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Shock Formation for 2D Isentropic Euler Equations with Self-similar Variables*

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Abstract The author studies the 2D isentropic Euler equations with the ideal gas law. He exhibits a set of smooth initial data that give rise to shock formation at a single point near the planar symmetry. These solutions to the 2D isentropic Euler equations are associated with non-zero vorticity at the shock and have uniform-in-time $\frac{1}{3}$-Hölder bound. Moreover, these point shocks are of self-similar type and share the same profile, which is a solution to the 2D self-similar Burgers equation. The proof of the solutions, following the 3D construction of Buckmaster, Shkoller and Vicol (in 2023), is based on the stable 2D self-similar Burgers profile and the modulation method.

Keywords 2D isentropic Euler equations, Shock formation, Self-similar solution
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1 Introduction

The two-dimensional compressible isentropic Euler equations read

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p = 0, \\ p = \dfrac{1}{\gamma}\rho^\gamma, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$ are space and time coordinates, respectively. The unknown scalar ρ is the fluid density, $u = (u_1, u_2)$ is the velocity field of the fluid, $p = \frac{1}{\gamma}\rho^\gamma$ is the pressure with adiabatic index $\gamma > 1$. This system describes the evolution of a two-dimensional compressible ideal gas without viscosity.

We define the vorticity $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$ and the specific vorticity $\zeta = \frac{\omega}{\rho}$ at those points, where $\rho > 0$. One can deduce from (1.1) that ζ is purely transported by the velocity field:

$$\partial_t \zeta + u \cdot \nabla_x \zeta = 0. \tag{1.2}$$

Our main result can be stated roughly as follows.

Theorem 1.1 (Rough statement of the main theorem) *There exists a set of initial data* (u_0, ρ_0) *with* $|\nabla(u_0, \rho_0)| = \mathcal{O}\big(\frac{1}{\varepsilon}\big)$, *such that their corresponding solutions to* (1.1) *develop a shock-type singularity within time* $\mathcal{O}(\varepsilon)$.

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It is well known that governed by compressible Euler equations, shock can develop from smooth initial data. In the one-dimensional case, this fact can be obtained by studying the dynamics of the Riemann invariants, which were first introduced in Riemann's foundational work in [32]. See the discussion in John [20], Liu [21], and Majda [24].

In multi-dimensional cases, Sideris [33] proved a blow-up result. However, the shock formation remained open. In 2007, Christodoulou [14] studied the relativistic fluids and he found a set of irrotational initial data that will eventually develop shock-type singularity, which was considered to be the first proof of shock formation for the compressible Euler equations in multi-dimensional cases. Later in [17] the authors established the shock formation for non-relativistic and irrotational flow. In the case of irrotational flow, one can rewrite the isentropic Euler equations as a scalar second-order quasilinear wave equation. Alinhac in [2–3] proved the first blow-up result for 2D quasilinear wave equations, which do not satisfy Klainerman's null condition. Using geometric method, shock formation for the 3D quasilinear wave equations was studied in [29, 34–35]. The first result on shock formation that admits non-zero vorticity for the compressible Euler equations was given by Luk-Speck [22]. They use the geometric framework and developed new methods to study the vorticity transport. Later in [23] proved shock formation for full compressible Euler equations in 3D with non-trivial vorticity and variable entropy. In [4–8], the authors proved the low regularity ill-posedness for elastic waves and MHD equations and showed that the ill-posedness is driven by shock formation. As to the shock development problem for the compressible Euler equations, one could refer to the discussions in [1, 9, 15–16].

In [11], Buckmaster, Shkoller, and Vicol utilized the modulation method to construct shock solutions to the 2D Euler equations with azimuthal symmetry. Later in [13], they extend this method to the 3D case with no symmetry assumptions. After a dynamical rescaling, the solutions which they constructed are close to a profile $\overline{W}$, which solves the self-similar Burgers equation. By a singular coordinate transformation controlled by several modulation variables, proving shock formation is equivalent to showing global existence in the self-similar coordinate. This approach, known as the modulation method or dynamical rescaling, was successfully applied in [25–27] for the blow-up of Schrödinger equations and in [28] for the nonlinear heat equation. The proof in [11] is L^∞ based since there is no derivative in the forcing term, whereas in [13], an additional L^2 based energy estimate was used to overcome the derivative loss in the L^∞-type argument. They also analyzed the non-isentropic case in [12].

Following the work in [13], we utilize the self-similar Burgers ansatz to construct shock solutions. To keep track of the curvature of shock formation while maintaining the solution's stationarity in the far field, we make a minor modification to the construction in [13]. Different from the construction in [11], we consider shock solutions without any symmetry.

The shock which we attempt to construct is of self-similar type. We introduce a self-similar coordinate transformation $(\mathrm{t},\mathrm{x}) \mapsto (s,y)$, where (t,x) is the original Cartesian coordinate and (s,y) is the self-similar coordinate. The new coordinate is aligned to the shock formation and will become singular when t approaches the blow-up time T_*. Roughly speaking, we have that

$$y_1 \approx (T_* - t)^{-\frac{3}{2}}\mathrm{x}_1, \quad y_2 \approx (T_* - t)^{-\frac{1}{2}}\mathrm{x}_2.$$

Thus y is a zoom-in version of x. In self-similar coordinates, the Riemann invariant W(will be defined in the next subsection) will converge to a profile $\overline{W}$, uniformly on any compact set of y. Moreover, $\overline{W}$ solves the self-similar Burgers equation:

$$-\frac{1}{2}\overline{W} + \Big(\frac{3}{2}y_1 + \overline{W}\Big)\partial_{y_1}\overline{W} + \frac{1}{2}y_2\partial_{y_2}\overline{W} = 0.$$

In this sense, the constructed blow-up solution of the Euler equations is close to a fixed shape on a smaller and smaller scale.

To better understand what happens, we shall examine the simplest 1D inviscid Burgers model, whose C_c^∞ solutions are proved to become singular in finite time. It is pointed out explicitly in [18–19] that as we are approaching the blow-up point, the blow-up solution can be well modeled by a dynamically rescaled version of a fixed profile, which belongs to a countable family $\mathcal{F}$ of functions, and the members in $\mathcal{F}$ are solutions to the self-similar Burgers equation. The choice of profile only depends on the derivatives of initial data at the point that achieves the minimum negative slope. Thus the family $\mathcal{F}$ of solutions to the self-similar Burgers equation plays an important role in the blow-up phenomenon of the Burgers equation. For a detailed discussion see [18] or the toy model in appendix A.

After the asymptotic blow-up behavior of the inviscid Burgers equation was clarified systematically in [18], the self-similar Burgers profiles have been used to explore blow-up phenomena in various systems (see [10, 13, 30–31, 36]). The modulation method that was developed in the context of nonlinear dispersive equations, is the suitable for the self-similar Burgers profiles.

2 Preliminaries

We introduce the scaled sound speed $\sigma = \frac{1}{\alpha}\rho^\alpha$, where $\alpha = \frac{\gamma-1}{2} > 0$. Then the system of (u, ρ) is transformed into a system of (u, σ), which reads

$$\begin{cases} \partial_{\mathrm{t}}\sigma + u\cdot\nabla_{\mathrm{x}}\sigma + \alpha\sigma\nabla_{\mathrm{x}}\cdot u = 0, \\ \partial_{\mathrm{t}}u + (u\cdot\nabla)u + \alpha\sigma\nabla_{\mathrm{x}}\sigma = 0, \end{cases} \tag{2.1}$$

By defining $t = \frac{1+\alpha}{2}\mathrm{t}$, the above equations become

$$\begin{cases} \dfrac{1+\alpha}{2}\partial_t\sigma + u\cdot\nabla_{\mathrm{x}}\sigma + \alpha\sigma\nabla_{\mathrm{x}}\cdot u = 0, \\ \dfrac{1+\alpha}{2}\partial_t u + (u\cdot\nabla)u + \alpha\sigma\nabla_{\mathrm{x}}\sigma = 0, \end{cases} \tag{2.2}$$

The vorticity is defined as

$$\omega = \partial_{\mathrm{x}_1}u_2 - \partial_{\mathrm{x}_2}u_1. \tag{2.3}$$

We also introduce the specific vorticity $\zeta := \frac{\omega}{\rho}$, which satisfies

$$\frac{1+\alpha}{2}\partial_t\zeta + u\cdot\nabla_{\mathrm{x}}\zeta = 0. \tag{2.4}$$

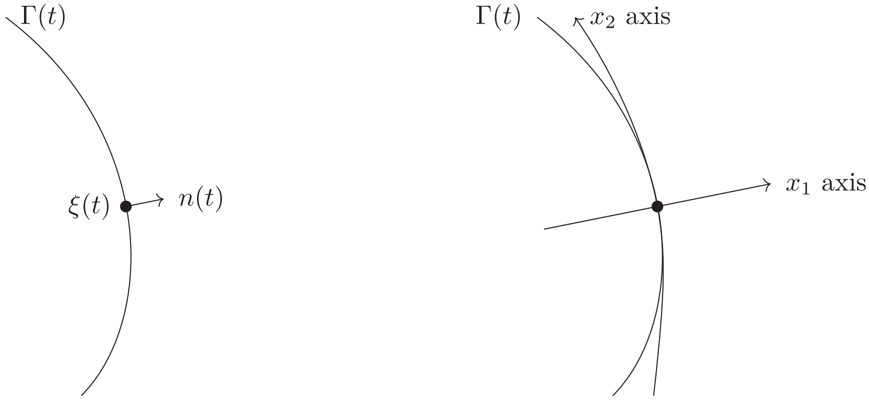
2.1 Coordinates adapted to the shock

In this subsection we introduce a series of coordinate transformations and the Riemann variables.

Prior to the formation of the shock wave, how can we identify the most likely location for its emergence? Suppose that w is a time-varying scalar field. To dynamically track the point where $|\nabla w|$ is maximized (this point varies with time), we employ a position $\xi(t) = (\xi_1(t), \xi_2(t))$.

Unlike the one-dimensional case where there is no shock wave front, and a single position is sufficient to capture all geometric information of the discontinuity, in two dimensions, discontinuities commonly occur along curves. Hence, we consider the "shock wave front" $\Gamma(t)$. It is the level set of w and passes through $\xi(t)$. Apart from ξ, we also need a unit vector $n(t) = (n_1(t), n_2(t))$ to record the normal vector of Γ at ξ and use $\phi(t)$ to denote the curvature of Γ at $\xi(t)$.

What we aim to achieve is the determination of a curvilinear coordinate system (x_1, x_2), where the x_2 coordinate axis, denoted by $\{x_1 = 0\}$, mimics the behavior of $\Gamma(t)$.



Moreover, we introduce $\tau(t), \kappa(t) \in \mathbb{R}$:

$$-\frac{1}{\tau(t) - t} := |\nabla w(\xi(t), t)|, \quad \kappa(t) := w(\xi(t), t). \tag{2.5}$$

In summary, to keep track of shock formation, we introduce six time-dependent modulation variables $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2, n = (n_1, n_2) \in \mathbb{S}^1, \tau \in \mathbb{R}, \phi \in \mathbb{R}$, and $\kappa \in \mathbb{R}$. ξ records the location of the shock; n records the direction of the shock; τ records the slope of w; ϕ records the curvature of the "shock front"; κ records the value of the w at $\xi(t)$.

Using modulation variables ξ, n, ϕ, we define the coordinate (x_1, x_2) which is adapted to the shock formation. Employing the parameter τ, we then define the self-similar coordinate. κ is used to normalize the value of w at ξ.

2.1.1 Tracing the location and direction of the shock

With the time-dependent vector $\xi(t) = (\xi_1(t), \xi_2(t))$, and the normal vector

$$n(t) = (n_1(t), n_2(t)),$$

we define a coordinate transformation $\widetilde{x} = R(t)^{\mathrm{T}}(\mathrm{x} - \xi(t))$, where

$$R(t) = \begin{bmatrix} n_1 & -n_2 \\ n_2 & n_1 \end{bmatrix} \in SO(2) \tag{2.6}$$

The origin of the $\widetilde{x}$ coordinate coincides with $\xi(t)$, which dynamically tracks the spatial location of the shock formation, and $\widetilde{e}_1$ aligns with $n(t)$, direction of the shock.

The functions should also be rewritten in the new coordinate:

$$\begin{cases} \widetilde{u}(\widetilde{x}, t) = R(t)^{\mathrm{T}} u(\mathrm{x}, t), \\ \widetilde{\rho}(\widetilde{x}, t) = \rho(\mathrm{x}, t), \\ \widetilde{\sigma}(\widetilde{x}, t) = \sigma(\mathrm{x}, t), \\ \widetilde{\zeta}(\widetilde{x}, t) = \zeta(\mathrm{x}, t). \end{cases} \tag{2.7}$$

Then $(\widetilde{u}, \widetilde{\sigma})$ satisfies

$$\begin{cases} \dfrac{1+\alpha}{2}\partial_t \widetilde{\sigma} + (\widetilde{u} + \widetilde{v}) \cdot \nabla_{\widetilde{x}} \widetilde{\sigma} + \alpha \widetilde{\sigma} \nabla_{\widetilde{x}} \cdot \widetilde{u} = 0, \\ \dfrac{1+\alpha}{2}\partial_t \widetilde{u} - \dfrac{1+\alpha}{2} Q \widetilde{u} + [(\widetilde{u} + \widetilde{v}) \cdot \nabla_{\widetilde{x}}] \widetilde{u} + \alpha \widetilde{\sigma} \nabla_{\widetilde{x}} \widetilde{\sigma} = 0, \end{cases} \tag{2.8}$$

where $Q(t) = \frac{\mathrm{d}R(t)^{\mathrm{T}}}{\mathrm{d}t} R(t) = \dot{R}(t)^{\mathrm{T}} R(t)$, and $\widetilde{v}(\widetilde{x}, t) = \frac{1+\alpha}{2}(Q\widetilde{x} - R^{\mathrm{T}}\dot{\xi})$.

The equation of specific vorticity is transformed into

$$\frac{1+\alpha}{2}\partial_t \widetilde{\zeta} + (\widetilde{u} + \widetilde{v}) \cdot \nabla_{\widetilde{x}} \widetilde{\zeta} = 0. \tag{2.9}$$

2.1.2 Tracking the curvature of shock front

In order to track the curvature of the shock, we introduce a time-dependent scalar function $\widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, t)$.

We denote $\phi(t) \in \mathbb{R}$ as the "curvature" of the "wavefront of the shock formation" at the origin, and we assume that $\widetilde{f}$ satisfies $\partial^2_{\widetilde{x}_2} \widetilde{f}(0, 0, t) = \phi(t)$. In particular, we construct $\widetilde{f}$ as follows. Let $\theta \in C_c^\infty\big(-\frac{5}{4}, \frac{5}{4}\big)$ be a bump function such that $\theta(\widetilde{x}_2) \equiv 1$ when $|\widetilde{x}_2| \le 1$. Then we define

$$\widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, t) = \theta(\varepsilon^{-\frac{1}{2}}\widetilde{x}_1) \int_0^{\widetilde{x}_2} \phi(t) \widetilde{x}_2' \theta(\varepsilon^{-\frac{1}{6}} \widetilde{x}_2') \mathrm{d}\widetilde{x}_2', \tag{2.10}$$

where ε is a small constant to be specified. Note that $\widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, t) = \frac{1}{2}\phi \widetilde{x}_2^2$ when $|\widetilde{x}|$ is small. This gurantees that in the forcing terms of W, Z, A (to be defined in (2.25)) those related to the coordinate transformation vanish when y is far from the origin, while not affecting the computation near the origin.

Now we introduce the coordinate transformation that adapted to the shock front:

$$\begin{cases} x_1 = \widetilde{x}_1 - \widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, t), \\ x_2 = \widetilde{x}_2. \end{cases} \tag{2.11}$$

Letting $f(x_1, x_2, t) := \widetilde{f}(\widetilde{x}_1, \widetilde{x}_2, t)$, then we have

$$\begin{cases} \widetilde{x}_1 = x_1 + f(x_1, x_2, t), \\ \widetilde{x}_2 = x_2. \end{cases} \tag{2.12}$$

We define

$$J(\widetilde{x}_1, \widetilde{x}_2, t) = |\nabla_{\tilde{x}} x_1| = \sqrt{(1 - \widetilde{f}_{\tilde{x}_1})^2 + \widetilde{f}_{\tilde{x}_2}^2} = \frac{\sqrt{1 + f_{x_2}^2}}{1 + f_{x_1}}, \tag{2.13}$$

$$N = J^{-1} \nabla_{\widetilde{x}} x_1 = \frac{(1 - \widetilde{f}_{\tilde{x}_1}, -\widetilde{f}_{\tilde{x}_2})}{\sqrt{(1 - \widetilde{f}_{\tilde{x}_1})^2 + \widetilde{f}_{\tilde{x}_2}^2}} = \frac{1}{\sqrt{1 + f_{x_2}^2}} (1, -f_{x_2}), \tag{2.14}$$

$$T = N^{\perp} = \frac{(\widetilde{f}_{\tilde{x}_2}, 1 - \widetilde{f}_{\tilde{x}_1})}{\sqrt{(1 - \widetilde{f}_{\tilde{x}_1})^2 + \widetilde{f}_{\tilde{x}_2}^2}} = \frac{1}{\sqrt{1 + f_{x_2}^2}} (f_{x_2}, 1). \tag{2.15}$$

Note that $\{N, T\}$ forms an orthonormal basis.

J, N, T can also be viewed as functions of (x_1, x_2, t) and we overload their names for the sake of convenience. One can verify that

$$\text{supp}_x (N - \widetilde{e}_1, T - \widetilde{e}_2) \subset \Big\{ |x_1| \le \frac{3}{2} \varepsilon^{\frac{1}{2}}, |x_2| \le \frac{3}{2} \varepsilon^{\frac{1}{6}} \Big\}. \tag{2.16}$$

Now the functions are redefined as

$$\begin{cases} \mathring{u}(x, t) = \widetilde{u}(\widetilde{x}, t), \\ \mathring{\rho}(x, t) = \widetilde{\rho}(\widetilde{x}, t), \\ \mathring{\sigma}(x, t) = \widetilde{\sigma}(\widetilde{x}, t), \\ \mathring{\zeta}(x, t) = \widetilde{\zeta}(\widetilde{x}, t), \\ v(x, t) = \widetilde{v}(\widetilde{x}, t), \end{cases}$$

and the system can be written as

$$\begin{cases} \quad \partial_t \mathring{u} - Q \mathring{u} + \Big[- \dfrac{\partial_t f}{1 + f_{x_1}} + 2\beta_1 (\mathring{u} + v) \cdot JN \Big] \partial_{x_1} \mathring{u} + 2\beta_1 (\mathring{u}_2 + v_2) \partial_{x_2} \mathring{u} \\ = -2\beta_3 JN \mathring{\sigma} \partial_{x_1} \mathring{\sigma} - 2\beta_3 \mathring{\sigma} \partial_{x_2} \mathring{\sigma} \widetilde{e}_2, \\ \quad \partial_t \mathring{\sigma} + \Big[- \dfrac{\partial_t f}{1 + f_{x_1}} + 2\beta_1 (\mathring{u} + v) \cdot JN \Big] \partial_{x_1} \mathring{\sigma} + 2\beta_1 (\mathring{u}_2 + v_2) \partial_{x_2} \mathring{\sigma} \\ = -2\beta_3 \mathring{\sigma} JN \cdot \partial_{x_1} \mathring{u} - 2\beta_3 \mathring{\sigma} \partial_{x_2} \mathring{u}_2, \end{cases} \tag{2.17}$$

where

$$\beta_1 = \frac{1}{1 + \alpha}, \quad \beta_2 = \frac{1 - \alpha}{1 + \alpha}, \quad \beta_3 = \frac{\alpha}{1 + \alpha}. \tag{2.18}$$

We can also deduce the equation governing the evolution of $\mathring{\zeta}$:

$$\partial_t \mathring{\zeta} + \Big[- \frac{\partial_t f}{1 + f_{x_1}} + 2\beta_1 (\mathring{u} + v) \cdot JN \Big] \partial_{x_1} \mathring{\zeta} + 2\beta_1 (\mathring{u}_2 + v_2) \partial_{x_2} \mathring{\zeta} = 0. \tag{2.19}$$

2.1.3 Riemann variables

We define the Riemann variables by

$$\begin{cases} w(x,t) = \mathring{u}(x,t)\cdot N + \mathring{\sigma}(x,t), \\ z(x,t) = \mathring{u}(x,t)\cdot N - \mathring{\sigma}(x,t), \\ a(x,y) = \mathring{u}(x,t)\cdot T. \end{cases} \tag{2.20}$$

Then the system of $(\mathring{u},\mathring{\sigma})$ can be rewritten in terms of (w,z,a) as

$$\begin{aligned} &\partial_t w + \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + Jw + \beta_2 Jz\Big)\partial_{x_1} w \\ &+ (2\beta_1 v_2 + N_2 w + \beta_2 N_2 z + 2\beta_1 a T_2)\partial_{x_2} w \\ =& -2\beta_3\mathring{\sigma}\partial_{x_2} a T_2 + aT\cdot(\partial_t)_x N + aQ_{ij}T_jN_i + 2\beta_1(\mathring{u}\cdot NN_2 + aT_2 + v_2)aT\cdot\partial_{x_2}N \\ &- 2\beta_3\sigma(a\partial_{x_2}T_2 + \mathring{u}\cdot N\partial_{x_2}N_2) - \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + 2\beta_1 J\mathring{u}\cdot N\Big)a\partial_{x_1}T\cdot N, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\partial_t z + \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + \beta_2 Jw + Jz\Big)\partial_{x_1} z \\ &+ (2\beta_1 v_2 + \beta_2 N_2 w + N_2 z + 2\beta_1 a T_2)\partial_{x_2} w \\ =& 2\beta_3\mathring{\sigma}\partial_{x_2} a T_2 + aT\cdot(\partial_t)_x N + aQ_{ij}T_jN_i + 2\beta_1(\mathring{u}\cdot NN_2 + aT_2 + v_2)aT\cdot\partial_{x_2}N \\ &+ 2\beta_3\mathring{\sigma}(a\partial_{x_2}T_2 + \mathring{u}\cdot N\partial_{x_2}N_2) - \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + 2\beta_1 J\mathring{u}\cdot N\Big)a\partial_{x_1}T\cdot N, \end{aligned} \tag{2.22}$$

$$\begin{aligned} &\partial_t a + \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + \beta_1 Jw + \beta_1 Jz\Big)\partial_{x_1} a \\ &+ 2\beta_1\Big(v_2 + \frac{w+z}{2}N_2 + aT_2\Big)\partial_{x_2} a \\ =& -2\beta_3\mathring{\sigma}T_2\partial_{x_2}\mathring{\sigma} + \mathring{u}\cdot TN\cdot(\partial_t)_x T + \mathring{u}\cdot NQ_{ij}N_jT_i + 2\beta_1(\mathring{u}\cdot NN_2 + aT_2 + v_2)\mathring{u}\cdot NN\cdot\partial_{x_2}T \\ &- \Big(-\frac{\partial_t f}{1+f_{x_1}} + 2\beta_1 v\cdot JN + 2\beta_1 J\mathring{u}\cdot N\Big)\mathring{u}\cdot N\partial_{x_1}N\cdot T. \end{aligned} \tag{2.23}$$

2.1.4 Self-similar transformation

We introduce self-similar variables as follows

$$\begin{cases} s(t) = -\log(\tau(t)-t), \\ y_1 = \dfrac{x_1}{(\tau-t)^{\frac{3}{2}}} = x_1 \mathrm{e}^{\frac{3}{2}s}, \\ y_2 = \dfrac{x_2}{(\tau-t)^{\frac{1}{2}}} = x_2 \mathrm{e}^{\frac{s}{2}}, \end{cases} \tag{2.24}$$

where $\tau(t)$ is a parameter to be determined.

Now the original time t is transformed into the self-similar time s, and the space variable x is transformed into the self-similar space variable y. At each fixed time t, y is a dilation of x. In the y coordinate, we can closely observe the behavior of the solution around the shock location.

Now we assume that

$$\begin{cases} w(x,t) = \mathrm{e}^{-\frac{s}{2}} W(y,s) + \kappa(t), \\ z(x,t) = Z(y,s), \\ a(x,t) = A(y,s), \end{cases} \tag{2.25}$$

where κ is also a modulation parameter to be determined.

