Chin. Ann. Math. Ser. B
45(2), 2024, 265–278
DOI: 10.1007/s11401-024-0016-6

Chinese Annals of Mathematics, Series B

Integral Operators Between Fock Spaces*

Yongqing LIU[1] Shengzhao HOU[2]

Abstract In this paper, the authors study the integral operator

$$S_\phi f(z) = \int_{\mathbb{C}} \phi(z, \overline{w}) f(w) \mathrm{d}\lambda_\alpha(w)$$

induced by a kernel function $\phi(z, \cdot) \in F_\alpha^\infty$ between Fock spaces. For $1 \le p \le \infty$, they prove that $S_\phi : F_\alpha^1 \to F_\alpha^p$ is bounded if and only if

$$\sup_{a \in \mathbb{C}} \|S_\phi k_a\|_{p,\alpha} < \infty, \tag{$\dagger$}$$

where k_a is the normalized reproducing kernel of F_α^2; and, $S_\phi : F_\alpha^1 \to F_\alpha^p$ is compact if and only if

$$\lim_{|a| \to \infty} \|S_\phi k_a\|_{p,\alpha} = 0.$$

When $1 < q \le \infty$, it is also proved that the condition (†) is not sufficient for boundedness of $S_\phi : F_\alpha^q \to F_\alpha^p$.

In the particular case $\phi(z, \overline{w}) = \mathrm{e}^{\alpha z \overline{w}} \varphi(z - \overline{w})$ with $\varphi \in F_\alpha^2$, for $1 \le q < p < \infty$, they show that $S_\phi : F_\alpha^p \to F_\alpha^q$ is bounded if and only if $\varphi = 0$; for $1 < p \le q < \infty$, they give sufficient conditions for the boundedness or compactness of the operator $S_\phi : F_\alpha^p \to F_\alpha^q$.

Keywords Fock spaces, Integral operators, Normalized reproducing kernel

2000 MR Subject Classification 32A36, 45P05

1 Introduction

Fock spaces are deeply connected with quantum mechanics, partial differential equations, and harmonic analysis. For instance, creation and annihilation operators in quantum mechanics are multiplication and, respectively, differential operators on Fock spaces. Characterizing the zero sequence of Fock space is equivalent to studying the completeness of coherent state systems in quantum mechanics (see [4, 11] and the references therein).

Let α be a positive number, and $\mathrm{d}\lambda_\alpha(z) = \frac{\alpha}{\pi} \mathrm{e}^{-\alpha|z|^2} \mathrm{d}A(z)$ be the Gaussian measure on the complex plane $\mathbb{C}$, where $\mathrm{d}A$ is the Euclidean area measure on $\mathbb{C}$. For $0 < p < \infty$, the Fock

Manuscript received February 21, 2022. Revised September 30, 2022.
[1]School of Mathematics and Statistics, Changshu Institute of Technology, Changshu 215500, Jiangsu, China. E-mail: yongqingliu@cslg.edu.cn
[2]Corresponding author. School of Mathematical Sciences, Soochow University, Suzhou 215006, Jiangsu, China. E-mail: shou@suda.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11971340).

space F^p_α consists of all entire functions f such that

$$\|f\|_{p,\alpha}=\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|f(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2}|^p\mathrm{d}A(z)\Big)^{\frac{1}{p}}<\infty. \tag{1.1}$$

The Fock space F^∞_α (i.e., $p=\infty$) consists of all entire functions f such that

$$\|f\|_{\infty,\alpha}=\sup_{z\in\mathbb{C}}|f(z)|\mathrm{e}^{-\frac{\alpha}{2}|z|^2}<\infty. \tag{1.2}$$

When $1\le p\le\infty$, Fock space F^p_α is a Banach space. In particular, F^2_α is a Hilbert space. $\big\{e_n(z)=\sqrt{\frac{\alpha^n}{n!}}z^n,\ n=0,1,2,3,\cdots\big\}$ is an orthonormal basis for F^2_α. The reproducing kernel of F^2_α is given by $K_\alpha(z,w)=\sum\limits_{n=0}^{\infty}e_n(z)\overline{e_n(w)}=\mathrm{e}^{\alpha z\overline{w}}$, and the normalized reproducing kernel at $w\in\mathbb{C}$ is $k_w(z)=\mathrm{e}^{\alpha z\overline{w}-\frac{\alpha}{2}|w|^2}$.

Integral operators on Fock spaces have been studied widely (see [1–3, 5, 11]). We know in [4] that every bounded operator T on Fock space F^2_α can be expressed as the integral operator:

$$Tf(z)=\int_{\mathbb{C}}\phi(z,\overline{w})f(w)\mathrm{d}\lambda_\alpha(w),$$

where the kernel function $\phi(z,\overline{w})=TK_\alpha(\cdot,w)(z)$, and $\phi(\cdot,w)\in F^2_\alpha$ for all $w\in\mathbb{C}$ and $\phi(z,\cdot)\in F^2_\alpha$ for all $z\in\mathbb{C}$. On the other hand, it is a difficult problem to obtain conditions on ϕ under which the corresponding integral operator is bounded or compact. In this paper, we study the boundedness and compactness of the integral operators induced by a kernel function $\phi(z,\cdot)\in F^\infty_\alpha$ between Fock spaces.

