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# A Generalized Blow up Criteria with One Component of Velocity for 3D Incompressible MHD System*

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**Abstract** In this paper, the authors study the global regularity of the 3D magnetohydrodynamics system in terms of one velocity component. In particular, they establish a new Prodi-Serrin type regularity criterion in the framework of weak Lebesgue spaces both in time and space variables.

**Keywords** MHD, Regularity, Blow-up criterion
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## 1 Introduction

In this work, we investigate the Cauchy problem of the following incompressible magnetohydrodynamics (MHD for short) system on $\mathbb{R}^3$,

$$\begin{cases}\partial_t v + v\cdot\nabla v - b\cdot\nabla b + \nabla p = \Delta v,\\ \partial_t b + v\cdot\nabla b - b\cdot\nabla v = \Delta b,\\ \operatorname{div} v = \operatorname{div} b = 0,\\ v|_{t=0} = v_0,\quad b|_{t=0} = b_0,\end{cases} \tag{1.1}$$

where $b = (b^1, b^2, b^3)$, $v = (v^1, v^2, v^3)$ and $p$ denote the magnetic fields, velocity fields and scalar pressure of fluid, respectively. $(v_0, b_0)$ is the prescribed initial data which satisfy $\operatorname{div} v_0 = \operatorname{div} b_0 = 0$ in the sense of distribution. Physically, system (1.1) governs the dynamics of velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals and salt water. Moreover, the first equation reflects the conservation of momentum, and the second one is the induction equation. Since universal physical laws should be independent of the underlying units (dimension), system (1.1) remains invariant under natural scaling transformations. Indeed, if $(v, b, p)$ is a solution of (1.1), then for any $\lambda > 0$,

$$v_\lambda(t,x) = \lambda v(\lambda^2 t, \lambda x),\quad b_\lambda(t,x) = \lambda b(\lambda^2 t, \lambda x),\quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

is also a solution corresponding to rescaled initial data $v_{0,\lambda}(x) = \lambda v_0(\lambda x), b_{0,\lambda}(x) = \lambda b_0(\lambda x)$. As the classical Navier-Stokes system, such a scaling transformation determines the critical space (norm) for MHD system and plays a fundamental role in the well-posedness theory.

It is obvious that (1.1) is the system with full viscosity and diffusion, and it is globally well-posed in two dimension. In the general case $\mathbb{R}^d$, Duvaut and Lions in [8] established the local

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existence and uniqueness of solution in the classical Sobolev space $H^s(\mathbb{R}^d)$, $s \geq d$; they also proved the global existence of solutions to this system with small initial data. Abidi and Paicu [1] proved similar result as that in [8] for the so called inhomogeneous MHD system with initial data in the critical spaces. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [4] (see also [3]) proved that such a system is globally well-posed for any data in $H^2(\mathbb{R}^2)$. Chemin et al. [5] proved the local well-posedness of (1.1) with initial data in the critical Besov spaces.

On the other hand, there exists a vast literature on finite time blowup or non-blowup criteria for local strong solutions. By discovering some special structures of the nonlinear coupling terms with the magnetic field, the authors of [9, 14–18, 20] were able to provide us some regularity criteria involving lesser components of the velocity field and the magnetic field.

One can easily find out that, when $b$ is a constant, (1.1) is nothing but the classical Navier-Stokes equations. Most of the regularity and uniqueness criteria of Navier-Stokes equations can be extended to MHD equations. It is also well known that system (1.1) has a local weak solution. Similarly, we can impose additional conditions on the weak solution of the MHD equations to obtain its global regularity. Motivated by [6], Yamazaki in [19] established the following blow up criteria for MHD system: Let $(v, b)$ be the solution of 3D incompressible MHD system on $[0, T^*)$ and if

$$\int_0^{T^*} (\|v \cdot e\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b\|^{r_1}_{L^{p_1}} + \|\nabla b\|^{r_2}_{L^{p_2}})\mathrm{d}t < \infty \tag{1.2}$$

holds for $e \in \mathcal{S}^2$, $p \in (4, 6)$ and $p_1 > 9$, $p_2 > \frac{9}{2}$, then $T^* = \infty$. Later, corresponding to [7], Liu successfully extended $p$ in [19] to $(4, \infty)$ and also got rid of the terms $\|b\|^{r_1}_{L^{p_1}} + \|\nabla b\|^{r_2}_{L^{p_2}}$ in (1.2). In other words, he proved that if for some $p \in (4, \infty)$ there holds

$$\int_0^{T^*} (\|v \cdot e\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}})\mathrm{d}t < \infty,$$

then $T^* = \infty$. The general case $p \in [2, \infty)$ was proved by the authors in [11]. While in [10], the authors proved an improved blow up criterion that concerns with the $\dot{H}^{\frac{1}{2}+\frac{2}{p}}$ norm of $(v \cdot e)$ and the Lebesgue norm of $b$ for all $2 \leq p < \infty$.

