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Global Solvability for a Predator-Prey Model with Prey-Taxis and Rotational Flux Terms*

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Abstract In this paper the author investigates the following predator-prey model with prey-taxis and rotational flux terms

$$\begin{cases} u_t = \Delta u - \nabla\cdot(uS(x,u,v)\nabla v) + \gamma uF(v) - uh(u), & x\in\Omega,\quad t>0,\\ v_t = D\Delta v - uF(v) + f(v), & x\in\Omega,\quad t>0 \end{cases} \qquad (*)$$

in a bounded domain with smooth boundary. He presents the global existence of generalized solutions to the model $(*)$ in any dimension.

Keywords Predator-prey, Prey-taxis, Global existence, Rotational flux
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1 Introduction

In complex ecosystems, predator pursuit of prey is a fundamental binary interaction, and various mathematical models have been built to describe such predator-prey relation to predict long term outcome and impact on the entire ecosystem (see [28, 39]). The relationship between predators and their preys has long been and will continue to be one of the research hotspots in both ecology and mathematical ecology because of its universal existence and important significance (see [12]). Predator-prey theory is one of the most mature theories in population ecology. In particular, the predator-prey system is a typical inhibition model, which greatly changes the understanding of the species diversity in the biome (see [24]).

Numerous reaction-diffusion equations have been applied to model the spatial predator-prey distributions (see [7, 9, 15, 30, 47–48]). In the spatial predator-prey interaction, besides the random diffusion of predator and prey, the predator has the tendency to move towards the region with higher density of prey population. Prey-taxis, the movement of predators towards the area with higher-density of prey population, plays an extremely important part in biological control and ecological balance such as maintaining the pest population below some economic threshold or leading to outbreaks of pest density (see [27, 40]). Karevia and Odell first derived a

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PDE prey-taxis model to illustrate a population model of spatially heterogeneous predator-prey interactions (see [20]). The prototypical prey-taxis model can be written as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\rho(u,v)\nabla v) + \gamma u F(v) - u h(u), & x \in \Omega,\ t > 0, \\ v_t = D\Delta v - uF(v) + f(v), & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, where $u = u(x,t)$ denotes the predator density at position x and time $t > 0$ and $v = v(x,t)$ denotes the prey population density; the term $-\nabla \cdot (\rho(u,v)\nabla v)$ stands for the prey-taxis with a coefficient $\rho(u,v)$ which may depend on the predator or prey density and D is the prey diffusion rate. $uF(v)$ stands for the inter-specific interaction, $uh(u)$ accounts for the intra-specific interaction. $F(v)$ accounts for the intake rate of predators, $h(u)$ is the predator mortality rate function and $f(v)$ is the prey growth function. The parameter $\gamma > 0$ denotes the intrinsic predation rate. Since then, various reaction-diffusion models have been proposed to interpret the prey-taxis phenomenon (see [3, 10, 14, 16, 18, 21–22, 40–41, 43–45, 49, 51, 55, 57]): That is, Wang et al. [49] studied nonconstant positive steady states of a wide class of prey-taxis systems with general functional responses over 1D domain. Linearized stability of the positive equilibrium is analyzed to show that prey-taxis destabilizes prey-predator homogeneity when prey repulsion (e.g., due to volume-filling effect in predator species or group defense in prey species) is presented, and prey-taxis stabilizes the homogeneity otherwise. When $F(v) = \frac{v}{\lambda+v}$, $h(v) = \theta$, $f(v) = \mu v\big(1 - \frac{v}{K}\big)$ and $\rho(u,v) = \rho_1(u)$ depends only on u but is truncated at some number $u_* > 0$ (i.e., $\rho_1(u_*) = 0$ and $\rho_1(u) > 0$ for $0 \leq u < u_*$), Ainseba et al. [3] obtained the global weak solutions of (1.1) for $N \geq 1$ by the Schauder fixed point theorem and duality technique, Tao [41] extended the global weak solutions to the global classical solutions for $N \leq 3$ via L^p-estimates and Schauder estimates, where the bound of solution depends on time. He and Zheng [16] improved the result of [41] by obtaining the uniform-in-time boundedness of solutions. When $\rho(u,v) = \chi u$, $h(u) = a - \mu u$, $f(v) = v(c - \beta v)$ and $\sup\limits_{s \in [0,\infty)} F(s) \leq K$, Wang and Wang [45] proved that system (1.1) admits a unique nonnegative global classical solution in two space dimensions with $\beta \equiv 0$. Moreover, if $\mu > \chi K(3 + \frac{\sqrt{10}}{2} + \frac{2+\sqrt{10}}{D})$, system (1.1) also possesses a unique nonnegative global classical solution in three-dimensional bounded domain with $\beta \equiv 0$. When $\rho(u,v) = \chi\rho_2(u)$ and $\rho_2(u) \leq u$, Wu et al. [55] considered various functional forms of $F(v)$, $h(u)$ and $f(v)$, and showed that the solution was globally bounded if χ is small. The asymptotic behavior of solutions was derived for some particularized predator-prey interactions under certain conditions. When $\rho(u,v) = \chi u$, by deriving an entropy-like equality and a boundedness criterion, Jin and Wang [18] showed that the intrinsic interaction between predators and preys was sufficient to prevent the population overcrowding even the prey-taxis was included and strong. Furthermore, by constructing appropriate Lyapunov functionals, they showed that prey-only steady state was globally asymptotically stable if the predation is weak, and the co-existence steady state was globally asymptotically stable under some conditions (like the prey-taxis is weak or the prey diffuses fast) if the predation is strong. The convergence rates of solutions to the steady states were derived. When $\rho(u,v) = \chi u$, $F(v) = v$, $h(u) = \rho$ and $f(v) = \mu v(1 - \alpha v)$, Winkler [51] proved that if $N \leq 5$, then for all appropriately regular initial data system (1.1) admits a global weak solution at first. To the best of our knowledge, this provides the first result on global existence in a system (1.1) in a spatially three-dimensional setting when arbitrarily large initial data and parameters are involved. Then, under the additional hypotheses that $N \leq 3$, $\rho = 0$ and $\mu < \frac{16D\alpha}{\chi^2}$, it is shown that each of these solutions becomes eventually smooth and stabilizes toward a spatially homogeneous equilibrium in the sense that $u(\cdot,t) \to u_\infty$ in $L^\infty(\Omega)$ and $v(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$, where u_∞ is a constant fulfilling $u_\infty \geq \frac{1}{|\Omega|}\int_\Omega u_0$. When $\rho(u,v) = \chi u$, $F(v) = \gamma v$, $h(u) = \rho$ and $f(v) = \mu v(1 - v)$, Li [21] showed that the two-dimensional system (1.1) possesses a unique global-bounded classical solution. Furthermore, she used some higher-order estimates to obtain the classical solutions with uniform-in-time bounded for suitably small initial data. In addition, the asymptotic behavior of the solutions is studied. When $\rho(u,v) = \chi u$, $F(v) = v$, $h(u) = a_1$, $f(v) = a_2 v$ and Δu is replaced by $d_1 \Delta u$, Xiang [57] proved that, for any regular initial data, system (1.1) admits a unique global smooth solution for arbitrary size of χ, and it is uniformly bounded in time in the case of $a_2 \leq 0$. In the latter case, we further study its long time dynamics, which in particular imply that the prey-tactic cross-diffusion and even the linear instability of the semi-trivial constant steady states $(0, v_*)$ with $v_* > \frac{a_1}{\gamma}$, $\gamma > 0$ and $a_2 \equiv 0$ still cannot induce pattern formation. More specifically, it is shown that (u, v) converges exponentially to $(0, 0)$ in the case that the net growth rate of prey is negative, i.e., $a_2 < 0$.