In [10], the authors gave some sufficient conditions for a linear operator on F^2_α to be bounded and compact. The authors of [7] extended the work in [10] to F^p_α. Precisely, using the atomic decomposition of Fock spaces, the authors of [7] proved that for $0<p\le 1$ a linear operator $T:F^p_\alpha\to F^p_\alpha$ is bounded if and only if

$$\sup_{a\in\mathbb{C}}\|Tk_a\|_{p,\alpha}<\infty.$$

For $\phi(z,\cdot)\in F^\infty_\alpha$, let

$$S_\phi f(z)=\int_{\mathbb{C}}\phi(z,\overline{w})f(w)\mathrm{d}\lambda_\alpha(w)$$

be the integral operator induced by ϕ. In Section 2, for $1\le p\le\infty$, we prove that $S_\phi:F^1_\alpha\to F^p_\alpha$ is bounded if and only if

$$\sup_{a\in\mathbb{C}}\|S_\phi k_a\|_{p,\alpha}<\infty \tag{1.3}$$

(see Theorem 2.1); and, $S_\phi:F^1_\alpha\to F^p_\alpha$ is compact if and only if $\lim\limits_{|a|\to\infty}\|S_\phi k_a\|_{p,\alpha}=0$ (see Theorem 2.3). When $1<q\le\infty$, it is also proved that the condition (1.3) is not sufficient for boundedness of $S_\phi:F^q_\alpha\to F^p_\alpha$ (see Theorem 2.2).

In particular, for $\varphi \in F^2_\alpha$, let $\phi(z,\overline{w}) = \mathrm{e}^{\alpha z\overline{w}}\varphi(z-\overline{w})$, we consider corresponding integral operator

$$T_\varphi f(z) = \int_{\mathbb{C}} \mathrm{e}^{\alpha z\overline{w}}\varphi(z-\overline{w})f(w)\mathrm{d}\lambda_\alpha(w).$$

Bounded operators on Fock space F^2_α and on $L^2(\mathbb{R},\mathrm{d}x)$ can be connected through the unitary operator Bargmann transform. Using Bargmann transform, Zhu [12] transferred the classical Hilbert transform to the integral operator T_φ on F^2_α, where $\varphi(z) = \int_0^{\frac{z}{\sqrt{2}}} \mathrm{e}^{u^2}\mathrm{d}u$, and proposed a question: Find necessary and sufficient conditions in terms of $\varphi \in F^2_\alpha$ such that T_φ is bounded on F^2_α. Recently, Cao et al. [1] proved that T_φ is bounded on F^2_α if and only if there exists an $m \in L^\infty(\mathbb{R})$ such that

$$\varphi(z) = \int_{\mathbb{R}} \mathrm{e}^{-2(x-\frac{\mathrm{i}}{2}z)^2} m(x)\mathrm{d}x, \quad z \in \mathbb{C}.$$

In Section 3, we study boundedness and compactness of the integral operator T_φ from F^p_α to F^q_α. For $1 \le q < p < \infty$, we proved that there are no non-zero bounded integral operators T_φ from F^p_α to F^q_α (see Theorem 3.1). But, when $1 < p \le q < \infty$, we give sufficient conditions for T_φ from F^p_α to F^q_α to be bounded (see Theorem 3.2) and compact (see Theorem 3.3).

In this paper, we use C to denote a positive number, which may vary from place to place. For two quantities A and B, $A \lesssim B$ means that there exists a constant $C > 0$, independent of the involved variables, such that $A \le CB$, and $A \simeq B$ if and only if $A \lesssim B$ and $B \lesssim A$.

2 Boundedness and Compactness of S_ϕ

In this section, we give necessary and sufficient conditions for integral operator S_ϕ from F^1_α to F^p_α to be bounded, and, respectively, compact, for $1 \le p \le \infty$.

In order to prove Theorem 2.1, we need the following Minkowski's integral inequality from [9].

Lemma 2.1 *Let (X,μ) and (Y,ν) be two σ-finite measure spaces and let $1 \le p < \infty$. For every nonnegative measurable function F on the product space $(X,\mu)\times(Y,\nu)$ we have*

$$\Big[\int_Y \Big(\int_X F(x,y)\mathrm{d}\mu(x)\Big)^p \mathrm{d}\nu(y)\Big]^{\frac{1}{p}} \le \int_X \Big[\int_Y F(x,y)^p \mathrm{d}\nu(y)v\Big]^{\frac{1}{p}}\mathrm{d}\mu(x).$$

The following result provides a necessary and sufficient condition for S_ϕ to be bounded.

Theorem 2.1 *Let $1 \le p \le \infty$. For any $z \in \mathbb{C}$, suppose $\phi(z,\cdot) \in F^\infty_\alpha$. Then S_ϕ is bounded from F^1_α to F^p_α if and only if*

$$\sup_{a\in\mathbb{C}} \|S_\phi k_a\|_{p,\alpha} < \infty,$$

or equivalently

$$\sup_{a\in\mathbb{C}} \int_{\mathbb{C}} |\phi(z,\overline{a})\mathrm{e}^{-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|a|^2}|^p \mathrm{d}A(z) < \infty, \quad 1 \le p < \infty;$$

$$\sup_{a\in\mathbb{C}}\sup_{z\in\mathbb{C}}|\phi(z,\overline{a})\mathrm{e}^{-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|a|^2}|<\infty,\quad p=\infty.$$

Proof Assume that S_ϕ is bounded from F^1_α to F^p_α. Since each k_a is a unit vector in F^1_α, we have

$$\|S_\phi k_a\|_{p,\alpha}\le\|S_\phi\|_{F^1_\alpha\to F^p_\alpha}.$$

By the reproducing formula (see [2, Lemma 2.15]), we get

$$\begin{aligned}S_\phi k_a(z)&=\int_{\mathbb{C}}\phi(z,\overline{w})k_a(w)\mathrm{d}\lambda_\alpha(w)\\&=\mathrm{e}^{-\frac{\alpha}{2}|a|^2}\int_{\mathbb{C}}\phi(z,\overline{w})\mathrm{e}^{\alpha w\overline{a}}\mathrm{d}\lambda_\alpha(w)\\&=\mathrm{e}^{-\frac{\alpha}{2}|a|^2}\phi(z,\overline{a}).\end{aligned}$$