Recently, an important class of regularity criteria was proved in [12], which was concerned with Lebesgue norm of one component of the velocity. More precisely, the authors proved that if the initial value $(v_0,\ b_0) \in H^3(\mathbb{R}^3)$ with $s \geq 3$, and $(v, b)$ satisfies

$$\begin{cases} b,\ v^3 \in L^{p_1}(0, T; L^{q_1}(\mathbb{R}^3)), & \dfrac{2}{p_1} + \dfrac{3}{q_1} \leq 1,\ 3 < q_1 \leq \infty, \\ \partial_3 b,\ \partial_3 v \in L^{p_2}(0, T; L^{q_2}(\mathbb{R}^3)), & \dfrac{2}{p_2} + \dfrac{3}{q_2} \leq 2,\ \dfrac{3}{2} < q_2 \leq \infty, \end{cases}$$

then the weak solution of (1.1) is smooth on $[0, T]$. Motivated by [2], the main work of this paper is to prove a new Prodi-Serrin type regularity criterion for the MHD equations. More precisely, we refine the criterion to Lorentz space, and have the following theorem.

**Theorem 1.1** *Let* $(v, b)$ *be a weak solution of the* MHD *system with initial condition* $(v_0, b_0) \in L^2(\mathbb{R}^3)$. *Then* $(v, b)$ *is smooth beyond* $T (T > 0)$ *if*

$$\|v^3\|_{L^{p,\infty}(0,T;L^{s,\infty}(\mathbb{R}^3))} + \|b\|_{L^{p,\infty}(0,T;L^{s,\infty}(\mathbb{R}^3))} < \infty$$

*with* $s > \frac{10}{3}$ *and*

$$\frac{2}{p} + \frac{3}{s} \leq \frac{3}{4} + \frac{1}{2s}.$$

**Notations** For any $1 \leq p \leq \infty$ and measurable $f : \mathbb{R}^n \to \mathbb{R}$, we will use $\|f\|_{L^p(\mathbb{R}^n)}$, $\|f\|_{L^p}$ or simply $\|f\|_p$ to denote the usual $L^p$ norm. For a vector valued function $f = (f^1, \cdots, f^m)$, we still denote $\|f\|_p := \sum\limits_{j=1}^{m} \|f^j\|_p$. We denote by $L^q_T(L^p_h(L^s_v))$ the space $L^q((0,T); L^p_h(\mathbb{R}^2_{x_h}; (L^s(\mathbb{R}_{x_3}))))$ with $x_h = (x_1, x_2)$.

For any $0 < T < \infty$ and any Banach space $\mathbb{B}$ with norm $\|\cdot\|_{\mathbb{B}}$, we will use the notation $C([0,T], \mathbb{B})$ or $C^0_t\mathbb{B}$ to denote the space of continuous $\mathbb{B}$-valued functions endowed with the norm

$$\|f\|_{C([0,T],\mathbb{B})} := \max_{0 \leq t \leq T} \|f(t)\|_{\mathbb{B}}.$$

Also for $1 \leq p \leq \infty$, we define

$$\|f\|_{L^p_t\mathbb{B}([0,T])} := \|\|f(t)\|_{\mathbb{B}}\|_{L^p_t([0,T])}.$$

## 2 Preparations

Let us grasp basic definitions and properties of Lorentz spaces.

**Definition 2.1** (Lorentz Spaces) *Let $(X, \mu)$ be a measure space. For $0 < p < \infty$, $0 < q \leq \infty$, we define the Lorentz space $L^{p,q}(X, \mu)$ as collection of measurable function for which*

$$\|f\|_{L^{p,q}} := \begin{cases} p^{\frac{1}{q}} \Big( \displaystyle\int_0^\infty (d_f(s)^{\frac{1}{p}} s)^q \frac{\mathrm{d}s}{s} \Big)^{\frac{1}{q}}, & q < \infty, \\ \sup\limits_{s>0} s d_f(s)^{\frac{1}{p}}, & q = \infty \end{cases}$$

*is finite, where the distribution function $d_f(s)$ is defined as*

$$d_f(s) := \mu(\{x \in X : |f(x) > s|\}).$$

**Proposition 2.1** (Properties of Lorentz Spaces)

(a) $L^p(X, \mu) = L^{p,p}(X, \mu)$.

(b) $\|f^r\|_{L^{p,q}} = \|f\|^r_{L^{pr,qr}}$ *for* $0 < p,\ r < \infty$ *and* $0 < q \leq \infty$.

(c) $\|f\|_{L^{p,q_1}} \leq c\|f\|_{L^{p,q_2}}$ *for* $0 < p \leq \infty$ *and* $0 < q_2 \leq q_1 \leq \infty$. *In particular, there holds*

$$\|f\|_{L^{p,\infty}} \leq \Big(\frac{q}{p}\Big)^{\frac{1}{q}} \|f\|_{L^{p,q}}$$

*for* $1 < p < \infty$ *and* $1 \leq q < \infty$.