In the self-similar variables, the system becomes

$$\begin{cases} \Big(\partial_s - \dfrac{1}{2}\Big) W + \Big(\dfrac{3}{2}y_1 + g_W\Big)\partial_1 W + \Big(\dfrac{1}{2}y_2 + h_W\Big)\partial_2 W = F_W, \\ \partial_s Z + \Big(\dfrac{3}{2}y_1 + g_Z\Big)\partial_1 Z + \Big(\dfrac{1}{2}y_2 + h_Z\Big)\partial_2 Z = F_Z, \\ \partial_s A + \Big(\dfrac{3}{2}y_1 + g_A\Big)\partial_1 A + \Big(\dfrac{1}{2}y_2 + h_A\Big)\partial_2 A = F_A. \end{cases} \tag{2.26}$$

Here and throughout the papar we use the notation $\partial_j = \partial_{y_j}$, and $\beta_\tau := \frac{1}{1-\dot{\tau}}$. The transport terms and and the forcing terms are given by

$$\begin{cases} g_W = \beta_\tau JW + \beta_\tau \mathrm{e}^{\frac{s}{2}} \Big[- \dfrac{\partial_t f}{1+f_{x_1}} + J(\kappa + \beta_2 Z + 2\beta_1 V\cdot N)\Big] = \beta_\tau JW + G_W, \\ g_Z = \beta_2\beta_\tau JW + \beta_\tau \mathrm{e}^{\frac{s}{2}} \Big[- \dfrac{\partial_t f}{1+f_{x_1}} + J(\beta_2\kappa + Z + 2\beta_1 V\cdot N)\Big] = \beta_2\beta_\tau JW + G_Z, \\ g_A = \beta_1\beta_\tau JW + \beta_\tau \mathrm{e}^{\frac{s}{2}} \Big[- \dfrac{\partial_t f}{1+f_{x_1}} + J(\beta_1\kappa + \beta_1 Z + 2\beta_1 V\cdot N)\Big] = \beta_1\beta_\tau JW + G_A, \end{cases} \tag{2.27}$$

$$\begin{cases} h_W = \beta_\tau \mathrm{e}^{-s} N_2 W + \beta_\tau \mathrm{e}^{-\frac{s}{2}} (2\beta_1 V_2 + N_2\kappa + \beta_2 N_2 Z + 2\beta_1 A T_2), \\ h_Z = \beta_2\beta_\tau \mathrm{e}^{-s} N_2 W + \beta_\tau \mathrm{e}^{-\frac{s}{2}} (2\beta_1 V_2 + \beta_2 N_2\kappa + N_2 Z + 2\beta_1 A T_2), \\ h_A = \beta_1\beta_\tau \mathrm{e}^{-s} N_2 W + \beta_\tau \mathrm{e}^{-\frac{s}{2}} (2\beta_1 V_2 + \beta_1 N_2\kappa + \beta_1 N_2 Z + 2\beta_1 A T_2) \end{cases} \tag{2.28}$$

and

$$\begin{cases} F_W = -2\beta_3\beta_\tau S\partial_2 A T_2 + \beta_\tau \mathrm{e}^{-\frac{s}{2}} AT\cdot(\partial_t)_x N + \beta_\tau \mathrm{e}^{-\frac{s}{2}} Q_{ij} A T_j N_i \\ \qquad + 2\beta_1\beta_\tau (V_2 + U\cdot N N_2 + AT_2) AT\cdot\partial_2 N \\ \qquad - 2\beta_3\beta_\tau S(U\cdot N\partial_2 N_2 + A\partial_2 T_2) \\ \qquad - \beta_\tau \mathrm{e}^{s}\Big(- \dfrac{\partial_t f}{1+f_{x_1}} + 2\beta_1 V\cdot JN + 2\beta_1 JU\cdot N\Big) A\partial_1 T\cdot N - \beta_\tau \mathrm{e}^{-\frac{s}{2}}\dot{\kappa}, \\ F_Z = 2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}} S\partial_2 A T_2 + \beta\tau \mathrm{e}^{-s} AT\cdot(\partial_t)_x N + \beta \mathrm{e}^{-s} Q_{ij} A T_j N_i \\ \qquad + 2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}} (V_2 + U\cdot N N_2 + AT_2) AT\cdot\partial_2 N \\ \qquad + 2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}} (A\partial_2 T_2 + U\cdot N\partial_2 N_2) \\ \qquad - \beta_\tau \mathrm{e}^{\frac{s}{2}}\Big(- \dfrac{\partial_t f}{1+f_{x_1}} + 2\beta_1 V\cdot JN + 2\beta_1 JU\cdot N\Big) A\partial_1 T\cdot N, \\ F_A = -2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}} S T_2 \partial_2 S + \beta_\tau \mathrm{e}^{-s} U\cdot N N\cdot(\partial_t)_x T \\ \qquad + \beta_\tau \mathrm{e}^{-s} Q_{ij}(U\cdot N N_j + AT_j) T_i \\ \qquad + 2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}} (V_2 + U\cdot N N_2 + AT_2) U\cdot N N\cdot\partial_2 T \\ \qquad - \beta_\tau \mathrm{e}^{\frac{s}{2}}\Big(- \dfrac{\partial_t f}{1+f_{x_1}} + 2\beta_1 V\cdot JN + 2\beta_1 JU\cdot N\Big) U\cdot N\partial_1 N\cdot T, \end{cases} \tag{2.29}$$

where U, V, S are the self-similar versions of $\mathring{u}$, v, $\mathring{\sigma}$, for example $S(y,s)=\mathring{\sigma}(x,t)$.

If we write the transport terms as

$$\begin{cases}\mathcal{V}_W=\Big(\frac{3}{2}y_1+g_W,\frac{1}{2}y_2+h_W\Big),\\ \mathcal{V}_Z=\Big(\frac{3}{2}y_1+g_Z,\frac{1}{2}y_2+h_Z\Big),\\ \mathcal{V}_A=\Big(\frac{3}{2}y_1+g_A,\frac{1}{2}y_2+h_A\Big),\end{cases}\tag{2.30}$$

then the equation of (W,Z,A) can be written in a compact form

$$\begin{cases}\partial_s W-\frac{1}{2}W+\mathcal{V}_W\cdot\nabla W=F_W,\\ \partial_s Z+\mathcal{V}_Z\cdot\nabla Z=F_Z,\\ \partial_s A+\mathcal{V}_A\cdot\nabla A=F_A.\end{cases}\tag{2.31}$$

We also deduce the equations of (U,S):

$$\begin{cases}\partial_s U_i-\beta_\tau \mathrm{e}^{-s}Q_{ij}U_j+\mathcal{V}_A\cdot\nabla U=-2\beta_3\beta_\tau \mathrm{e}^{\frac{s}{2}}S\partial_1 SJN_i-2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}S\partial_2 S\delta_{i2},\\ \partial_s S+\mathcal{V}_A\cdot\nabla S=-2\beta_3\beta_\tau \mathrm{e}^{\frac{s}{2}}S\partial_1 U\cdot JN-2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}S\partial_2 U_2.\end{cases}\tag{2.32}$$

We can see that (U,S) are transported in the same way as A. The transport terms g_A, h_A in the equation of A can also be expressed in terms of U, S:

$$\begin{cases}g_A=\beta_\tau \mathrm{e}^{\frac{s}{2}}\Big[2\beta_1(U+V)\cdot JN-\frac{\partial_t f}{1+f_{x_1}}\Big],\\ h_A=2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}(U_2+V_2).\end{cases}\tag{2.33}$$

Here we record the relation between (U,S) and (W,Z,A):

$$\begin{cases}U=\frac{1}{2}(\mathrm{e}^{-\frac{s}{2}}W+Z+\kappa)N+AT,\\ S=\frac{1}{2}(\mathrm{e}^{-\frac{s}{2}}W-Z+\kappa),\end{cases}\tag{2.34}$$

and

$$\begin{cases}W=\mathrm{e}^{\frac{s}{2}}(U\cdot N+S-\kappa),\\ Z=U\cdot N-S,\\ A=U\cdot T.\end{cases}\tag{2.35}$$

Although we introduce the self-similar version of functions like $V(y,s)$ of $v(x,t)$, we overload the functions f, J, N, T as functions of (y,s). For example, in the self-similar coordinates, we view N as the map $y\mapsto N(x(y),t(s))$, and $\partial_2 N(y)=\partial_{y_2}[N(x(y),t(s))]$.

2.2 Self-similar 2D Burgers profile

We first introduce the 1D self-similar Burgers profile

$$W_{1d}(y_1)=\Big(-\frac{y_1}{2}+\Big(\frac{1}{27}+\frac{y_1^2}{4}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}-\Big(\frac{y_1}{2}+\Big(\frac{1}{27}+\frac{y_1^2}{4}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{3}},\tag{2.36}$$

which solves the 1D self-similar Burgers equation (see [18]):

$$-\frac{1}{2}W_{1d}+\Big(\frac{3}{2}y_1+W_{1d}\Big)\partial_{y_1}W_{1d}=0. \tag{2.37}$$

Moreover, we introduce

$$\overline{W}(y_1,y_2)=\langle y_2\rangle W_{1d}(\langle y_2\rangle^{-3}y_1), \tag{2.38}$$

where $\langle y_2\rangle=\sqrt{1+y_2^2}$. One can verify that $\overline{W}$ is a solution to the 2D self-similar Burgers equation:

$$-\frac{1}{2}\overline{W}+\Big(\frac{3}{2}y_1+\overline{W}\Big)\partial_{y_1}\overline{W}+\frac{1}{2}y_2\partial_{y_2}\overline{W}=0. \tag{2.39}$$

2.2.1 Properties of $\overline{\mathbf{W}}$

It can be checked via the explicit formula of W_{1d} that

$$|W_{1d}(y_1)|\le\min\Big(|y_1|,\frac{|y_1|}{\frac{1}{3}+|y_1|^{\frac{2}{3}}}\Big)\le\min(|y_1|,|y_1|^{\frac{1}{3}}), \tag{2.40}$$

$$|W_{1d}'(y_1)|\le\langle y_1\rangle^{-\frac{2}{3}},\quad |W_{1d}''(y_1)|\le\langle y_1\rangle^{-\frac{5}{3}}, \tag{2.41}$$

$$|W_{1d}(y_1)W_{1d}'(y_1)|\le\frac{1}{3}\langle y_1\rangle^{-\frac{1}{3}},\quad |(W_{1d}W_{1d}')'(y_1)|\le\min(\langle y_1\rangle^{-\frac{4}{3}},\frac{1}{7}|y_1|^{-1}\langle y_1\rangle^{-\frac{1}{3}}). \tag{2.42}$$

Define $\eta(y)=1+y_1^2+y_2^6$, $\widetilde{\eta}(y)=1+|y|^2+y_2^6$, then the above inequalities imply that

$$|\overline{W}|\le(1+y_1^2)^{\frac{1}{6}}\le\eta^{\frac{1}{6}}, \tag{2.43}$$

$$|\partial_1\overline{W}|\le\widetilde{\eta}^{-\frac{1}{3}},\quad |\partial_2\overline{W}|\le\frac{2}{3}, \tag{2.44}$$

$$|\partial_{11}\overline{W}|\le\widetilde{\eta}^{-\frac{5}{6}},\quad |\partial_{12}\overline{W}|\le 2\eta^{-\frac{1}{2}},\quad |\partial_{22}\overline{W}|\le\frac{6}{7}\eta^{-\frac{1}{6}}. \tag{2.45}$$

At the origin we can check by the expression of $\overline{W}$ that

$$\overline{W}(0)=0,\quad \nabla\overline{W}(0)=\begin{pmatrix}-1\\0\end{pmatrix},\quad \nabla^2\overline{W}(0)=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad \partial_1\nabla^2\overline{W}(0)=\begin{pmatrix}6&0\\0&2\end{pmatrix}. \tag{2.46}$$

2.3 Evolution of $\widetilde{W}$ and higher order derivatives of the unknowns

If we define $\widetilde{W}=W-\overline{W}$, then $\widetilde{W}$ satisfies

$$\Big(\partial_s-\frac{1}{2}+\beta_\tau J\partial_1\overline{W}\Big)\widetilde{W}+\mathcal{V}_W\cdot\nabla\widetilde{W}=\widetilde{F}_W, \tag{2.47}$$

where

$$\widetilde{F}_W=F_W+[(1-\beta_\tau J)\overline{W}-G_W]\partial_1\overline{W}-h_W\partial_2\overline{W}. \tag{2.48}$$

For a multi-index $\gamma = (\gamma_1, \gamma_2)$ satisfying $|\gamma| \geq 1$, we have the evolution equation for $(\partial^\gamma W, \partial^\gamma Z, \partial^\gamma A)$:

$$\begin{cases} \Big(\partial_s + \frac{3\gamma_1 + \gamma_2 - 1}{2} + \beta_\tau(1 + \gamma_1 \mathbb{1}_{\gamma_1 \geq 2} J\partial_1 W)\Big)\partial^\gamma W + \mathcal{V}_W \cdot \nabla \partial^\gamma W = F_W^{(\gamma)}, \\ \Big(\partial_s + \frac{3\gamma_1 + \gamma_2}{2} + \beta_2\beta_\tau\gamma_1 J\partial_1 W\Big)\partial^\gamma Z + \mathcal{V}_Z \cdot \nabla \partial^\gamma Z = F_Z^{(\gamma)}, \\ \Big(\partial_s + \frac{3\gamma_1 + \gamma_2}{2} + \beta_2\beta_\tau\gamma_1 J\partial_1 W\Big)\partial^\gamma A + \mathcal{V}_A \cdot \nabla \partial^\gamma A = F_A^{(\gamma)}, \end{cases} \tag{2.49}$$

where the forcing terms are

$$\begin{aligned} F_W^{(\gamma)} = {} & \partial^\gamma F_W - \beta_\tau \partial_1 W [\partial^\gamma, J] W - \beta_\tau \mathbb{1}_{|\gamma| \geq 2} \sum_{\substack{|\beta| = |\gamma| - 1 \\ \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta W \\ & - \beta_\tau \mathbb{1}_{|\gamma| \geq 3} \sum_{\substack{1 \leq |\beta| \leq |\gamma| - 1 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta W \\ & - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma - \beta} G_W \partial_1 \partial^\beta W + \partial^{\gamma - \beta} h_W \partial_2 \partial^\beta W), \end{aligned} \tag{2.50}$$

$$\begin{aligned} F_Z^{(\gamma)} = {} & \partial^\gamma F_Z - \beta_2 \beta_\tau \sum_{\substack{|\beta| = |\gamma| - 1 \\ \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta Z \\ & - \beta_2 \beta_\tau \mathbb{1}_{|\gamma| \geq 2} \sum_{\substack{0 \leq |\beta| \leq |\gamma| - 2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta Z \\ & - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma - \beta} G_Z \partial_1 \partial^\beta Z + \partial^{\gamma - \beta} h_Z \partial_2 \partial^\beta Z), \end{aligned} \tag{2.51}$$

$$\begin{aligned} F_A^{(\gamma)} = {} & \partial^\gamma F_A - \beta_2 \beta_\tau \sum_{\substack{|\beta| = |\gamma| - 1 \\ \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta A \\ & - \beta_2 \beta_\tau \mathbb{1}_{|\gamma| \geq 2} \sum_{\substack{0 \leq |\beta| \leq |\gamma| - 2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma - \beta}(JW) \partial_1 \partial^\beta A \\ & - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma - \beta} G_A \partial_1 \partial^\beta A + \partial^{\gamma - \beta} h_A \partial_2 \partial^\beta A). \end{aligned} \tag{2.52}$$

Similarly we can deduce the equation of $\partial^\gamma \widetilde{W}$:

$$\Big[\partial_s + \frac{3\gamma_1 + \gamma_2 - 1}{2} + \beta_\tau J(\partial_1 \overline{W} + \gamma_1 \partial_1 W)\Big]\partial^\gamma \widetilde{W} + \mathcal{V}_W \cdot \nabla \partial^\gamma \widetilde{W} = \widetilde{F}_W^{(\gamma)}, \tag{2.53}$$

where

$$\widetilde{F}_W^{(\gamma)} = \partial^\gamma \widetilde{F}_W - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} [\partial^{\gamma - \beta} G_W \partial_1 \partial^\beta \widetilde{W} + \partial^{\gamma - \beta} h_W \partial_2 \partial^\beta \widetilde{W} + \beta_\tau \partial^{\gamma - \beta}(J\partial_1 \overline{W}) \partial^\beta \widetilde{W}]$$

$$- \beta_\tau \gamma_2 \partial_2 (JW) \partial_1^{\gamma_1+1} \partial_2^{\gamma_2-1} \widetilde{W}$$
$$- \beta_\tau \mathbb{1}_{|\gamma|\geq 2} \sum_{\substack{0\leq|\beta|\leq|\gamma|-2 \\ \beta\leq\gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta}(JW)\partial_1 \partial^\beta \widetilde{W}. \tag{2.54}$$

3 Main Result

In this section, we state the initial conditions and the main shock formation result of the 2D compressible Euler equations. The proof of the main theorem will be given in Section 14.

3.1 Initial data in physical variables

The initial time is $t = -\varepsilon$ with ε to be determined. For modulation variables, we assume that

$$\kappa(-\varepsilon) = \kappa_0, \quad \xi(-\varepsilon) = 0, \quad n_2(-\varepsilon) = 0, \quad \tau(-\varepsilon) = 0, \quad \phi(-\varepsilon) = \phi_0 = 0, \tag{3.1}$$

where

$$\kappa_0 \geq \frac{3}{1 - \max(\beta_1, \beta_2)}. \tag{3.2}$$

Since $n_2(-\varepsilon) = 0$ and $\xi(-\varepsilon) = 0$, x-coordinate and $\widetilde{x}$-coordinate coincide at $t = -\varepsilon$, and

$$\begin{cases} x_1 = \widetilde{f}(\mathrm{x}_1, \mathrm{x}_2, -\varepsilon), \\ x_2 = \mathrm{x}_2. \end{cases} \tag{3.3}$$

Now we prescribe the initial data:

$$u_0(\mathrm{x}) := u(\mathrm{x}, -\varepsilon), \quad \rho_0(\mathrm{x}) := \rho(\mathrm{x}, -\varepsilon), \quad \sigma_0 := \frac{\rho_0^\alpha}{\alpha}. \tag{3.4}$$

We choose u_0 and ρ_0 such that the corresponding Riemann varibles satisfy the conditions stated in this section. The initial data of the Riemann variables are denoted as

$$\begin{aligned} \widetilde{w}_0(\mathrm{x}) &:= u_0(\mathrm{x}) \cdot N(\mathrm{x}, -\varepsilon) + \sigma_0(\mathrm{x}) =: w_0(x), \\ \widetilde{z}_0(\mathrm{x}) &:= u_0(\mathrm{x}) \cdot N(\mathrm{x}, -\varepsilon) - \sigma_0(\mathrm{x}) =: z_0(x), \\ \widetilde{a}_0(\mathrm{x}) &:= u_0(\mathrm{x}) \cdot T(\mathrm{x}, -\varepsilon) =: w_0(x). \end{aligned} \tag{3.5}$$

First we assume that

$$\mathrm{supp}_{\mathrm{x}}\ (\widetilde{w}_0 - \kappa_0, \widetilde{z}_0, \widetilde{a}_0) \subset \mathcal{X}_0 := \left\{ |\mathrm{x}_1| \leq \frac{1}{2}\varepsilon^{\frac{1}{2}}, |\mathrm{x}_2| \leq \varepsilon^{\frac{1}{6}} \right\}. \tag{3.6}$$

This implies that

$$\mathrm{supp}_x\ (w_0 - \kappa_0, z_0, a_0) \subset \{|x_1| \leq \varepsilon^{\frac{1}{2}}, |x_2| \leq \varepsilon^{\frac{1}{6}}\}. \tag{3.7}$$

The function $\widetilde{w}_0(\mathrm{x})$ is chosen such that

$$\begin{aligned} &\text{the minimum negative slope of } \widetilde{w}_0 \text{ occurs in the } \mathrm{x}_1 \text{ direction,} \\ &\partial_{\mathrm{x}_1} \widetilde{w}_0 \text{ attains its global minimum at } \mathrm{x} = 0. \end{aligned} \tag{3.8}$$

and

$$\nabla_{\mathrm{x}}\partial_{\mathrm{x}_1}\widetilde{w}_0(0)=0. \tag{3.9}$$

We also assume that

$$\widetilde{w}_0(0)=\kappa_0,\quad \partial_{\mathrm{x}_1}\widetilde{w}_0(0)=-\frac{1}{\varepsilon},\quad \partial_{\mathrm{x}_2}\widetilde{w}_0(0)=0. \tag{3.10}$$

Define

$$\overline{w}_\varepsilon(x):=\varepsilon^{\frac{1}{2}}\overline{W}(\varepsilon^{-\frac{3}{2}}x_1,\varepsilon^{-\frac{1}{2}}x_2), \tag{3.11}$$

and we set

$$\widehat{w}_0(\mathrm{x}):=\widetilde{w}_0(\mathrm{x})-\overline{w}_\varepsilon(\mathrm{x}_1-\widetilde{f}(\mathrm{x},t),\mathrm{x}_2)=w_0(x)-\overline{w}_\varepsilon(x)=\varepsilon^{\frac{1}{2}}\widetilde{W}(y,-\log\varepsilon)+\kappa_0. \tag{3.12}$$

We assume that for x such that $|(\varepsilon^{-\frac{3}{2}}\mathrm{x}_1,\varepsilon^{-\frac{1}{2}}\mathrm{x}_2)|\le 2\varepsilon^{-\frac{1}{10}}$, the following bounds hold

$$\begin{aligned}
|\widehat{w}_0(\mathrm{x})-\kappa_0|&\le \varepsilon^{\frac{1}{10}}(\varepsilon^3+\mathrm{x}_1^2+\mathrm{x}_2^6)^{\frac{1}{6}},\\
|\partial_{\mathrm{x}_1}\widehat{w}_0(\mathrm{x})|&\le \varepsilon^{\frac{1}{11}}(\varepsilon^3+\mathrm{x}_1^2+\mathrm{x}_2^6)^{-\frac{1}{3}},\\
|\partial_{\mathrm{x}_2}\widehat{w}_0(\mathrm{x})|&\le \frac{1}{2}\varepsilon^{\frac{1}{12}}.
\end{aligned} \tag{3.13}$$

For x such that $|(\varepsilon^{-\frac{3}{2}}\mathrm{x}_1,\varepsilon^{-\frac{1}{2}}\mathrm{x}_2)|\le 1$, we assume that

$$|\partial_{\mathrm{x}}^{\gamma}\widehat{w}_0(\mathrm{x})|\overset{|\gamma|=4}{\le}\frac{1}{2}\varepsilon^{\frac{5}{8}-\frac{1}{2}(3\gamma_1+\gamma_2)}. \tag{3.14}$$

At x = 0, we assume that

$$|\partial_{\mathrm{x}}^{\gamma}\widehat{w}_0(0)|\overset{|\gamma|=3}{\le}\frac{1}{2}\varepsilon^{1-\frac{1}{2}(3\gamma_1+\gamma_2)-\frac{4}{2k-7}}. \tag{3.15}$$

For $\mathrm{x}\in\mathcal{X}_0$ such that $|(\varepsilon^{-\frac{3}{2}}\mathrm{x}_1,\varepsilon^{-\frac{1}{2}}\mathrm{x}_2)|\ge \frac{1}{2}\varepsilon^{-\frac{1}{10}}$, we assume that

$$\begin{aligned}
|\widetilde{w}_0(\mathrm{x})-\kappa_0|&\le (1+\varepsilon^{\frac{1}{11}})(\varepsilon^4+\mathrm{x}_1^2+\mathrm{x}_2^6)^{\frac{1}{6}},\\
|\partial_{\mathrm{x}_1}\widetilde{w}_0(\mathrm{x})|&\le (1+\varepsilon^{\frac{1}{12}})(\varepsilon^4+\mathrm{x}_1^2+\mathrm{x}_2^6)^{-\frac{1}{3}},\\
|\partial_{\mathrm{x}_2}\widetilde{w}_0(\mathrm{x})|&\le \frac{2}{3}+\varepsilon^{\frac{1}{13}}.
\end{aligned} \tag{3.16}$$

For all $\mathrm{x}\in\mathcal{X}_0$, we assume that

$$\begin{aligned}
|\partial_{\mathrm{x}_1}^2\widetilde{w}_0(\mathrm{x})|&\le \varepsilon^{-\frac{3}{2}}(\varepsilon^3+\mathrm{x}_1^2+\mathrm{x}_2^6)^{-\frac{1}{3}},\\
|\partial_{\mathrm{x}_1\mathrm{x}_2}\widetilde{w}_0(\mathrm{x})|&\le \frac{1}{2}\varepsilon^{-\frac{1}{2}}(\varepsilon^3+\mathrm{x}_1^2+\mathrm{x}_2^6)^{-\frac{1}{3}},\\
|\partial_{\mathrm{x}_2}^2\widetilde{w}_0(\mathrm{x})|&\le \frac{1}{2}(\varepsilon^3+\mathrm{x}_1^2+\mathrm{x}_2^6)^{-\frac{1}{6}}.
\end{aligned} \tag{3.17}$$