For the prey-predator model with indirect prey-taxis, the model is given by

$$\begin{cases} u_t = d_u \Delta u + u f(u) - v g(u,v), & x \in \Omega,\ t > 0, \\ c_t = d_c \Delta c + \alpha u - \beta c, & x \in \Omega,\ t > 0, \\ v_t = d_v \Delta v - \nabla \cdot (\chi v \nabla c) - r v g(u,v) - k v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary, u and v represent population densities of prey and predator, respectively. In addition, c is the concentration of chemical density secreted by the prey with a constant rate $\alpha > 0$. Moreover, c has constant diffusivity d_c and decays with constant rate $\beta > 0$. The conversion rate of the species and the decay rate of v are specified as $r > 0$ and $k > 0$, respectively. Ahn and Yoon [1] proved the global existence and uniform boundedness of solutions to the model for general functional responses in any spatial dimensions. Moreover, through linear stability analysis, it turns out that prey-taxis is an essential factor in generating pattern formations. This result differs in that the destabilizing effect of taxis does not occur in the direct prey-taxis case. In addition, they showed the global stability of the semi-trivial steady state and coexistence steady state for some specific functional responses, and gave numerical examples to support the analytic results. Wang and Wang [46] considered the following model

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(u\chi(w)\nabla w) + bug(v) - uh(u), & x\in\Omega,\ t>0,\\ w_t = d_2\Delta w + rv - \mu w, & x\in\Omega,\ t>0,\\ v_t = d_3\Delta v + f(v) - ug(v), & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

they investigated the global existence and boundedness of the unique classical solution. Then they studied the asymptotic stabilities of nonnegative spatially homogeneous equilibria. Moreover, the convergence rates were also studied. And other related model, we recommend that readers refer to the literature [2, 8, 13, 19, 25, 29, 31, 34–35, 37–38, 52, 56].

Based on recent experiments, the movement of cells is not directed toward the concentration of chemical signal, but with a rotational motion. Consequently, the chemotactic sensitivity is a tensor, see [23, 58–59] for more details. In this paper we investigate the following predator-prey model with prey-taxis and rotational flux terms:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(uS(x,u,v)\nabla v) + \gamma uF(v) - uh(u), & x\in\Omega,\ t>0,\\ v_t = D\Delta v - uF(v) + f(v), & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, where $u=u(x,t)$ denotes the predator density at position x and time $t>0$ and $v=v(x,t)$ the prey population density. The parameters $D,\gamma>0$. Throughout this paper, we assume that $F(v)$, $f(v)$ and $h(u)$ fulfill the following hypotheses:

(H1) $F(v)\in C^2([0,\infty))$, $F(0)=0$, $F(v)>0$ in $(0,\infty)$ and $F'(v)>0$, $F''(v)\le 0$ on $[0,\infty)$.

(H2) The function $h:[0,\infty)\to(0,\infty)$ is continuously differentiable and there exist three constants $\alpha,\beta,\theta>0$ such that $h(u)\ge\beta u^\alpha+\theta$ for any $u\ge 0$.

(H3) The function $f:[0,\infty)\to\mathbb{R}$ is continuously differentiable satisfying $f(0)=0$, and there exist two constants $\mu,K>0$ such that $f(v)\le\mu v$ for any $v\ge 0$, $f(K)=0$ and $f(v)<0$ for all $v>K$. Moreover the ratio $\frac{f(v)}{F(v)}$ is continuous on $(0,\infty)$ and $\lim\limits_{v\to 0}\frac{f(v)}{F(v)}$ exists.

(H4) The chemotactic sensitivity tensor $S=(S_{ij})_{i,j\in\{1,\cdots,N\}}$ fulfills $S_{ij}\in C^2(\overline{\Omega}\times[0,\infty)\times[0,\infty))$ for $i,j\in\{1,\cdots,N\}$ and with some nondecreasing function S_0 on $[0,\infty)$ such that $|S(x,u,v)|\le S_0(v)$ for all $(x,u,v)\in\overline{\Omega}\times[0,\infty)\times[0,\infty)$.

(H5) The initial data (u_0,v_0) satisfies $u_0\in L^1(\Omega)$, $v_0\in W^{1,\infty}(\Omega)$ and $u_0,v_0\ge 0$.

Remark 1.1 In order to avoid the tedious discussion of different situations, in what follows, we may assume that $0 < \alpha < 1$ since that the global existence had been proved for the case $\alpha \geq 1$ in some previous papers (see [18–19]).

Now, we state the main result in this paper.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ *be a bounded domain with smooth boundary,* $D, \gamma > 0$*. Assume that* F, h, f *and* S *satisfy* (H1)–(H4)*. Then for any choice of initial data* (u_0, v_0) *satisfying* (H5)*, the problem* (1.4) *possesses at least one global generalized solution*

$$u \in L^1_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)),$$

$$v \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)) \cap L^2_{\mathrm{loc}}([0, \infty); W^{1,2}(\Omega))$$

in the sense of Definition 2.1 *below. This solution can be obtained as the limit a.e. in* $\Omega \times (0, \infty)$ *of a sequence* $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon = \varepsilon_j \searrow 0}$ *of smooth classical solutions to the regularized problems* (2.10) *below.*

Remark 1.2 Theorem 1.1 partly generalizes the results in [18–19, 55].

In this paper, we use symbols C_i and c_i $(i = 1, 2, \cdots)$ as some generic positive constants which may vary in the context. For simplicity, $u(x, t)$ is written as u, the integral $\int_\Omega u(x)\mathrm{d}x$ is written as $\int_\Omega u(x)$ and $\int_0^t \int_\Omega u(x, t)\mathrm{d}x\mathrm{d}t$ is written as $\int_0^t \int_\Omega u(x, t)$.

The contents of the present paper as follows. In Section 2, we first introduce the concept of generalized solutions and then give the global existence result for system (1.4). In Section 3, we give some fundamental estimates for the solution to system (1.4). In Section 4, we prove Theorem 1.1.

2 Preliminaries

In this section, motivated by the thought from [51], and also [26, 32, 36, 53–54, 60], we first introduce the concept of generalized solution and then give the global existence result for the system (2.10) below.