**Proposition 2.2** (cf. [13]) *Let $1 < p < \infty$, $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Further assume that $1 < p_1 < p'$, $q' \leq q \leq \infty$, $\frac{1}{p_2} + 1 = \frac{1}{p} + \frac{1}{p_1}$ and $\frac{1}{q_2} = \frac{1}{q} + \frac{1}{q_1}$. Then the convolution operator*

$$*: \ L^{p,q}(\mathbb{R}^d) \times L^{p_1,q_1}(\mathbb{R}^d) \mapsto L^{p_2,q_2}(\mathbb{R}^d)$$

*is a bounded bilinear operator.*

**Proposition 2.3** (cf. [2]) *Let $n \geq 2$, $1 \leq p < n$ and $q = \frac{np}{n-p}$. Then for any $f \in W^{1,p}(\mathbb{R}^n)$, we have*

$$\|f\|_{L^{q,p}} \leq C \prod_{k=1}^{n} \|\partial_k f\|_{L^p}^{\frac{1}{n}}.$$

**Proposition 2.4** (Hölder's Inequality in Lorentz Spaces) *If $0 < p_1,\ p_2,\ p < \infty$ and $0 < q_1,\ q_2,\ q \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|fg\|_{L^{p,q}} \leq C(p_1, p_2, q_1, q_2)\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},$$

*whenever the right-hand side norms are finite.*

**Lemma 2.1** *Let $T > 0$ and $\psi \in L^\infty_{\rm loc}([0,T))$ be non-negative. Further assume that*

$$\psi(t) \leq C_0 + C_1 \int_0^t \mu(s)\psi(s)\mathrm{d}s + \kappa \int_0^t \lambda(s)^{1-\varepsilon}\psi(s)^{1+A(\varepsilon)}\mathrm{d}s, \quad \forall\ 0 < \varepsilon < \varepsilon_0,$$

*where $\kappa,\ \varepsilon_0 > 0$ are constants, $\mu \in L^1(0,T)$, and $A(\varepsilon) > 0$ which satisfies $\lim\limits_{\varepsilon \searrow 0} \frac{A(\varepsilon)}{\varepsilon} = c_0 > 0$. Then $\psi$ stays bounded on $[0,T]$ if $\|\lambda\|_{L^{1,\infty}(0,T)} < c_0^{-1}\kappa^{-1}$.*

## 3 Proof of Theorem 1.1

By denoting $V = v + b$, $W = v - b$, and starting from system (1.1), we obtain

$$\begin{cases} \partial_t W + V \cdot \nabla W - \Delta W + \nabla p = 0, \\ \partial_t V + W \cdot \nabla V - \Delta V + \nabla p = 0, \\ \operatorname{div} W = \operatorname{div} V = 0. \end{cases} \tag{3.1}$$

Setting $\nabla_h = (\partial_1,\ \partial_2)$, and for some $t_0 \in (0,t)$ which will be specified later, we define

$$\begin{aligned} J_1(t) &:= 1 + \sup_{\tau \in (t_0,t)} \|\nabla_h W\|_{L^2}^2(\tau) + 2\int_{t_0}^t \|\nabla\nabla_h W\|_{L^2}^2\,\mathrm{d}\tau, \\ J_2(t) &:= 1 + \sup_{\tau \in (t_0,t)} \|\nabla_h V\|_{L^2}^2(\tau) + 2\int_{t_0}^t \|\nabla\nabla_h V\|_{L^2}^2\,\mathrm{d}\tau, \\ L_1(t) &:= 1 + \sup_{\tau \in (t_0,t)} \|\nabla W\|_{L^2}^2(\tau) + 2\int_{t_0}^t \|\nabla^2 W\|_{L^2}^2\,\mathrm{d}\tau, \\ L_2(t) &:= 1 + \sup_{\tau \in (t_0,t)} \|\nabla V\|_{L^2}^2(\tau) + 2\int_{t_0}^t \|\nabla^2 V\|_{L^2}^2\,\mathrm{d}\tau, \\ e_1(t_0) &:= \int_{t_0}^t \|\nabla W\|_{L^2}^2\,\mathrm{d}\tau, \quad e_2(t_0) := \int_{t_0}^t \|\nabla V\|_{L^2}^2\,\mathrm{d}\tau. \end{aligned}$$

Then we set $J = J_1 + J_2,\ L = L_1 + L_2$ *and* $e = e_1 + e_2$. It is easy to verify that $J(t), L(t)$ are non-decreasing, and satisfy $J(t) \leq L(t)$. $e(t_0) \leq e(0) < \infty$ and their exactly value can be made arbitrarily small through choosing $t_0$ close to $T$.

### 3.1 The relationship between $L(t)$ and $J(t)$

We will focus on the first equation of system (3.1), and deal with the second equation in a similar way. We multiply the first equation by $-\Delta W$ and integrate over $\mathbb{R}^3$. Then integration by parts implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W\|_2^2+\|\nabla^2 W\|_2^2=-\int_{\mathbb{R}^3}\partial_k V^i\partial_i W^j\partial_k W^j\,\mathrm{d}x,\tag{3.2}$$

where $i,j,k=1,2,3$.

