omega, T, \mu)$ be a toric origami and the following Figure 2 is the associated origami template, whose resulting space is homeomorphic to $M/T$ as a manifold with corners. The outward normal vectors of vertical facets from left to right are $v_1 = (-1, 0), v_2 = (1, 0), v_3 = (-1, 0), v_4 = (1, 0)$ and outward normal vectors of horizontal facets from up to down are $v_5 = (0, 1), v_6 = (0, -1), v_7 = (0, 1), v_8 = (0, -1)$. Therefore we have

$$\tau_1 - \tau_2 + \tau_3 - \tau_4 \in \mathrm{im}(\pi^*), \quad \tau_5 - \tau_6 + \tau_7 - \tau_8 \in \mathrm{im}(\pi^*)$$

by Corollary 6.1. Hence $\tau_1 + \tau_2 + \cdots + \tau_8 \in \mathrm{im}(\pi^*)$ by taking $\mathbb{Z}_2$ coefficient. This implies that $M$ is a spin manifold by Corollary 6.2.

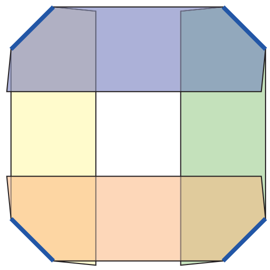

Figure 2 The origami template with four polygons

## 7 Appendix

In this appendix, we show Lemma 5.1 by applying the arguments in [2, Section 3] to $H_T^*(M)$.

Let $M$ be an orientable toric origami manifold of dimension $2n$ with a fold $Z$. Let $F$ be the corresponding folded facet in the origami template of $M$ and let $B$ be the toric symplectic manifold corresponding to $F$. Since the normal line bundle of $Z$ to $M$ is trivial, the equivariant neighborhood $\mathcal{U}$ of $Z$ in Theorem 2.1 can be identified with $Z \times (-\varepsilon, \varepsilon)$. We define

$$\widetilde{M} := M \setminus \mathcal{U}.$$

Therefore $\widetilde{M}$ is a manifold with boundary $Z \sqcup Z$. We close $\widetilde{M}$ by gluing two copies of the $T$-invariant disk bundle associated to the principal $S^1$-bundle $Z \to B$ along their boundaries. The resulting closed manifold (possibly disconnected), denoted by $M'$, is again a toric origami manifold by [4]. The graph $G'$ associated to $M'$ is obtained by removing an edge $e$ corresponding to the folded facet $F$ in $G$. We assume that $b_1(G) \geq 1$, where $b_1(G)$ is the first Betti number of $G$, and we choose the edge $e$ in a (non-trivial) cycle of $G$ then $M'$ is connected and $b_1(G') = b_1(G) - 1$.

Two copies of $B$ lie in $M'$ as closed submanifolds, denoted by $B_+$ and $B_-$. Let $N_+$ (resp. $N_-$) be a $T$-invariant closed tubular neighborhood of $B_+$ (resp. $B_-$) and $Z_+$ (resp. $Z_-$) be the boundary of $N_+$ (resp. $N_-$). Note that $M' \setminus \mathbf{Int}(N_+ \cup N_-)$ can naturally be identified with $\widetilde{M}$, so that

$$\widetilde{M} = M' \setminus \mathbf{Int}(N_+ \cup N_-) = M \setminus \mathbf{Int}(Z \times [-\varepsilon, \varepsilon])$$

and

$$M' = \widetilde{M} \cup (N_+ \cup N_-), \quad \widetilde{M} \cap (N_+ \cup N_-) = Z_+ \cup Z_-, \tag{7.1}$$

$$M = \widetilde{M} \cup (Z \times [-\varepsilon, \varepsilon]), \quad \widetilde{M} \cap (Z \times [-\varepsilon, \varepsilon]) = Z_+ \cup Z_-. \tag{7.2}$$

**Proof** (Proof of Lemma 5.1) We prove this lemma by induction on $b_1(G)$. If $b_1(G) = 0$, then the origami template is acyclic. So each face of $M/T$ is acyclic by [10, Theorem 5.3]. Hence $H_T^{\text{even}}(M)$ is torsion free by [14, Theorem 8.3] or [10, Theorem 4.3]. We assume by induction that the lemma holds when $b_1(G) \leq p$ for some nonnegtive integer $p$ and we prove the lemma when $b_1(G) = p + 1$.

We consider the Mayer-Vietoris exact sequence in equivariant cohomology for the triple $(M', \widetilde{M}, N_+ \cup N_-)$ :

$$\begin{aligned} &\to H_T^{2i-2}(M') \to H_T^{2i-2}(\widetilde{M}) \oplus H_T^{2i-2}(N_+ \cup N_-) \to H_T^{2i-2}(Z_+ \cup Z_-) \\ &\xrightarrow{\delta^{2i-2}} H_T^{2i-1}(M') \to H_T^{2i-1}(\widetilde{M}) \oplus H_T^{2i-1}(N_+ \cup N_-) \to H_T^{2i-1}(Z_+ \cup Z_-). \end{aligned}$$

Note that $H_T^*(Z_+) \cong H_{T/S^1}^*(B_+)$ and $H_T^*(N_+) \cong H_T^*(B_+)$. Hence $H_T^*(Z_+)$ is torsion free and $H_T^{2i-2}(N_+ \cup N_-) \to H_T^{2i-2}(Z_+ \cup Z_-)$ is surjective. Since $H_{T/S^1}^{\text{odd}}(B_+) = 0$ we have the following short exact sequence:

$$0 \to H_{T^n}^{2i-2}(M') \to H_{T^n}^{2i-2}(\widetilde{M}) \oplus H_{T^n}^{2i-2}(N_+ \cup N_-) \to H_{T^n}^{2i-2}(Z_+ \cup Z_-) \to 0. \tag{7.3}$$

Since $b_1(G') = b_1(G) - 1 = p$, $H_T^{\text{even}}(M')$ is torsion free by induction hypothesis. Due to $H_T^{\text{even}}(Z_+ \cup Z_-)$ is also torsion free, $H_T^{\text{even}}(\widetilde{M})$ is torsion free by (7.3).

Next we consider the Mayer-Vietoris exact sequence in equivariant cohomology for the triple $(M, \widetilde{M}, \mathcal{U})$:

$$\begin{aligned} &\to H_T^{2i-3}(M) \to H_T^{2i-3}(\widetilde{M}) \oplus H_T^{2i-3}(\mathcal{U}) \to H_T^{2i-3}(Z_+ \cup Z_-) \\ &\to H_T^{2i-2}(M) \to H_T^{2i-2}(\widetilde{M}) \oplus H_T^{2i-2}(\mathcal{U}) \to H_T^{2i-2}(Z_+ \cup Z_-). \end{aligned}$$

Since $H_T^{\text{odd}}(Z_+ \cup Z_-) = 0$ by Lemma 4.2, $H_T^{2i-2}(M)$ is a subgroup of $H_T^{2i-2}(\widetilde{M}) \oplus H_T^{2i-2}(\mathcal{U})$. Since $H_T^*(\mathcal{U}) \cong H_T^*(Z)$, $H_T^{\text{even}}(\mathcal{U})$ is torsion free. Therefore $H_T^{\text{even}}(M)$ is torsion free.

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## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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