Definition 2.1 *Assume that* F, h, f *and* S *satisfy* (H1)–(H4)*, and that the initial data* (u_0, v_0) *fulfills* (H5)*. Let*

$$\begin{cases} u \in L^1_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)), \\ v \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)) \cap L^2_{\mathrm{loc}}([0, \infty); W^{1,2}(\Omega)) \end{cases} \tag{2.1}$$

be nonnegative and satisfy

$$\gamma u F(v) - u h(u) \in L^1_{\mathrm{loc}}(\Omega \times [0, \infty)). \tag{2.2}$$

Then (u,v) *will be called a global generalized solution of* (1.4) *if*

$$\begin{aligned}&\int_0^\infty\int_\Omega v\psi_t-\int_\Omega v_0\psi(\cdot,0)\\ =&D\int_0^\infty\int_\Omega \nabla v\cdot\nabla\psi+\int_0^\infty\int_\Omega uF(v)\psi-\int_0^\infty\int_\Omega f(v)\psi\end{aligned}\tag{2.3}$$

for all $\psi\in C_0^\infty(\Omega\times[0,\infty))$, *if*

$$\int_\Omega u(\cdot,t)\le\int_\Omega u_0+\gamma\int_0^t\int_\Omega uF(v)-\int_0^t\int_\Omega uh(u)\quad \textit{for a.e. } t>0\tag{2.4}$$

and if there exists a function $\phi\in C^2([0,\infty))$ *fulfilling*

$$\begin{aligned}&\phi(u),\ \phi''(u)|\nabla u|^2,\ uF(v)\phi'(u),\ uh(u)\phi'(u)\in L^1_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty)),\\ &u\phi''(u)\nabla u,\ u\phi'(u)\in L^2_{\mathrm{loc}}(\overline{\Omega}\times[0,\infty))\end{aligned}$$

and that for all nonnegative $\psi\in C_0^\infty(\Omega\times[0,\infty))$, *the inequality*

$$\begin{aligned}&-\int_0^\infty\int_\Omega\phi(u)\psi_t-\int_\Omega\phi(u_0)\psi(\cdot,0)\\ \ge&-\int_0^\infty\int_\Omega\phi''(u)|\nabla u|^2\psi+\int_0^\infty\int_\Omega u\phi'(u)(\nabla u\cdot(S(x,u,v)\nabla v))\psi\\ &+\int_0^\infty\int_\Omega\phi(u)\Delta\psi+\int_0^\infty\int_\Omega u\phi'(u)(S(x,u,v)\nabla v)\cdot\nabla\psi\\ &+\int_0^\infty\int_\Omega uF(v)\phi'(u)\psi-\int_0^\infty\int_\Omega uh(u)\phi'(u)\psi\end{aligned}\tag{2.5}$$

holds.

In order to introduce an appropriate regularization of (1.4), now let us fix families $\{u_{0\varepsilon}\}_{\varepsilon\in(0,1)}\subset C^0(\overline{\Omega})$ and $(v_{0\varepsilon})_{\varepsilon\in(0,1)}\subset W^{1,\infty}(\Omega)$ such that $u_{0\varepsilon}$ and $v_{0\varepsilon}$ are nonnegative for all $\varepsilon\in(0,1)$, and that as $\varepsilon\to 0$ we have

$$u_{0\varepsilon}\to u_0\quad\text{and}\quad v_{0\varepsilon}\to v_0\quad\text{in } L^1(\Omega) \text{ a.e. in }\Omega$$

as well as

$$\|u_{0\varepsilon}-u_0\|_{L^1(\Omega)}\le 1\quad\text{and}\quad\|v_{0\varepsilon}-v_0\|_{L^1(\Omega)}\le 1\quad\text{for all }\varepsilon\in(0,1).\tag{2.6}$$

Moreover, we fix nonnegative families $\{\xi_\varepsilon\}_{\varepsilon\in(0,1)}\subset C_0^\infty(\Omega)$ and $\{\zeta_\varepsilon\}_{\varepsilon\in(0,1)}\subset C_0^\infty([0,\infty))$ with

$$0\le\xi_\varepsilon\le 1\quad\text{in }\Omega\quad\text{and}\quad\xi_\varepsilon\nearrow 1\quad\text{in }\Omega\ \text{ as }\varepsilon\searrow 0\tag{2.7}$$

as well as

$$\zeta_\varepsilon(s)\begin{cases}=1, & 0\le s\le\dfrac{1}{\varepsilon},\\ \le 1, & \dfrac{1}{\varepsilon}<s<\dfrac{2}{\varepsilon},\\ =0, & s\ge\dfrac{2}{\varepsilon},\end{cases}$$

which fulfills

$$0 \le \zeta_\varepsilon \le 1 \quad \text{in } [0,\infty) \quad \text{and} \quad \zeta_\varepsilon \nearrow 1 \quad \text{in } [0,\infty) \quad \text{as } \varepsilon \searrow 0. \tag{2.8}$$

Let

$$S_\varepsilon(x,u,v) = \xi_\varepsilon(x)\zeta_\varepsilon(u)S(x,u,v), \quad x \in \overline{\Omega},\ u,v \ge 0, \tag{2.9}$$

then we consider the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon) + \gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon), & x \in \Omega,\ t>0, \\ v_{\varepsilon t} = D\Delta v_\varepsilon - u_\varepsilon F(v_\varepsilon) + f(v_\varepsilon), & x \in \Omega,\ t>0. \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x),\ v_\varepsilon(x,0) = v_{0\varepsilon}(x), & x \in \Omega \end{cases} \tag{2.10}$$

for $\varepsilon \in (0,1)$. All of these problems (2.10) are indeed global solvable in the classical sense.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a bounded domain with smooth boundary, $D,\gamma > 0$. Assume that F,h,f and S satisfy* (H1)–(H4). *Then for any choice of initial data (u_0,v_0) satisfying* (H5) *and for each $\varepsilon \in (0,1)$, there exist functions*

$$u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$$

and

$$v_\varepsilon \in \bigcap_{p>\max\{2,N\}} C^0([0,\infty);W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$$

such that $(u_\varepsilon,v_\varepsilon)$ solves (2.10) *classically.*

Proof Local existence and uniqueness of a smooth solution in $\overline{\Omega} \times [0,T_{\max,\varepsilon})$ can be constructed by a well-established contraction mapping argument for suitably small $T_{\max,\varepsilon} > 0$ as in [17, 33] or by Amann's theorem (see [4–5]), where $T_{\max,\varepsilon}$ denotes the maximal existence time. From the maximum principle, the nonnegativity of u_ε and v_ε are obtained. Since $S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \equiv 0$ for all sufficiently large u_ε, standard estimation techniques yield extensibility of this local solution for all times as in [17]. The proof is complete.

3 Some Lemmas

In this section, we shall give some lemmas which will be used in proving the main theorem in the next section.