Therefore, we have

$$\sup_{a\in\mathbb{C}}\int_{\mathbb{C}}|\phi(z,\overline{a})\mathrm{e}^{-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|a|^2}|^p\mathrm{d}A(z)<\infty,\quad 1\le p<\infty$$

and

$$\sup_{a\in\mathbb{C}}\sup_{z\in\mathbb{C}}|\phi(z,\overline{a})\mathrm{e}^{-\frac{\alpha}{2}|z|^2-\frac{\alpha}{2}|a|^2}|<\infty,\quad p=\infty.$$

Conversely, when $1\le p<\infty$, let

$$C=\sup_{a\in\mathbb{C}}\int_{\mathbb{C}}|\phi(z,\overline{a})\mathrm{e}^{-\frac{\alpha|z|^2}{2}-\frac{\alpha|a|^2}{2}}|^p\mathrm{d}A(z)<\infty.$$

For $f\in F^1_\alpha$, by Lemma 2.1 and Fubini's Theorem, we have

$$\begin{aligned}\|S_\phi f\|_{p,\alpha}&=\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|S_\phi f(z)\mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^p\mathrm{d}A(z)\Big)^{\frac1p}\\&=\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}\Big|\int_{\mathbb{C}}\phi(z,\overline{w})f(w)\mathrm{d}\lambda_\alpha(w)\Big|^p\mathrm{e}^{-\frac{p\alpha|z|^2}{2}}\mathrm{d}A(z)\Big)^{\frac1p}\\&\le\int_{\mathbb{C}}\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|\phi(z,\overline{w})f(w)|^p\mathrm{e}^{-\frac{p\alpha|z|^2}{2}}\mathrm{d}A(z)\Big)^{\frac1p}\mathrm{d}\lambda_\alpha(w)\\&\lesssim\frac{\alpha}{2\pi}\int_{\mathbb{C}}|f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\mathrm{d}A(w)\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|\phi(z,\overline{w})|^p\mathrm{e}^{-\frac{p\alpha|z|^2}{2}-\frac{p\alpha|w|^2}{2}}\mathrm{d}A(z)\Big)^{\frac1p}\\&\lesssim C^{\frac1p}\frac{\alpha}{2\pi}\int_{\mathbb{C}}|f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\mathrm{d}A(w).\end{aligned}$$

Hence, we obtain

$$\|S_\phi f\|_{p,\alpha}\le C^{\frac1p}\|f\|_{1,\alpha}.$$

For $p=\infty$, by Fubini's Theorem, we have

$$\begin{aligned}&\sup_{z\in\mathbb{C}}|S_\phi f(z)\mathrm{e}^{-\frac{\alpha|z|^2}{2}}|\\&\le\int_{\mathbb{C}}|\phi(z,\overline{w})f(w)|\mathrm{d}\lambda_\alpha(w)\mathrm{e}^{-\frac{\alpha|z|^2}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{\pi} \int_{\mathbb{C}} |\phi(z, \overline{w}) f(w)| \mathrm{e}^{-\alpha|w|^2} \mathrm{d}A(w) \mathrm{e}^{-\frac{\alpha|z|^2}{2}} \\
&\lesssim \sup_{w\in\mathbb{C}} \sup_{z\in\mathbb{C}} |\phi(z, \overline{w})| \mathrm{e}^{-\frac{\alpha|z|^2}{2}} \mathrm{e}^{-\frac{\alpha|w|^2}{2}} \frac{\alpha}{2\pi} \int_{\mathbb{C}} |f(w) \mathrm{e}^{-\frac{\alpha|w|^2}{2}}| \mathrm{d}A(w) \\
&= \sup_{w\in\mathbb{C}} \sup_{z\in\mathbb{C}} |\phi(z, \overline{w}) \mathrm{e}^{-\frac{\alpha|z|^2}{2}} \mathrm{e}^{-\frac{\alpha|w|^2}{2}} | \|f\|_{1,\alpha}.
\end{aligned}
$$

By the assumption, we obtain

$$\|S_\phi f\|_{\infty,\alpha} \le C\|f\|_{1,\alpha}.$$

This completes the proof.

The following result shows that the condition in Theorem 2.1 is no longer sufficient for S_ϕ to be bounded, when $q > 1$.

Theorem 2.2 *Let $1 < q \le \infty$ and $1 \le p \le \infty$. For any $z \in \mathbb{C}$, there exists a function $\phi(z, \cdot) \in F_\alpha^\infty$ such that the integral operator S_ϕ satisfies*

$$\sup_{a\in\mathbb{C}} \|S_\phi k_a\|_{p,\alpha} < \infty,$$

but the operator S_ϕ is not bounded from F_α^q to F_α^p.

Proof When $1 \le p \le \infty$ and $1 < q \le \infty$, we can choose some

$$\delta \in \Big(\frac{1}{2q} - \frac{1}{2p}, \frac{1}{2} - \frac{1}{2p}\Big).$$

Let

$$a_{n_k} = \begin{cases} \dfrac{n_k^\delta \alpha^{n_k}}{n_k!}, & n_k \ge 2^{\frac{k}{(\frac{1}{2} - \frac{1}{2p}) - \delta}}; \\ 0, & \text{others} \end{cases}$$

and

$$\phi(z, \overline{w}) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k} \overline{w}^{n_k}.$$

Then for $z \in \mathbb{C}$, we have

$$\phi(z, \cdot) \in F_\alpha^2 \subset F_\alpha^\infty.$$

Define the integral operator

$$S_\phi f(z) = \int_{\mathbb{C}} \phi(z, \overline{w}) f(w) \mathrm{d}\lambda_\alpha(w), \quad z \in \mathbb{C}.$$

One can check that the operator S_ϕ is well-defined on some dense subset of F_α^q.