When $k=1,2$ or $i=1,2$, we have

$$\int_{\mathbb{R}^3}\partial_k V^i\partial_i W^j\partial_k W^j\,\mathrm{d}x\le C\int_{\mathbb{R}^3}|\nabla_h V||\nabla W|^2\,\mathrm{d}x.$$

By using of $\operatorname{div} W=0$, we can gain the same estimation when $k=i=3$,

$$\int_{\mathbb{R}^3}\partial_3 V^3\partial_3 W^j\partial_3 W^j\,\mathrm{d}x=-\int_{\mathbb{R}^3}(\partial_1 V^1+\partial_2 V^2)(\partial_3 W^j)^2\,\mathrm{d}x\le C\int_{\mathbb{R}^3}|\nabla_h V||\nabla W|^2\,\mathrm{d}x.$$

Here and after, $C$ denotes a pure positive constant. Thus, we reach that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla W\|_2^2+\|\nabla^2 W\|_2^2\le C\int_{\mathbb{R}^3}|\nabla_h V||\nabla W|^2\,\mathrm{d}x.\tag{3.3}$$

Application of Hölder and Troisi inequalities now gives

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla_h V||\nabla W|^2\,\mathrm{d}x&\le\|\nabla_h V\|_2\|\nabla W\|_4^2\\&\le\|\nabla_h V\|_2\|\nabla W\|_2^{\frac12}\|\nabla W\|_6^{\frac32}\\&\le\|\nabla_h V\|_2\|\nabla W\|_2^{\frac12}\|\nabla_h\nabla W\|_2\|\nabla^2 W\|_2^{\frac12}.\end{aligned}\tag{3.4}$$

Performing the time integration over the interval $[t_0,t]$ for any $t>t_0$ for (3.3), one has

$$\begin{aligned}&\|\nabla W(t)\|_2^2-\|\nabla W(t_0)\|_2^2+2\int_{t_0}^t\|\nabla^2 W\|_2^2\,\mathrm{d}\tau\\&\le C\sup_{(t_0,t)}\|\nabla_h V\|_2\int_{t_0}^t\|\nabla W\|_2^{\frac12}\|\nabla_h\nabla W\|_2\|\nabla^2 W\|_2^{\frac12}\,\mathrm{d}\tau\\&\le CJ_2(t)^{\frac12}\Big(\int_{t_0}^t\|\nabla W\|_2^2\,\mathrm{d}\tau\Big)^{\frac14}\Big(\int_{t_0}^t\|\nabla_h\nabla W\|_2^2\,\mathrm{d}\tau\Big)^{\frac12}\Big(\int_{t_0}^t\|\nabla^2 W\|_2^2\,\mathrm{d}\tau\Big)^{\frac14}\\&\le Ce_1(t_0)^{\frac14}J_1(t)^{\frac12}J_2(t)^{\frac12}L_1(t)^{\frac14}.\end{aligned}\tag{3.5}$$

By the definition of $L_1(t)$, we can obtain

$$L_1(t)\le C(t_0)+Ce_1(t_0)^{\frac14}J_1(t)^{\frac12}J_2(t)^{\frac12}L_1(t)^{\frac14}.$$

Application of Young's inequality now gives

$$L_1(t)\le C(t_0)+Ce_1(t_0)^{\frac13}J_1(t)^{\frac23}J_2(t)^{\frac23}.$$

Similarly, the operation on the section equation of system (3.1) implies that

$$L_2(t)\le C(t_0)+Ce_2(t_0)^{\frac13}J_1(t)^{\frac23}J_2(t)^{\frac23}.$$

Thus, we have

$$L(t)\le C(t_0)+Ce(t_0)^{\frac13}J(t)^{\frac43}.$$

In what follows, we are going to prove the uniform boundedness of $L(t)$ on $(t_0,T)$.

### 3.2 $H^1$ energy estimate

We set

$$r := \frac{8s}{3s-10}$$

which satisfies $\frac{2}{r} + \frac{3}{s} = \frac{3}{4} + \frac{1}{2s}$.

We multiply the first equation in system (3.1) by $-\Delta_h W$ and integrate over $\mathbb{R}^3$. Then integration by parts implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h W\|_2^2 + \|\nabla\nabla_h W\|_2^2 = -\int_{\mathbb{R}_3} \partial_k V^i \partial_i W^j \partial_k W^j \,\mathrm{d}x,$$

where $i, j = 1, 2, 3$; $k = 1, 2$. The right hand side of the above equation can be decomposed into four parts.

**Case I** $j = 3$; $i = 1, 2, 3$; $k = 1, 2$. We integrate by parts to reach

$$\begin{aligned} J_{11} :\triangleq & -\int_{\mathbb{R}^3} \partial_k V^i \partial_i W^3 \partial_k W^3 \,\mathrm{d}x \\ \leq & \int_{\mathbb{R}^3} |W^3 \partial_{kk} V^i \partial_i W^3| \,\mathrm{d}x + \int_{\mathbb{R}^3} |W^3 \partial_k V^i \partial_{ik} W^3| \,\mathrm{d}x \\ \leq & \, C\int_{\mathbb{R}^3} |W^3||\nabla W||\nabla\nabla_h V| \,\mathrm{d}x + \int_{\mathbb{R}^3} |W^3||\nabla V||\nabla\nabla_h W| \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

**Case II** $i = 3$; $j, k = 1, 2$. Similarly, integration by parts gives

$$\begin{aligned} J_{12} :\triangleq & -\int_{\mathbb{R}^3} \partial_k V^3 \partial_3 W^j \partial_k W^j \,\mathrm{d}x \\ \leq & \int_{\mathbb{R}^3} |V^3 \partial_{3k} W^j \partial_k W^j| \,\mathrm{d}x + \int_{\mathbb{R}^3} |V^3 \partial_3 W^j \partial_{kk} W^j| \,\mathrm{d}x \\ \leq & \, C\int_{\mathbb{R}^3} |V^3||\nabla W||\nabla\nabla_h W| \,\mathrm{d}x. \end{aligned} \tag{3.7}$$