Lemma 3.1 *Let $\varepsilon \in (0,1)$ and $T \in (0,\infty]$ as well as nonnegative functions u_ε and v_ε belong to $C^{2,1}(\overline{\Omega} \times (0,T))$ and such that* (2.10) *is satisfied in $\Omega \times (0,T)$. If $\phi \in C^2([0,\infty))$, then for* *any arbitrary* $\psi \in C^\infty(\overline{\Omega} \times (0,T))$ *with* $\frac{\partial \psi}{\partial \nu} = 0$ *on* $\partial\Omega \times (0,T)$*, the equality*

$$\begin{aligned}
&\int_\Omega \partial_t\{\phi(u_\varepsilon)\}\psi \\
=& -\int_\Omega \phi''(u_\varepsilon)|\nabla u_\varepsilon|^2\psi + \int_\Omega u_\varepsilon\phi''(u_\varepsilon)\nabla u_\varepsilon \cdot (S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\psi \\
&- \int_\Omega \phi'(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\psi + \int_\Omega u_\varepsilon\phi'(u_\varepsilon)(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\nabla\psi \\
&+ \gamma\int_\Omega u_\varepsilon F(v_\varepsilon)\phi'(u_\varepsilon)\psi - \int_\Omega u_\varepsilon h(u_\varepsilon)\phi'(u_\varepsilon)\psi
\end{aligned} \tag{3.1}$$

holds for all $t \in (0,T)$ *and each* $\varepsilon \in (0,1)$*.*

Proof By the straightforward calculation, we have

$$\begin{aligned}
&\int_\Omega \partial_t\{\phi(u_\varepsilon)\}\psi \\
=& \int_\Omega \phi'(u_\varepsilon)\psi[\Delta u_\varepsilon - \nabla\cdot(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon) \\
&+ \gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)] \\
&= -\int_\Omega \nabla\{\phi(u_\varepsilon)\psi\}\cdot(\nabla u_\varepsilon - u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon) \\
&+ \gamma\int_\Omega u_\varepsilon F(v_\varepsilon)\phi'(u_\varepsilon)\psi - \int_\Omega u_\varepsilon h(u_\varepsilon)\phi'(u_\varepsilon)\psi \\
=& -\int_\Omega \phi''(u_\varepsilon)|\nabla u_\varepsilon|^2\psi + \int_\Omega u_\varepsilon\phi''(u_\varepsilon)\nabla u_\varepsilon \cdot (S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\psi \\
&- \int_\Omega \phi'(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\psi + \int_\Omega u_\varepsilon\phi'(u_\varepsilon)(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\nabla\psi \\
&+ \gamma\int_\Omega u_\varepsilon F(v_\varepsilon)\phi'(u_\varepsilon)\psi - \int_\Omega u_\varepsilon h(u_\varepsilon)\phi'(u_\varepsilon)\psi
\end{aligned}$$

for all $t \in (0,T)$ and each $\varepsilon \in (0,1)$. The proof is complete.

Lemma 3.2 *Assume that the conditions of Theorem* 1.2 *hold, the solution* $(u_\varepsilon, v_\varepsilon)$ *of* (2.10) *fulfills*

$$0 < v_\varepsilon(x,t) \le K_0 \quad for\ all\ (x,t) \in \Omega \times (0,\infty) \tag{3.2}$$

and each $\varepsilon \in (0,1)$*, where*

$$K_0 := \max\{\|v_0\|_{L^\infty(\Omega)} + 1, K\}.$$

In particular,

$$\|v_\varepsilon\|_{L^\infty(\Omega)} \le K_0 \quad for\ any\ \varepsilon \in (0,1)\ and\ all\ t > 0. \tag{3.3}$$

Proof The proof can be found in [18]. To avoid repetition, we omit giving details on this here.

Lemma 3.3 *There exists $C_1 > 0$ such that*

$$\gamma sF(\widetilde{s}) - sh(s) \leq C_1 - s \quad for\ all\ s, \widetilde{s} \geq 0. \tag{3.4}$$

Proof From (H1) and (H2), we obtain

$$\gamma sF(\widetilde{s}) - sh(s) \leq \gamma sF(K_0) - \beta s^{1+\alpha} - \theta s,$$

where $\alpha > 0$. By Young's inequality, it is easy to see that (3.4) is valid with

$$C_1 = \frac{\alpha\beta[\gamma F(K_0) + 1 - \theta]^{\frac{1+\alpha}{\alpha}}}{(1+\alpha)^{\frac{1+\alpha+\alpha^2}{\alpha^2}}}.$$

The proof is complete.

Lemma 3.4 *There exists $C_2 > 0$ independent of t such that*

$$\|u_\varepsilon\|_{L^1(\Omega)} < C_2 \quad for\ all\ \varepsilon \in (0,1) \tag{3.5}$$

and all $t > 0$. For any $T > 0$ there exists $C_3 = C_3(T) > 0$ such that

$$\int_0^T \int_\Omega |\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)| \leq C_3 \tag{3.6}$$

for all $\varepsilon \in (0,1)$. Moreover, $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\left\{\frac{\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)}{u_\varepsilon + 1}\right\}$ are uniformly integrable over $\Omega \times (0,T)$.

Proof By Lemma 3.3, there exists $c_1 > 0$ such that

$$\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon) \leq c_1 - u_\varepsilon \quad \text{for all } \varepsilon \in (0,1). \tag{3.7}$$

Integrating the first equation of (2.10) and using (3.7), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_\varepsilon \leq c_1|\Omega| - \int_\Omega u_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

which yields

$$\|u_\varepsilon\|_{L^1(\Omega)} \leq \max\{\|u_0\|_{L^1(\Omega)} + 1, c_1|\Omega|\}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Let

$$g_\varepsilon(u_\varepsilon, v_\varepsilon) = \gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon).$$

It is well-known that

$$|g_\varepsilon| = g_{\varepsilon+} + g_{\varepsilon-}$$

with

$$g_{\varepsilon+} := \max\{g_\varepsilon, 0\}$$

and

$$g_{\varepsilon-} := \max\{-g_\varepsilon, 0\}.$$

Once more employing Lemma 3.3, there exist $c_2, c_3 > 0$ and $s_* > 0$ such that

$$g_{\varepsilon+}(s, \widetilde{s}) \le c_2 \quad \text{for all } s, \widetilde{s} \ge 0$$

and

$$g_{\varepsilon-} \ge s - c_3 \quad \text{for all } s \ge s_*.$$

Then, we get

$$\begin{aligned}
&\int_0^T \int_\Omega |g_\varepsilon(u_\varepsilon, v_\varepsilon)| \\
&= \int_0^T \int_\Omega g_{\varepsilon+}(u_\varepsilon, v_\varepsilon) + \int_0^T \int_\Omega g_{\varepsilon-}(u_\varepsilon, v_\varepsilon) \\
&\le 2c_1|\Omega|T + \int_\Omega u_{0\varepsilon} \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Then, for any $T > 0$ there exists $c_4 > 0$ such that

$$\int_0^T \int_\Omega |\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)| \le c_4 \quad \text{for all } \varepsilon \in (0,1).$$