We first show that S_ϕ is not bounded from F_α^q to F_α^p. By direct calculation, we get

$$S_\phi e_{n_k}(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{C}} \phi(z,\overline{w}) e_{n_k}(w) \mathrm{d}\lambda_\alpha(w) \\
&= a_{n_k} \sqrt{\frac{\alpha^{n_k}}{n_k!}} z^{n_k} \int_{\mathbb{C}} |w|^{2n_k} \mathrm{d}\lambda_\alpha(w) \\
&= a_{n_k} n_k! e_{n_k}(z).
\end{aligned}
$$

We calculate the norm of $S_\phi e_{n_k}$ in F_α^p for $1 \le p \le \infty$. By Stirling's formula, we have

$$
\begin{aligned}
&\|S_\phi e_{n_k}\|_{p,\alpha}^p \\
=& \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |a_{n_k} n_k! e_{n_k}(z) \mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^p \mathrm{d}A(z) \\
=& (|a_{n_k}| n_k!)^p \Big(\frac{\alpha^{n_k}}{n_k!}\Big)^{\frac{p}{2}} \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |z^{n_k} \mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^p \mathrm{d}A(z) \\
=& (|a_{n_k}| n_k!)^p \Big(\frac{\alpha^{n_k}}{n_k!}\Big)^{\frac{p}{2}} \Big(\frac{2}{\alpha p}\Big)^{\frac{np}{2}} \Gamma\Big(\frac{n_k p}{2} + 1\Big) \\
\sim& (|a_{n_k}| n_k!)^p n_k^{\frac{1}{2} - \frac{p}{4}}.
\end{aligned}
$$

Similarly, for $p = \infty$, we have $\|S_\phi e_{n_k}\|_{\infty,\alpha} \sim |a_{n_k}| n_k! n_k^{-\frac{1}{4}}$. Therefore,

$$
\|S_\phi e_{n_k}\|_{p,\alpha} \sim |a_{n_k}| n_k! n_k^{\frac{1}{2p} - \frac{1}{4}}. \tag{2.1}
$$

When $1 < q \le \infty$, the norm estimate of e_n in F_α^q is

$$
\|e_{n_k}\|_{q,\alpha} \sim n_k^{\frac{1}{2q} - \frac{1}{4}}. \tag{2.2}
$$

Now combining (2.1) with (2.2), we get

$$
\begin{aligned}
\frac{\|S_\phi e_{n_k}\|_{p,\alpha}}{\|e_{n_k}\|_{q,\alpha}} &\sim |a_{n_k}| n_k! n_k^{\frac{1}{2p} - \frac{1}{2q}} \\
&= \alpha^{n_k} n_k^{\delta - (\frac{1}{2q} - \frac{1}{2p})} \to \infty, \quad k \to \infty.
\end{aligned}
$$

So S_ϕ is not bounded from F_α^q to F_α^p, when $1 < q \le \infty$ and $1 \le p \le \infty$.

Next we prove that

$$
\sup_{a \in \mathbb{C}} \|S_\phi k_a\|_{p,\alpha} < \infty.
$$

Using the reproducing formula, we have

$$
\begin{aligned}
&S_\phi k_a(z) \\
=& \int_{\mathbb{C}} \phi(z,\overline{w}) k_a(w) \mathrm{d}\lambda_\alpha(w) \\
=& \sum_{k=1}^{\infty} a_{n_k} z^{n_k} \int_{\mathbb{C}} \overline{w}^{n_k} k_a(w) \mathrm{d}\lambda_\alpha(w) \\
=& \sum_{k=1}^{\infty} a_{n_k} z^{n_k} \overline{a}^{n_k} \mathrm{e}^{-\frac{\alpha}{2}|a|^2}.
\end{aligned}
$$

By computation, we get

$$\begin{aligned}
&\|S_\phi k_a\|_{p,\alpha}\\
\le&\sum_{k=1}^{\infty}|a_{n_k}||a|^{n_k}\|z^{n_k}\|_{p,\alpha}\mathrm{e}^{-\frac{\alpha}{2}|a|^2}\\
=&\sum_{k=1}^{\infty}|a_{n_k}||a|^{n_k}\mathrm{e}^{-\frac{\alpha|a|^2}{2}}\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|z^{n_k}\mathrm{e}^{-\frac{\alpha}{2}|z|^2}|^p\mathrm{d}A(z)\Big)^{\frac{1}{p}}\\
\lesssim&\sum_{k=1}^{\infty}|a_{n_k}|\Big(\max_{a\in\mathbb{C}}|a|^{n_k}\mathrm{e}^{-\frac{\alpha|a|^2}{2}}\Big)\Big(\frac{n_k}{\alpha\mathrm{e}}\Big)^{\frac{n_k}{2}}n_k^{\frac{1}{2p}}.
\end{aligned}$$

Note that

$$\max_{a\in\mathbb{C}}|a|^n\mathrm{e}^{-\frac{\alpha|a|^2}{2}}=\big(\frac{n}{\alpha\mathrm{e}}\big)^{\frac{n}{2}}.$$

By Stirling's formula, we have

$$\begin{aligned}
&\|S_\phi k_a\|_{p,\alpha}\\
\lesssim&\sum_{k=1}^{\infty}|a_{n_k}|\Big(\frac{n_k}{\alpha\mathrm{e}}\Big)^{n_k}n_k^{\frac{1}{2p}}\\
\sim&\sum_{k=1}^{\infty}|a_{n_k}|\frac{n_k!}{\alpha^{n_k}}n^{\frac{1}{2p}-\frac{1}{2}}=\sum_{k=1}^{\infty}n_k^{\delta-(\frac{1}{2}-\frac{1}{2p})}\\
\le&\sum_{k=1}^{\infty}\frac{1}{2^k}<\infty.
\end{aligned}$$

This completes the proof.

We can characterize the compactness of S_ϕ completely.