Also at x = 0 we assume that

$$|\partial_{\mathrm{x}_2}^2\widetilde{w}_0(0)|\le 1. \tag{3.18}$$

For the initial data of $\widetilde{z}_0$ and $\widetilde{a}_0$ we assume that

$$\begin{aligned}
&|\widetilde{z}_0(\mathrm{x})|\le\varepsilon,\quad |\partial_{\mathrm{x}_1}\widetilde{z}_0(\mathrm{x})|\le 1,\quad |\partial_{\mathrm{x}_2}\widetilde{z}_0(\mathrm{x})|\le\frac{1}{2}\varepsilon^{\frac{1}{2}},\\
&|\partial_{\mathrm{x}_1}^2\widetilde{z}_0(\mathrm{x})|\le\varepsilon^{-\frac{3}{2}},\quad |\partial_{\mathrm{x}_1\mathrm{x}_2}\widetilde{z}_0(\mathrm{x})|\le\frac{1}{2}\varepsilon^{-\frac{1}{2}},\quad |\partial_{\mathrm{x}_2}^2\widetilde{z}_0(\mathrm{x})|\le\frac{1}{2},
\end{aligned} \tag{3.19}$$

and

$$|\widetilde{a}_0(\mathrm{x})| \le \varepsilon, \quad |\partial_{\mathrm{x}_1}\widetilde{a}_0(\mathrm{x})| \le 1, \quad |\partial_{\mathrm{x}_2}\widetilde{a}_0(\mathrm{x})| \le \frac{1}{2}\varepsilon^{\frac{1}{2}}, \quad |\partial^2_{\mathrm{x}_2}\widetilde{a}_0(\mathrm{x})| \le \frac{1}{2}. \tag{3.20}$$

For the initial specific vorticity, we assume that

$$\Big\| \frac{\operatorname{curl} u_0(\mathrm{x})}{\rho_0(\mathrm{x})} \Big\|_{L^\infty} \le 1. \tag{3.21}$$

Finally for the Sobolev norm of the initial data we assume that for a fixed k with $k \ge 18$ the following holds

$$\sum_{\gamma=k} \varepsilon^2 \|\partial_{\mathrm{x}}^\gamma \widetilde{w}_0\|_{L^2}^2 + \|\partial_{\mathrm{x}}^\gamma \widetilde{z}_0\|_{L^2}^2 + \|\partial_{\mathrm{x}}^\gamma \widetilde{a}_0\|_{L^2}^2 \le \frac{1}{2}\varepsilon^{\frac{7}{2}-3\gamma_1-\gamma_2}. \tag{3.22}$$

Theorem 3.1 (Main result in physical variables) *There exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, and*

- *the initial values of the modulation variables satisfy* (3.1)–(3.2);
- *the initial datum (u_0, ρ_0) of the Euler equations is smooth, and it gurantees that the corresponding Riemann variables (w_0, z_0, a_0) satisfies the initial conditions* (3.8)–(3.22), *then the corresponding solution (u, ρ) to* (1.1) *blows up in finite time $-\varepsilon < T_* = O(\varepsilon^2) < +\infty$. Moreover, we have the following description of the shock*:

(1) *Blow-up speed. For $t \in [-\varepsilon, T_*)$, we have the following inequalities for (u, σ):*

$$\frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t} \le \|\nabla_{\mathrm{x}} u(t)\|_{L^\infty} \le \frac{1 + 2\varepsilon^{\frac{1}{2}}}{T_* - t}, \tag{3.23}$$

$$\frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t} \le \|\nabla_{\mathrm{x}} \sigma(t)\|_{L^\infty} \le \frac{1 + 2\varepsilon^{\frac{1}{2}}}{T_* - t}. \tag{3.24}$$

(2) *Blow-up location. For arbitrary $\delta \in (0,1)$ and $t \in [-\varepsilon, T_*)$, there holds that*

$$\|\nabla_{\mathrm{x}} u(t)\|_{L^\infty(B_\delta^c(\xi(t)))} + \|\nabla_{\mathrm{x}} \sigma(t)\|_{L^\infty(B_\delta^c(\xi(t)))} \le C\delta^{-2}, \tag{3.25}$$

where C is a universal constant. And we have the unboundedness of gradient along $\xi(t)$

$$|\nabla_{\mathrm{x}} u(\xi(t), t)| \ge \frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t}, \qquad |\nabla_{\mathrm{x}} \sigma(\xi(t), t)| \ge \frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t}. \tag{3.26}$$

Moreover, the limit of $\xi(t)$ exists

$$\lim_{t \to T_*} \xi(t) = \xi_* \in \mathbb{R}^2. \tag{3.27}$$

(3) *Direction of the shock. The gradient of (u, σ) blows up only in one direction*

$$|[(R(t)N) \cdot \nabla_{\mathrm{x}}] u(\xi(t), t)| \ge \frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t}, \qquad |(R(t)N) \cdot \nabla_{\mathrm{x}} \sigma(\xi(t), t)| \ge \frac{\frac{1}{2} - \varepsilon^{\frac{1}{2}}}{T_* - t}; \tag{3.28}$$

$$\|[(R(t)T) \cdot \nabla_{\mathrm{x}}] u(t)\|_{L^\infty} + \|[(R(t)T) \cdot \nabla_{\mathrm{x}}] \sigma(t)\|_{L^\infty} \le 1 + \varepsilon^{\frac{1}{2}}. \tag{3.29}$$

Moreover, we have $n(t) = R(t)N(0, t)$, and the limit of $n(t)$ exists

$$\lim_{t \to T_*} n(t) = n_* \in \mathbb{S}^1. \tag{3.30}$$

(4) $\frac{1}{3}$ *-Hölder continuity. The solution has a uniform-in-time $C^{\frac{1}{3}}$ bound. More precisely, we have*

$$(u,\sigma)\in L_t^\infty([-\varepsilon,T_*),C_{\mathrm{x}}^{\frac{1}{3}}). \tag{3.31}$$

Proof of the main result will be given in Section 14.

3.2 Initial data in self-similar variables

Since $\tau(-\varepsilon)=0$, we have that the initial self-similar time is $s=-\log\varepsilon$.

When $s=-\log\varepsilon$, $y_1=x_1\varepsilon^{-\frac{3}{2}}$, $y_2=x_2\varepsilon^{-\frac{1}{2}}$, from (3.7) we have that the initial data of W,Z,A are supported in

$$\mathcal{X}_0=\{|y_1|\le\varepsilon^{-1},\quad |y_2|\le\varepsilon^{-\frac{1}{3}}\}. \tag{3.32}$$

Now we introduce a large constant $M=M(\alpha,\kappa_0,k)$ to absorb universal constants, where k is the order of the energy estimate that will be established in Section 6. Subsequently, the choice of the small parameter ε will be made in relation to M, such that we ensure the following hierarchy

$$1\ll\log M\ll M\ll\varepsilon^{-\frac{1}{1000}}. \tag{3.33}$$

In terms of M and ε, we define a small scale l and a large scale L by

$$l=(\log M)^{-5}, \tag{3.34a}$$

$$L=\varepsilon^{-\frac{1}{10}}. \tag{3.34b}$$

From (3.13)–(3.15) we know that $\widetilde{W}(y,-\log\varepsilon)$ satisfies

$$\eta^{-\frac{1}{6}}\big|\widetilde{W}(y,-\log\varepsilon)\big|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{10}}, \tag{3.35a}$$

$$\eta^{\frac{1}{3}}\big|\partial_1\widetilde{W}(y,-\log\varepsilon)\big|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{11}}, \tag{3.35b}$$

$$\big|\partial_2\widetilde{W}(y,-\log\varepsilon)\big|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{12}}, \tag{3.35c}$$

$$\big|\partial^\gamma\widetilde{W}(y,-\log\varepsilon)\big|\mathbb{1}_{|y|\le l}\overset{|\gamma|=4}{\le}\varepsilon^{\frac{1}{8}}, \tag{3.35d}$$

$$\big|\partial^\gamma\widetilde{W}(0,-\log\varepsilon)\big|\overset{|\gamma|=3}{\le}\varepsilon^{\frac{1}{2}-\frac{1}{k-3}}. \tag{3.35e}$$

For $W(y,-\log\varepsilon)$, from (3.16), for all $y\in\mathcal{X}_0\in\{|y|\ge L\}$, we have that

$$\begin{aligned}\eta^{-\frac{1}{6}}|W(y,-\log\varepsilon)|&\le 1+\varepsilon^{\frac{1}{11}},\\ \eta^{\frac{1}{3}}|\partial_1W(y,-\log\varepsilon)|&\le 1+\varepsilon^{\frac{1}{12}},\\ |\partial_2W(y,-\log\varepsilon)|&\le\frac{3}{4}.\end{aligned} \tag{3.36}$$

and from (3.17) we have that for all $y\in\mathcal{X}_0$,

$$\begin{aligned}\eta^{\frac{1}{3}}|\partial_{11}W(y,-\log\varepsilon)|&\le 1,\\ \eta^{\frac{1}{3}}|\partial_{12}W(y,-\log\varepsilon)|&\le 1,\\ \eta^{\frac{1}{6}}|\partial_{22}W(y,-\log\varepsilon)|&\le 1.\end{aligned} \tag{3.37}$$

From (3.19)–(3.20), we have that the initial data of Z and A satisfy

$$|\partial^\gamma Z(y,-\log\varepsilon)| \le \begin{cases} \varepsilon^{\frac{3}{2}}, & \gamma_1>0,\ |\gamma|=2, \\ \varepsilon, & \gamma_1=0,\ |\gamma|\le 2, \end{cases} \tag{3.38}$$

$$|\partial^\gamma A(y,-\log\varepsilon)| \le \begin{cases} \varepsilon^{\frac{3}{2}}, & \gamma=(1,0), \\ \varepsilon, & \gamma_1=0,\ |\gamma|\le 2. \end{cases} \tag{3.39}$$

Furthermore, from (3.21) we know the specific vorticity satisfies

$$\|\Omega(\cdot,-\log\varepsilon)\|_{L^\infty} \le 1. \tag{3.40}$$

Finally from (3.22) we have

$$\varepsilon\|W(\cdot,-\log\varepsilon)\|^2_{\dot H^k} + \|Z(\cdot,-\log\varepsilon)\|^2_{\dot H^k} + \|A(\cdot,-\log\varepsilon)\|^2_{\dot H^k} \le \varepsilon. \tag{3.41}$$

Theorem 3.2 (Main theorem in self-similar coordinate) *Suppose that $W(y,-\log\varepsilon)$, $Z(y,-\log\varepsilon)$, $A(y,-\log\varepsilon)\in H^k(\mathbb{R}^2)$ with integer k large enough, and they satisfy* (3.32)–(3.41), *and the initial data of modulation variables $(\kappa,\xi,n_2,\tau,\phi)$ satisfy* (3.1)–(3.2)*. Then there exists a choice of $\varepsilon\ll 1$, such that the system* (2.26) *coupled with* $(7.8)-(7.9), (7.15)$*and* (7.17) *admits a global solution, and the solution $(W,Z,A,\kappa,\phi,\tau,\xi,n)$ satisfies the bootstrap assumptions* (*which are stated in the next section*) *for all time.*

4 Bootstrap Argument

To establish global existence in self-similar coordinate, we set up a bootstrap argument.

4.1 Bootstrap assumption

We first state the bootstrap assumptions.

(1) Assumptions on modulation variables. For the modulation variables, we assume that

$$\begin{cases} \frac{1}{2}\kappa_0 \le \kappa \le 2\kappa_0, & |\dot\kappa|\le M, \\ |\tau|\le M\varepsilon^2, & |\dot\tau|\le M\mathrm{e}^{-s}, \\ |\xi|\le M^{\frac{1}{4}}\varepsilon, & |\dot\xi|\le M^{\frac{1}{4}}, \\ |n_2|\le M^2\varepsilon^{\frac{3}{2}}, & |\dot n_2|\le M^2\varepsilon^{\frac{1}{2}}, \\ |\phi|\le M^2\varepsilon, & |\dot\phi|\le M^2. \end{cases} \tag{B-M}$$

(2) Assumptions on Spatial support bootstrap. We define $\mathcal{X}(s) := \{|y_1|\le 2\varepsilon^{\frac{1}{2}}\mathrm{e}^{\frac{3}{2}s}, |y_2|\le 2\varepsilon^{\frac{1}{6}}\mathrm{e}^{\frac{s}{2}}\}$, and assume

$$\mathrm{supp}(DW,DZ,DA)\subset\mathcal{X}(s). \tag{B-S}$$

We will show that this assumption together with (2.16) imply that $\mathrm{supp}(DU,DS)\subset\mathcal{X}(s)$ in Lemma 5.1.

(3) Assumptions on W and $\widetilde{W}$. For $|\gamma|\le 2$, we assume that either $\partial^\gamma W$ is close to $\partial^\gamma \overline{W}$, or it behaves like $\partial^\gamma\overline{W}$. More precisely, we assume

$$\begin{cases}|W|\le (1+\varepsilon^{\frac{1}{20}})\eta^{\frac16}, & |\partial_1 W|\le 2\eta^{-\frac13}, & |\partial_2 W|\le 1,\\ |\partial_{11}W|\le M^{\frac13}\eta^{-\frac13}, & |\partial_{12}W|\le M^{\frac23}\eta^{-\frac13}, & |\partial_{22}W|\le M\eta^{-\frac16}.\end{cases} \tag{B-W}$$

Noting that by supp $DW\subset\mathcal{X}(s)$ and $W(0)=\overline{W}(0)=0$, we have

$$|W(y)|\le\int_0^{y_1}2\eta^{-\frac13}(y_1',0)\mathrm{d}y_1'+\|\partial_2 W\|_{L^\infty}|y_2|\lesssim\varepsilon^{\frac16}\mathrm{e}^{\frac{s}{2}}. \tag{4.1}$$

For $\widetilde{W}$ we assume

$$\begin{cases}|\widetilde{W}|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{11}}\eta^{\frac16},\\ |\partial_1\widetilde{W}|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{12}}\eta^{-\frac13},\\ |\partial_2\widetilde{W}|\mathbb{1}_{|y|\le L}\le\varepsilon^{\frac{1}{13}},\end{cases} \tag{B-$\widetilde{W}$-1}$$

where $L=\varepsilon^{-\frac{1}{10}}$. And

$$|\partial^\gamma\widetilde{W}|\mathbb{1}_{|y|\le l}\le\log^4 M\varepsilon^{\frac{1}{10}}|y|^{4-|\gamma|}+M\varepsilon^{\frac14}|y|^{3-|\gamma|},\quad\forall|\gamma|\le 3, \tag{B-$\widetilde{W}$-2}$$

$$|\partial^\gamma\widetilde{W}|\mathbb{1}_{|y|\le l}\le\frac12\log^{|\check{\gamma}|}M\varepsilon^{\frac{1}{10}},\quad\forall|\gamma|=4, \tag{B-$\widetilde{W}$-3}$$

where $l=(\log M)^{-5}$, and

$$|\partial^\gamma\widetilde{W}(0,s)|\le\varepsilon^{\frac14},\quad\forall|\gamma|=3,\ \forall s\ge s_0. \tag{B-$\widetilde{W}^0$}$$

(4) Assumptions on Z and A. For Z, A and their derivatives up to second order, we assume that they are small or have decay properties. More precisely, we assume

$$\begin{cases}|Z|\le M\varepsilon, & |\partial_1 Z|\le M^{\frac12}\mathrm{e}^{-\frac32 s}, & |\partial_2 Z|\le M\varepsilon^{\frac12}\mathrm{e}^{-\frac{s}{2}},\\ |\partial_{11}Z|\le M^{\frac12}\mathrm{e}^{-\frac32 s}, & |\partial_{12}Z|\le M\mathrm{e}^{-\frac32 s}, & |\partial_{22}Z|\le M\mathrm{e}^{-s}\end{cases} \tag{B-Z}$$

and

$$\begin{cases}|A|\le M\varepsilon, & |\partial_1 A|\le M\mathrm{e}^{-\frac32 s},\\ |\partial_2 A|\le M\varepsilon^{\frac12}\mathrm{e}^{-\frac{s}{2}}, & |\partial_{22}A|\le M\mathrm{e}^{-s}.\end{cases} \tag{B-A}$$

Remark 4.1 While the bootstrap assumptions remain valid, it follows that $S>0$. This assertion is supported by the inequality

$$S=\frac12(\mathrm{e}^{-\frac{s}{2}}W+\kappa-Z)\ge\frac{\kappa_0}{4}-C\varepsilon^{\frac16}-M\varepsilon>0, \tag{4.2}$$

provided that ε is sufficiently small. In particular, since the initial data satisfy the bootstrap assumptions, it can be deduced that $\rho_0>0$.

4.2 Bootstrap procedure

Now we state the improved bootstrap inequality (IB), which supposedly can be deduced from the bootstrap assumptions and the initial conditions:

$$\begin{cases} \frac{3}{4}\kappa_0 \le \kappa \le \frac{5}{4}\kappa_0, & |\dot{\kappa}| \le \frac{1}{2}M, \\ |\tau| \le \frac{1}{4}M\varepsilon^2, & |\dot{\tau}| \le \frac{1}{4}M\mathrm{e}^{-s}, \\ |\xi| \le \frac{1}{2}M^{\frac{1}{4}}\varepsilon, & |\dot{\xi}| \le \frac{1}{2}M^{\frac{1}{4}}, \\ |n_2| \le \frac{1}{2}M\varepsilon, & |\dot{n}_2| \le \frac{1}{2}M^2\varepsilon^{\frac{1}{2}}, \\ |\phi| \le \frac{1}{2}M^2\varepsilon, & |\dot{\phi}| \le \frac{1}{10}M^2, \end{cases} \tag{IB-M}$$

$$\operatorname{supp}(DW, DZ, DA) \subset \frac{7}{8}\mathcal{X}(s), \tag{IB-S}$$

$$\begin{cases} |W| \le (1+\varepsilon^{\frac{1}{19}})\eta^{\frac{1}{6}}, & |\partial_1 W| \le (1+\varepsilon^{\frac{1}{13}})\eta^{-\frac{1}{3}}, & |\partial_2 W| \le \frac{5}{6}, \\ |\partial_{11} W| \le \frac{1}{2}M^{\frac{1}{3}}\eta^{-\frac{1}{3}}, & |\partial_{12} W| \le \frac{1}{2}M^{\frac{2}{3}}\eta^{-\frac{1}{3}}, & |\partial_{22} W| \le \frac{1}{2}M\eta^{-\frac{1}{6}}, \end{cases} \tag{IB-W}$$

$$\begin{cases} \big|\widetilde{W}\big|\mathbb{1}_{|y|\le L} \le \frac{1}{2}\varepsilon^{\frac{1}{11}}\eta^{\frac{1}{6}}, \\ \big|\partial_1\widetilde{W}\big|\mathbb{1}_{|y|\le L} \le \frac{1}{2}\varepsilon^{\frac{1}{12}}\eta^{-\frac{1}{3}}, \\ \big|\partial_2\widetilde{W}\big|\mathbb{1}_{|y|\le L} \le \frac{1}{2}\varepsilon^{\frac{1}{13}}, \end{cases} \tag{IB-$\widetilde{W}$-1}$$

$$\big|\partial^\gamma\widetilde{W}\big|\mathbb{1}_{|y|\le l} \le \frac{1}{2}\log^4 M\varepsilon^{\frac{1}{10}}|y|^{4-|\gamma|} + \frac{1}{2}M\varepsilon^{\frac{1}{4}}|y|^{3-|\gamma|}, \quad \forall|\gamma| \le 3, \tag{IB-$\widetilde{W}$-2}$$

$$\big|\partial^\gamma\widetilde{W}\big|\mathbb{1}_{|y|\le l} \le \frac{1}{4}\log^{|\check{\gamma}|} M\varepsilon^{\frac{1}{10}}, \quad \forall|\gamma| = 4, \tag{IB-$\widetilde{W}$-3}$$

$$\big|\partial^\gamma\widetilde{W}(0,s)\big| \le \frac{1}{10}\varepsilon^{\frac{1}{4}}, \quad \forall|\gamma| = 3, \quad \forall s \ge s_0, \tag{IB-$\widetilde{W}^0$}$$

$$\begin{cases} |Z| \le \frac{1}{2}M\varepsilon, & |\partial_1 Z| \le \frac{1}{2}M^{\frac{1}{2}}\mathrm{e}^{-\frac{3}{2}s}, & |\partial_2 Z| \le \frac{1}{2}M\varepsilon^{\frac{1}{2}}\mathrm{e}^{-\frac{s}{2}}, \\ |\partial_{11} Z| \le \frac{1}{2}M^{\frac{1}{2}}\mathrm{e}^{-\frac{3}{2}s}, & |\partial_{12} Z| \le \frac{1}{2}M\mathrm{e}^{-\frac{3}{2}s}, & |\partial_{22} Z| \le \frac{1}{2}M\mathrm{e}^{-s}, \end{cases} \tag{IB-Z}$$

$$\begin{cases} |A| \le M\varepsilon, & |\partial_1 A| \le \frac{1}{2}M\mathrm{e}^{-\frac{3}{2}s}, \\ |\partial_2 A| \le \frac{1}{2}M\varepsilon^{\frac{1}{2}}\mathrm{e}^{-\frac{s}{2}}, & |\partial_{22} A| \le \frac{1}{2}M\mathrm{e}^{-s}. \end{cases} \tag{IB-A}$$

Compare to the 3D case in [13], we carefully close the bootstrap argument of spatial support in subsection 10.1. To prove that W, Z, A are constant outside $\frac{7}{8}\mathcal{X}(s)$, we define two rectangles $Q_{\mathrm{big}} = \{|y_1| \le M', |y_2| \le M'\}$ and $Q_{\mathrm{small}}(s)$ satisfying

$$\frac{3}{4}\mathcal{X}(s) \subset Q_{\mathrm{small}}(s) \subset \frac{7}{8}\mathcal{X}(s) \subset Q_{\mathrm{big}},$$

where M' can be chosen arbitrarily large. Then we consider the quantity

$$\int_{Q_{\rm big}\backslash Q_{\rm small}} E(y,s)\mathrm{d}y,$$

where $E(y,s)=\frac{1}{2}(\mathrm{e}^{-s}(W-W_\infty)^2+(Z-Z_\infty)+2(A-A_\infty)^2)$. From the equations of W,Z,A and bootstrap assumptions, we find that

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{Q_{\rm big}\backslash Q_{\rm small}} E\le C\int_{Q_{\rm big}\backslash Q_{\rm small}} E.$$

By Gronwall's inequality and the initial conditions, we can deduce that W,Z,A are constant outside $Q_{\rm small}$.

5 Immediate Corollaries of Bootstrap Assumptions

5.1 Blow-up time

By the definition of s, we have $t=\tau-\mathrm{e}^{-s}$. From the bootstrap assumption of τ and $s\ge-\log\varepsilon$, we can see that if the bootstrap assumptions hold on the interval $[t_0,t]=[-\varepsilon,t]$, then t satisfies

$$|t-t_0|=|t+\varepsilon|\le\varepsilon+M\varepsilon^2+\mathrm{e}^{\log\varepsilon}\le3\varepsilon. \tag{5.1}$$

The blow-up time T_* is defined to be $T_*=\tau(T_*)$.

5.2 Closure of bootstrap argument for W,$\widetilde{W}$ near the origin

From estimates (2.43)–(2.44) of $\overline{W}$ and bootstrap assumptions (B-$\widetilde{W}$-1), we have

$$\begin{cases}|W|\mathbb{1}_{|y|\le L}\le(1+\varepsilon^{\frac{1}{11}})\eta^{\frac16},\\ |\partial_1W|\mathbb{1}_{|y|\le L}\le(1+\varepsilon^{\frac{1}{12}})\eta^{-\frac13},\\ |\partial_2W|\mathbb{1}_{|y|\le L}\le\dfrac23+\varepsilon^{\frac{1}{13}}.\end{cases} \tag{5.2}$$

Thus we closed the bootstrap argument for W and DW in the region $\{|y|\le L\}$, and by $D^2\widetilde{W}(0,s)=0$, the bootstrap argument for D^2W in $\{|y|\le l\}$ is automatically closed.

Note that by (5.2) and (B-W), for ε small enough, we have

$$|\partial_1W|\le(1+\varepsilon^{\frac{1}{12}})\eta^{-\frac13}\mathbb{1}_{|y|\le L}+2\eta^{-\frac13}\mathbb{1}_{|y|>L}\le1+\varepsilon^{\frac{1}{12}}. \tag{5.3}$$

This bound will be used to estimate the damping terms.