**Case III** $i = j = k = 1$ and $i = j = k = 2$.

$$\begin{aligned} J_{13} :\triangleq & -\int_{\mathbb{R}^3} \partial_1 V^1 \partial_1 W^1 \partial_1 W^1 \,\mathrm{d}x - \int_{\mathbb{R}^3} \partial_2 V^2 \partial_2 W^2 \partial_2 W^2 \,\mathrm{d}x \\ = & \int_{\mathbb{R}^3} \partial_1 V^1 \partial_1 W^1 \partial_2 W^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_1 V^1 \partial_2 W^2 \partial_2 W^2 \,\mathrm{d}x \\ & - \int_{\mathbb{R}^3} \partial_1 V^1 \partial_2 W^2 \partial_2 W^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \partial_2 V^2 \partial_2 W^2 \partial_2 W^2 \,\mathrm{d}x \\ & + \int_{\mathbb{R}^3} \partial_1 V^1 \partial_1 W^1 \partial_3 W^3 \,\mathrm{d}x \\ = & -\int_{\mathbb{R}^3} \partial_1 V^1 \partial_2 W^2 \partial_3 W^3 \,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_3 V^3 \partial_2 W^2 \partial_2 W^2 \,\mathrm{d}x \\ & + \int_{\mathbb{R}^3} \partial_1 V^1 \partial_1 W^1 \partial_3 W^3 \,\mathrm{d}x \\ \leq & \, C\int_{\mathbb{R}^3} |W^3||\nabla W||\nabla\nabla_h V| \,\mathrm{d}x + C\int_{\mathbb{R}^3} |V^3||\nabla W||\nabla\nabla_h W| \,\mathrm{d}x. \end{aligned} \tag{3.8}$$

**Case IV** The fourth part is for the remaining items:

$$J_{14} :\triangleq -\int_{\mathbb{R}^3} \partial_k V^i \partial_i W^j \partial_k W^j \,\mathrm{d}x = -\int_{\mathbb{R}^3} \partial_k (v^i + b^i) \partial_i (v^j - b^j) \partial_k (v^j - b^j) \,\mathrm{d}x.$$

Thus, the terms in $J_{14}$ can be estimated by

$$\begin{aligned}
&\int_{\mathbb{R}^3} \partial_k v^i \partial_i v^j \partial_k v^j \,\mathrm{d}x \le C \int_{\mathbb{R}^3} |v^3||\nabla v||\nabla\nabla_h v| \,\mathrm{d}x,\\
&\int_{\mathbb{R}^3} \partial_k b^i \partial_i b^j \partial_k b^j \,\mathrm{d}x \le C \int_{\mathbb{R}^3} |b^h||\nabla b||\nabla\nabla_h b| \,\mathrm{d}x,\\
&\int_{\mathbb{R}^3} \partial_k b^i \partial_i v^j \partial_k v^j \,\mathrm{d}x \le C \int_{\mathbb{R}^3} |b^h||\nabla v||\nabla\nabla_h v| \,\mathrm{d}x,\\
&\int_{\mathbb{R}^3} \partial_k v^i \partial_i b^j \partial_k b^j \,\mathrm{d}x \le C \int_{\mathbb{R}^3} |b^h||\nabla v||\nabla\nabla_h b| \,\mathrm{d}x + \int_{\mathbb{R}^3} |b^h||\nabla b||\nabla\nabla_h v| \,\mathrm{d}x.
\end{aligned}$$