Theorem 2.3 *Let $1\le p\le\infty$. For any $z\in\mathbb{C}$, suppose $\phi(z,\cdot)\in F_\alpha^\infty$. Then S_ϕ is a compact operator from F_α^1 to F_α^p if and only if*

$$\lim_{|a|\to\infty}\|S_\phi k_a\|_{p,\alpha}=0.$$

Proof We firstly prove that S_ϕ is a compact operator. Let f_n be a sequence in F_α^1 such that $\sup_n\|f_n\|_{1,\alpha}<\infty$ and $f_n\to0$ uniformly on compact sets in $\mathbb{C}$. When $1\le p<\infty$, by the proof in Theorem 2.1, we get

$$\begin{aligned}
&\|S_\phi f_n\|_{p,\alpha}\\
=&\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|S_\phi f_n(z)\mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^p\mathrm{d}A(z)\Big)^{\frac{1}{p}}\\
\le&\frac{\alpha}{2\pi}\int_{\mathbb{C}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\mathrm{d}A(w)\Big(\frac{p\alpha}{2\pi}\int_{\mathbb{C}}|\phi(z,\overline{w})|^p\mathrm{e}^{-\frac{p\alpha|z|^2}{2}-\frac{p\alpha|w|^2}{2}}\mathrm{d}A(z)\Big)^{\frac{1}{p}}\\
=&\frac{\alpha}{2\pi}\int_{\mathbb{C}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w).
\end{aligned}$$

For any $0 < r < \infty$, we denote

$$D_r = \{w \in \mathbb{C} : |w| > r\}.$$

Then,

$$\begin{aligned}
&\|S_\phi f_n\|_{p,\alpha}\\
\leq&\frac{\alpha}{2\pi}\int_{\mathbb{C}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w)\\
=&\frac{\alpha}{2\pi}\int_{D_r}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w)\\
&+\frac{\alpha}{2\pi}\int_{\{w\in\mathbb{C}:|w|\leq r\}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w)\\
=&I_1+I_2.
\end{aligned}$$

We estimate each item independently. We first estimate I_1.

$$\begin{aligned}
I_1 &= \frac{\alpha}{2\pi}\int_{D_r}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w)\\
&\leq \frac{\alpha}{2\pi}\int_{\mathbb{C}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\mathrm{d}A(w)\sup_{w\in D_r}\|S_\phi k_w\|_{p,\alpha}\\
&= \|f_n\|_{1,\alpha}\sup_{w\in D_r}\|S_\phi k_w\|_{p,\alpha}.
\end{aligned}$$

Since $\sup_n \|f_n\|_{1,\alpha} < \infty$, we see that $I_1 \to 0$ as $r \to \infty$. We need to make a similar conclusion for I_2, and it follows from $\sup_{w\in\mathbb{C}}\|S_\phi k_w\|_{p,\alpha} < \infty$ that

$$\begin{aligned}
I_2 &= \frac{\alpha}{2\pi}\int_{\{w\in\mathbb{C}:|w|\leq r\}}|f_n(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|\|S_\phi k_w\|_{p,\alpha}\mathrm{d}A(w)\\
&\lesssim \sup_{\{w\in\mathbb{C}:|w|\leq r\}}|f_n(w)|.
\end{aligned}$$

Since $f_n \to 0$ uniformly on compact sets in $\mathbb{C}$, we get that $I_2 \to 0$ as $n \to \infty$. Therefore, $\|S_\phi f_n\|_{p,\alpha}$ converges to 0 as $n \to \infty$. So, S_ϕ is a compact operator.

Conversely, suppose that S_ϕ is a compact operator. Since k_a is a unit vector in F^1_α and converges uniformly to 0 on any compact subset of $\mathbb{C}$ as $|a| \to \infty$, we have

$$\|S_\phi k_a\|_{p,\alpha} \to 0, \quad |a| \to \infty.$$

For $p = \infty$, the proof is similar above, we omit it. This completes the proof.

3 Boundedness and Compactness of T_φ

The section is devoted to studying the integral operator

$$T_\varphi f(z) = \int_{\mathbb{C}} \mathrm{e}^{\alpha z\overline{w}}\varphi(z-\overline{w})f(w)\mathrm{d}\lambda_\alpha(w)$$

between Fock spaces.

Let $B(z, r)$ be the open Euclidean disk centred at z with radius r. The following lemma gives us the optimal pointwise estimates for functions in Fock spaces.

Lemma 3.1 (see [5, Lemma 2.1]) *For any $r > 0$ and $p > 0$, there exists a constant $C > 0$ such that*

$$|f(z)\mathrm{e}^{-\frac{\alpha}{2}|z|^2}|^p \le C \int_{B(z,r)} |f(w)\mathrm{e}^{-\frac{\alpha}{2}|w|^2}|^p \mathrm{d}A(w)$$

for all entire function f and all $z \in \mathbb{C}$.

For our next lemma, we need the concept of a lattice. For $r > 0$, a sequence $\{a_k\}$ in $\mathbb{C}$ is called an r-lattice if the following conditions are satisfied:

(1) $\bigcup\limits_{k=1}^{\infty} B(a_k, r) = \mathbb{C}$.

(2) $\big\{B\big(a_k, \frac{r}{2}\big)\big\}_{k=1}^{\infty}$ are mutually disjoint.

If $\{a_k\}$ is an r-lattice, then for any $\delta > 0$ there exists a positive integer m (depending only on r and δ) such that every point in $\mathbb{C}$ belongs to at most m of the sets $B(a_k, \delta)$.

The following lemma is some partial result about atomic decomposition of Fock spaces.

Lemma 3.2 (see [6, Theorem 8.2]) *Let $r > 0$ and $\{a_k\}$ be an r-lattice. For $1 \le p \le \infty$ and $\{c_k\} \in l^p$, set*

$$f(z) = \sum_{k=1}^{\infty} c_k \mathrm{e}^{\alpha z \overline{a_k} - \frac{\alpha}{2}|a_k|^2}. \tag{3.1}$$

Then $f \in F_\alpha^p$. Moreover,

$$\|f\|_{p,\alpha} \lesssim \inf \|\{c_k\}\|_{l^p},$$

where the infimum is taken over all sequences $\{c_k\}$ that give rise to the representation of f in (3.1).