Now we prove (IB-$\widetilde{W}$-2) for $\widetilde{W}$:

$$\begin{aligned}|\partial^\gamma\widetilde{W}|\mathbb{1}_{|y|\le l}&\overset{|\gamma|=3}{\le}|\partial^\gamma\widetilde{W}(0,s)|+\left\|D\partial^\gamma\widetilde{W}\right\|_{L^\infty(|y|\le l)}|y|\\&\le\varepsilon^{\frac14}+\frac12\log^4M\varepsilon^{\frac{1}{10}}|y|;\end{aligned} \tag{5.4}$$

if $|\gamma|\le2$, we have that

$$|\partial^\gamma\widetilde{W}|\mathbb{1}_{|y|\le l}\overset{|\gamma|\le2}{\le}\left\|D\partial^\gamma\widetilde{W}(\cdot,s)\right\|_{L^\infty(|\cdot|\le|y|)}|y|. \tag{5.5}$$

5.3 Spatial support of unknowns

For the support of unknowns, we have the following lemma.

Lemma 5.1 supp $(DU, DS) \subset \mathcal{X}(s)$.

Proof According to the spatial support assumption of (DW, DZ, DA), it suffices to show $\mathrm{supp}_x(D_xN, D_xT) \subset \{|x_1| \le 2\varepsilon^{\frac{1}{2}}, |x_2| \le 2\varepsilon^{\frac{1}{6}}\}$. Now by the expression of N, T, we only need to show that $\mathrm{supp}_x\ f_{x_2} \subset \{|x_1| \le 2\varepsilon^{\frac{1}{2}}, |x_2| \le 2\varepsilon^{\frac{1}{6}}\}$. Note that $f_{x_2} = \widetilde{f}_{\tilde{x}_2}\Big(1 + \frac{\widetilde{f}_{\tilde{x}_1}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)$, and $\mathrm{supp}_{\tilde{x}}\widetilde{f}_{\tilde{x}_2} \subset \{|\widetilde{x}_1| \le \frac{5}{4}\varepsilon^{\frac{1}{2}}, |\widetilde{x}_2| \le \frac{5}{4}\varepsilon^{\frac{1}{6}}\}$, thus we have $\mathrm{supp}_x(D_xN, D_xT) \subset \{|x_1| \le \frac{3}{2}\varepsilon^{\frac{1}{2}}, |x_2| \le \frac{3}{2}\varepsilon^{\frac{1}{6}}\}$ by choosing ε small enough in terms of M.

From (3.7), we know that in the original x coordinate, we have

$$\lim_{|\mathrm{x}|\to\infty} u(\mathrm{x}, -\varepsilon) = \frac{\kappa_0}{2}e_1, \quad \lim_{|\mathrm{x}|\to\infty} \sigma(\mathrm{x}, -\varepsilon) = \frac{\kappa_0}{2}. \tag{5.6}$$

From the finite propagation speed of the Euler equations, we have that for all $t \in [-\varepsilon, T_*)$, there hold

$$\lim_{|\mathrm{x}|\to\infty} u(\mathrm{x}, t) = \frac{\kappa_0}{2}e_1, \quad \lim_{|\mathrm{x}|\to\infty} \sigma(\mathrm{x}, t) = \frac{\kappa_0}{2}. \tag{5.7}$$

Noting that the coordinate transformation is determined by the modulation variables, and from bootstrap assumptions we can deduce that

$$y \notin \mathcal{X}(s) \text{ implies that } \begin{cases} W(y,s) = W_\infty(s), \\ Z(y,s) = Z_\infty(s), \\ A(y,s) = A_\infty(s), \\ S(y,s) = S_\infty(s), \\ U(y,s) = U_\infty(s), \end{cases} \tag{5.8}$$

where

$$\begin{cases} W_\infty(s) := \Big[\frac{\kappa_0}{2}(n_1+1) - \kappa\Big]\mathrm{e}^{\frac{s}{2}}, \\ Z_\infty(s) := \frac{\kappa_0}{2}(n_1 - 1), \\ A_\infty(s) := -\frac{\kappa_0}{2}n_2, \\ S_\infty(s) := \frac{\mathrm{e}^{-\frac{s}{2}}W_\infty + \kappa - Z_\infty}{2} = \frac{\kappa_0}{2}, \\ U_\infty(s) := \frac{\mathrm{e}^{-\frac{s}{2}}W_\infty + \kappa + Z_\infty}{2}\widetilde{e}_1 + A_\infty\widetilde{e}_2 = \frac{\kappa_0 n_1}{2}\widetilde{e}_1 - \frac{\kappa_0 n_2}{2}\widetilde{e}_2. \end{cases} \tag{5.9}$$

5.4 Estimates related to coordinate transformation

In this section we will estimate the functions f, J, N, T, Q, V, which only depend on modulation variables.

Lemma 5.2 *For any multi-index $\gamma \in \mathbb{Z}^2_{\geq 0}$, we have*

$$\begin{cases} |\partial_x^\gamma f| \leq C_\gamma M^2 \varepsilon^{\frac{4}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma (J-1)| \leq C_\gamma M^2 \varepsilon^{\frac{5}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma (N-\widetilde{e}_1)| \leq C_\gamma M^2 \varepsilon^{\frac{7}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma (T-\widetilde{e}_2)| \leq C_\gamma M^2 \varepsilon^{\frac{7}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma (JN-\widetilde{e}_1)| \leq C_\gamma M^2 \varepsilon^{\frac{5}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma \partial_t f| \leq C_\gamma M^2 \varepsilon^{\frac{1}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ \Big|\partial_x^\gamma \dfrac{\partial_t f}{1+f_{x_1}}\Big| \leq C_\gamma M^2 \varepsilon^{\frac{1}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma \partial_t N| \leq C_\gamma M^2 \varepsilon^{\frac{1}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}, \\ |\partial_x^\gamma \partial_t T| \leq C_\gamma M^2 \varepsilon^{\frac{1}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}. \end{cases} \tag{5.10}$$

Proof From the expression of $\widetilde{f}$ and the bootstrap assumption for ϕ and $\dot{\phi}$, it is not hard to see that $|\partial_{\tilde{x}}^\gamma \widetilde{f}| \leq C_\gamma M^2 \varepsilon^{\frac{4}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}$, $|\partial_{\tilde{x}}^\gamma \partial_t \widetilde{f}| \leq C_\gamma M^2 \varepsilon^{\frac{1}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}$.

Using chain rule, one can see that

$$\begin{cases} \partial_{x_1} = \dfrac{\partial_{\tilde{x}_1}}{1-\widetilde{f}_{\tilde{x}_1}}, \\ \partial_{x_2} = \dfrac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}} \partial_{\tilde{x}_1} + \partial_{\tilde{x}_2}. \end{cases} \tag{5.11}$$

By Faà di Bruno's formula, we have

$$\begin{aligned} \Big|\partial_{\tilde{x}}^\gamma \Big(\frac{1}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big| &\overset{\gamma>0}{\lesssim_{|\gamma|}} \sum_{\sum_{\beta\leq\gamma} \beta m_\beta = \gamma} |1-\widetilde{f}_{\tilde{x}_1}|^{-1-\sum_{\beta\leq\gamma} m_\beta} \prod_{\beta\leq\gamma} |\partial_{\tilde{x}}^\beta \widetilde{f}_{\tilde{x}_1}|^{m_\beta} \\ &\overset{\varepsilon\ll 1}{\lesssim} \sum_{\sum_{\beta\leq\gamma} \beta m_\beta = \gamma} (1-\varepsilon^{\frac{1}{2}})^{-\sum_{\beta\leq\gamma} m_\beta} \prod_{\beta\leq\gamma} (M^2 \varepsilon^{\frac{4}{3}-\frac{\beta_1+1}{2}-\frac{\beta_2}{6}})^{m_\beta} \\ &\lesssim \varepsilon^{-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}} \sum_{\sum_{\beta\leq\gamma} \beta m_\beta = \gamma} ((1-\varepsilon^{\frac{1}{2}}) M^2 \varepsilon^{\frac{5}{6}})^{\sum_{\beta\leq\gamma} m_\beta} \lesssim M^2 \varepsilon^{\frac{5}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}. \end{aligned} \tag{5.12}$$

And by Leibniz rule, we have

$$\begin{aligned} \Big|\partial_{\tilde{x}}^\gamma \Big(\frac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big| &\lesssim \sum_{0<\beta\leq\gamma} |\partial^{\gamma-\beta} \widetilde{f}_{\tilde{x}_2}| \Big|\partial_{\tilde{x}}^\beta \Big(\frac{1}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big| + M^2 \varepsilon^{\frac{4}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2+1}{6}} \\ &\lesssim M^2 \varepsilon^{\frac{7}{6}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}. \end{aligned} \tag{5.13}$$

Note that

$$\partial_{x_2}^k = \Big(\frac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}}\partial_{\tilde{x}_1} + \partial_{\tilde{x}_2}\Big)^k$$

$$= \sum_{\substack{\sum_{|\beta|\le k} n_\beta = \gamma_1 + \sum_{|\beta|\le k}\beta_1 n_\beta \\ \gamma_1+\gamma_2+\sum_{|\beta|\le k} n_\beta|\beta| = k}} C(k,\gamma,n_\beta)\prod_{|\beta|\le k}\Big(\partial_{\tilde{x}}^\beta\Big(\frac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big)^{n_\beta}\partial_{\tilde{x}}^\gamma. \tag{5.14}$$

Thus, we have

$$|\partial_{\tilde{x}_1}^j(\partial_{x_2}^k f)| \lesssim \Big|\partial_{\tilde{x}_1}^j \sum_{\substack{\sum_{|\beta|\le k} n_\beta = \gamma_1 + \sum_{|\beta|\le k}\beta_1 n_\beta \\ \gamma_1+\gamma_2+\sum_{|\beta|\le k} n_\beta|\beta| = k}} C(k,\gamma,n_\beta)\prod_{|\beta|\le k}\Big(\partial_{\tilde{x}}^\beta\Big(\frac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big)^{n_\beta}\partial_{\tilde{x}}^\gamma \widetilde{f}\Big|$$

$$\lesssim_{j,k} \sum_{\substack{\sum_{|\beta|\le k+j} n_\beta + j = \gamma_1 + \sum_{|\beta|\le k+j}\beta_1 n_\beta \\ \gamma_1+\gamma_2+\sum_{|\beta|\le k+j} n_\beta|\beta| = k+j}} \prod_{|\beta|\le k+j}\Big|\partial_{\tilde{x}}^\beta\Big(\frac{\widetilde{f}_{\tilde{x}_2}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big|^{n_\beta}|\partial_{\tilde{x}}^\gamma f|$$

$$\lesssim \sum_{\substack{\sum_{|\beta|\le k+j} n_\beta + j = \gamma_1 + \sum_{|\beta|\le k+j}\beta_1 n_\beta \\ \gamma_1+\gamma_2+\sum_{|\beta|\le k+j} n_\beta|\beta| = k+j}} \prod_{|\beta|\le k+j}(M^2\varepsilon^{\frac{7}{6}-\frac{\beta_1}{2}-\frac{\beta_2}{6}})^{n_\beta} M^2\varepsilon^{\frac{4}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}$$

$$\lesssim M^2\varepsilon^{\frac{4}{3}-\frac{j}{2}-\frac{k}{6}} \sum_{\substack{\sum_{|\beta|\le k+j} n_\beta + j = \gamma_1 + \sum_{|\beta|\le k+j}\beta_1 n_\beta \\ \gamma_1+\gamma_2+\sum_{|\beta|\le k+j} n_\beta|\beta| = k+j}} (M^2\varepsilon)^{\sum_{|\beta|\le k+j} n_\beta} \lesssim M^2\varepsilon^{\frac{4}{3}-\frac{j}{2}-\frac{k}{6}}. \tag{5.15}$$

Finally, we have

$$|\partial_x^\gamma f| = \Big|\Big(\frac{\partial_{\tilde{x}_1}}{1-\widetilde{f}_{\tilde{x}_1}}\Big)^{\gamma_1}\partial_{x_2}^{\gamma_2} f\Big|$$

$$\overset{\gamma_1\ge 1}{\lesssim} \sum_{j=1}^{\gamma_1}\sum_{\substack{n_1+2n_2+\cdots+\gamma_1 n_{\gamma_1} = \gamma_1 - j \\ n_0+n_1+\cdots+n_{\gamma_1}=\gamma_1}} \Big|\frac{1}{1-\widetilde{f}_{\tilde{x}_1}}\Big|^{n_0}\Big|\partial_{\tilde{x}_1}\Big(\frac{1}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big|^{n_1}\cdots\Big|\partial_{\tilde{x}_1}^{\gamma_1}\Big(\frac{1}{1-\widetilde{f}_{\tilde{x}_1}}\Big)\Big|^{n_{\gamma_1}}|\partial_{\tilde{x}_1}^j\partial_{x_2}^{\gamma_2} f|$$

$$\lesssim \sum_{j=1}^{\gamma_1}\sum_{\substack{n_1+2n_2+\cdots+\gamma_1 n_{\gamma_1} = \gamma_1 - j \\ n_0+n_1+\cdots+n_{\gamma_1}=\gamma_1}} (1-\varepsilon^{\frac{1}{2}})^{-n_0}(M^2\varepsilon^{\frac{5}{6}-\frac{1}{2}})^{n_1}\cdots(M^2\varepsilon^{\frac{5}{6}-\frac{\gamma_1}{2}})^{n_{\gamma_1}}|\partial_{\tilde{x}_1}^j\partial_{x_2}^{\gamma_2} f|$$

$$\lesssim \sum_{j=1}^{\gamma_1}\varepsilon^{-\frac{\gamma_1-j}{2}}|\partial_{\tilde{x}_1}^j\partial_{x_2}^{\gamma_2} f|\sum_{\substack{n_1+2n_2+\cdots+\gamma_1 n_{\gamma_1} = \gamma_1 - j \\ n_0+n_1+\cdots+n_{\gamma_1}=\gamma_1}} [(1-\varepsilon^{\frac{1}{2}})M^2\varepsilon^{\frac{5}{6}}]^{\gamma_1-n_0}$$

$$\lesssim \varepsilon^{-\frac{\gamma_1}{2}}\sum_{j=1}^{\gamma_1}\varepsilon^{\frac{j}{2}}M^2\varepsilon^{\frac{4}{3}-\frac{j}{2}-\frac{\gamma_2}{6}} \lesssim M^2\varepsilon^{\frac{4}{3}-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}. \tag{5.16}$$

One can check that the same estimate holds when $\gamma_1 = 0$.

Also from Faà di Bruno's formula one can see that for $\alpha \in \mathbb{R}$ and $\gamma > 0$, we have $|\partial_x^\gamma (1 + f_{x_2}^2)^\alpha| \lesssim_{\alpha,\gamma} M^4 \varepsilon^{\frac{7}{3} - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}}$. This estimate combining with Leibniz rule gives that $|\partial_x^\gamma N| \lesssim M^2 \varepsilon^{\frac{7}{6} - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}}$ for $\gamma > 0$. $|N - \widetilde{e}_1| \lesssim M^4 \varepsilon^{\frac{7}{6}}$ can be checked separately. The estimates of N implies $|\partial_x^\gamma (T - \widetilde{e}_2)| \lesssim M^2 \varepsilon^{\frac{7}{6} - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}}$ for $\gamma \geq 0$ since $T = N^\perp$. The estimate of JN is similar.

As for $J = \frac{\sqrt{1+f_{x_2}^2}}{1+f_{x_1}}$, we use Leibniz rule to deduce that $|\partial^\gamma J| \lesssim M^2 \varepsilon^{\frac{5}{6} - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}}$ holds for $\gamma > 0$, then one can check $|J - 1| \lesssim M^2 \varepsilon^{\frac{5}{6}}$.

The estimates of $\partial_t f$ and $\frac{\partial_t f}{1+f_{x_1}}$ are much the same and rely on the facts that $|\partial_{\widetilde{x}}^\gamma \partial_t \widetilde{f}| \lesssim M^2 \varepsilon^{\frac{1}{3} - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}}$ and $(\partial_t)_x f = \frac{(\partial_t)_{\widetilde{x}} \widetilde{f}}{1 - \widetilde{f}_{\widetilde{x}_1}}$.

Here we emphasize that C_γ in Lemma 5.2 grows at least exponentially since f is compactly supported and cannot be analytic.

Lemma 5.3 *For $\varepsilon \ll 1$ small enough and $M \gg 1$ large enough we have*

$$|Q| \leq M^2 \varepsilon^{\frac{1}{2}}. \tag{5.17}$$

Proof Since we have

$$Q = \dot{R}^{\mathrm{T}} R = \begin{bmatrix} 0 & -n_1 \dot{n}_2 + n_2 \dot{n}_1 \\ -n_2 \dot{n}_1 + n_1 \dot{n}_2 & 0 \end{bmatrix}, \tag{5.18}$$

the rest is appealing to $n_1 = \sqrt{1 - n_2^2}$ and the bootstrap assumptions (B-M) for n_2 and $\dot{n}_2$.

Lemma 5.4 *For $y \in 10\mathcal{X}(s) = \{|y_1| \leq 20\varepsilon^{\frac{1}{2}} \mathrm{e}^{\frac{3}{2}s}, |y_2| \leq 20\varepsilon^{\frac{1}{6}} \mathrm{e}^{\frac{s}{2}}\}$, we have*

$$|V| \lesssim M^{\frac{1}{4}}, \tag{5.19}$$

and for $\forall y \in \mathbb{R}^2$, it holds that

$$\begin{cases} |\partial_1 V| \lesssim M^2 \varepsilon^{\frac{1}{2}} \mathrm{e}^{-\frac{3}{2}s}, \\ |\partial_2 V| \lesssim M^2 \varepsilon^{\frac{1}{2}} \mathrm{e}^{-\frac{s}{2}}, \\ |\partial_{11} V| \lesssim M^4 \varepsilon^{\frac{5}{6}} \mathrm{e}^{-3s}, \\ |\partial_{12} V| \lesssim M^4 \varepsilon^{\frac{7}{6}} \mathrm{e}^{-2s}, \\ |\partial_{22} V| \lesssim M^4 \varepsilon^{\frac{3}{2}} \mathrm{e}^{-s}, \\ |\partial^\gamma V| \overset{|\gamma| \geq 3}{\lesssim} M^4 \varepsilon^{\frac{11}{6}} \mathrm{e}^{-(\gamma_1 + \frac{\gamma_2}{3})s}, \\ |\partial^\gamma V| \overset{|\gamma| \geq 1}{\lesssim} M^4 \varepsilon^{\frac{2}{3}} \mathrm{e}^{-(\gamma_1 + \frac{\gamma_2}{3})s}. \end{cases} \tag{5.20}$$

Proof Note that

$$V(y,s) = \frac{1+\alpha}{2} \Big(Q \begin{bmatrix} y_1 \mathrm{e}^{-\frac{3}{2}s} + f \\ y_2 \mathrm{e}^{-\frac{s}{2}} \end{bmatrix} - R^{\mathrm{T}} \dot{\xi} \Big). \tag{5.21}$$

By $R \in SO(2)$, (B-M) and (5.10), we have the above estimates.

5.5 Estimates for $\boldsymbol{U}, \boldsymbol{S}$

Lemma 5.5 *For $U \cdot N$ and S, we have*

$$|\partial^\gamma (U\cdot N)| + |\partial^\gamma S| \lesssim \begin{cases} M^{\frac14}, & \gamma=(0,0),\\ \mathrm{e}^{-\frac{s}{2}}\eta^{-\frac13}, & \gamma=(1,0),\\ \mathrm{e}^{-\frac{s}{2}}, & \gamma=(0,1)\\ M^{\frac13}\mathrm{e}^{-\frac{s}{2}}\eta^{-\frac13}, & \gamma=(2,0),\\ M^{\frac23}\mathrm{e}^{-\frac{s}{2}}\eta^{-\frac13}, & \gamma=(1,1),\\ M\mathrm{e}^{-\frac{s}{2}}\eta^{-\frac16}, & \gamma=(0,2). \end{cases} \tag{5.22}$$

Proof One can express $U \cdot N$, S in terms of W, Z, A as in (2.34). Then by directly appealing to the bootstrap assumptions we obtain the desired estimates.

Lemma 5.6 *By taking ε sufficiently small, we have*

$$\begin{cases} |U| \lesssim M^{\frac14},\\ |\partial_1 U| \le (1+\varepsilon^{\frac34})\mathrm{e}^{-\frac{s}{2}},\\ |\partial_2 U| \le \mathrm{e}^{-\frac{s}{2}},\\ |\partial_1 S| \le (1+\varepsilon)\mathrm{e}^{-\frac{s}{2}},\\ |\partial_2 S| \le \Big(\dfrac12+\varepsilon^{\frac12}\Big)\mathrm{e}^{-\frac{s}{2}}. \end{cases} \tag{5.23}$$

Proof Express U in terms of W, Z, A, then use bootstrap assumptions and the estimates (5.10) of N, T.

5.6 Transport estimates

Lemma 5.7 *For $\varepsilon \ll 1$ and $\forall y \in 10\mathcal{X}(s)$, we have*

$$\begin{cases} |\partial_1 G_A| \lesssim M^2 \mathrm{e}^{-\frac56 s}, \quad |\partial_2 G_A| \lesssim M^2\varepsilon^{\frac16}\\ |\partial_{11} G_A| \lesssim M^{\frac12}\mathrm{e}^{-s}, \quad |\partial_{12} G_A| \lesssim M\mathrm{e}^{-s}, \quad |\partial_{22} G_A| \lesssim M^2\mathrm{e}^{-\frac{s}{2}}. \end{cases} \tag{5.24}$$

Proof We first deal with $\partial_1 G_A$. Using the definition (2.27) of G_A, the estimates (5.10) for functions of coordinate transformation, estimates (5.19)–(5.20) for V, and the bootstrap assumptions, we have by Leibniz rule that

$$\begin{aligned} |\partial_1 G_A| &\lesssim \mathrm{e}^{\frac{s}{2}}\Big|\partial_1 \frac{\partial_t f}{1+f_{x_1}}\Big| + \mathrm{e}^{\frac{s}{2}}|\partial_1 J|(\kappa_0+|Z|+|V|) + \mathrm{e}^{\frac{s}{2}}|\partial_1 Z| + \mathrm{e}^{\frac{s}{2}}|\partial_1 (V\cdot N)|\\ &\lesssim \mathrm{e}^{\frac{s}{2}}M^2\varepsilon^{-\frac16}\mathrm{e}^{-\frac32 s} + \mathrm{e}^{\frac{s}{2}}\varepsilon^{\frac13}\mathrm{e}^{-\frac32 s}M^{\frac14} + \mathrm{e}^{\frac{s}{2}}(M^{\frac12}\mathrm{e}^{-\frac32 s} + M^2\varepsilon^{\frac12}\mathrm{e}^{-\frac32 s} + M^{2+\frac14}\varepsilon\mathrm{e}^{-\frac32 s})\\ &\lesssim M^2\varepsilon^{-\frac16}\mathrm{e}^{-s} \lesssim M^2\mathrm{e}^{-\frac56 s}. \end{aligned} \tag{5.25}$$

The other derivatives of G_A are estimated in a similar way.

Lemma 5.8 *For $\varepsilon \ll 1$ and $\forall y \in \mathcal{X}(s)$, we have*

$$\begin{cases} |g_A| \lesssim M^{\frac{1}{4}} \mathrm{e}^{\frac{s}{2}}, \\ |\partial_1 g_A| \le 3, \\ |\partial_2 g_A| \le 2, \\ |D^2 g_A| \lesssim M\eta^{-\frac{1}{6}} + M^2 \mathrm{e}^{-\frac{s}{2}}, \\ |\partial_1 h_A| \lesssim \mathrm{e}^{-s}, \\ |\partial_2 h_A| \lesssim \mathrm{e}^{-s}. \end{cases} \tag{5.26}$$

Proof Use the definition (2.27) and the estimates (B-W), (5.10) and (5.24), calculate similarly as we did in the proof of (5.24) with more care since there is no room of a universal constant in some of the inequalities.

6 Energy Estimate

To overcome the loss of derivative in L^∞ estimates of W, Z, and A, we will establish an additional energy estimate to control the $\dot{H}^k (k \ll 1)$ norms of W, Z, and A. It is crutial that in the proof of energy estimate we only use the bootstrap assumptions, not requiring any information on higher order derivatives.

Proposition 6.1 (Energy estimate for W, Z, A) *For an integer $k \ge 18$, and a constant $\lambda = \lambda(k)$,*

$$\|Z(\cdot, s)\|_{\dot{H}^k}^2 + \|A(\cdot, s)\|_{\dot{H}^k}^2 \le 2\lambda^{-k} \mathrm{e}^{-s} + M^{4k} \mathrm{e}^{-s}(1 - \varepsilon^{-s} \mathrm{e}^{-s}) \lesssim M^{4k} \mathrm{e}^{-s}, \tag{6.1}$$

$$\|W(\cdot, s)\|_{\dot{H}^k}^2 \le 2\lambda^{-k} \varepsilon^{-1} \mathrm{e}^{-s} + M^{4k}(1 - \varepsilon^{-s} \mathrm{e}^{-s}). \tag{6.2}$$

We will prove this by using the $\dot{H}^k$ bound for (U, S), and the fact that the $\dot{H}^k$ norm of (W, Z, A) can be controlled by the $\dot{H}^k$ norm of (U, S). More precisely, we have the following lemma.