For any $\varsigma_1 > 0$ we take $L_1 = L_1(\varsigma_1, T) > 0$ large enough satisfying $\frac{c_4}{L_1} \le \frac{\varsigma_1}{2}$. By (H1)–(H2), there exists $s_{**} = s_{**}(\varsigma_1, T) > 0$ such that $g(s, \widetilde{s}) \le -L_1 s$ for all $s > s_{**}$ and $\widetilde{s} \ge 0$, and thus,

$$|g(s, \widetilde{s})| \ge L_1 s \quad \text{for all } s_{**} \text{ and } \widetilde{s} \ge 0,$$

then, we can choose $\eta = \eta(\varsigma_1, T) > 0$ appropriately small such that $s_{**}\eta \le \frac{\varsigma_1}{2}$. For any measurable $\Lambda \subset \Omega \times (0,T)$ with $|\Lambda| \le \eta$, we estimate

$$\begin{aligned}
\int\int_\Lambda u_\varepsilon &= \int\int_{\Lambda \cap \{u_\varepsilon \le s_{**}\}} u_\varepsilon + \int\int_{\Lambda \cap \{u_\varepsilon > s_{**}\}} u_\varepsilon \\
&\le s_{**} \cdot |\Lambda \cap \{u_\varepsilon \le s_{**}\}| + \int\int_{\Lambda \cap \{u_\varepsilon > s_{**}\}} \frac{|g_\varepsilon(u_\varepsilon, v_\varepsilon)|}{L} \\
&\le s_{**} \cdot |\Lambda| + \frac{1}{L_1} \int_0^T \int_\Omega |g_\varepsilon(u_\varepsilon, v_\varepsilon)| \\
&\le s_{**}\eta + \frac{c_4}{L_1} \le \frac{\varsigma_1}{2} + \frac{\varsigma_1}{2} \quad \text{for all } \varepsilon \in (0,1).
\end{aligned}$$

By the definition of uniformly integrable (see [11]), we know that $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ is uniformly integrable over $\Omega \times (0,T)$. Given $\varsigma_2 > 0$, we take $c_5 = c_5(\varsigma_2, T) > 0$ large enough such that $\frac{c_4}{c_5+1} \le \frac{\varsigma_2}{2}$, whereupon the continuity of F and h on $[0, c_5]$, we can find $c_6 = c_6(\varsigma_2, T) > 0$ fulfilling

$$|\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)| \le c_6 \quad \text{for all } u_\varepsilon \in [0, c_5].$$

Then each measurable $\widetilde{\Lambda} \subset \Omega \times (0,T)$ satisfying $|\widetilde{\Lambda}| \leq \frac{\varsigma_2}{2c_6}$ has the property that

$$\begin{aligned}
&\int\int_{\widetilde{\Lambda}} \frac{|\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)|}{u_\varepsilon + 1} \\
=&\int\int_{\widetilde{\Lambda}\cap\{u_\varepsilon \leq c_5\}} \frac{|\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)|}{u_\varepsilon + 1} \\
&+\int\int_{\widetilde{\Lambda}\cap\{u_\varepsilon > c_5\}} \frac{|\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)|}{u_\varepsilon + 1} \\
\leq& c_6|\widetilde{\Lambda}| + \frac{1}{c_5+1}\int\int_{\widetilde{\Lambda}\cap\{u_\varepsilon > c_5\}} |\gamma u_\varepsilon F(v_\varepsilon) - u_\varepsilon h(u_\varepsilon)| \\
\leq& \frac{c_4}{c_5+1} + \frac{\varsigma_2}{2} \leq \frac{\varsigma_2}{2} + \frac{\varsigma_2}{2} = \varsigma_2 \quad \text{for all } \varepsilon \in (0,1).
\end{aligned}$$

The proof is complete.

Lemma 3.5 *For any $T > 0$, there exists $C_3 = C_3(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \leq C_3 \quad \textit{for all } \varepsilon \in (0,1) \tag{3.8}$$

and

$$\int_0^T \int_\Omega F(v_\varepsilon) u_\varepsilon \leq C_3 \quad \textit{for all } \varepsilon \in (0,1). \tag{3.9}$$

Proof We test the second equation in (2.10) by v_ε and integrate by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_\varepsilon^2 + D\int_\Omega |\nabla v_\varepsilon|^2 \\
=& -\int_\Omega u_\varepsilon v_\varepsilon F(v_\varepsilon) + \int_\Omega f(v_\varepsilon) v_\varepsilon \\
\leq& -\int_\Omega u_\varepsilon v_\varepsilon F(v_\varepsilon) + \mu \int_\Omega v_\varepsilon^2 \\
\leq& -\int_\Omega u_\varepsilon v_\varepsilon F(v_\varepsilon) + \mu K_0^2 |\Omega|
\end{aligned}$$

for all $\varepsilon \in (0,1)$ and all $t > 0$. Integrating the above inequality over $(0,T)$, we obtain

$$D\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega (v_0+1)^2 + \mu K_0^2 T |\Omega|$$

for all $\varepsilon \in (0,1)$, here we used the nonnegativity of $u_\varepsilon, v_\varepsilon$ and F. Then, integrating the second equation in (2.10), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_\varepsilon + \int_\Omega u_\varepsilon F(v_\varepsilon) = \int_\Omega f(v_\varepsilon) \leq \mu \int_\Omega v_\varepsilon \leq \mu K_0 |\Omega|$$

for all $\varepsilon \in (0,1)$ and all $t \in (0,T)$. Integrating the above inequality over $(0,T)$, we deduce

$$\int_0^T \int_\Omega u_\varepsilon F(v_\varepsilon) \leq \int_\Omega (v_\varepsilon + 1) + \mu K_0 T |\Omega|$$

for all $\varepsilon \in (0,1)$. The proof is complete.

Lemma 3.6 *For any $T>0$, there exists $C_4=C_4(T)>0$ such that*

$$\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\le C_4 \quad \textit{for all } \varepsilon\in(0,1). \tag{3.10}$$

Proof Letting $\psi\equiv 1$ and $\phi(s)=\ln(s+1)$ on $[0,\infty)$ in Lemma 3.1, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln(u_\varepsilon+1) &= \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\ &\quad+\gamma\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)-\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\end{aligned} \tag{3.11}$$

for all $\varepsilon\in(0,1)$ and all $t\in(0,T)$. Integrating (3.11) over $t\in(0,T)$, we obtain

$$\begin{aligned}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} &\le \int_\Omega \ln(u_\varepsilon(\cdot,T)+1)+\int_0^T\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\ &\quad+\int_0^T\int_\Omega |\gamma u_\varepsilon F(v_\varepsilon)-u_\varepsilon h(u_\varepsilon)|\end{aligned}$$

for all $\varepsilon\in(0,1)$. Using Young's inequality, we get

$$\begin{aligned}&\Big|\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\Big|\\ &\le \frac12\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\frac12\int_\Omega\frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}|S_\varepsilon(x,u_\varepsilon,v_\varepsilon)|^2|\nabla v_\varepsilon|^2\\ &\le \frac12\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\frac12 S_0^2(K_0)\int_\Omega|\nabla v_\varepsilon|^2\end{aligned}$$

for all $\varepsilon\in(0,1)$ and all $t\in(0,T)$. Since $0\le\ln(s+1)\le s$ for all $s\ge0$,

$$\begin{aligned}\frac12\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} &\le \int_\Omega u_\varepsilon(\cdot,T)+\frac12 S_0^2(K_0)\int_0^T\int_\Omega|\nabla v_\varepsilon|^2\\ &\quad+\int_0^T\int_\Omega|\gamma u_\varepsilon F(v_\varepsilon)-u_\varepsilon h(u_\varepsilon)|.\end{aligned}$$

Combining Lemma 3.4 with Lemma 3.5, we draw the conclusion immediately. The proof is complete.