We shall use the following technique (due to Luecking [8]) in our proof of boundedness of the integral operator T_φ. Recall that the Rademacher functions r_k are defined by

$$r_0(t) = \begin{cases} 1, & \text{if } 0 \le t - [t] < \dfrac{1}{2}, \\ -1, & \text{if } \dfrac{1}{2} \le t - [t] < 1 \end{cases}$$

and $r_k(t) = r_0(2^k t)$ for $k = 1, 2, \cdots$, where $[t]$ denotes the largest integer not greater than t. An important property of Rademacher functions is the Khinchine's inequality: For any $0 < p < \infty$, there exist some positive constants C_1 and C_2 depending only on p such that

$$C_1 \Big(\sum_{k=1}^{\infty} |b_k|^2 \Big)^{\frac{p}{2}} \le \int_0^1 \Big| \sum_{k=1}^{\infty} b_k r_k(t) \Big|^p \mathrm{d}t \le C_2 \Big(\sum_{k=1}^{\infty} |b_k|^2 \Big)^{\frac{p}{2}}$$

for any complex number sequences $\{b_k\}_{k=1}^{\infty}$.

The following result shows that there are no non-zero bounded integral operators T_φ from F_α^p to F_α^q if $1 \le q < p < \infty$.

Theorem 3.1 *Let* $1 \le q < p < \infty$. *Suppose* $\varphi \in F_\alpha^2$, *then the integral operator* T_φ *is bounded from* F_α^p *to* F_α^q *if and only if* $\varphi = 0$.

Proof We need only to prove that $\varphi = 0$ if T_φ is bounded from F_α^p to F_α^q. Let $r > 0$ and let $\{a_k\}$ be an r-lattice. For any $\{c_k\} \in l^p$, then $\{c_k r_k(t)\} \in l^p$, where $r_k(t)$ is a Rademacher function. By Lemma 3.2, the function

$$f_t(z) = \sum_{k=1}^{\infty} c_k r_k(t) k_{a_k}(z) = \sum_{k=1}^{\infty} c_k r_k(t) \mathrm{e}^{\alpha z \overline{a_k} - \frac{\alpha}{2}|a_k|^2}$$

belongs to F_α^p and $\|f_t\|_{p,\alpha} \lesssim \|\{c_k\}\|_{l^p}$. Since T_φ is bounded from F_α^p to F_α^q, we get

$$\|T_\varphi f_t\|_{q,\alpha} \le \|T_\varphi\|_{F_\alpha^p \to F_\alpha^q} \|f_t\|_{p,\alpha} \lesssim \|T_\varphi\|_{F_\alpha^p \to F_\alpha^q} \|\{c_k\}\|_{l^p}.$$

In the above inequality, integrate with respect to t from 0 to 1. By Fubini's theorem and Khinchine's inequality, we have

$$\begin{aligned}
&\int_0^1 \|T_\varphi f_t\|_{q,\alpha}^q \mathrm{d}t \\
=& \int_0^1 \int_{\mathbb{C}} |T_\varphi f_t(z) \mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^q \mathrm{d}A(z)\mathrm{d}t \\
=& \int_0^1 \int_{\mathbb{C}} \Big|\sum_{k=1}^{\infty} c_k r_k(t) T_\varphi k_{a_k}(z)\Big|^q \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z)\mathrm{d}t \\
=& \int_{\mathbb{C}} \int_0^1 \Big|\sum_{k=1}^{\infty} c_k r_k(t) T_\varphi k_{a_k}(z)\Big|^q \mathrm{d}t \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z) \\
\gtrsim& \int_{\mathbb{C}} \Big(\sum_{k=1}^{\infty} |c_k|^2 |T_\varphi k_{a_k}(z)|^2\Big)^{\frac{q}{2}} \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z) \\
\gtrsim& \sum_{j=1}^{\infty} \int_{B(a_j,r)} \Big(\sum_{k=1}^{\infty} |c_k|^2 |T_\varphi k_{a_k}(z)|^2\Big)^{\frac{q}{2}} \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z).
\end{aligned}$$

For j fixed, we have

$$\sum_{k=1}^{\infty} |c_k|^2 |T_\varphi k_{a_k}(z)|^2 \ge |c_j|^2 |T_\varphi k_{a_j}(z)|^2.$$

This, together with Lemma 3.1, shows that

$$\begin{aligned}
&\sum_{j=1}^{\infty} \int_{B(a_j,r)} \Big(\sum_{k=1}^{\infty} |c_k|^2 |T_\varphi k_{a_k}(z)|^2\Big)^{\frac{q}{2}} \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z) \\
\ge& \sum_{j=1}^{\infty} \int_{B(a_j,r)} |c_j|^q |T_\varphi k_{a_j}(z)|^q \mathrm{e}^{-\frac{q\alpha|z|^2}{2}} \mathrm{d}A(z) \\
=& \sum_{j=1}^{\infty} |c_j|^q \int_{B(a_j,r)} |T_\varphi k_{a_j}(z)|^q \mathrm{e}^{-\frac{q|z|^2}{2}} \mathrm{d}v(z) \\
\gtrsim& \sum_{j=1}^{\infty} |c_j|^q |T_\varphi k_{a_j}(a_j)|^q \mathrm{e}^{-\frac{q|a_j|^2}{2}}.
\end{aligned}$$

By direct calculation, we have

$$T_\varphi k_{a_j}(a_j)\mathrm{e}^{-\frac{|a_j|^2}{2}} = \varphi(a_j - \overline{a_j}).$$