Lemma 6.1 *The following inequalities hold*

$$\begin{aligned} \|W\|_{\dot{H}^k} &\lesssim_k \mathrm{e}^{\frac{s}{2}} \Big(\|U\|_{\dot{H}^k} + \|S\|_{\dot{H}^k} + M^{\frac{9}{4}} \varepsilon^{\frac{3}{2}} \mathrm{e}^{-\frac{k-3}{3}s} \Big), \\ \|Z\|_{\dot{H}^k} + \|A\|_{\dot{H}^k} &\lesssim_k \|U\|_{\dot{H}^k} + \|S\|_{\dot{H}^k} + M^{\frac{9}{4}} \varepsilon^{\frac{3}{2}} \mathrm{e}^{-\frac{k-3}{3}s}. \end{aligned} \tag{6.3}$$

Proof We first estimate $\|W\|_{\dot{H}^k}$. Note that by (2.35), $\mathrm{supp}(DU, DS) \subset \mathcal{X}(s)$. We have

$$\begin{aligned} &\mathrm{e}^{-\frac{s}{2}} \|\partial^\gamma W\|_{L^2(\mathbb{R}^2)} \\ &\overset{|\gamma|=k}{\lesssim_k} \|\partial^\gamma S\|_{L^2} + \sum_{\beta \le \gamma} \|\partial^{\gamma-\beta} U \cdot \partial^\beta N\|_{L^2(\mathcal{X}(s))} \\ &\lesssim \|S\|_{\dot{H}^k} + \|U\|_{L^\infty} \|\partial^\gamma N\|_{L^\infty} |\mathcal{X}(s)|^{\frac{1}{2}} + \|\partial^\gamma U\|_{L^2} + \sum_{0<\beta<\gamma} \|\partial^{\gamma-\beta} U\|_{L^2} \|\partial^\beta N\|_{L^\infty} \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{Poincar}}{\lesssim_k} \|S\|_{\dot H^k} + \|U\|_{\dot H^k} + M^{\frac14} M^2 \varepsilon^{\frac76 - \frac{\gamma_1}{2} - \frac{\gamma_2}{6}} \mathrm{e}^{-(\frac32\gamma_1 + \frac12\gamma_2)s} \varepsilon^{\frac13} e^s \\
&\quad + \sum_{0<\beta<\gamma} (\varepsilon^{\frac16} \mathrm{e}^{\frac{s}{2}})^{|\beta|} \|D^k U\|_{L^2} M^2 \varepsilon^{\frac76 - \frac{\beta_1}{2} - \frac{\beta_2}{6}} \mathrm{e}^{-(\frac32\beta_1 + \frac12\beta_2)s} \\
&\lesssim \|S\|_{\dot H^k} + \|U\|_{\dot H^k} + M^{\frac94} \varepsilon^{\frac32} \mathrm{e}^{-\frac{|\gamma|-3}{3}s}.
\end{aligned}
\tag{6.4}
$$

The estimates of Z and A are similar.

Definition 6.1 (Modified $\dot H^k$ norm) *We define*

$$
E_k^2(s) := \sum_{|\gamma|=k} \lambda^{\gamma_2} (\|\partial^\gamma U(\cdot, s)\|_{L^2}^2 + \|\partial^\gamma S(\cdot, s)\|_{L^2}^2),
\tag{6.5}
$$

where $\lambda \in (0,1)$ is to be specified below. Clearly we have the norm equivalence:

$$
\lambda^k (\|U\|_{\dot H^k}^2 + \|S\|_{\dot H^k}^2) \le E_k^2 \le \|U\|_{\dot H^k}^2 + \|S\|_{\dot H^k}^2.
\tag{6.6}
$$

6.1 Evolution of derivatives of (U, S)

Applying ∂^γ to both sides of the (U,S) equation (2.32), we see that

$$
\begin{aligned}
&\partial_s \partial^\gamma U_i - \beta_\tau \mathrm{e}^{-s} Q_{ij} \partial^\gamma U_j + \mathcal{V}_A \cdot \nabla \partial^\gamma U_i + D_\gamma \partial^\gamma U_i + \beta_3 \beta_\tau (1+\gamma_1) J N_i \partial^\gamma S \partial_1 W \\
&+ 2\beta_3\beta_\tau S (\mathrm{e}^{\frac{s}{2}} J N_i \partial_1 \partial^\gamma S + \mathrm{e}^{-\frac{s}{2}} \delta_{i2} \partial_2 \partial^\gamma S) = F_{U_i}^{(\gamma)},
\end{aligned}
\tag{6.7a}
$$

$$
\begin{aligned}
&\partial_s \partial^\gamma S + \mathcal{V}_A \cdot \nabla \partial^\gamma S + D_\gamma \partial^\gamma S + \beta_\tau (\beta_1 + \beta_3 \gamma_1) J N \cdot \partial^\gamma U \partial_1 W \\
&+ 2\beta_3\beta_\tau S (\mathrm{e}^{\frac{s}{2}} J N \cdot \partial_1 \partial^\gamma U + \mathrm{e}^{-\frac{s}{2}} \partial_2 \partial^\gamma U_2) = F_S^{(\gamma)},
\end{aligned}
\tag{6.7b}
$$

where $D_\gamma = \frac12|\gamma| + \gamma_1(1+\partial_1 g_U)$, and the forcing terms are $F_{U_i}^{(\gamma)} = F_{U_i}^{(\gamma,U)} + F_{U_i}^{(\gamma-1,U)} + F_{U_i}^{(\gamma,S)} + F_{U_i}^{(\gamma-1,S)}$, $F_S^{(\gamma)} = F_S^{(\gamma,U)} + F_S^{(\gamma-1,U)} + F_S^{(\gamma,S)} + F_S^{(\gamma-1,S)}$. Here

$$
\begin{aligned}
F_{U_i}^{(\gamma,U)} =& - 2\beta_1\beta_\tau (\mathrm{e}^{\frac{s}{2}} J N_j \partial^\gamma U_j \partial_1 U_i + \mathrm{e}^{-\frac{s}{2}} \partial^\gamma U_2 \partial_2 U_i) \\
&- \gamma_2 \partial_2 g_A \partial_1 \partial^{\gamma - e_2} U_i - \sum_{\substack{|\beta|=|\gamma|-1\\ \beta\le\gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} h_A \partial_2 \partial^\beta U_i \\
=& F_{U_i,(1)}^{(\gamma,U)} + F_{U_i,(2)}^{(\gamma,U)} + F_{U_i,(3)}^{(\gamma,U)},
\end{aligned}
\tag{6.8a}
$$

$$
\begin{aligned}
F_{U_i}^{(\gamma-1,U)} =& - \sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} g_A \partial_1 \partial^\beta U_i + \partial^{\gamma-\beta} h_A \partial_2 \partial^\beta U_i) \\
&- 2\beta_1\beta_\tau \mathrm{e}^{\frac{s}{2}} [\partial^\gamma, JN] \cdot U \partial_1 U_i - \beta_\tau \mathrm{e}^{\frac{s}{2}} \partial^\gamma \Big(2\beta_1 V \cdot JN - \frac{\partial_t f}{1+f_{x_1}} \Big) \partial_1 U_i \\
&- 2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}} \partial^\gamma V_2 \partial_2 U_i \\
=& F_{U_i,(1)}^{(\gamma-1,U)} + F_{U_i,(2)}^{(\gamma-1,U)} + F_{U_i,(3)}^{(\gamma-1,U)} + F_{U_i,(4)}^{(\gamma-1,U)},
\end{aligned}
\tag{6.8b}
$$

$$
\begin{aligned}
F_{U_i}^{(\gamma,S)} =& - 2\beta_3\beta_\tau \gamma_2 \mathrm{e}^{\frac{s}{2}} \partial_2 (S J N_i) \partial_1 \partial^{\gamma - e_2} S - \beta_3\beta_\tau (1+\gamma_1) \mathrm{e}^{\frac{s}{2}} J N_i \partial_1 Z \partial^\gamma S \\
&- 2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}} \delta_{i2} \sum_{\substack{|\beta|=|\gamma|-1\\ \beta\le\gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} S \partial_2 \partial^\beta S - 2\beta_3\beta_\tau \delta_{i2} \mathrm{e}^{-\frac{s}{2}} \partial^\gamma S \partial_2 S
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2\beta_3\beta_\tau\gamma_1 \mathrm{e}^{\frac{s}{2}}\partial_1(JN_i)S\partial^\gamma S\\
&= F_{U_i,(1)}^{(\gamma,S)} + F_{U_i,(2)}^{(\gamma,S)} + F_{U_i,(3)}^{(\gamma,S)} + F_{U_i,(4)}^{(\gamma,S)} + F_{U_i,(5)}^{(\gamma,S)}, \qquad (6.8c)\\
F_{U_i}^{(\gamma-1,S)} &= - 2\beta_3\beta_\tau \sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}} \binom{\gamma}{\beta}(\mathrm{e}^{\frac{s}{2}}\partial^{\gamma-\beta}(SJN_i)\partial_1\partial^\beta S + \mathrm{e}^{-\frac{s}{2}}\delta_{i2}\partial^{\gamma-\beta}S\partial_2\partial^\beta S)\\
&\quad - 2\beta_3\beta_\tau \mathrm{e}^{\frac{s}{2}}[\partial^\gamma, JN_i]S\partial_1 S\\
&= F_{U_i,(1)}^{(\gamma-1,S)} + F_{U_i,(2)}^{(\gamma-1,S)}, \qquad (6.8d)\\
F_S^{(\gamma,S)} &= - 2\beta_3\beta_\tau(\mathrm{e}^{\frac{s}{2}}\partial^\gamma SJN_j\partial_1 U_j + \mathrm{e}^{-\frac{s}{2}}\partial^\gamma S\partial_2 U_2)\\
&\quad - \gamma_2\partial_2 g_A\partial_1\partial^{\gamma-e_2}S - \sum_{\substack{|\beta|=|\gamma|-1\\ \beta\le\gamma}}\binom{\gamma}{\beta}\partial^{\gamma-\beta}h_A\partial_2\partial^\beta S, \qquad (6.8e)\\
F_S^{(\gamma-1,S)} &= - \sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}}\binom{\gamma}{\beta}(\partial^{\gamma-\beta}g_A\partial_1\partial^\beta S + \partial^{\gamma-\beta}h_A\partial_2\partial^\beta S)\\
&\quad - 2\beta_3\beta_\tau \sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}}\binom{\gamma}{\beta}(\mathrm{e}^{\frac{s}{2}}\partial^{\gamma-\beta}(SJN)\cdot\partial_1\partial^\beta U + \mathrm{e}^{-\frac{s}{2}}\partial^{\gamma-\beta}S\partial_2\partial^\beta U_2)\\
&\quad - 2\beta_3\beta_\tau \mathrm{e}^{\frac{s}{2}}\partial_1 U_j[\partial^\gamma, JN_j]S - \beta_\tau \mathrm{e}^{\frac{s}{2}}\partial^\gamma\Big(2\beta_1 V\cdot JN - \frac{\partial_t f}{1+f_{x_1}}\Big)\partial_1 S\\
&\quad - 2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial^\gamma V_2\partial_2 S, \qquad (6.8f)\\
F_S^{(\gamma,U)} &= - 2\beta_3\beta_\tau\gamma_2 \mathrm{e}^{\frac{s}{2}}\partial_2(SJN)\cdot\partial_1\partial^{\gamma-e_2}U + \beta_\tau(\beta_1+\beta_3\gamma_1)\mathrm{e}^{\frac{s}{2}}JN\cdot\partial^\gamma U\partial_1 Z\\
&\quad - 2\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}} \sum_{\substack{|\beta|=|\gamma|-1\\ \beta\le\gamma}}\binom{\gamma}{\beta}\partial^{\gamma-\beta}S\partial_2\partial^\beta U_2 - 2\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial^\gamma U_2\partial_2 S\\
&\quad - 2\beta_3\beta_\tau\gamma_1 \mathrm{e}^{\frac{s}{2}}S\partial^\gamma U_j\partial_1(JN_j), \qquad (6.8g)\\
F_S^{(\gamma-1,U)} &= - 2\beta_1\beta_\tau \mathrm{e}^{\frac{s}{2}}\partial_1 S[\partial^\gamma, JN_j]U_j. \qquad (6.8h)
\end{aligned}
$$

6.2 Estimates for forcing terms

Lemma 6.2 *Let $k \gg 1$ and $\delta \in (0, \frac{1}{32}]$, $\lambda = \frac{\delta^2}{12k^2}$. Then for $\varepsilon \ll 1$ we have*

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_{U_i}^{(\gamma)}\partial^\gamma U_i| \le (4+8\delta)E_k^2 + \mathrm{e}^{-s}M^{4k-4}, \qquad (6.9a)$$

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_S^{(\gamma)}\partial^\gamma S| \le (4+8\delta)E_k^2 + \mathrm{e}^{-s}M^{4k-4}. \qquad (6.9b)$$

Proof We begin with (6.9a).

We first deal with the term $F_{U_i}^{(\gamma,U)}$ involving the top order derivatives of U, this term is decomposed as a sum $F_{U_i,(1)}^{(\gamma,U)} + F_{U_i,(2)}^{(\gamma,U)} + F_{U_i,(3)}^{(\gamma,U)}$. From (B-M), $0 < \beta_1\beta_\tau < 1$, and (5.10), we have

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_{U_i,(1)}^{(\gamma)}\partial^\gamma U_i|$$

$$\overset{(6.8a)}{\leq} 4\beta_1\beta_\tau \sum_{|\gamma|=k} \lambda^{\gamma_2}[\mathrm{e}^{\frac{s}{2}}(1+\varepsilon^{\frac{3}{4}})\|\partial_1 U\|_{L^\infty} + \mathrm{e}^{-\frac{2}{2}}\|\partial_2 U\|_{L^\infty}]\|\partial^\gamma U\|_{L^2}^2$$
$$\leq (4+\varepsilon^{\frac{1}{2}})E_k^2. \tag{6.10}$$

By (5.26) and Young's inequality, we can see that

$$\begin{aligned}
&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_{U_i,(2)}^{(\gamma)}\partial^\gamma U_i| \\
&\overset{(6.8a)}{\leq} 2\sum_{|\gamma|=k}\lambda^{\gamma_2}\gamma_2\|\partial_2 g_A\|_{L^\infty(\mathcal{X}(s))}\|\partial_1\partial^{\gamma-e_2}U_i\|_{L^2}\|\partial^\gamma U_i\|_{L^2} \\
&\leq 2\sum_{|\gamma|=k}\left(\frac{\gamma_2^2}{\delta}\lambda^{\gamma_2+1}\|\partial^\gamma U\|_{L^2}^2 + \mathbb{1}_{\gamma_2>0}\delta\lambda^{\gamma_2-1}\|\partial_1\partial^{\gamma-e_2}U\|_{L^2}^2\right) \\
&\leq \lambda\frac{2k^2}{\delta}E_k^2 + 2\delta E_k^2 \overset{\lambda=\frac{\delta^2}{12k^2}}{\leq} 3\delta E_k^2
\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_{U_i,(3)}^{(\gamma)}\partial^\gamma U_i| \\
&\overset{(6.8a)}{\lesssim} \sum_{|\gamma|=k}\int\sum_{\substack{|\beta|=|\gamma|-1\\ \beta\leq\gamma}}|\partial^{\gamma-\beta}h_A||\partial_2\partial^\beta U||\partial^\gamma U| \\
&\lesssim \varepsilon\sum_{|\gamma|=k}\sum_{\substack{|\beta|=|\gamma|-1\\ \beta\leq\gamma}}(\|\partial^\gamma U\|_{L^2}^2 + \|\partial_2\partial^\beta U\|_{L^2}^2) \leq \varepsilon^{\frac{1}{2}}E_k^2.
\end{aligned} \tag{6.12}$$

Combining these three estimates, we have

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F_{U_i}^{(\gamma,U)}\partial^\gamma U_i| \leq (4+3\delta+\varepsilon^{\frac{1}{2}})E_k^2. \tag{6.13}$$

Next we deal with the forcing terms $F_{U_i}^{(\gamma-1,U)}$ involving lower order derivatives of U. We decompose its first part as $F_{U_i,(1)}^{(\gamma-1,U)} = I_{i1} + I_{i2} + I_{i3}$ where

$$\begin{aligned}
I_{i1} &= -\sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}\binom{\gamma}{\beta}\partial^{\gamma-\beta}g_A\partial^\beta\partial_1(U\cdot NN_i), \\
I_{i2} &= -\sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}\binom{\gamma}{\beta}\partial^{\gamma-\beta}g_A\partial^\beta\partial_1(AT_i), \\
I_{i3} &= -\sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}\binom{\gamma}{\beta}\partial^{\gamma-\beta}h_A\partial^\beta\partial_2 U_i.
\end{aligned} \tag{6.14}$$

Since $D(U\cdot N)$ is supported in $\mathcal{X}(s)$, we introduce a positive cut-off function $\widetilde{\theta}\in C_c(5\mathcal{X}(0))$ such that $\widetilde{\theta}\equiv 1$ on $\mathcal{X}(0)$. Let $\widetilde{\theta}_s(y) = \widetilde{\theta}(y_1\mathrm{e}^{-\frac{3}{2}s}, y_2\mathrm{e}^{-\frac{s}{2}})$. Then $\widetilde{\theta}_s\in C_c^\infty(5\mathcal{X}(s))$, $\widetilde{\theta}_s\equiv 1$ on

$\mathcal{X}(s)$, and

$$\|\partial^\gamma \widetilde{\theta}_s\|_{L^\infty} \lesssim \varepsilon^{-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}} \mathrm{e}^{-\frac{3}{2}\gamma_1 s-\frac{\gamma_2}{2}s} \lesssim \mathrm{e}^{-\frac{|\gamma|}{3}s}. \tag{6.15}$$

By the interpolation inequality (B.3), we have

$$\begin{aligned}&\|I_{i1}\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|D^k(\widetilde{\theta}_s g_A)\|^a_{L^2_y(\mathbb{R}^2)}\|D^2(\widetilde{\theta}_s g_A)\|^{1-a}_{L^q(\mathbb{R}^2)}\|D^k(U\cdot NN)\|^b_{L^2(\mathbb{R}^2)}\|D^2(U\cdot NN)\|^{1-b}_{L^q(\mathbb{R}^2)}.\end{aligned} \tag{6.16}$$

We estimate each factor. The $D^2 g_A$ term can be bounded by

$$\begin{aligned}&\|D^2(\widetilde{\theta}_s g_A)\|_{L^q(\mathbb{R}^2)}\\ &\overset{(5.26)}{\lesssim} M^{\frac{1}{4}}\mathrm{e}^{\frac{s}{2}}\mathrm{e}^{-\frac{2}{3}s}(\varepsilon^{\frac{2}{3}}\mathrm{e}^{2s})^{\frac{1}{q}} + \mathrm{e}^{-\frac{s}{3}}(\varepsilon^{\frac{2}{3}}\mathrm{e}^{2s})^{\frac{1}{q}} + \|M\eta^{-\frac{1}{6}} + M^2\mathrm{e}^{-\frac{s}{2}}\|_{L^q(5\mathcal{X}(s))}\\ &\lesssim M\|\eta^{-1}\|^{\frac{1}{6}}_{L^{\frac{q}{6}}(\mathbb{R}^2)} + M^2\mathrm{e}^{-\frac{s}{6}}\varepsilon^{\frac{2}{3q}}\mathrm{e}^{\frac{2}{q}s}\lesssim M.\end{aligned} \tag{6.17}$$

In the last inequality we require $q \geq 12$ and use the fact that $(1+|y_1|^{\alpha_1}+\cdots+|y_d|^{\alpha_d})^{-1} \in L^1(R^d)$ as long as $\Sigma\alpha_i^{-1} < 1$. From estimates (5.22) of $U\cdot N$ and estimates (5.10) of N, we have

$$\|D^2(U\cdot NN)\|_{L^q} \lesssim M\mathrm{e}^{-\frac{s}{2}}. \tag{6.18}$$

Then, as we did in the proof of Lemma 6.1, we have

$$\|D^k(U\cdot JN)\|_{L^2(5\mathcal{X}(s))}\lesssim\|D^k U\|_{L^2(\mathbb{R}^2)} + M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{k-3}{3}}, \tag{6.19}$$

$$\begin{aligned}&\mathrm{e}^{-\frac{s}{2}}\|D^m g_A\|_{L^2(5\mathcal{X}(s))}\\ &\lesssim \|D^m(U\cdot JN)\|_{L^2(5\mathcal{X}(s))} + \|D^m(V\cdot JN)\|_{L^2(5\mathcal{X}(s))}\\ &\quad + \left\|D^m\Big(\frac{\partial_t f}{1+f_{x_1}}\Big)\right\|_{L^2(5\mathcal{X}(s))}\\ &\overset{m>0}{\lesssim} \|D^m U\|_{L^2(\mathbb{R}^2)} + M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{m-3}{3}},\end{aligned} \tag{6.20}$$

$$\begin{aligned}\left\|\partial^\gamma\left(\widetilde{\theta}_s g_A\right)\right\|_{L^2(\mathbb{R}^2)} &\lesssim_\gamma \varepsilon^{-\frac{\gamma_1}{2}-\frac{\gamma_2}{6}}\mathrm{e}^{-\frac{3}{2}\gamma_1 s-\frac{\gamma_2}{2}s}\|g_A\|_{L^\infty}|5\mathcal{X}(s)|^{\frac{1}{2}}\\ &\quad + \sum_{\beta<\gamma}\varepsilon^{-\frac{\beta_1}{2}-\frac{\beta_2}{6}}\mathrm{e}^{-\frac{3}{2}\beta_1 s-\frac{\beta_2}{2}s}\|\partial^{\gamma-\beta}g_A\|_{L^2(5\mathcal{X}(s))}\\ &\lesssim \mathrm{e}^{\frac{s}{2}}(\|D^{|\gamma|}U\|_{L^2(\mathbb{R}^2)} + M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{|\gamma|-3}{3}s}).\end{aligned} \tag{6.21}$$

For $k \geq 5$, we have $a+b \geq \frac{1}{2}$, $\frac{2-a-b}{1-a-b} \leq 2k-4$. Hence, by taking M to be large enough in terms of λ and k, we have

$$\begin{aligned}&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|I_{i1}\partial^\gamma U_i|\\ &\lesssim \sum_{|\gamma|=k}\lambda^{\gamma_2}\|D^k U\|_{L^2}[\|D^k U\|^{a+b}_{L^2} + (M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{k-3}{3}s})^{a+b}]M^{2-a-b}\mathrm{e}^{\frac{a+b-1}{2}s}\\ &\lesssim \sum_{|\gamma|=k}\lambda^{\gamma_2}(\lambda^{-\frac{k}{2}}E_k)^{1+a+b}M^{2-a-b}\mathrm{e}^{\frac{a+b-1}{2}s}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\gamma|=k}\lambda^{\gamma_2}M^{2+3a+3b}\varepsilon^{\frac{a+b}{3}}\mathrm{e}^{-\frac{a+b}{3}ks+\frac{a+b+1}{2}s}\lambda^{-\frac{k}{2}}E_k\\
&\overset{a+b<1}{\leq} 2\delta E_k^2+C(a,b,\delta)\mathrm{e}^{-s}M^{\frac{2(2-a-b)}{1-a-b}}\lambda^{-\frac{1+a+b}{1-a-b}k}\\
&\quad +C(\delta)M^{10}\varepsilon^{\frac{2}{3}(a+b)}\lambda^{-k}\mathrm{e}^{-\frac{2}{3}(a+b)ks+(a+b+1)s}\\
&\leq 2\delta E_k^2+C(a,b,\delta)\mathrm{e}^{-s}M^{4k-8}\lambda^{-\frac{1+a+b}{1-a-b}k}\\
&\leq 2\delta E_k^2+\mathrm{e}^{-s}M^{4k-6}. \qquad (6.22)
\end{aligned}
$$

Next, we estimate the L^2 norm of I_{i2}:

$$
\begin{aligned}
\|I_{i2}\|_{L^2} &\lesssim \mathrm{e}^{\frac{s}{2}}\sum_{j=1}^{k-2}\|D^{k-j}(U\cdot JN)D^j\partial_1(AT)\|_{L^2}\\
&\quad +\mathrm{e}^{\frac{s}{2}}\sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}\left(\left|\partial^{\gamma-\beta}(V\cdot JN)\right|+\left|\partial^{\gamma-\beta}\frac{\partial_t f}{1+f_{x_1}}\right|\right)\|\partial^\beta\partial_1(AT)\|_{L^2}\\
&= I_{i2,1}+I_{i2,2}. \qquad (6.23)
\end{aligned}
$$