Applying Propositions 2.2 and 2.4, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |W^3||\nabla W||\nabla\nabla_h V| \,\mathrm{d}x &\le C\|W^3\|_{s,\infty} \|\nabla W \nabla\nabla_h V\|_{\frac{s}{s-1},1}\\
&\le C\|W^3\|_{s,\infty} \|\nabla W\|_{\frac{2s}{s-2},2} \|\nabla\nabla_h V\|_2\\
&\le C\|W^3\|_{s,\infty} \|\nabla W\|_2^{1-\frac{3}{s}} \|\nabla^2 W\|_2^{\frac{1}{s}} \|\nabla\nabla_h W\|_2^{\frac{2}{s}} \|\nabla\nabla_h V\|_2\\
&\le C\|W^3\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla W\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 W\|_2^{\frac{2}{s-2}}\\
&\quad + \frac{2}{s}\|\nabla\nabla_h W\|_2^2 + \frac{1}{2}\|\nabla\nabla_h V\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_3} |W^3||\nabla V||\nabla\nabla_h W| \,\mathrm{d}x &\le C\|W^3\|_{s,\infty} \|\nabla V \nabla\nabla_h W\|_{\frac{s}{s-1},1}\\
&\le C\|W^3\|_{s,\infty} \|\nabla V\|_{\frac{2s}{s-2},2} \|\nabla\nabla_h W\|_2\\
&\le C\|W^3\|_{s,\infty} \|\nabla V\|_2^{1-\frac{3}{s}} \|\nabla^2 V\|_2^{\frac{1}{s}} \|\nabla\nabla_h V\|_2^{\frac{2}{s}} \|\nabla\nabla_h W\|_2\\
&\le C\|W^3\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla V\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 V\|_2^{\frac{2}{s-2}}\\
&\quad + \frac{2}{s}\|\nabla\nabla_h V\|_2^2 + \frac{1}{2}\|\nabla\nabla_h W\|_2^2.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |V^3||\nabla W||\nabla\nabla_h W| \,\mathrm{d}x \le C\|V^3\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla W\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 W\|_2^{\frac{2}{s-2}} + \frac{1}{2}\|\nabla\nabla_h W\|_2^2,\\
&\int_{\mathbb{R}^3} |v^3||\nabla v||\nabla\nabla_h v| \,\mathrm{d}x \le C\|v^3\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla v\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 v\|_2^{\frac{2}{s-2}} + \frac{1}{2}\|\nabla\nabla_h v\|_2^2,\\
&\int_{\mathbb{R}^3} |b^3||\nabla b||\nabla\nabla_h b| \,\mathrm{d}x \le C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla b\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 b\|_2^{\frac{2}{s-2}} + \frac{1}{2}\|\nabla\nabla_h b\|_2^2,\\
&\int_{\mathbb{R}^3} |b^h||\nabla v||\nabla\nabla_h v| \,\mathrm{d}x \le C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla v\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 v\|_2^{\frac{2}{s-2}} + \frac{1}{2}\|\nabla\nabla_h v\|_2^2,\\
&\int_{\mathbb{R}^3} |b^h||\nabla b||\nabla\nabla_h v| \,\mathrm{d}x \le C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla b\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 b\|_2^{\frac{2}{s-2}} + \frac{2}{s}\|\nabla\nabla_h b\|_2^2 + \frac{1}{2}\|\nabla\nabla_h v\|_2^2,\\
&\int_{\mathbb{R}^3} |b^h||\nabla v||\nabla\nabla_h b| \,\mathrm{d}x \le C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}} \|\nabla v\|_2^{\frac{2s-6}{s-2}} \|\nabla^2 v\|_2^{\frac{2}{s-2}} + \frac{2}{s}\|\nabla\nabla v\|_2^2 + \frac{1}{2}\|\nabla\nabla_h b\|_2^2.
\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}
J_1 &= J_{11}+J_{12}+J_{13}+J_{14}\\
&\le C\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+C\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}\\
&\quad+C\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+C\|v^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\\
&\quad+C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla b\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 b\|_2^{\frac{2}{s-2}}+C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\\
&\quad+\Big(1+\frac{2}{s}\Big)\|\nabla\nabla_h W\|_2^2+\Big(\frac{1}{2}+\frac{2}{s}\Big)\|\nabla\nabla_h V\|_2^2\\
&\quad+\Big(1+\frac{3}{2}\Big)\|\nabla\nabla_h v\|_2^2+\Big(1+\frac{2}{s}\Big)\|\nabla\nabla_h b\|_2^2. \qquad (3.9)
\end{aligned}$$

Similarly, for $J_2$, we get

$$\begin{aligned}
J_2 &\le C\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+C\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}\\
&\quad+C\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+C\|v^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\\
&\quad+C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla b\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 b\|_2^{\frac{2}{s-2}}+C\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\\
&\quad+\Big(1+\frac{2}{s}\Big)\|\nabla\nabla_h V\|_2^2+\Big(\frac{1}{2}+\frac{2}{s}\Big)\|\nabla\nabla_h W\|_2^2\\
&\quad+\Big(1+\frac{3}{2}\Big)\|\nabla\nabla_h v\|_2^2+\Big(1+\frac{2}{s}\Big)\|\nabla\nabla_h b\|_2^2. \qquad (3.10)
\end{aligned}$$

It can be seen from the definition of $V$ that

$$\|b\|_2^2\le C(\|V\|_2^2+\|W\|_2^2).$$

Integrating the equation $\big(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h W\|_2^2+\|\nabla\nabla_h W\|_2^2=J_1\big)$ from $t_0$ to $t$ we can estimate $J_1$ as

$$\begin{aligned}
J_1 &\le C\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+C\int_{t_0}^{t}(\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}+\|v^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+C\int_{t_0}^{t}(\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla b\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 b\|_2^{\frac{2}{s-2}}+\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+\Big(\frac{1}{2}+\frac{2}{s}\Big)\int_{t_0}^{t}\|\nabla\nabla_h V\|_2^2\,\mathrm{d}\tau+C(t_0). \qquad (3.11)
\end{aligned}$$

Similarly, we estimate $J_2$ as

$$\begin{aligned}
J_2 &\le C\int_{t_0}^{t}(\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}+\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+C\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}+\|v^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+C\int_{t_0}^{t}(\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla b\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 b\|_2^{\frac{2}{s-2}}+\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}})\,\mathrm{d}\tau\\
&\quad+\Big(\frac{1}{2}+\frac{2}{s}\Big)\int_{t_0}^{t}\|\nabla\nabla_h W\|_2^2\,\mathrm{d}\tau+C(t_0). \qquad (3.12)
\end{aligned}$$

It is suffice to estimate one term of the right hand side of (3.11).