Lemma 3.7 *Let $m>\frac{N}{2}$. Then for any $T>0$ there exists $C_5=C_5(T)>0$ such that*

$$\int_0^T \|\partial_t\ln(u_\varepsilon+1)\|_{(W_0^{m,2}(\Omega))^*}\mathrm{d}t\le C_5 \quad \textit{for all } \varepsilon\in(0,1). \tag{3.12}$$

Proof Letting $\phi(s)=\ln(s+1)$ on $[0,\infty)$ in Lemma 3.1, we have

$$\begin{aligned}&\int_\Omega \partial_t\ln(u_\varepsilon+1)\cdot\psi\\ &=\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\psi-\int_\Omega\frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\psi\\ &\quad-\int_\Omega\frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi+\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\nabla\psi\end{aligned}$$

$$+\gamma\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)\psi-\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\psi \tag{3.13}$$

for all $\varepsilon\in(0,1)$ and all $t>0$ as well as arbitrary $\psi\in W_0^{m,2}(\Omega)$. Now, we estimate the right hand side of (3.13) one by one. By Hölder's inequality, we obtain

$$\Big|\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\psi\Big|\le\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)\|\psi\|_{L^\infty(\Omega)}, \tag{3.14}$$

$$\begin{aligned}&\Big|-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\psi\Big|\\ &\le S_0(K_0)\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac12}\Big(\int_\Omega|\nabla v_\varepsilon|^2\Big)^{\frac12}\|\psi\|_{L^\infty(\Omega)},\end{aligned} \tag{3.15}$$

$$\Big|-\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi\Big|\le\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac12}\|\nabla\psi\|_{L^2(\Omega)}, \tag{3.16}$$

$$\Big|\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\nabla\psi\Big|\le S_0(K_0)\Big(\int_\Omega|\nabla v_\varepsilon|^2\Big)^{\frac12}\|\psi\|_{L^\infty(\Omega)}, \tag{3.17}$$

$$\Big|\gamma\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)\psi-\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\psi\Big|\le\Big(\int_\Omega|\gamma u_\varepsilon F(v_\varepsilon)-u_\varepsilon h(u_\varepsilon)|\Big)\|\psi\|_{L^\infty(\Omega)} \tag{3.18}$$

for all $\varepsilon\in(0,1)$ and all $t>0$. Inserting (3.14)–(3.18) into (3.13), we deduce

$$\begin{aligned}&\Big|\int_\Omega \partial_t\ln(u_\varepsilon+1)\cdot\psi\Big|\\ &\le\Big\{\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+S_0(K_0)\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac12}\Big(\int_\Omega|\nabla v_\varepsilon|^2\Big)^{\frac12}\|\psi\|_{L^\infty(\Omega)}\\ &\quad+\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac12}+S_0(K_0)\Big(\int_\Omega|\nabla v_\varepsilon|^2\Big)^{\frac12}\\ &\quad+\int_\Omega|\gamma u_\varepsilon F(v_\varepsilon)-u_\varepsilon h(u_\varepsilon)|\Big\}(\|\nabla\psi\|_{L^2(\Omega)}+\|\psi\|_{L^\infty(\Omega)})\end{aligned}$$

for all $\varepsilon\in(0,1)$ and all $t>0$ as well as arbitrary $\psi\in W_0^{m,2}(\Omega)$. Owing to $m>\frac{N}{2}$, it is easy to see that $W_0^{m,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ is continuous. Thus, by Young's inequality, there exists $c_1>0$ such that

$$\begin{aligned}\Big|\int_\Omega \partial_t\ln(u_\varepsilon+1)\cdot\psi\Big|&\le c_1\Big\{1+\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_\Omega|\nabla v_\varepsilon|^2\\ &\quad+\int_\Omega|\gamma u_\varepsilon F(v_\varepsilon)-u_\varepsilon h(u_\varepsilon)|\Big\}\|\nabla\psi\|_{W_0^{m,2}(\Omega)}\end{aligned}$$

for all $\varepsilon\in(0,1)$ and all $t>0$. In accordance with Lemmas 3.4–3.6, integrating the above inequality over $(0,T)$, we obtain (3.12) immediately. The proof is complete.

Lemma 3.8 *Let $T>0$. Then*

(i) $\{\ln(u_\varepsilon+1)\}_{\varepsilon\in(0,1)}$ *is relatively compact in $L^2((0,T);W^{1,2}(\Omega))$ with respect to the weak topology, and relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology;*

(ii) $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ *is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology;*

(iii) $\{F(v_\varepsilon)u_\varepsilon\}_{\varepsilon\in(0,1)}$ *is relatively compact in $L^1(\Omega\times(0,T))$ with respect to the weak topology.*

Proof (i) Let $z_\varepsilon := \ln(u_\varepsilon + 1)$. By means of Lemmas 3.6–3.7, we have

$$\{z_\varepsilon\}_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T);W^{1,2}(\Omega))$$

and

$$\{z_{\varepsilon t}\}_{\varepsilon\in(0,1)} \text{ is bounded in } L^1((0,T);(W^{N,2}(\Omega))^*)$$

for all $T > 0$. According to Aubin-Lions lemma (see [42]), we obtain the claimed strong precompactness property.

(ii) Let $m > \frac{N}{2}$ and an arbitrary $\varphi \in W_0^{m,2}(\Omega)$. We test the second equation in (2.10) by φ, integrating by parts and using the Cauchy-Schwarz inequality, for each fixed $t \in (0,T)$, we get

$$\begin{aligned}\Big|\int_\Omega v_{\varepsilon t}\varphi\Big| \le{}& D\Big(\int_\Omega |\nabla v_\varepsilon|^2\Big)^{\frac{1}{2}} \cdot \|\nabla\psi\|_{L^2(\Omega)} + \Big(\int_\Omega u_\varepsilon F(v_\varepsilon)\Big) \cdot \|\psi\|_{L^\infty(\Omega)}\\ &+ \mu K_0|\Omega| \cdot \|\psi\|_{L^\infty(\Omega)}.\end{aligned}$$

Once more employing the embedding $W_0^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, we can find a constant $c_1 > 0$ such that

$$\int_0^T \|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^*}\mathrm{d}t \le c_1\int_0^T (1 + |\nabla v_\varepsilon|^2 + u_\varepsilon F(v_\varepsilon))\mathrm{d}t.$$

In accordance with Lemma 3.5, there exists $c_2 > 0$ such that

$$\int_0^T \|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^*}\mathrm{d}t \le c_2(1+T).$$

Thus, together with Aubin-Lions lemma (see [42]), Lemmas 3.2 and 3.5, we draw the claim immediately.