First, for $I_{i2,1}$, we have

$$
\begin{aligned}
I_{i2,1} &\overset{\text{Hölder}}{\lesssim} \mathrm{e}^{\frac{s}{2}}\sum_{j=1}^{k-2}\|D^{k-j-1}D(\widetilde{\theta}_sU\cdot JN)\|_{L^{\frac{2(k-1)}{k-1-j}}(\mathbb{R}^2)}\|D^j\partial_1(AT)\|_{L^{\frac{2(k-1)}{j}}}\\
&\overset{(B.2)}{\lesssim} \mathrm{e}^{\frac{s}{2}}\sum_{j=1}^{k-2}\|D(\widetilde{\theta}_sU\cdot JN)\|_{\dot H^{k-1}}^{\frac{k-j-1}{k-1}}\|D(\widetilde{\theta}_sU\cdot JN)\|_{L^\infty}^{\frac{j}{k-1}}\|\partial_1(AT)\|_{\dot H^{k-1}}^{\frac{j}{k-1}}\|\partial_1(AT)\|_{L^\infty}^{\frac{k-j-1}{k-1}}\\
&\lesssim \mathrm{e}^{\frac{s}{2}}\sum_{j=1}^{k-2}(\|D^kU\|_{L^2}+M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{k-3}{3}s})^{\frac{k-j-1}{k-1}}(M\mathrm{e}^{-\frac{s}{2}})^{-\frac{j}{k-1}}\\
&\quad\times(\|D^kA\|_{L^2}+M^4\varepsilon\mathrm{e}^{-\frac{k-3}{3}s})^{\frac{j}{k-1}}(M\mathrm{e}^{-\frac{3}{2}s})^{\frac{k-j-1}{k-1}}\\
&\lesssim M^{\frac{1}{k-1}}\mathrm{e}^{-\frac{1}{k-1}s}(\lambda^{-\frac{k}{2}}E_k+M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{k-3}{3}s}). \qquad (6.24)
\end{aligned}
$$

Then

$$
\begin{aligned}
I_{i2,2} &\lesssim \mathrm{e}^{\frac{s}{2}}\sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}M^2\varepsilon^{\frac{1}{3}-\frac{\gamma_1-\beta_1}{2}-\frac{\gamma_2-\beta_2}{6}}\mathrm{e}^{-\frac{3}{2}(\gamma_1-\beta_1)s-\frac{1}{2}(\gamma_2-\beta_2)s}(\varepsilon^{\frac{1}{6}}\mathrm{e}^{\frac{s}{2}})^{|\gamma|-|\beta|-1}\|D^k(AT)\|_{L^2}\\
&\lesssim \sum_{\substack{1\leq|\beta|\leq|\gamma|-2\\ \beta\leq\gamma}}M^2\varepsilon^{\frac{1}{6}-\frac{\gamma_1-\beta_1}{3}}\mathrm{e}^{-(\gamma_1-\beta_1)s}(\|D^kA\|_{L^2}+M^{\frac{9}{4}}\varepsilon^{\frac{3}{2}}\mathrm{e}^{-\frac{k-3}{3}s})\\
&\lesssim M^2\varepsilon^{\frac{1}{6}}(\lambda^{-\frac{k}{2}}E_k+M^{\frac{9}{4}}\varepsilon^{\frac{3}{2}}\mathrm{e}^{-\frac{k-3}{3}s}). \qquad (6.25)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|I_{i2}\partial^\gamma U_i| &\overset{k\geq 7}{\lesssim} \lambda^{-\frac{k}{2}}E_kM^{\frac{1}{k-1}}\mathrm{e}^{-\frac{1}{k-1}s}(\lambda^{-\frac{k}{2}}E_k+M^2\varepsilon^{\frac{1}{3}}\mathrm{e}^{-\frac{k-3}{3}s})\\
&\lesssim (M\varepsilon)^{\frac{1}{k-1}}\lambda^{-k}E_k^2+M^{\frac{1}{k-1}}\mathrm{e}^{-\frac{1}{k-1}s}M^6\varepsilon^{\frac{2}{3}}\mathrm{e}^{-\frac{2}{3}(k-3)s}
\end{aligned}
$$

$$\leq \varepsilon^{\frac{1}{k}} E_k^2 + \mathrm{e}^{-s}. \tag{6.26}$$

Like the estimate of I_{i2}, we can estimate I_{i3} as

$$2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int |I_{i3} \partial^\gamma U_i| \leq \varepsilon^{\frac{1}{2}} E_k^2 + \mathrm{e}^{-s}. \tag{6.27}$$

Summing up these estimates, we obtain

$$2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int |F_{U_i,(1)}^{(\gamma-1,U)} \partial^\gamma U_i| \leq (2\delta + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{k}}) E_k^2 + \mathrm{e}^{-s} M^{4k-5}. \tag{6.28}$$

Now we turn to the estimate of $F_{U_i,(2)}^{(\gamma-1,U)}$. Using the same method in the proof of Lemma 6.1, we have

$$\|[\partial^\gamma, JN]U\|_{L^2} \leq \varepsilon^{\frac{1}{2}} \|D^k U\|_{L^2} + \varepsilon \mathrm{e}^{-(\gamma_1 + \frac{\gamma_2}{3} - 1)s}. \tag{6.29}$$

Thus, upon choosing ε small enough in terms of λ and k, we have

$$\begin{aligned}
&2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int |F_{U_i,(2)}^{(\gamma-1,U)} \partial^\gamma U_i| \\
&\lesssim \sum_{|\gamma|=k} \mathrm{e}^{\frac{s}{2}} \|[\partial^\gamma, JN]U\|_{L^2} \|\partial^\gamma U\|_{L^2} \|\partial_1 U\|_{L^\infty} \\
&\overset{(5.23)}{\lesssim} \mathrm{e}^{\frac{s}{2}} (\varepsilon^{\frac{1}{2}} \|D^k U\|_{L^2} + \varepsilon \mathrm{e}^{-\frac{k-3}{3}s}) \|D^k U\|_{L^2} \mathrm{e}^{-\frac{s}{2}} \\
&\lesssim \lambda^{-k} \varepsilon^{\frac{1}{2}} E_k^2 + \varepsilon \lambda^{-\frac{k}{2}} E_k \mathrm{e}^{-\frac{k-3}{3}s} \\
&\lesssim \lambda^{-k} \varepsilon^{\frac{1}{2}} E_k^2 + \varepsilon^{\frac{1}{2}} \mathrm{e}^{-\frac{2(k-3)}{3}s} \leq \varepsilon^{\frac{1}{4}} E_k^2 + \mathrm{e}^{-s}.
\end{aligned} \tag{6.30}$$

From the estimates (5.19)–(5.10) of V and J, N, we can see that

$$|\partial^\gamma (V \cdot JN)| + |\partial^\gamma \frac{\partial_t f}{1 + f_{x_1}}| \lesssim M^2 \varepsilon^{\frac{1}{3}} \mathrm{e}^{-(\gamma_1 + \frac{\gamma_2}{3})s}. \tag{6.31}$$

Therefore, we have

$$\begin{aligned}
2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int \big|F_{U_i,(3)}^{(\gamma-1,U)} \partial^\gamma U_i\big| &\lesssim \mathrm{e}^{\frac{s}{2}} \sum_{|\gamma|=k} M^2 \varepsilon^{\frac{1}{3}} \mathrm{e}^{-(\gamma_1 + \frac{\gamma_2}{3})s} \|\partial_1 U\|_{L^\infty} \|\partial^\gamma U\|_{L^2} |\mathcal{X}(s)|^{\frac{1}{2}} \\
&\lesssim M^2 \varepsilon^{\frac{2}{3}} \mathrm{e}^{-\frac{k-3}{3}s} \|D^k U\|_{L^2} \\
&\lesssim \varepsilon^{\frac{2}{3}} \|D^k U\|_{L^2}^2 + M^4 \varepsilon^{\frac{2}{3}} \mathrm{e}^{-\frac{2(k-3)}{3}s} \\
&\leq \varepsilon^{\frac{1}{2}} E_k^2 + \mathrm{e}^{-s}.
\end{aligned} \tag{6.32}$$

The estimate of $F_{U_i,(4)}^{(\gamma-1,U)}$ is much the same. We have

$$2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int |F_{U_i,(4)}^{(\gamma-1,U)} \partial^\gamma U_i| \leq \varepsilon^{\frac{1}{2}} E_k^2 + \mathrm{e}^{-s}. \tag{6.33}$$

Combining the above estimates, we arrive at

$$2 \sum_{|\gamma|=k} \lambda^{\gamma_2} \int |F_{U_i}^{(\gamma-1,U)} \partial^\gamma U_i| \leq 2(\delta + \varepsilon^{\frac{1}{4}}) E_k^2 + \mathrm{e}^{-s} M^{4k-4}. \tag{6.34}$$

Now we estimate the terms involving k order derivatives of S.

$$\begin{aligned}2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int\left|(F_{U_i,(2)}^{(\gamma,S)}+F_{U_i,(4)}^{(\gamma,S)})\partial^\gamma U_i\right|\\ \lesssim(\mathrm{e}^{\frac{s}{2}}\|\partial_1 Z\|_{L^\infty}+\mathrm{e}^{-\frac{s}{2}}\|\partial_2 S\|_{L^\infty})\lambda^{-k}E_k^2\le\varepsilon^{\frac12}E_k^2.\end{aligned}\tag{6.35}$$

$$\begin{aligned}2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|F_{U_i,(3)}^{(\gamma,S)}\partial^\gamma U_i|&\lesssim\sum_{|\gamma|=k}\lambda^{\gamma_2}\mathrm{e}^{-\frac{s}{2}}\sum_{\substack{|\beta|=|\gamma|-1\\ \beta\le\gamma}}\|\nabla S\|_{L^\infty}\|\partial_2\partial^\beta S\|_{L^2}\|\partial^\gamma U\|_{L^2}\\ &\lesssim\mathrm{e}^{-s}\lambda^{-k}E_k^2\le\varepsilon^{\frac12}E_k^2,\end{aligned}\tag{6.36}$$

$$\begin{aligned}2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|F_{U_i,(1)}^{(\gamma,S)}\partial^\gamma U_i|&\lesssim\sum_{|\gamma|=k}\lambda^{\frac{\gamma_2+1}{2}}\gamma_2\|\partial_2(SJN)\|_{L^\infty}\|\partial^\gamma U\|_{L^2}\|\partial_1\partial^{\gamma-e_2}S\|_{L^2}\lambda^{\frac{\gamma_2-1}{2}}\\ &\lesssim\sum_{|\gamma|=k}\mathrm{e}^{-\frac{s}{2}}(\lambda^{\gamma_2+1}\|\partial^\gamma U\|_{L^2}^2+\lambda^{\gamma_2-1}\gamma_2^2\|\partial_1\partial^{\gamma-e_2}S\|_{L^2}^2)\\ &\le\varepsilon^{\frac14}E_k^2,\end{aligned}\tag{6.37}$$

$$\begin{aligned}2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|F_{U_i,(5)}^{(\gamma,S)}\partial^\gamma U_i|&\lesssim\mathrm{e}^{\frac{s}{2}}\|\partial_1(JN)\|_{L^\infty}\|S\|_{L^\infty}\lambda^{-k}E_k^2\\ &\lesssim\mathrm{e}^{\frac{s}{2}}M^2\varepsilon^{\frac56-\frac12}\mathrm{e}^{-\frac32 s}M^{\frac14}\lambda^{-k}E_k^2\le\varepsilon E_k^2.\end{aligned}\tag{6.38}$$

Summing up the above inequalities, we get

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|F_{U_i}^{(\gamma,S)}\partial^\gamma U_i|\le 2\varepsilon^{\frac14}E_k^2.\tag{6.39}$$

Now we look at the terms involving lower order derivatives of S. We decompose $F_{U_i,(1)}^{(\gamma-1,S)}=I_{i1}+I_{i2}+I_{i3}$, where

$$\begin{aligned}I_{i1}&=-2\beta_3\beta_\tau\sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}}\binom{\gamma}{\beta}\mathrm{e}^{\frac{s}{2}}\partial^{\gamma-\beta}((S-S_\infty)JN_i)\partial_1\partial^\beta S,\\ I_{i2}&=-2\beta_3\beta_\tau\sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}}\binom{\gamma}{\beta}\mathrm{e}^{\frac{s}{2}}S_\infty\partial^{\gamma-\beta}(JN_i)\partial_1\partial^\beta S,\\ I_{i3}&=-2\beta_3\beta_\tau\sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}}\binom{\gamma}{\beta}\mathrm{e}^{-\frac{s}{2}}\delta_{i2}\partial^{\gamma-\beta}S\partial_2\partial^\beta S.\end{aligned}\tag{6.40}$$

For the first part I_{i_1}, we have that

$$\begin{aligned}&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|I_{i1}\partial^\gamma U_i|\\ &\lesssim\mathrm{e}^{\frac{s}{2}}\|D^kU\|_{L^2}\sum_{j=1}^{k-2}\|D^{k-1-(j-1)}((S-S_\infty)JN)D^{j-1}D^2S\|_{L^2}\end{aligned}$$

$$\lesssim e^{\frac{s}{2}}\|D^kU\|_{L^2}\sum_{j=1}^{k-2}\|D^k((S-S_\infty)JN)\|_{L^2}^a\|D^2((S-S_\infty)JN)\|_{L^q}^{1-a}\|D^kS\|_{L^2}^b\|D^2S\|_{L^q}^{1-b}. \quad (6.41)$$

As before, we use Leibniz rule, estimates (5.10) of J,N and the Poincaré inequality in y_2 direction to deduce that

$$\begin{aligned}\|D^k((S-S_\infty)JN)\|_{L^2(\mathbb{R}^2)} &\lesssim \|D^kS\|_{L^2},\\ |D^2(JN)| &\lesssim \varepsilon^{\frac14}e^{-s},\\ \|D^2((S-S_\infty)JN)\|_{L^q(\mathbb{R}^2)} &\lesssim Me^{-\frac{s}{2}}.\end{aligned} \quad (6.42)$$

In the last inequality we used the fact that $q>4\Rightarrow \|\eta^{-1}\|_{L^{\frac{q}{6}}(\mathbb{R}^2)}<\infty$. Thus we have

$$\begin{aligned}&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|I_{i1}\partial^\gamma U_i|\\ &\lesssim e^{\frac{s}{2}}\|D^kU\|_{L^2}\sum_{j=1}^{k-2}\|D^kS\|_{L^2}^{a+b}(Me^{-\frac{s}{2}})^{2-a-b}\\ &\lesssim \sum_{j=1}^{k-2}\lambda^{-\frac{k}{2}(1+a+b)}M^{2-a-b}e^{-\frac{1-a-b}{2}s}E_k^{1+a+b}\\ &\le \sum_{j=1}^{k-2}(\delta E_k^2+C(\delta)\lambda^{-\frac{2k(1+a+b)}{2(1-a-b)}}M^{\frac{2(2-a-b)}{1-a-b}}e^{-s})\le \delta E_k^2+e^{-s}M^{4k-6}.\end{aligned} \quad (6.43)$$

I_{i2} is estimated as

$$\begin{aligned}\|I_{i2}\|_{L^2}&\lesssim \sum_{\substack{1\le|\beta|\le|\gamma|-2\\ \beta\le\gamma}} e^{\frac{s}{2}}M^3\varepsilon^{\frac56-\frac{\gamma_1-\beta_1}{2}-\frac{\gamma_2-\beta_2}{6}}e^{-\frac32(\gamma_1-\beta_1)s-\frac12(\gamma_2-\beta_2)s}\cdot(\varepsilon^{\frac16}e^{\frac{s}{2}})^{k-1-|\beta|}\|D^kS\|_{L^2}\\ &\overset{|\gamma|=k}{\lesssim} M^3\varepsilon^{\frac23}\|D^kS\|_{L^2}.\end{aligned} \quad (6.44)$$

And I_{i3} is estimated as

$$\begin{aligned}2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|I_{i3}\partial^\gamma U_i| &\lesssim \sum_{|\gamma|=k}e^{-\frac{s}{2}}\|D^kU\|_{L^2}\sum_{j=1}^{k-2}\|S\|_{\dot H^k}^{\frac{k-1-j}{k-1}}\|DS\|_{L^\infty}^{\frac{j}{k-1}}\|S\|_{\dot H^k}^{\frac{j}{k-1}}\|\partial_2S\|_{L^\infty}^{\frac{k-1-j}{k-1}}\\ &\lesssim e^{-\frac{s}{2}}\|U\|_{\dot H^k}\|S\|_{\dot H^k}e^{-\frac{s}{2}}\le \varepsilon^{\frac12}E_k^2.\end{aligned} \quad (6.45)$$

Hence, we have

$$2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int\left|F_{U_i,(1)}^{(\gamma-1,S)}\partial^\gamma U_i\right|\le(\delta+2\varepsilon^{\frac12})E_k^2+e^{-s}M^{4k-6}. \quad (6.46)$$

Next, we turn to $F_{U_i,(2)}^{(\gamma-1,S)}$. From Leibniz rule we have

$$\|[\partial^\gamma,JN_i]S\|_{L^2(\mathcal{X}(s))}\lesssim \varepsilon^{\frac12}\|D^kS\|_{L^2(\mathbb{R}^2)}+\varepsilon e^{-(\gamma_1+\frac{\gamma_2}{3}-1)s} \quad (6.47)$$

and

$$
\begin{aligned}
&2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int|F^{(\gamma-1,S)}_{U_i,(2)}\partial^\gamma U_i|\\
&\lesssim\sum_{|\gamma|=k}\mathrm{e}^{\frac{s}{2}}\,\|[\partial^\gamma,JN_i]S\|_{L^2(\mathcal{X}(s))}\,\|\partial_1 S\|_{L^\infty}\|D^kU_i\|_{L^2}\\
&\lesssim\mathrm{e}^{\frac{s}{2}}(\varepsilon^{\frac12}\|D^kS\|_{L^2}+\varepsilon\mathrm{e}^{-\left(\gamma_1+\frac{\gamma_2}{3}-1\right)s})\mathrm{e}^{-\frac{s}{2}}\|D^kU\|_{L^2}\le\varepsilon^{\frac14}E_k^2+\mathrm{e}^{-s}.
\end{aligned}\tag{6.48}
$$

Thus we have

$$
2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F^{(\gamma-1,S)}_{U_i}\partial^\gamma U_i|\le(\delta+2\varepsilon^{\frac12}+\varepsilon^{\frac14})E_k^2+\mathrm{e}^{-s}M^{4k-5}.\tag{6.49}
$$

Summing all the estimates together leads us to

$$
2\sum_{|\gamma|=k}\lambda^{\gamma_2}\int_{\mathbb{R}^3}|F^{(\gamma)}_{U_i}\partial^\gamma U_i|\le(4+C\varepsilon^{\frac14}+6\delta)E_k^2+\mathrm{e}^{-s}M^{4k-4}.\tag{6.50}
$$

The inequality (6.9b) can be proved in the same way.

Proof of $\dot{H}^k$ estimates of U, S We multiply the equations of $\partial^\gamma U_i$, $\partial^\gamma S$ by $\partial^\gamma U_i$, $\partial^\gamma S$ respectively and sum over to obtain

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}s}\|\partial^\gamma U\|^2_{L^2}\\
&\le\frac12\int|\partial^\gamma U|^2(\operatorname{div}\mathcal{V}_A-2D_\gamma)+\frac12(1+\gamma_1)\beta_3\beta_\tau(1+\varepsilon^{\frac1{13}})(\|\partial^\gamma S\|^2_{L^2}+\|\partial^\gamma U\|^2_{L^2})\\
&-2\beta_3\beta_\tau\int S(\mathrm{e}^{\frac{s}{2}}JN_i\partial_1\partial^\gamma S+\mathrm{e}^{-\frac{s}{2}}\delta_{i2}\partial_2\partial^\gamma S)\partial^\gamma U_i+\int|F^{(\gamma)}_{U_i}\partial^\gamma U_i|,
\end{aligned}\tag{6.51a}
$$

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}s}\|\partial^\gamma S\|^2_2\\
&\le\frac12\int|\partial^\gamma S|^2(\operatorname{div}\mathcal{V}_A-2D_\gamma)+\frac12\beta_\tau(\beta_1+\beta_3\gamma_1)(1+\varepsilon^{\frac1{13}})(\|\partial^r S\|^2_2+\|\partial^\gamma U\|^2_2)\\
&-2\beta_3\beta_\tau\int S(\mathrm{e}^{\frac{s}{2}}\partial_1\partial^\gamma U_jJN_j+\mathrm{e}^{-\frac{s}{2}}\partial_2\partial^\gamma U_2)\partial^\gamma S+\int|F^{(\gamma)}_S\partial^\gamma S|.
\end{aligned}\tag{6.51b}
$$

Here we used the fact that $|JN\partial_1W|\le|JN||\partial_1W|\le(1+\varepsilon^{\frac23})(1+\varepsilon^{\frac1{12}})\le1+\varepsilon^{\frac1{13}}$. By summing up the above two inequalities and integrating by part, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(\|\partial^\gamma U\|^2_{L^2}+\|\partial^\gamma S\|^2_{L^2})+\int(2D_\gamma-\operatorname{div}\mathcal{V}_A-\beta_\tau(1+2\gamma_1\beta_3)(1+\varepsilon^{\frac1{13}}))(|\partial^\gamma U|^2+|\partial^\gamma S|^2)\\
&\le2\int|F^{(\gamma)}_{U_i}\partial^\gamma U_i|+2\int|F^{(\gamma)}_S\partial^\gamma S|+4\beta_3\beta_\tau\int[\mathrm{e}^{\frac{s}{2}}\partial^\gamma S\partial^\gamma U\cdot\partial_1(SJN)+\mathrm{e}^{-\frac{s}{2}}\partial^\gamma S\partial_2U_2\partial_2S]\\
&\le2\int|F^{(\gamma)}_{U_i}\partial^\gamma U_i|+2\int|F^{(\gamma)}_S\partial^\gamma S|+2\beta_3\beta_\tau(1+2\varepsilon^{\frac12})(\|\partial^\gamma U\|^2_{L^2}+\|\partial^\gamma S\|^2_{L^2}).
\end{aligned}\tag{6.52}
$$

In the last inequality we used the facts that $|\partial_1(SJN)|\le(1+\varepsilon^{\frac12})\mathrm{e}^{-\frac{s}{2}}$ and the estimate (5.23) of S. The first fact can be obtained from (5.23) and the estimates (5.10) of J, N.

Now we estimate the damping term

$$
2D_\gamma-\operatorname{div}\mathcal{V}_A-\beta_\tau(1+2\gamma_1\beta_3)(1+\varepsilon^{\frac1{13}})-2\beta_3\beta_\tau(1+2\varepsilon^{\frac12})
$$

$$\begin{aligned}&\geq |\gamma| + 2\gamma_1(1+\beta_1\beta_\tau\partial_1(JW)+\partial_1 G_A) - 2 - \beta_1\beta_\tau\partial_1(JW) - \partial_1 G_A - \partial_2 h_A\\ &\quad - 2\beta_3\beta_\tau(1+\varepsilon^{\frac{1}{13}})\gamma_1 - \underbrace{[\beta_\tau(1+\varepsilon^{\frac{1}{13}}) + 2\beta_3\beta_\tau(1+2\varepsilon^{\frac{1}{2}})]}_{\leq 3}\\ &\overset{(5.3)}{\geq} |\gamma| + 2\gamma_1(1-\beta_1\beta_\tau-\beta_3\beta_\tau) - 6 - C\varepsilon^{\frac{1}{13}} \geq k-7. \end{aligned} \tag{6.53}$$

Multiply (6.52) by λ^{γ_2} and take the sum with respect to $|\gamma| = k$. Then using Lemma 6.2, we find

$$\frac{\mathrm{d}}{\mathrm{d}s}E_k^2 + (k-7)E_k^2 \leq (8+16\delta)E_k^2 + \mathrm{e}^{-s}M^{4k-3}. \tag{6.54}$$

By taking $k \geq 18$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}E_k^2 + 2E_k^2 \leq \mathrm{e}^{-s}M^{4k-3}, \tag{6.55}$$

which results in

$$E_k^2(s) \leq \mathrm{e}^{-2(s-s_0)}E_k^2(s_0) + (1-\mathrm{e}^{-(s-s_0)})\mathrm{e}^{-s}M^{4k-3}. \tag{6.56}$$

By Leibniz rule, we have

$$\begin{cases} \|WN\|_{\dot{H}^k} \leq (1+C\varepsilon^{\frac{1}{2}})\|W\|_{\dot{H}^k} + CM^2\varepsilon^{\frac{5}{3}}\mathrm{e}^{-(\frac{k}{3}-\frac{3}{2})s},\\ \|AT\|_{\dot{H}^k}, \|ZN\|_{\dot{H}^k} \leq (1+C\varepsilon^{\frac{1}{2}})\|A \text{ or } Z\|_{\dot{H}^k} + CM^3\varepsilon^{\frac{5}{3}}\mathrm{e}^{-\frac{k-3}{3}s} \end{cases} \tag{6.57}$$

and

$$\begin{cases} \|U\|_{\dot{H}^k} \leq (1+C\varepsilon^{\frac{1}{2}})\Big[\dfrac{1}{2}(\mathrm{e}^{-\frac{s}{2}}\|W\|_{\dot{H}^k} + \|Z\|_{\dot{H}^k}) + \|A\|_{\dot{H}^k}\Big] + CM^3\varepsilon^{\frac{5}{3}}\mathrm{e}^{-\frac{k-3}{3}s},\\ \|S\|_{\dot{H}^k} \leq \dfrac{1}{2}(1+C\varepsilon^{\frac{1}{2}})(\mathrm{e}^{-\frac{s}{2}}\|W\|_{\dot{H}^k} + \|Z\|_{\dot{H}^k}) + CM^3\varepsilon^{\frac{5}{3}}\mathrm{e}^{-\frac{k-3}{3}s}. \end{cases} \tag{6.58}$$

According to the assumption (3.41) of $\dot{H}^k$ norm of W,Z,A, we have

$$E_k^2(s_0) \leq (2+C\varepsilon^{\frac{1}{2}})\varepsilon. \tag{6.59}$$

Thus we finally obtain that

$$\lambda^k(\|U\|_{\dot{H}^k} + \|S\|_{\dot{H}^k}) \leq E_k^2 \leq (2+C\varepsilon^{\frac{1}{2}})\varepsilon^{-1}\mathrm{e}^{-2s} + M^{4k-3}\mathrm{e}^{-s}(1-\varepsilon^{-1}\mathrm{e}^{-s}). \tag{6.60}$$

This finishes the proof of energy estimate.