Therefore, we get

$$\sum_{j=1}^{\infty} |c_j|^q |\varphi(a_j - \overline{a_j})|^q \lesssim \int_0^1 \|T_\varphi f_t\|_{q,\alpha}^q \mathrm{d}t \lesssim \|T_\varphi\|_{F_\alpha^p \to F_\alpha^q}^q \|\{c_k\}\|_{l^p}^q.$$

Denote $d_j = |c_j|^q$, then

$$\sum_{j=1}^{\infty} |d_j||\varphi(a_j - \overline{a_j})|^q \lesssim \|T_\varphi\|_{F_\alpha^p \to F_\alpha^q}^q \|\{d_j\}\|_{l^{\frac{p}{q}}}.$$

Since the conjugate exponent of $\frac{p}{q}$ is $\frac{p}{p-q}$, by a duality argument $(l^{\frac{p}{q}})^* = l^{\frac{p}{p-q}}$, we imply that

$$\sum_{j=1}^{\infty} |\varphi(a_j - \overline{a_j})|^{\frac{pq}{p-q}} \lesssim \|T_\varphi\|_{F_\alpha^p \to F_\alpha^q}^{\frac{pq}{p-q}}. \tag{3.2}$$

Notice that the above inequality (3.2) holds for any r-lattices. Choose a point $\xi_j \in \overline{B(a_j, r)}$ such that

$$|\varphi(\xi_j - \overline{\xi_j})| = \sup_{z \in B(a_j,r)} |\varphi(z - \overline{z})|.$$

Hence, we conclude

$$\begin{aligned}
&\int_{\mathbb{C}} |\varphi(z - \overline{z})|^{\frac{pq}{p-q}} \mathrm{d}A(z) \\
\leq &\sum_{j=1}^{\infty} \int_{B(a_j,r)} |\varphi(z - \overline{z})|^{\frac{pq}{p-q}} \mathrm{d}A(z) \\
\lesssim &\sum_{j=1}^{\infty} \sup_{z \in B(a_j,r)} |\varphi(z - \overline{z})|^{\frac{pq}{p-q}} \\
\leq &\sum_{j=1}^{\infty} |\varphi(\xi_j - \overline{\xi_j})|^{\frac{pq}{p-q}}.
\end{aligned}$$

Notice that there exists some $\delta > 0$ such that $\{\xi_j\}$ is a finite union of δ-lattices. This together with (3.2) shows

$$\int_{\mathbb{C}} |\varphi(z - \overline{z})|^{\frac{pq}{p-q}} \mathrm{d}A(z) < \infty,$$

which is impossible unless $\varphi = 0$. In fact, if $\varphi(z_0 - \overline{z_0}) \neq 0$, set $z = x + \mathrm{i}y$, $z_0 = x_0 + \mathrm{i}y_0$, there exists $r > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{C}} |\varphi(z - \overline{z})|^{\frac{pq}{p-q}} \mathrm{d}A(z) \\
= &\int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(2\mathrm{i}y)|^{\frac{pq}{p-q}} \mathrm{d}y \mathrm{d}x
\end{aligned}$$

$$\geq \int_{\mathbb{R}} \int_{y_0-r}^{y_0+r} |\varphi(2\mathrm{i}y)|^{\frac{pq}{p-q}} \mathrm{d}y\mathrm{d}x = \infty.$$

This completes the proof.

The next result is a sufficient condition for T_φ to be bounded from F_α^p to F_α^q.

Theorem 3.2 *Let* $1 < p \leq q < \infty$. *If*

$$\sup_{a\in\mathbb{C}} \int_{\mathbb{C}} |\varphi(z-\overline{a})| \mathrm{e}^{-\frac{\alpha}{2}|z-a|^2} \mathrm{d}A(z) < \infty,$$

then T_φ *is bounded from* F_α^p *to* F_α^q.

Proof By Hölder's Inequality, we get

$$\begin{aligned}
&|T_\varphi f(z)\mathrm{e}^{-\frac{\alpha|z|^2}{2}}| \\
\leq &\int_{\mathbb{C}} |\mathrm{e}^{\alpha z\overline{w}} \varphi(z-\overline{w}) f(w) \mathrm{e}^{-\frac{\alpha|z|^2}{2}}| \mathrm{d}\lambda_\alpha(w) \\
= &\frac{\alpha}{\pi} \int_{\mathbb{C}} |f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}||\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha|z-w|^2}{2}} \mathrm{d}A(w) \\
\leq &\frac{\alpha}{\pi} \Big(\int_{\mathbb{C}} |\varphi(z-\overline{w})| \mathrm{e}^{-\frac{\alpha|z-w|^2}{2}} \mathrm{d}A(w) \Big)^{\frac{p-1}{p}} \\
&\cdot \Big(\int_{\mathbb{C}} |f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p |\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha|z-w|^2}{2}} \mathrm{d}A(w) \Big)^{\frac{1}{p}}.
\end{aligned}$$

Using hypothesis and Fubini's Theorem, we have

$$\begin{aligned}
&\|T_\varphi f\|_{p,\alpha}^p \\
= &C \int_{\mathbb{C}} |T_\varphi f(z) \mathrm{e}^{-\frac{\alpha|z|^2}{2}}|^p \mathrm{d}A(z) \\
\lesssim &\int_{\mathbb{C}} \int_{\mathbb{C}} |f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p |\varphi(z-\overline{w})| \mathrm{e}^{-\frac{\alpha|z-w|^2}{2}} \mathrm{d}A(w)\mathrm{d}A(z) \\
= &\int_{\mathbb{C}} |f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p \mathrm{d}A(w) \int_{\mathbb{C}} |\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha|z-w|^2}{2}} \mathrm{d}A(z) \\
\lesssim &\int_{\mathbb{C}} |f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p \mathrm{d}A(w).
\end{aligned}$$

Therefore,

$$\|T_\varphi f\|_{p,\alpha} \leq C\|f\|_{p,\alpha}.$$

Since $\|f\|_{q,\alpha} \lesssim \|f\|_{p,\alpha}$ if $p \leq q$ for any $f \in F_\alpha^p$, we see that T_φ is bounded from F_α^p to F_α^q. This completes the proof.