(iii) Let $z_\varepsilon := u_\varepsilon F(v_\varepsilon)$. In light of Lemma 3.2 and (H1), there exists a constant $c_3 > 0$ such that

$$\begin{aligned}&\int_0^T\int_\Omega z_\varepsilon \ln(z_\varepsilon + 1)\\ \le&\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon)\ln(F(K_0)u_\varepsilon + 1)\\ \le&\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon)\ln(c_3(u_\varepsilon + 1))\\ \le&\ln c_3\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon) + \int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon)\ln(u_\varepsilon + 1).\end{aligned}\tag{3.19}$$

By the straightforward calculation, we derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_\varepsilon \ln(u_\varepsilon + 1)\\ =&D\int_\Omega \Delta v_\varepsilon \cdot \ln(u_\varepsilon + 1) - \int_\Omega u_\varepsilon F(v_\varepsilon)\ln(u_\varepsilon + 1)\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega f(v_\varepsilon)\ln(u_\varepsilon+1)+\int_\Omega \frac{v_\varepsilon}{u_\varepsilon+1}\cdot\Delta u_\varepsilon\\
&-\int_\Omega \frac{v_\varepsilon}{u_\varepsilon+1}\nabla\cdot(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\
&+\gamma\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)-\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\quad \text{for all } t>0.
\end{aligned}
$$

Integrating the above equation over $t\in(0,T)$, we deduce

$$
\begin{aligned}
&\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon)\ln(u_\varepsilon+1)\\
&\le \int_\Omega v_0\ln(u_0+1)-(D+1)\int_0^T\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad+\int_0^T\int_\Omega \frac{v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2+\int_0^T\int_\Omega f(v_\varepsilon)\ln(u_\varepsilon+1)\\
&\quad+\int_0^T\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\nabla v_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\
&\quad-\int_0^T\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\
&\quad+\gamma\int_0^T\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon).
\end{aligned}
\tag{3.20}
$$

Applying Young's inequality, we estimate

$$
-(D+1)\int_0^T\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \le \int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+(D+1)^2\int_0^T\int_\Omega|\nabla v_\varepsilon|^2, \tag{3.21}
$$

$$
\int_0^T\int_\Omega \frac{v_\varepsilon}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\le K_0\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}, \tag{3.22}
$$

$$
\int_0^T\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\nabla v_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\le S_0(K_0)\int_0^T\int_\Omega|\nabla v_\varepsilon|^2, \tag{3.23}
$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\\
&\le K_0S_0(K_0)\Big(\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_0^T\int_\Omega|\nabla v_\varepsilon|^2\Big),
\end{aligned}
\tag{3.24}
$$

$$
\int_0^T\int_\Omega f(v_\varepsilon)\ln(u_\varepsilon+1)\le \mu K_0\int_0^T\int_\Omega u_\varepsilon \tag{3.25}
$$

and

$$
\gamma\int_0^T\int_\Omega \frac{u_\varepsilon v_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)\le \gamma K_0\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon). \tag{3.26}
$$

Substituting (3.21)–(3.26) into (3.20) and in light of Lemmas 3.4–3.6, there exists a constant $c_4>0$ such that

$$
\int_0^T\int_\Omega u_\varepsilon F(v_\varepsilon)\ln(u_\varepsilon+1)\le c_4.
$$

Together with Lemma 3.5 and (3.19), we get

$$\int_0^T \int_\Omega z_\varepsilon \ln(z_\varepsilon + 1) \le c_5$$

with $c_5 > 0$ is a constant. Given $\delta > 0$, we take $c_6 > 0$ sufficiently large fulfilling

$$\frac{c_5}{\ln(1 + c_6)} \le \frac{\delta}{2}.$$

Let $\Lambda \subset \Omega \times (0, T)$ is an arbitrary measurable set satisfying $|\Lambda| \le \frac{\delta}{2c_6}$ for $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}\int\int_\Lambda z_\varepsilon &= \int\int_{\Lambda\cap\{z_\varepsilon \ge c_6\}} z_\varepsilon + \int\int_{\Lambda\cap\{z_\varepsilon < c_6\}} z_\varepsilon \\ &\le \frac{1}{\ln(1 + c_6)} \int\int_{\Lambda\cap\{z_\varepsilon \ge c_6\}} z_\varepsilon \ln(z_\varepsilon + 1) + c_6|\Lambda| \\ &\le \frac{c_5}{\ln(1 + c_6)} + c_6|\Lambda| \le \delta.\end{aligned}$$

Owing to $\delta > 0$ is arbitrary, we know that $\{z_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly integrable over $\Omega \times (0, T)$. According to Pettis theorem (see [6]), it is easy to see that $\{F(v_\varepsilon)u_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^1(\Omega \times (0, T))$ with respect to the weak topology. The proof is complete.

Now, we are preparing to extract a suitable sequence of number ε along with the respective solutions approach a limit in appropriate topologies.

Lemma 3.9 *Assume that the conditions of Theorem* 1.1 *hold. Then there are* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, 1)$ *and nonnegative functions*

$$u \in L^1_{\mathrm{loc}}([0, \infty); L^1(\Omega)), \quad v \in L^2_{\mathrm{loc}}([0, \infty); W^{1,2}(\Omega)) \tag{3.27}$$

such that $\varepsilon_j \searrow 0$ *as* $j \to \infty$ *and*

$$u_\varepsilon \to u \quad \textit{a.e. in } \Omega \times (0, \infty), \tag{3.28}$$

$$\ln(u_\varepsilon + 1) \rightharpoonup \ln(u + 1), \quad L^2_{\mathrm{loc}}([0, \infty); W^{1,2}(\Omega)), \tag{3.29}$$

$$v_\varepsilon \to v \quad \textit{a.e. in } \Omega \times (0, \infty), \tag{3.30}$$

$$v_\varepsilon \to v \quad \textit{in } L^2_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)), \tag{3.31}$$

$$v_\varepsilon \overset{*}{\rightharpoonup} v \quad \textit{in } L^\infty_{\mathrm{loc}}(\Omega \times (0, \infty)), \tag{3.32}$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \quad \textit{in } L^2_{\mathrm{loc}}(\Omega \times (0, \infty)), \tag{3.33}$$

$$F(v_\varepsilon)u_\varepsilon \to F(v)u \quad \textit{in } L^1_{\mathrm{loc}}(\overline{\Omega} \times [0, \infty)), \tag{3.34}$$

as well as

$$\frac{\gamma F(v_\varepsilon)u_\varepsilon - u_\varepsilon h(u_\varepsilon)}{u_\varepsilon + 1} \to \frac{\gamma F(v)u - uh(u)}{u + 1} \quad \textit{in } L^1_{\mathrm{loc}}(\Omega \times (0, \infty)) \tag{3.35}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Proof By means of Lemma 3.8, (3.27)–(3.28) can be obtained through a straightforward extraction process. Together with (3.3), (3.8) and Lemma 3.8 yield that (3.30)–(3.33) hold along a further subsequence. In particular, we have

$$F(v_\varepsilon)u_\varepsilon \to F(v)u \quad a.e.\ in\ \Omega \times [0,\infty) \tag{3.36}$$

as $\varepsilon = \varepsilon_j \searrow 0$. The combination of Lemma 3.8 and Egorov's theorem guarantees that

$$F(v_\varepsilon)u_\varepsilon \rightharpoonup F(v)u \quad \text{in } L^1_{\text{loc}}(\overline{\Omega} \times [0,\infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$. According to [50, Lemma A.3] and (3.36), we know that (3.34) holds. By (3.28), (3.30), Lemma 3.4 and Vitali convergence theorem, we obtain (3.35). The proof is complete.