6.3 Higher order estimates for W, Z, A

Using the energy estimate, we can further obtain higher order estimates for W, Z, A.

Lemma 6.3 *For $k \gg 1$, we have that*

$$|\partial^\gamma W| \lesssim \begin{cases} \eta^{-\frac{1}{6}}\mathrm{e}^{\frac{s}{2(k-3)}}, & \gamma_1 = 0, \quad |\gamma| = 3,\\ \eta^{-\frac{1}{3}}\mathrm{e}^{\frac{s}{k-3}}, & \gamma_1 > 0, \quad |\gamma| = 3, \end{cases} \tag{6.61a}$$

$$|\partial^\gamma Z| \lesssim \begin{cases} \mathrm{e}^{-(\frac{3}{2}-\frac{1}{2(k-3)})s}, & \gamma_1 \geq 1, \quad |\gamma| = 3,\\ \mathrm{e}^{-(1-\frac{|\gamma|-1}{2(k-3)})s}, & |\gamma| = 3,4,5, \end{cases} \tag{6.61b}$$

$$|\partial^\gamma A| \lesssim \begin{cases} \mathrm{e}^{-\left(\frac{3}{2}-\frac{|\gamma|}{k-2}\right)s}, & \gamma_1 \geq 1, \quad |\gamma| = 2,3, \\ \mathrm{e}^{-\left(1-\frac{|\gamma|-1}{2(k-3)}\right)s}, & |\gamma| = 3,4,5. \end{cases} \tag{6.61c}$$

Proof The proof is similar to the interpolation in [13], still for the reader's convinience we recap the proof here.

First we deal with A. For $\gamma_1 \geq 1$, $|\gamma| = 2,3$, we have

$$\begin{aligned} |\partial^\gamma A| &\lesssim \|\partial_1 A\|_{\dot{H}^{k-1}}^{\frac{|\gamma|-1}{k-2}} \|\partial_1 A\|_{L^\infty}^{1-\frac{|\gamma|-1}{k-2}} \lesssim (M^{2k}\mathrm{e}^{-\frac{s}{2}})^{\frac{|\gamma|-1}{k-2}} (M\mathrm{e}^{-\frac{3}{2}s})^{1-\frac{|\gamma|-1}{k-2}} \\ &\lesssim M^{2k}\mathrm{e}^{-\left(\frac{3}{2}-\frac{|\gamma|-1}{k-2}\right)s} \lesssim \mathrm{e}^{-\left(\frac{3}{2}-\frac{|\gamma|}{k-2}\right)s}. \end{aligned} \tag{6.62}$$

For $|\gamma| = 3,4,5$, we have

$$\begin{aligned} |\partial^\gamma A| &\lesssim \|D^k A\|_{L^2}^{\frac{|\gamma|-2}{k-3}} \|D^2 A\|_{L^\infty}^{1-\frac{|\gamma|-2}{k-3}} \lesssim (M^{2k}\mathrm{e}^{-\frac{s}{2}})^{\frac{|\gamma|-2}{k-3}} (M\mathrm{e}^{-s})^{1-\frac{|\gamma|-2}{k-3}} \\ &\lesssim M^{2k}\mathrm{e}^{-\left(1-\frac{|\gamma|-2}{2(k-3)}\right)s} \lesssim \mathrm{e}^{-\left(1-\frac{|\gamma|-1}{2(k-3)}\right)s}. \end{aligned} \tag{6.63}$$

Next we estimate Z. For $\gamma_1 \geq 1$, $|\gamma| = 3$, we have

$$\begin{aligned} |\partial^\gamma Z| &\lesssim \|\partial_1 \nabla Z\|_{\dot{H}^{k-2}}^{\frac{1}{k-3}} \|\partial_1 \nabla Z\|_{L^\infty}^{1-\frac{1}{k-3}} \lesssim (M^{2k}\mathrm{e}^{-\frac{s}{2}})^{\frac{1}{k-3}} (M\mathrm{e}^{-\frac{3}{2}s})^{1-\frac{1}{k-3}} \\ &\lesssim M^{2k}\mathrm{e}^{-(\frac{3}{2}-\frac{1}{k-3})s} \lesssim \mathrm{e}^{-\left(\frac{3}{2}-\frac{1}{2(k-3)}\right)}. \end{aligned} \tag{6.64}$$

For $|\gamma| = 3,4,5$, the estimates for Z are the same as A.

Now we turn to W. Since $|\gamma| = 3$, we can split γ as $\gamma = \gamma' + \gamma''$, where $|\gamma'| = 1$ and $\gamma_1'' = \min(\gamma_1, 2)$, then $\eta^\mu \partial^\gamma W = \partial^{\gamma'}(\eta^\mu \partial^{\gamma''} W) - \partial^{\gamma'}(\eta^\mu)\partial^{\gamma''} W = I_1 + I_2$. Let

$$\mu = \begin{cases} \dfrac{1}{6}, & \gamma_1 = 0, \\ \dfrac{1}{3}, & \text{otherwise}. \end{cases} \tag{6.65}$$

Note that $|\partial_1(\eta^\mu)| \lesssim \eta^{\mu-\frac{1}{2}}$, $|\partial_2(\eta^\mu)| \lesssim \eta^{\mu-\frac{1}{6}}$. Thus when $\gamma_1 = 0$ we have $|I_2| \lesssim \eta^{\mu-\frac{1}{6}}|\partial_{22} W| \lesssim M$; when $\gamma_1 > 0$ we have $|I_2| \lesssim M\eta^{-\frac{1}{6}} \lesssim M$. By interpolation and bootstrap assumptions for W, we have

$$|I_1| \lesssim \|D(\eta^\mu \partial^{\gamma''} W)\|_{L^\infty} \lesssim \|\eta^\mu \partial^{\gamma''} W\|_{\dot{H}^{k-2}}^{\frac{1}{k-3}} \|\eta^\mu \partial^{\gamma''} W\|_{L^\infty}^{1-\frac{1}{k-3}} \lesssim M\|\eta^\mu \partial^{\gamma''} W\|_{\dot{H}^{k-2}}^{\frac{1}{k-3}}. \tag{6.66}$$

We estimate the $\dot{H}^{k-2}$ norm as follows

$$\begin{aligned} \|\eta^\mu \partial^{\gamma''} W\|_{\dot{H}^{k-2}} &\lesssim \sum_{m=0}^{k-2} \|D^m \partial^{\gamma''} W D^{k-2-m} \eta^\mu\|_{L^2} \\ &\lesssim \sum_{m=0}^{k-2} \|D^m \partial^{\gamma''} W\|_{L^{\frac{2(k-1)}{m+1}}} \|D^{k-2-m}\eta^\mu\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))} \\ &\lesssim \sum_{m=0}^{k-2} \|W\|_{\dot{H}^k}^{\frac{m+1}{k-1}} \|\nabla W\|_{L^\infty}^{1-\frac{m+1}{k-1}} \|D^{k-2-m}\eta^\mu\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))} \end{aligned}$$

$$\lesssim \sum_{m=0}^{k-2} (M^{2k})^{\frac{m+1}{k-1}} \|D^{k-2-m}\eta^{\mu}\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))}. \tag{6.67}$$

Simple calculation yields $|D^k(\eta^\mu)| \lesssim \eta^{\mu-\frac{k}{6}}$, thus we have that

$$\begin{aligned}
&\|D^{k-2-m}\eta^{\mu}\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))} \\
&\lesssim \|\eta^{\mu-\frac{k-2-m}{6}}\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))} \lesssim \|\eta^{\mu}\|_{L^\infty(\mathcal{X}(s))}\|\eta^{-\frac{k-2-m}{6}}\|_{L^{\frac{2(k-1)}{k-2-m}}(\mathcal{X}(s))} \\
&\lesssim \varepsilon e^{3\mu s} \times \begin{cases} 1, & m = k-2, \\ \|\eta^{-1}\|_{L^{\frac{k-1}{3}}(\mathcal{X}(s))}, & m < k-2 \end{cases} \\
&\overset{k>3}{\lesssim} \varepsilon^{\mu} e^{3\mu s}.
\end{aligned} \tag{6.68}$$

Consequently, we obtain $|I_1| \lesssim M(M^{2k}\varepsilon^\mu e^{3\mu s})^{\frac{1}{k-3}} \lesssim e^{\frac{3\mu s}{k-3}}$, and $|\eta^\mu \partial^\gamma W| \lesssim e^{\frac{3\mu s}{k-3}} + M \lesssim e^{\frac{3\mu s}{k-3}}$.

7 Constraints and Evolution of Modulation Variables

In this section we close the bootstrap argument for the modulation variables $\xi, n, \phi, \tau, \kappa$. The equation of these variables are deduced from the constraints that we impose on W.

7.1 Constraints

We impose constraints on W and its derivatives up to second order at the origin, i.e.

$$\begin{aligned}
W(0,s) &= \overline{W}(0) = 0, \\
\nabla W(0,s) &= \nabla\overline{W}(0) = (-1,0)^{\mathrm{T}}, \\
\nabla^2 W(0,s) &= \nabla^2\overline{W}(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{7.1}$$

It is possible to impose these constraints. In fact, as long as the initial datum $W(y, -\log\varepsilon)$ satisfies these constraints, we can choose 6 modulation variables ξ, n_2, ϕ, τ, κ in a continuous manner with respect to time in terms of $w(x,t)$, ensuring that $W(y,s)$ still satisfies these constraints.

7.2 The functions G_W, h_W, F_W and their derivatives, evaluated at $y=0$

In a neighborhood of the origin, $\widetilde{f}$ reduces to $\widetilde{f}(\widetilde{x},t) = \frac{1}{2}\phi\widetilde{x}_2^2$, and as a consequence in a neighborhood of 0, $f(x,t) = \frac{1}{2}\phi x_2^2$. Note that any derivatives with respect to x_1 or $\widetilde{x}_1$ of those function vanish at the origin. We can conveniently evaluated the f-related functions at the origin:

$$\widetilde{f}^0 = 0, \quad \partial_{\tilde{x}_2}\widetilde{f}^0 = 0, \quad \partial_{\tilde{x}_2}^2\widetilde{f}^0 = 0; \tag{7.2a}$$

$$(\partial_t)_{\tilde{x}}\widetilde{f}^0 = 0, \quad \partial_{\tilde{x}_2}(\partial_t)_{\tilde{x}}\widetilde{f}^0 = 0, \quad \partial_{\tilde{x}_2}^2(\partial_t)_{\tilde{x}}\widetilde{f}^0 = \dot{\phi}; \tag{7.2b}$$

$$f^0 = 0, \quad \partial_{x_2} f^0 = 0, \quad \partial_{x_2}^2 f^0 = 0; \tag{7.2c}$$

$$J^0 = 0,\ \ \partial_{x_2} J^0 = 0,\ \ \partial_{x_2}^2 J^0 = \phi^2,\ \ \partial_{x_2}^3 J^0 = 0; \tag{7.2d}$$

$$N^0 = (1,0)^{\mathrm{T}},\ \ \partial_{x_2} N^0 = (0,-\phi)^{\mathrm{T}},\ \ \partial_{x_2}^2 N^0 = (-\phi^2,0)^{\mathrm{T}},\ \ \partial_{x_2}^3 N^0 = (0,2\phi^3)^{\mathrm{T}}; \tag{7.2e}$$

$$T^0 = (0,1)^{\mathrm{T}},\ \ \partial_{x_2} T^0 = (\phi,0)^{\mathrm{T}},\ \ \partial_{x_2}^2 T^0 = (0,\phi^2)^{\mathrm{T}},\ \ \partial_{x_2}^3 T^0 = (-2\phi^3,0)^{\mathrm{T}}; \tag{7.2f}$$

$$(\partial_t)_x f^0 = 0,\ \ \partial_{x_2}(\partial_t)_x f^0 = 0,\ \ \partial_{x_2}^0(\partial_t)_x f^0 = \dot{\phi}; \tag{7.2g}$$

$$\partial_t J^0 = 0,\ \ \partial_{x_2}\partial_t J^0 = 0,\ \ \partial_{x_2}^2\partial_t J^0 = 2\phi\dot{\phi}; \tag{7.2h}$$

$$\partial_t N^0 = (0,0)^{\mathrm{T}},\ \ \partial_{x_2}\partial_t N^0 = (0,-\dot{\phi})^{\mathrm{T}},\ \ \partial_{x_2}^2\partial_t N^0 = (-2\phi\dot{\phi},0)^{\mathrm{T}}; \tag{7.2i}$$

$$\partial_t T^0 = (0,0)^{\mathrm{T}},\ \ \partial_{x_2}\partial_t T^0 = (\dot{\phi},0)^{\mathrm{T}},\ \ \partial_{x_2}^2\partial_t T^0 = (0,-2\phi\dot{\phi})^{\mathrm{T}}. \tag{7.2j}$$

By the definition of V, we have

$$V_i^0 = -\frac{1+\alpha}{2} R_{ji}\dot{\xi}_j, \tag{7.3a}$$

$$\partial_1 V^0 = \frac{1+\alpha}{2}\mathrm{e}^{-\frac{3}{2}s}(0,Q_{21})^{\mathrm{T}},\ \ \partial_2 V^0 = \frac{1+\alpha}{2}\mathrm{e}^{-\frac{s}{2}}(Q_{12},0)^{\mathrm{T}}, \tag{7.3b}$$

$$\partial_{11} V^0 = \frac{1+\alpha}{2}\phi\mathrm{e}^{-3s}(0,Q_{21})^{\mathrm{T}},\ \ \partial_{12}V^0 = 0,\ \ \partial_{22}V^0 = 0. \tag{7.3c}$$

From the definition of G_W and (7.2)–(7.3), we have

$$\frac{1}{\beta_\tau} G_W^0 = \mathrm{e}^{\frac{s}{2}}[\kappa + \beta_2 Z^0 - (1+\alpha)\beta_1 R_{j1}\dot{\xi}_j], \tag{7.4a}$$

$$\frac{1}{\beta_\tau}\partial_1 G_W^0 = \beta_2 \mathrm{e}^{\frac{s}{2}}\partial_1 Z^0, \tag{7.4b}$$

$$\frac{1}{\beta_\tau}\partial_2 G_W^0 = \beta_2 \mathrm{e}^{\frac{s}{2}}\partial_2 Z^0 + (1+\alpha)\beta_1 Q_{12} + \beta_1(1+\alpha)\phi R_{j2}\dot{\xi}_j, \tag{7.4c}$$

$$\frac{1}{\beta_\tau}\partial_{11} G_W^0 = \beta_2 \mathrm{e}^{\frac{s}{2}}\partial_{11} Z^0, \tag{7.4d}$$

$$\frac{1}{\beta_\tau}\partial_{12} G_W^0 = \beta_2 \mathrm{e}^{\frac{s}{2}}\partial_{12} Z^0 - (1+\alpha)\beta_1 \mathrm{e}^{-\frac{3}{2}s}\phi Q_{21}, \tag{7.4e}$$

$$\frac{1}{\beta_\tau}\partial_{22} G_W^0 = -\phi\mathrm{e}^{-\frac{s}{2}} + \phi^2\mathrm{e}^{-s}\frac{G_W^0}{\beta_\tau} + \mathrm{e}^{-\frac{s}{2}}\beta_2\partial_{22}Z^0 - (1+\alpha)\beta_1\phi^2\mathrm{e}^{-\frac{s}{2}}R_{j1}\dot{\xi}_j. \tag{7.4f}$$

Similarly for h_W, we have

$$\frac{1}{\beta_\tau} h_W^0 = \beta_1 e^{-\frac{s}{2}}(2A^0 - (1+\alpha)R_{j2}\dot{\xi}_j). \tag{7.5}$$

And for the forcing terms, we also plug the above evaluation into the definition (2.29) to see that

$$\begin{aligned} F_W^0 &= -\beta_3\beta_\tau(\kappa - Z^0)\partial_2 A^0 + \beta_\tau \mathrm{e}^{-\frac{s}{2}} Q_{12} A^0 \\ &\quad - 2\phi\beta_1\beta_\tau e^{-\frac{s}{2}}\Big(-\frac{1+\alpha}{2}R_{j2}\dot{\xi}_j + A^0\Big)A^0 + \frac{1}{2}\phi\beta_3\beta_\tau e^{-\frac{s}{2}}(\kappa + Z^0)(Z^0 - \kappa), \end{aligned} \tag{7.6a}$$

$$\begin{aligned} \partial_1 F_W^0 &= \beta_3\beta_\tau\partial_2 A^0(\partial_1 Z^0 + \mathrm{e}^{-\frac{s}{2}}) - \beta_3\beta_\tau\partial_{12}A^0(\kappa - Z^0) + \beta_\tau\mathrm{e}^{\frac{s}{2}}Q_{12}\partial_1 A^0 \\ &\quad - \phi\partial_1 A h_W^0 - \phi\beta_2\beta_\tau\mathrm{e}^{-\frac{s}{2}}A^0((1+\alpha)Q_{21}\mathrm{e}^{-\frac{3}{2}s} + 2\partial_1 A^0) \\ &\quad - \frac{1}{2}\phi\beta_3\beta_\tau e^{-\frac{s}{2}}(\mathrm{e}^{-\frac{s}{2}} + \partial_1 Z^0)(\kappa + Z^0) + \frac{1}{2}\phi\beta_3\beta_\tau e^{-\frac{s}{2}}(\kappa - Z^0)(\partial_1 Z^0 - \mathrm{e}^{-\frac{s}{2}}), \end{aligned} \tag{7.6b}$$

$$\begin{aligned}\partial_2 F_W^0 &= -\beta_3\beta_\tau(\kappa - Z^0)\partial_{22}A^0 + \beta_3\beta_\tau\partial_2 Z^0\partial_2 A^0 - \dot{\phi}\beta_\tau \mathrm{e}^{-s}A^0 + \beta_\tau \mathrm{e}^{-\frac{s}{2}}Q_{12}\partial_2 A^0\\ &\quad - \phi\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial_2 Z^0 Z^0 + \phi^2\beta_3\beta_\tau \mathrm{e}^{-s}A^0(\kappa - Z^0)\\ &\quad - \phi\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}A^0\Big(2\partial_2 A^0 - \phi \mathrm{e}^{-\frac{s}{2}}(\kappa + Z^0)\Big) - \phi\partial_2 A^0 h_W^0, \end{aligned}\tag{7.6c}$$

$$\begin{aligned}\partial_{11} F_W^0 &= 2\beta_3\beta_\tau(\mathrm{e}^{-\frac{s}{2}} + \partial_{12}Z^0)\partial_{12}A^0 - \beta_3\beta_\tau(\kappa - Z^0)\partial_{112}A^0 + \beta_\tau \mathrm{e}^{-\frac{s}{2}}Q_{12}\partial_{11}A^0\\ &\quad - 2\phi\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial_{11}A^0\Big(2A^0 - \frac{1+\alpha}{2}R_{j2}\dot{\xi}_j\Big)\\ &\quad - 4\phi\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial_1 A^0\Big(\frac{1+\alpha}{2}Q_{21}\mathrm{e}^{-\frac{3}{2}s} + \partial_1 A^0\Big)\\ &\quad - \phi\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}((\partial_1 Z^0)^2 - \mathrm{e}^{-s} + Z^0\partial_{11}Z^0) + \beta_3\beta_\tau\partial_{11}Z^0\partial_2 A^0, \end{aligned}\tag{7.6d}$$

$$\begin{aligned}\partial_{12} F_W^0 &= -2\beta_3\beta_\tau(\kappa - Z^0)\partial_{122}A^0 + \beta_3\beta_\tau\partial_{12}Z^0\partial_2 A^0 + \beta_3\beta_\tau\partial_2 Z^0\partial_{12}A^0\\ &\quad + \beta_3\beta_\tau(\mathrm{e}^{-\frac{s}{2}} + \partial_1 Z^0)\partial_{22}A^0 - \beta_\tau\dot{\phi}\mathrm{e}^{-s}\partial_1 A^0 + \beta_\tau \mathrm{e}^{-\frac{s}{2}}Q_{12}\partial_{12}A^0\\ &\quad - \phi\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}(\partial_{12}Z^0 Z^0 + \partial_1 Z^0\partial_2 Z^0) + \phi^2\beta_3\beta_\tau \mathrm{e}^{-s}((\kappa - Z^0)\partial_1 A^0 - (\mathrm{e}^{-\frac{s}{2}} + \partial_1 Z^0)A^0)\\ &\quad - 2\phi\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}\Big[\partial_1 A^0\partial_2 A^0 + \Big(\frac{1+\alpha}{2}Q_{21}\mathrm{e}^{-\frac{3}{2}s} + \partial_1 A^0\Big)\partial_2 A^0 + A^0\partial_{12}A^0\Big]\\ &\quad - \phi\partial_{12}A^0 h_W^0 + \phi^2\beta_1\beta_\tau \mathrm{e}^{-s}[\partial_1 A^0(\kappa + Z^0) + A^0(\partial_1 Z^0 - \mathrm{e}^{-\frac{s}{2}})], \end{aligned}\tag{7.6e}$$

$$\begin{aligned}\partial_{22} F_W^0 &= \beta_3\beta_\tau[\partial_{22}Z^0\partial_2 A^0 - (\kappa - Z^0)\partial_{222}A^0 + \partial_2 Z^0\partial_{22}A^0] + \phi^2\beta_3\beta_\tau \mathrm{e}^{-s}(\kappa - Z^0)\partial_2 A^0\\ &\quad - 2\dot{\phi}\beta_\tau \mathrm{e}^{-s}\partial_2 A^0 - \phi\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial_2 Z^0\partial_2 Z^0 + \beta_\tau \mathrm{e}^{-\frac{s}{2}}\partial_{22}A^0 Q_{12}\\ &\quad + 2\phi^2\beta_3\beta_\tau \mathrm{e}^{-s}[(\kappa - Z^0)\partial_2 A^0 - A^0\partial_2 Z^0] - \phi^3\beta_3\beta_\tau \mathrm{e}^{-\frac{3}{2}s}(\kappa - Z^0)(\kappa + Z^0)\\ &\quad - 2\phi\beta_1\beta_\tau \mathrm{e}^{-\frac{s}{2}}[2\partial_2 A^0\partial_2 A^0 + A^0\partial_{22}A^0] + 2\phi^3\beta_1\beta_\tau \mathrm{e}^{-\frac{3}{2}s}(A^0)^2\\ &\quad - 2\phi^2\beta_1\beta_\tau \mathrm{e}^{-s}\partial_2[A(U\cdot N)]^0 - \phi h_W^0\partial_{22}A^0 + \phi^3 \mathrm{e}^{-s}h_W^0 A^0\\ &\quad - (1+\alpha)\phi^2\beta_1\beta_\tau \mathrm{e}^{-\frac{3}{2}s}Q_{21}A^0 - \phi\beta_3\beta_\tau \mathrm{e}^{-\frac{s}{2}}Z^0\partial_{22}Z^0. \end{aligned}\tag{7.6f}$$

Also note that if $|\gamma| = 1, 2$, we have

$$F_W^{(\gamma),0} = \partial^\gamma F_W^0 + \partial^\gamma G_W^0. \tag{7.7}$$

7.3 Evolution of modulation variables

Setting $y = 0$ in the equation of W, we can see that

$$\dot{\kappa} = \frac{1}{\beta_\tau}\mathrm{e}^{\frac{s}{2}}(F_W^0 + G_W^0). \tag{7.8}$$