$$\begin{aligned}
&\int_{t_0}^{t}\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau\\
&\le\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{\frac{2s}{s-3}}\|\nabla V\|_2^2\,\mathrm{d}\tau\Big)^{\frac{s-3}{s-2}}\Big(\int_{t_0}^{t}\|\nabla^2 V\|_2^2\,\mathrm{d}\tau\Big)^{\frac{1}{s-2}}\\
&=\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{\frac{2s}{s-3}}\|\nabla V\|_2^{\frac{3s-10}{2s-6}}\|\nabla V\|_2^{\frac{s-2}{2s-6}}\,\mathrm{d}\tau\Big)^{\frac{s-3}{s-2}}\Big(\int_{t_0}^{t}\|\nabla^2 V\|_2^2\,\mathrm{d}\tau\Big)^{\frac{1}{s-2}}\\
&\le e_2(t_0)^{\frac{1}{4}}\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r}\|\nabla V\|_2^2\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}\Big(\int_{t_0}^{t}\|\nabla^2 V\|_2^2\,\mathrm{d}\tau\Big)^{\frac{1}{s-2}}\\
&\le C\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_2(\tau)^{\frac{1}{s-2}}.
\end{aligned}\tag{3.13}$$

Other items in (3.11) and (3.12) can be estimated by

$$\begin{aligned}
\int_{t_0}^{t}\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r}L_1(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_1(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|W^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_1(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla V\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 V\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|V^3\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_2(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|V^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla W\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 W\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r}L_1(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_1(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|v^3\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|v^3\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_2(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla b\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 b\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|b^h\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_2(\tau)^{\frac{1}{s-2}},\\
\int_{t_0}^{t}\|b^h\|_{s,\infty}^{\frac{2s}{s-2}}\|\nabla v\|_2^{\frac{2s-6}{s-2}}\|\nabla^2 v\|_2^{\frac{2}{s-2}}\,\mathrm{d}\tau&\le C\Big(\int_{t_0}^{t}\|b^h\|_{s,\infty}^{r}L_2(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L_2(\tau)^{\frac{1}{s-2}}.
\end{aligned}$$

Thus, we reach that

$$J(t)\le C(t_0)+C\Big(\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|v^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L(\tau)^{\frac{1}{s-2}}.$$

Using the inequality $(a+b)^{\frac{4}{3}}\le(2a)^{\frac{4}{3}}+(2b)^{\frac{4}{3}}$ and Young's inequality, we obtain

$$\begin{aligned}
J(t)^{\frac{4}{3}}&\le\Big[C(t_0)+\Big(\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{4s-8}}L(\tau)^{\frac{1}{s-2}}\Big]^{\frac{4}{3}}\\
&\le C(t_0)+C\Big(\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau\Big)^{\frac{3s-10}{3s-6}}L(\tau)^{\frac{4}{3s-6}}\\
&\le C(t_0)+C\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau+Ce(t_0)^{\frac{1}{3}}L(\tau).
\end{aligned}$$

Thus, we have

$$L(t)\le C(t_0)+Ce(t_0)^{\frac{1}{3}}J(t)^{\frac{4}{3}}$$

$$\leq C(t_0)+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau+Ce(t_0)^{\frac{1}{3}}L(\tau). \tag{3.14}$$

When $t_0$ infinitely approaches $T$, we have

$$L(t)\leq C(t_0)+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r}+\|V^3\|_{s,\infty}^{r}+\|b^h\|_{s,\infty}^{r})L(\tau)\,\mathrm{d}\tau.$$

In the following, we discuss the right hand side of the above inequality in two cases.

**Case I** When $s\leq 6$, there exists a small positive constant $c$ such that

$$\begin{aligned}
\|W^3\|_{s,\infty}^{r\varepsilon}&\leq C\|W^3\|_{s}^{r\varepsilon}\leq C\|W\|_2^{\alpha r\varepsilon}\|\nabla W\|_2^{(1-\alpha)r\varepsilon}\leq CL(t)^{c\varepsilon},\\
\|V^3\|_{s,\infty}^{r\varepsilon}&\leq C\|V^3\|_{s}^{r\varepsilon}\leq C\|V\|_2^{\alpha r\varepsilon}\|\nabla V\|_2^{(1-\alpha)r\varepsilon}\leq CL(t)^{c\varepsilon},\\
\|b^h\|_{s,\infty}^{r\varepsilon}&\leq C\|b^h\|_{s}^{r\varepsilon}\leq C\|b\|_{s}^{r\varepsilon}\leq C(\|V\|_s^{r\varepsilon}+\|W\|_s^{r\varepsilon})\leq CL(t)^{c\varepsilon},
\end{aligned}$$

where $\alpha=\frac{6-s}{2s}$.