We also give a sufficient condition for T_φ to be a compact operator from F_α^p to F_α^q.

Theorem 3.3 *Let* $1 < p \leq q < \infty$. *If*

$$\lim_{|a|\to\infty} \int_{\mathbb{C}} |\varphi(z-\overline{a})| \mathrm{e}^{-\frac{\alpha}{2}|z-a|^2} \mathrm{d}A(z) = 0,$$

then T_φ *is compact from* F_α^p *to* F_α^q.

Proof By the argument in the proof of Theorem 3.2, we have

$$\begin{aligned}&\|T_\varphi f\|_{q,\alpha}^q\\ \le& C\int_{\mathbb{C}}\int_{\mathbb{C}}|f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha|z-w|^2}{2}}\mathrm{d}A(w)\mathrm{d}A(z)\\ =&C\int_{\mathbb{C}}|f(w)\mathrm{e}^{-\frac{\alpha|w|^2}{2}}|^p\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha|z-w|^2}{2}}\mathrm{d}A(z)\mathrm{d}A(w).\end{aligned}$$

Let f_n be a sequence in F_α^p such that

$$\sup_n \|f_n\|_{p,\alpha}<\infty$$

and

$$f_n\to 0$$

uniformly on compact sets in $\mathbb{C}$. For any $0<r<\infty$, we denote

$$D_r=\{w\in\mathbb{C}:|w|>r\}.$$

Then,

$$\begin{aligned}&\|T_\varphi f_n\|_{q,\alpha}^q\\ \le& C\int_{D_r}|f_n(w)\mathrm{e}^{-\frac{\alpha}{2}|w|^2}|^p\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha}{2}|z-w|^2}\mathrm{d}A(z)\mathrm{d}A(w)\\ &+\int_{\{w\in\mathbb{C}:|w|\le r\}}|f_n(w)\mathrm{e}^{-\frac{\alpha}{2}|w|^2}|^p\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha}{2}|z-w|^2}\mathrm{d}A(z)\mathrm{d}A(w)\\ =&J_1+J_2.\end{aligned}$$

We estimate each item independently. We first estimate J_1.

$$\begin{aligned}J_1&=\int_{D_r}|f_n(w)\mathrm{e}^{-\frac{\alpha}{2}|w|^2}|^p\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha}{2}|z-w|^2}\mathrm{d}A(z)\mathrm{d}A(w)\\ &\le \|f_n\|_{p,\alpha}^p\sup_{w\in D_r}\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha}{2}|z-w|^2}\mathrm{d}A(z).\end{aligned}$$

Since $\sup_n\|f_n\|_{p,\alpha}<\infty$, we see that $J_1\to 0$ as $r\to\infty$. We next estimate J_2.

$$\begin{aligned}J_2&=\int_{\{w\in\mathbb{C}:|w|\le r\}}|f_n(w)\mathrm{e}^{-\frac{\alpha}{2}|w|^2}|^p\int_{\mathbb{C}}|\varphi(z-\overline{w})|\mathrm{e}^{-\frac{\alpha}{2}|z-w|^2}\mathrm{d}A(z)\mathrm{d}A(w)\\ &\le C\sup_{\{w\in\mathbb{C}:|w|\le r\}}|f_n(w)|.\end{aligned}$$

Since $f_n\to 0$ uniformly on compact sets in $\mathbb{C}$, we get that $J_2\to 0$ as $n\to\infty$. Therefore, $\|T_\varphi f_n\|_{q,\alpha}$ converges to 0 as $n\to\infty$. So, T_φ is a compact operator. This completes the proof.

Acknowledgement The authors would like to thank the referee for his/her careful reading and valuable comments.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Cao, G., Li, J., Shen, M., et al., A boundedness cirterion for singular integral operators of convolution type on the Fock space, *Adv. Math.*, **363**, 2020, 1–33.

[2] Cascante, C., Fàbrega, J. and Pascuas, D., Boundedness of the Bergman projection on generalized Fock-Sobolev spaces on $\mathbb{C}^n$, *Complex Anal. Oper. Theory*, **14**(2), 2020, 26 pp.

[3] Coburn, L., Isralowitz, J. and Li, B., Toeplitz operators with BMO symbols on the Segal-Bargmann space, *Trans. Amer. Math. Soc.*, **363**(6), 2011, 3015–3030.

[4] Folland, G., Harmonic Analysis in Phrase Space, Ann. of Math. Stud., 122, Princeton, NJ, Princeton University Press, 1989.

[5] Isralowitz, J. and Zhu, K., Toeplitz operators on the Fock space, *Integral Equations Operator Theory*, **66**(4), 2010, 593–611.

[6] Janson, S., Peetre, J. and Rochberg, R., Hankel forms and the Fock space, *Rev. Mat. Iberoam.*, **3**(1), 1987, 61–138.

[7] Lou, Z., Zhu, K. and Zhu, S., Linear operators on Fock spaces, *Integral Equations Operator Theory*, **88**(2), 2017, 1–14.

[8] Luecking, D., Embedding theorems for spaces of analytic functions via Khinchine's inequality, *Michigan Math. J.*, **40**(2), 1993, 333–358.

[9] Okikiolu, G. O., Aspects of the Theory of Bounded Integral Operators in L^p-Spaces, Academic, London, 1971.

[10] Wang, X., Cao, G. and Zhu, K., Boundedness and compactness of operators on the Fock space, *Integral Equations Operator Theory*, **77**(3), 2013, 355–370.

[11] Zhu, K., Analysis on Fock Spaces, Springer-Verlag, New York, 2012.

[12] Zhu, K., Singular integral operators on the Fock space, *Integral Equations Operator Theory*, **81**(4), 2015, 451–454.