Lemma 3.10 *Assume that the conditions of Theorem* 1.1 *hold, and let* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ *be as in Lemma* 3.9. *Then there exists a subsequence, again denoted by* $(\varepsilon_j)_{j\in\mathbb{N}}$, *such that for a.e.* $T > 0$ *we have*

$$\nabla v_\varepsilon \to \nabla v \quad in\ L^2(\Omega \times (0,T))\ as\ \varepsilon = \varepsilon_j \searrow 0.$$

Proof In light of (3.33), we obtain

$$\int_0^T \int_\Omega |\nabla v|^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \quad \text{for all } T > 0. \tag{3.37}$$

Then, multiplying the second equation in (2.10) by v_ε and integrating over $\Omega \times (0,T)$ implies

$$\begin{aligned} D\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 = &-\int_0^T \int_\Omega u_\varepsilon v_\varepsilon F(v_\varepsilon) + \int_0^T \int_\Omega f(v_\varepsilon)v_\varepsilon \\ &+ \frac{1}{2}\int_\Omega v_0^2 - \frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,T). \end{aligned} \tag{3.38}$$

By (3.32) and (3.34), we get

$$\int_0^T \int_\Omega u_\varepsilon v_\varepsilon F(v_\varepsilon) \to \int_0^T \int_\Omega uvF(v) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{3.39}$$

(3.31) yields that

$$\int_0^T \int_\Omega f(v_\varepsilon)v_\varepsilon \to \int_0^T \int_\Omega f(v)v \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{3.40}$$

and

$$\int_\Omega v_\varepsilon^2(\cdot,T) \to \int_\Omega v^2(\cdot,T) \quad \text{for all } T \in (0,\infty)\backslash\Lambda_1 \tag{3.41}$$

with some null set $\Lambda_1 \subset (0,\infty)$. Thus, collecting (3.38)–(3.41) shows

$$D\lim_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 = -\int_0^T \int_\Omega uvF(v) + \int_0^T \int_\Omega f(v)v$$

$$+\frac{1}{2}\int_\Omega v_0^2-\frac{1}{2}\int_\Omega v^2(\cdot,T)\quad \text{for all } T\in(0,\infty)\backslash\Lambda_1,$$

which in conjunction with the fact of [50, Lemma 8.1] with a null set $\Lambda_2\subset(0,\infty)$,

$$-\int_0^T\int_\Omega uvF(v)+\frac{1}{2}\int_\Omega v_0^2-\frac{1}{2}\int_\Omega v^2(\cdot,T)\le D\int_0^T\int_\Omega|\nabla v|^2\quad \text{for all } T\in(0,\infty)\backslash\Lambda_1$$

gives the desired consequence. The proof is complete.

4 Proof of Main Theorem

Now, we can prove Theorem 1.1 on the basis of the above lemmas.

Proof of Theorem 1.1 We only verify the inequality (2.5). For any arbitrary $\psi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, we take $T>0$ such that $\operatorname{supp}\psi\subset\Omega\times[0,T]$ and choose $\phi(s)=\ln(s+1)$ on $[0,\infty)$ in Lemma 3.1, we have

$$\begin{aligned}
&-\int_0^\infty\int_\Omega\ln(u_\varepsilon+1)\psi_t-\int_\Omega\ln(u_{0\varepsilon}+1)\psi(\cdot,0)\\
&=\int_0^\infty\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\psi-\int_0^\infty\int_\Omega\frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\psi\\
&\quad+\int_0^\infty\int_\Omega\ln(u_\varepsilon+1)\Delta\psi+\int_0^\infty\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\cdot\nabla\psi\\
&\quad+\gamma\int_0^\infty\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)\psi-\int_0^\infty\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\psi\quad \text{for all } \varepsilon\in(0,1).
\end{aligned}\tag{4.1}$$

An application of (3.29) infers that

$$-\int_0^\infty\int_\Omega\ln(u_\varepsilon+1)\psi_t\to-\int_0^\infty\int_\Omega\ln(u+1)\psi_t\tag{4.2}$$

and

$$\int_0^\infty\int_\Omega\ln(u_\varepsilon+1)\Delta\psi\to\int_0^\infty\int_\Omega\ln(u+1)\Delta\psi\tag{4.3}$$

as $\varepsilon=\varepsilon_j\searrow0$. Thanks to (3.35), we get

$$\begin{aligned}
&\gamma\int_0^\infty\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}F(v_\varepsilon)\psi-\int_0^\infty\int_\Omega\frac{u_\varepsilon}{u_\varepsilon+1}h(u_\varepsilon)\psi\\
&\to\gamma\int_0^\infty\int_\Omega\frac{u}{u+1}F(v)\psi-\int_0^\infty\int_\Omega\frac{u}{u+1}h(u)\psi
\end{aligned}\tag{4.4}$$

as $\varepsilon=\varepsilon_j\searrow0$. Combining (3.28) with (3.30), we deduce

$$\frac{u_\varepsilon}{u_\varepsilon+1}S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\to\frac{u}{u+1}S(x,u,v)\qquad\text{a.e. in } \Omega\times(0,\infty),$$

along with Lemma 3.10 implies that

$$\frac{u_\varepsilon}{u_\varepsilon+1}(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon)\to\frac{u}{u+1}(S(x,u,v)\nabla v)\quad\text{in } L^2_{\text{loc}}(\overline{\Omega}\times[0,\infty))\tag{4.5}$$

as $\varepsilon = \varepsilon_j \searrow 0$, this ensures that

$$\int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla\psi \to \int_0^\infty \int_\Omega \frac{u}{u+1}(S(x, u, v)\nabla v) \cdot \nabla\psi \tag{4.6}$$

as $\varepsilon = \varepsilon_j \searrow 0$. The combination of (3.29) and (3.46) yields

$$\begin{aligned} &\int_0^\infty \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)^2}\nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \psi \\ =& \int_0^\infty \int_\Omega \nabla \ln(u_\varepsilon + 1) \cdot \Big(\frac{u_\varepsilon}{u_\varepsilon + 1} S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon\Big) \cdot \psi \\ \to& \int_0^\infty \int_\Omega \nabla \ln(u + 1) \cdot \Big(\frac{u}{u+1} S(x, u, v)\nabla v\Big) \cdot \psi \\ =& \int_0^\infty \int_\Omega \frac{u}{(u+1)^2}(\nabla u \cdot S(x, u, v)\nabla v) \cdot \psi \end{aligned} \tag{4.7}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Applying Fatou's lemma, we obtain

$$\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{(u+1)^2}\psi \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2}\psi \tag{4.8}$$

and

$$\int_\Omega \ln(u_0 + 1)\psi(\cdot, 0) \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_\Omega \ln(u_{0\varepsilon} + 1)\psi(\cdot, 0). \tag{4.9}$$

Collecting (3.43)–(3.45) and (3.47)–(3.50), we know that u fulfills (2.5). The proof is complete.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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