This leads to

$$L(t)\leq C(t_0)+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r(1-\varepsilon)}+\|V^3\|_{s,\infty}^{r(1-\varepsilon)}+\|b^h\|_{s,\infty}^{r(1-\varepsilon)})L(\tau)^{1+c\varepsilon}\,\mathrm{d}\tau.$$

**Case II** When $s>6$, we have

$$\begin{aligned}
\|W^3\|_{s,\infty}^{r\varepsilon}&\leq C\|W^3\|_{s}^{r\varepsilon}\leq C\|\nabla W\|_2^{r\varepsilon(\frac{1}{2}+\frac{3}{s})}\|\nabla^2 W\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}\\
&\leq CL(t)^{r\varepsilon(\frac{1}{4}+\frac{3}{2s})}\|\nabla^2 W\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})},\\
\|V^3\|_{s,\infty}^{r\varepsilon}&\leq C\|V^3\|_{s}^{r\varepsilon}\leq C\|\nabla V\|_2^{r\varepsilon(\frac{1}{2}+\frac{3}{s})}\|\nabla^2 V\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}\\
&\leq CL(t)^{r\varepsilon(\frac{1}{4}+\frac{3}{2s})}\|\nabla^2 V\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})},\\
\|b^h\|_{s,\infty}^{r\varepsilon}&\leq C\|b^h\|_{s}^{r\varepsilon}\leq C\|b\|_{s}^{r\varepsilon}\leq C(\|V\|_s^{r\varepsilon}+\|W\|_s^{r\varepsilon})\\
&\leq CL(t)^{r\varepsilon(\frac{1}{4}+\frac{3}{2s})}(\|\nabla^2 V\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}+\|\nabla^2 W\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}).
\end{aligned}$$

Thus, for the sufficient small constant $c$, we finally get that

$$\begin{aligned}
L(t)&\leq C(t_0)+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}\|W^3\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon}\|\nabla^2 W\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}\,\mathrm{d}\tau\\
&\quad+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}\|V^3\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon}\|\nabla^2 V\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}\,\mathrm{d}\tau\\
&\quad+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}\|b^h\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon}(\|\nabla^2 V\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})}+\|\nabla^2 W\|_2^{r\varepsilon(\frac{1}{2}-\frac{3}{s})})\,\mathrm{d}\tau\\
&\leq Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|W^3\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon})^{\frac{1}{1-\frac{2s-12}{3s-10}\varepsilon}}\,\mathrm{d}\tau+\frac{e(t_0)^{\frac{1}{3}}}{8}\int_{t_0}^{t}\|\nabla^2 W\|_2^2\,\mathrm{d}\tau\\
&\quad+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|V^3\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon})^{\frac{1}{1-\frac{2s-12}{3s-10}\varepsilon}}\,\mathrm{d}\tau+\frac{e(t_0)^{\frac{1}{3}}}{8}\int_{t_0}^{t}\|\nabla^2 V\|_2^2\,\mathrm{d}\tau\\
&\quad+Ce(t_0)^{\frac{1}{3}}\int_{t_0}^{t}(\|b^h\|_{s,\infty}^{r(1-\varepsilon)}L(\tau)^{1+c\varepsilon})^{\frac{1}{1-\frac{2s-12}{3s-10}\varepsilon}}\,\mathrm{d}\tau+\frac{e(t_0)^{\frac{1}{3}}}{4}\int_{t_0}^{t}\|\nabla^2 V\|_2^2\,\mathrm{d}\tau\\
&\quad+C(t_0)
\end{aligned}$$

$$\leq Ce(t_0)^{\frac{1}{3}} \int_{t_0}^{t} [(\|W^3\|_{s,\infty}^{r(1-\varepsilon)} + \|V^3\|_{s,\infty}^{r(1-\varepsilon)} + \|b^h\|_{s,\infty}^{r(1-\varepsilon)}) L(\tau)^{1+c\varepsilon}]^{\frac{1}{1-\frac{2s-12}{3s-10}\varepsilon}} \, \mathrm{d}\tau$$
$$+ \frac{e(t_0)^{\frac{1}{3}}}{2} L(t) + C(t_0). \tag{3.15}$$

When $t_0$ infinitely approaches $T$, we have

$$L(t) \leq C(t_0) + Ce(t_0)^{\frac{1}{3}} \int_{t_0}^{t} (\|W^3\|_{s,\infty}^{r} + \|V^3\|_{s,\infty}^{r} + \|b^h\|_{s,\infty}^{r})^{\frac{1-\varepsilon}{1-\frac{2s-12}{3s-10}\varepsilon}} L(\tau)^{\frac{1+c\varepsilon}{1-\frac{2s-12}{3s-10}\varepsilon}} \, \mathrm{d}\tau.$$

Setting $1 - \frac{1-\varepsilon}{1-\frac{2s-12}{3s-10}\varepsilon}$ as a new $\varepsilon$ in the case $s > 6$, we can get the same form as in the first case

$$L(t) \leq C(t_0) + Ce(t_0)^{\frac{1}{3}} \int_{t_0}^{t} (\|W^3\|_{s,\infty}^{r(1-\varepsilon)} + \|V^3\|_{s,\infty}^{r(1-\varepsilon)} + \|b^h\|_{s,\infty}^{r(1-\varepsilon)}) L(\tau)^{1+A\varepsilon} \, \mathrm{d}\tau$$

for some $A(\varepsilon) = O(\varepsilon)$. Thus, the results of the two cases hold with the same inequality when $t_0$ closes enough to $T$ . Using Lemma 2.1, we can get that $(V, W)$ is smooth beyond $T$. By the fact that

$$\|V^3\|_{p,\infty} \lesssim \|v^3\|_{p,\infty} + \|b\|_{p,\infty}, \quad \|W^3\|_{p,\infty} \lesssim \|v^3\|_{p,\infty} + \|b\|_{p,\infty},$$

we then finished the proof of the main theorem.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

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