Setting $y = 0$ in the equation of $\partial_1 W$, we have

$$\dot{\tau} = \frac{1}{\beta_\tau}(\partial_1 F_W^0 + \partial_1 G_W^0). \tag{7.9}$$

Setting $y = 0$ in the equation of $\partial_2 W$, we have

$$0 = \partial_2 F_W^0 + \partial_2 G_W^0. \tag{7.10}$$

Combining this with (7.4c), we obtain

$$Q_{12} = -\frac{1}{\beta_1\beta_\tau(1+\alpha)}(\partial_2 F_W^0 + \beta_2\beta_\tau \mathrm{e}^{\frac{s}{2}}\partial_2 Z^0 + \beta_1\beta_\tau(1+\alpha)\mathrm{e}^{\frac{s}{2}}\phi R_{j2}\dot{\xi}_j). \tag{7.11}$$

Setting $y=0$ in the equation of $\partial_{11}W$ and $\partial_{12}W$, we have

$$\begin{pmatrix}\partial_{111}W^0 & \partial_{112}W^0\\ \partial_{112}W^0 & \partial_{122}W^0\end{pmatrix}\begin{pmatrix}G_W^0\\ h_W^0\end{pmatrix}=\begin{pmatrix}\partial_{11}F_W^0+\partial_{11}G_W^0\\ \partial_{12}F_W^0+\partial_{12}G_W^0\end{pmatrix}. \tag{7.12}$$

Denote the matrix $\partial_1\nabla^2W^0$ by $H^0(s)$, then we have

$$|G_W^0|+|h_W^0|\lesssim|(H^0)^{-1}|(|\partial_1\nabla F_W^0|+|\partial_1\nabla G_W^0|), \tag{7.13}$$

which shall be used to establish an upper bound for $|G_W^0|$ and $|h_W^0|$. Since $R\in SO(2)$, we have

$$\dot{\xi}_j=R_{ji}R_{ki}\dot{\xi}_k=R_{j1}R_{k1}\dot{\xi}_k+R_{j2}R_{k2}\dot{\xi}_k. \tag{7.14}$$

Combining this with (7.4a)(7.5), we have

$$\dot{\xi}_j=\frac{R_{j1}}{(1+\alpha)\beta_1}\Big(\kappa+\beta_2Z^0-\frac{1}{\beta_\tau}\mathrm{e}^{-\frac{s}{2}}G_W^0\Big)+\frac{R_{j2}}{1+\alpha}\Big(2A^0-\frac{\mathrm{e}^{\frac{s}{2}}}{\beta_1\beta_\tau}h_W^0\Big). \tag{7.15}$$

Setting $y=0$ in the equation of $\partial_{11}W$ and $\partial_{12}W$, we have

$$G_W^0\partial_{122}W^0+h_W^0\partial_{222}W^0=\partial_{22}F_W^0+\partial_{22}G_W^0. \tag{7.16}$$

Then from (7.4f), we have

$$\begin{aligned}\dot{\phi}=&\frac{\mathrm{e}^{\frac{s}{2}}}{\beta_\tau}(\partial_{122}W^0G_W^0+\partial_{222}W^0h_W^0-\partial_{22}F_W^0)+\beta_2e^s\partial_{22}Z^0\\&+\phi^2\Big(\kappa+\beta_2Z^0-\frac{\mathrm{e}^{-\frac{s}{2}}}{\beta_\tau}G_W^0\Big)+\frac{\phi^2}{\beta_\tau}\mathrm{e}^{-\frac{s}{2}}G_W^0.\end{aligned} \tag{7.17}$$

8 Closure of Bootstrap Argument for the Modulation Variables

From (2.46) and (B-$\widetilde{W}^0$), we can see that

$$H^0:=\partial_1\nabla^2W^0=\partial_1\nabla^2\overline{W}^0+\partial_1\nabla^2\widetilde{W}^0=\mathrm{diag}(6,2)+O(\varepsilon^{\frac14}). \tag{8.1}$$

As a consequence, we have

$$|(H^0)^{-1}|\le 1. \tag{8.2}$$

Next we estimate $|\partial_1\nabla F_W^0|$. From (7.6d)–(7.6e), bootstrap assumptions and (6.61c), we have $|\partial_{11}F_W^0|\lesssim\mathrm{e}^{-s}$ and $|\partial_{12}F_W^0|\lesssim\mathrm{e}^{-s}+\varepsilon^2|h_W^0|$. Then by invoking (7.13), one can see that

$$|G_W^0|+|h_W^0|\lesssim\mathrm{e}^{-s}. \tag{8.3}$$

Now we give a new estimate for $V_2=\frac{1+\alpha}{2}\Big[Q_{21}(y_1\mathrm{e}^{-\frac32s}+f)+\frac{\mathrm{e}^{\frac{s}{2}}}{(1+\alpha)\beta_1\beta_\tau}h_W^0+\frac{2}{1+\alpha}A^0\Big]$. Recall that in (5.19) we already have a bound $|V_2|\lesssim M^{\frac14}$, but now with the help of (8.3) one can see that for all $y\in\mathcal{X}(s)$, there holds that

$$|V_2|\lesssim M\varepsilon^{\frac12}. \tag{8.4}$$

8.1 The ξ estimate

From (7.15) we have

$$|\dot{\xi}_j| =\kappa_0 + M\varepsilon + \mathrm{e}^{-\frac{s}{2}} M\mathrm{e}^{-s} \le \frac{1}{10} M^{\frac{1}{4}}. \tag{8.5}$$

From (5.1) and $\xi(-\varepsilon) = 0$, we have

$$|\xi_j(t)| \le \int_{-\varepsilon}^{t} |\dot{\xi}_j| \mathrm{d}t \le \frac{1}{10} M^{\frac{1}{4}} \varepsilon. \tag{8.6}$$

8.2 The κ estimate

From (7.6a) and bootstrap assumptions, we have $|F_W^0| \lesssim \varepsilon^{\frac{1}{4}} \mathrm{e}^{-\frac{s}{2}}$. Thus according to (7.8), (8.3), we have that

$$|\dot{\kappa}| \lesssim \mathrm{e}^{\frac{s}{2}} (M\mathrm{e}^{-s} + \varepsilon^{\frac{1}{4}} \mathrm{e}^{-\frac{s}{2}}) \le \frac{1}{2} M \tag{8.7}$$

and

$$|\kappa - \kappa_0| \le \frac{1}{2} M |t + \varepsilon| \lesssim M\varepsilon \le \frac{1}{4} \kappa_0. \tag{8.8}$$

8.3 The ϕ estimate

From (7.6f), bootstrap assumptions and (6.61c), we have $|\partial_{22} F_W^0| \lesssim \mathrm{e}^{-\frac{s}{2}}$. Thus via (7.17), we obtain

$$\begin{aligned} |\dot{\phi}| &\lesssim \mathrm{e}^{\frac{s}{2}} (\varepsilon^{\frac{1}{4}} M\mathrm{e}^{-s} + \varepsilon^{\frac{1}{4}} M\mathrm{e}^{-s} + \mathrm{e}^{-\frac{s}{2}}) + e^{s} M\mathrm{e}^{-s} \\ &\quad + M^4 \varepsilon^2 (\kappa_0 + M\varepsilon + \mathrm{e}^{-\frac{s}{2}} M\mathrm{e}^{-s}) + M^4 \varepsilon^2 \mathrm{e}^{-\frac{s}{2}} M^{-s} \\ &\lesssim M \le \frac{1}{10} M^2. \end{aligned} \tag{8.9}$$

Since $|\phi(-\varepsilon)| = |\phi_0| \le \varepsilon$, we can further obtain that

$$|\phi| \le \varepsilon + |\dot{\phi}| |t + \varepsilon| \le \frac{1}{2} M^2 \varepsilon. \tag{8.10}$$

8.4 The τ estimate

Also from (7.6b) and (7.4b) and bootstrap assumptions, we have $|\partial_1 F_W^0| \lesssim \mathrm{e}^{-s}$ and $|\partial_1 G_W^0| \lesssim M^{\frac{1}{2}} \mathrm{e}^{-s}$, thus by (7.9), we have

$$|\dot{\tau}| \lesssim \mathrm{e}^{-s} + M^{\frac{1}{2}} \mathrm{e}^{-s} \le \frac{1}{4} M\mathrm{e}^{-s}. \tag{8.11}$$

Since $\tau(-\varepsilon) = 0$, we get

$$|\tau(t)| \le \int_{-\varepsilon}^{t} \frac{1}{4} M\varepsilon \mathrm{d}t \le \frac{1}{4} M\varepsilon^2. \tag{8.12}$$

8.5 The n_2 estimate

We first estimate Q_{12}. From (7.6c) and (8.3) and bootstrap assumptions, $|\partial_2 F_W^0| \lesssim M\kappa_0 \mathrm{e}^{-s}$. Thus via (7.11), we can bound Q_{12} by

$$|Q_{12}| \lesssim M\kappa_0 \mathrm{e}^{-s} + \mathrm{e}^{\frac{s}{2}} M\varepsilon^{\frac{1}{2}} \mathrm{e}^{-\frac{s}{2}} \leq 2M\varepsilon^{\frac{1}{2}}. \tag{8.13}$$

From the definition of Q, we have

$$Q_{12} = -\dot{n}_2\sqrt{1-n_2^2} - \frac{n_2^2\dot{n}_2}{\sqrt{1-n_2^2}}, \tag{8.14}$$

thus by bootstrap assumption of n_2, we finally can see that

$$|\dot{n}_2| = |Q_{12}|\Big(\sqrt{1-n_2^2} + \frac{n_2^2}{\sqrt{1-n_2^2}}\Big)^{-1} \leq (1+\varepsilon^{\frac{1}{2}})|Q_{12}| \leq \frac{1}{2}M^2\varepsilon^{\frac{1}{2}}. \tag{8.15}$$

By $n_2(-\varepsilon) = 0$, we improve the assumption of n_2 by a factor $\frac{1}{2}$.

9 Estimates for Transport and Forcing Terms

To close the bootstrap argument of the Riemann variables W, Z, A, we estimate each term in the transport-type equations of W, Z, A.

9.1 Transport estimates

Lemma 9.1 *For the transport terms in the equations of W, Z, A, we have the following inequalities:*

$$|\partial^\gamma G_W| \lesssim \begin{cases} M\mathrm{e}^{-s} + M^{\frac{1}{2}}\mathrm{e}^{-s}|y_1| + M^2\varepsilon^{\frac{1}{2}}|y_2| \lesssim \varepsilon^{\frac{1}{3}}\mathrm{e}^{\frac{s}{2}}, & \gamma = (0,0), \\ M^2\mathrm{e}^{-\frac{5}{6}s}, & \gamma = (1,0), \\ M^2\varepsilon^{\frac{1}{6}}, & \gamma = (0,1), \\ M^2\mathrm{e}^{-\frac{s}{2}}, & \gamma| = 2, \end{cases} \tag{9.1}$$

$$\big|\partial^\gamma(G_A + (1-\beta_1)\kappa_0\mathrm{e}^{\frac{s}{2}})\big| + \big|\partial^\gamma(G_Z + (1-\beta_2)\kappa_0\mathrm{e}^{\frac{s}{2}})\big| \lesssim \begin{cases} \varepsilon^{\frac{1}{3}}\mathrm{e}^{\frac{s}{2}}, & \gamma = (0,0), \\ M^2\mathrm{e}^{-\frac{5}{6}s}, & \gamma = (1,0), \\ M^2\varepsilon^{\frac{1}{6}}, & \gamma = (0,1), \\ M^2\mathrm{e}^{-\frac{s}{2}}, & |\gamma| = 2, \end{cases} \tag{9.2}$$

$$|\partial^\gamma h_W| + |\partial^\gamma h_Z| + |\partial^\gamma h_A| \lesssim \begin{cases} M\varepsilon^{\frac{1}{2}}\mathrm{e}^{-\frac{s}{2}}, & \gamma = (0,0), \\ M\varepsilon^{\frac{1}{3}}\mathrm{e}^{-s}\eta^{-\frac{1}{3}}, & \gamma = (1,0), \\ \varepsilon^{\frac{1}{3}}\mathrm{e}^{-s} & \gamma = (0,1), \\ \varepsilon^{\frac{1}{6}}\mathrm{e}^{-s}\eta^{-\frac{1}{6}}, & \gamma = (2,0), \\ \varepsilon^{\frac{1}{6}}\mathrm{e}^{-s}\eta^{-\frac{1}{6}}, & \gamma = (1,1), \\ \mathrm{e}^{-s}\eta^{-\frac{1}{6}} & \gamma = (0,2). \end{cases} \tag{9.3}$$

Furthermore, for $|\gamma|=3,4$, it holds that

$$\begin{cases} |\partial^\gamma G_W| \lesssim \mathrm{e}^{-\left(\frac{1}{2}-\frac{|\gamma|-1}{2(k-3)}\right)s} \\ |\partial^\gamma h_W| \lesssim \mathrm{e}^{-s}. \end{cases} \tag{9.4}$$

Proof For $\gamma>0$, from the definition of G_W in (2.27), we have

$$|\partial^\gamma G_W| \lesssim \mathrm{e}^{\frac{s}{2}}\Big|\partial^\gamma \frac{\partial_t f}{1+f_{x_1}}\Big| + \mathrm{e}^{\frac{s}{2}} \sum_{\beta\le\gamma} |\partial^\beta J|(\kappa_0 \mathbb{1}_{\beta=\gamma} + |\partial^{\gamma-\beta} Z| + |\partial^{\gamma-\beta}(V\cdot N)|). \tag{9.5}$$

Then appealing to bootstrap assumptions and (5.10), (5.19), (6.61c) and (6.61b), we obtain the desired estimates for G_W. For the case $\gamma=0$, we have that

$$\begin{aligned} |G_W| &\le \Big|\Big(G_W+\beta_\tau \mathrm{e}^{\frac{s}{2}}\frac{\partial_t f}{1+f_{x_1}}\Big)^0\Big| + \Big\|\partial_1\Big(G_W+\beta_\tau \mathrm{e}^{\frac{s}{2}}\frac{\partial_t f}{1+f_{x_1}}\Big)\Big\|_{L^\infty}|y_1| \\ &\quad + \Big\|\partial_2\Big(G_W+\beta_\tau \mathrm{e}^{\frac{s}{2}}\frac{\partial_t f}{1+f_{x_1}}\Big)\Big\|_{L^\infty}|y_2| + \Big|\beta_\tau \mathrm{e}^{\frac{s}{2}}\frac{\partial_t f}{1+f_{x_1}}\Big| \\ &\lesssim |G_W^0| + M^{\frac12}\varepsilon^{\frac12}\mathrm{e}^{-s}|y_1| + M^2\varepsilon^{\frac23}\mathrm{e}^{\frac{s}{2}} \\ &\lesssim M\mathrm{e}^{-s} + M^{\frac12}\varepsilon^{\frac{s}{2}} + M^2\varepsilon^{\frac23}\mathrm{e}^{\frac{s}{2}} \lesssim \varepsilon^{\frac13}\mathrm{e}^{\frac{s}{2}}. \end{aligned} \tag{9.6}$$

Once we have the bounds for G_W and its derivatives, the estimates of G_Z and G_A follow from the identities

$$\begin{aligned} G_Z + (1-\beta_2)\kappa_0 \mathrm{e}^{\frac{s}{2}} &= G_W + (1-\beta_2)\mathrm{e}^{\frac{s}{2}}\left[(\kappa_0-\kappa)+(1-\beta_\tau J)\kappa+\beta_\tau J\right], \\ G_A + (1-\beta_1)\mathrm{e}^{\frac{s}{2}}\kappa_0 &= G_W + (1-\beta_1)\mathrm{e}^{\frac{s}{2}}\left[(\kappa_0-\kappa)+(1-\beta_\tau J)\kappa\right] + (\beta_2-\beta_1)\beta_\tau \mathrm{e}^{\frac{s}{2}} JZ. \end{aligned} \tag{9.7}$$

The estimates of h_W, h_Z, h_A can be deduced by their definitions, the bootstrap assumptions and the inequalities (5.10), (5.19), (6.61a)–(6.61c).

9.2 Forcing estimates

Now we deal with the forcing terms that appear in the equations of W, Z, A.

Lemma 9.2 *For derivatives of the forcing terms, we have the following bounds*:

$$|\partial^\gamma F_W| + \mathrm{e}^{\frac{s}{2}}|\partial^\gamma F_Z| \lesssim \begin{cases} \mathrm{e}^{-\frac{s}{2}}, & \gamma=(0,0), \\ \mathrm{e}^{-s}\eta^{-\frac16+\frac{2}{3(k-2)}}, & \gamma=(1,0), \\ M^2\mathrm{e}^{-s}, & \gamma=(0,1), \\ \mathrm{e}^{-s}\eta^{-\frac16+\frac{1}{k-2}}, & \gamma=(2,0), \\ \mathrm{e}^{-s}\eta^{-\frac16+\frac{1}{k-2}}, & \gamma=(1,1), \\ M^{\frac14}\mathrm{e}^{-(1-\frac{1}{k-3})s}, & \gamma=(0,2), \end{cases} \tag{9.8}$$

$$|\partial^\gamma F_W| \lesssim \begin{cases} \mathrm{e}^{-\frac{s}{2}}, & |\gamma|=3, \\ \varepsilon^{\frac16}, & |\gamma|=4, \quad |y|\le l, \end{cases} \tag{9.9}$$

$$|\partial^\gamma F_A| \lesssim \begin{cases} M^{\frac{1}{2}}\mathrm{e}^{-s}, & \gamma=(0,0),\\ M^{\frac{1}{4}}\mathrm{e}^{-s}, & \gamma=(0,1),\\ M^{\frac{1}{4}}\mathrm{e}^{-\left(1-\frac{1}{k-3}\right)s}\eta^{-\frac{1}{6}}, & \gamma=(0,2), \end{cases} \tag{9.10}$$

$$|\partial^\gamma \widetilde{F}_W| \lesssim \begin{cases} M\varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{6}}, & \gamma=(0,0), \quad |y|\le L,\\ \varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{2}+\frac{2}{3(k-2)}}, & \gamma=(1,0), \quad |y|\le L,\\ M^2\varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{3}}, & \gamma=(0,1), \quad |y|\le L,\\ \varepsilon^{\frac{1}{7}}, & |\gamma|\le 4, \quad |y|\le l \end{cases} \tag{9.11}$$

and

$$|(\partial^\gamma \widetilde{F}_W)^0| \overset{|\gamma|=3}{\lesssim} \mathrm{e}^{-\left(\frac{1}{2}-\frac{1}{k-3}\right)s}. \tag{9.12}$$

Proof The proof of (9.8)–(9.11) is just taking derivatives of the forcing terms, then using the bootstrap assumptions and the estimates (5.10), (5.17), (5.19)–(5.20), (6.61a)–(6.61c) and (9.1)–(9.4) to estimate each term therein. Finally we prove (9.12). Since $\partial^\gamma \overline{W}^0 = 0$ when $|\gamma|$ is even, and $\partial_2 G_W^0 + \partial_2 F_W^0 = 0$, we have

$$\begin{aligned} |(\partial^\gamma \widetilde{F}_W)^0| &\lesssim \mathrm{e}^{-\frac{s}{2}} + |(1-\beta_\tau J)^0| + \sum_{m=1}^{3} |\nabla^m J^0| + |\nabla G_W^0| + |\nabla^3 G_W^0| + |\nabla h_W^0| + |\nabla^3 h_W^0| \\ &\lesssim M\mathrm{e}^{-\frac{s}{2}} + M^2\mathrm{e}^{-\frac{5}{6}s} + \mathrm{e}^{-\left(\frac{1}{2}-\frac{1}{k-3}\right)s} \lesssim \mathrm{e}^{-\left(\frac{1}{2}-\frac{1}{k-3}\right)s}. \end{aligned} \tag{9.13}$$

Lemma 9.3 *For the forcing terms of $\partial^\gamma W, \partial^\gamma Z, \partial^\gamma A$, we have that*

$$|F_W^{(\gamma)}| \lesssim \begin{cases} \mathrm{e}^{-\frac{s}{2}}, & \gamma=(0,0),\\ \varepsilon^{\frac{1}{4}}\eta^{-\frac{1}{2}+\frac{2}{3(k-2)}}, & \gamma=(1,0),\\ M^2\varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{3}}, & \gamma=(0,1),\\ \eta^{-\frac{1}{2}+\frac{1}{k-2}}, & \gamma=(2,0),\\ M^{\frac{1}{3}}\eta^{-\frac{1}{3}}, & \gamma=(1,1),\\ M^{\frac{2}{3}}\eta^{-\frac{1}{3}+\frac{1}{3(k-3)}}, & \gamma=(0,2), \end{cases} \tag{9.14}$$

$$|F_Z^{(\gamma)}| \lesssim \begin{cases} \mathrm{e}^{-s}, & \gamma=(0,0),\\ \mathrm{e}^{-\frac{3}{2}s}\eta^{-\frac{1}{6}+\frac{2}{3(k-2)}}, & \gamma=(1,0),\\ M^2\mathrm{e}^{-\frac{3}{2}s}, & \gamma=(0,1),\\ \mathrm{e}^{-\frac{3}{2}s}(1+M\eta^{-\frac{1}{3}}), & \gamma=(2,0),\\ \mathrm{e}^{-\frac{3}{2}s}(M^{\frac{1}{2}}+M^2\eta^{-\frac{1}{3}}), & \gamma=(1,1),\\ M^{\frac{1}{4}}\mathrm{e}^{-\left(\frac{3}{2}-\frac{1}{k-3}\right)s}, & \gamma=(0,2), \end{cases} \tag{9.15}$$

$$|F_A^{(\gamma)}| \lesssim \begin{cases} M^{\frac{1}{4}}\mathrm{e}^{-s}, & \gamma=(0,0),\\ M^{\frac{1}{4}}\mathrm{e}^{-s}, & \gamma=(0,1),\\ \mathrm{e}^{-\left(1-\frac{2}{k-3}\right)s}\eta^{-\frac{1}{6}}, & \gamma=(0,2), \end{cases} \tag{9.16}$$

$$|\widetilde{F}_W^{(\gamma)}| \lesssim \begin{cases} \varepsilon^{\frac{1}{11}}\eta^{-\frac{1}{2}}, & \gamma=(1,0), \quad |y|\le L, \\ \varepsilon^{\frac{1}{12}}\eta^{-\frac{1}{3}}, & \gamma=(0,1), \quad |y|\le L, \\ \varepsilon^{\frac{1}{7}}+\varepsilon^{\frac{1}{10}}(\log M)^{\gamma_2-1}, & |\gamma|\le 4, \quad |y|\le l. \end{cases} \tag{9.17}$$

And for $y=0$ and $|\gamma|=3$, we have

$$|\widetilde{F}_W^{(\gamma),0}| \lesssim \mathrm{e}^{-\left(\frac{1}{2}-\frac{1}{k-3}\right)s}, \qquad |\gamma|=3. \tag{9.18}$$

Proof Firstly, we have

$$|F_W^{(0,0)}| = |F_W| \lesssim \mathrm{e}^{-\frac{s}{2}}. \tag{9.19}$$

For the case $1\le|\gamma|\le 2$, we decompose the estimate for forcing term as

$$\begin{aligned} |F_W^{(\gamma)}| &\overset{1\le|\gamma|\le 2}{\lesssim} |\partial^\gamma F_W| + \sum_{0\le\beta<\gamma}(|\partial^{\gamma-\beta}G_W||\partial_1\partial^\beta W| + |\partial^{\gamma-\beta}h_W||\partial_2\partial^\beta W|) \\ &\quad + \mathbb{1}_{|\gamma|=2}\gamma_2|\partial_2(JW)||\partial_1^{\gamma_1+1}\partial_2^{\gamma_2-1}W| + |[\partial^\gamma, J]W\partial_1 W| \\ &= |\partial^\gamma F_W| + I_1^{(\gamma)} + I_2^{(\gamma)} + I_3^{(\gamma)}. \end{aligned} \tag{9.20}$$

Then one can check that each term do not exceed the proposed bound. $F_Z^{(\gamma)}$, $F_A^{(\gamma)}$ and $\widetilde{F}_W^{(\gamma)}$ can be estimated in a similar fashion.

10 Bounds on Lagrangian Trajectories

Given a point y_0 and an initial time $s_0 \ge -\log\varepsilon$, we define the Lagrangian trajectory $\Phi_W^{y_0}$ by

$$\begin{cases} \dfrac{\mathrm{d}\Phi_W^{y_0}(s)}{\mathrm{d}s} = \mathcal{V}_W \circ \Phi_W^{y_0}(s) \\ \Phi_W^{y_0}(s_0) = y_0. \end{cases} \tag{10.1}$$

Similarly we define $\Phi_Z^{y_0}$ and $\Phi_A^{y_0}$ using the transport terms in the equations of Z and A, respectively.

We now discuss the upper bound and the lower bound of these Lagrangian trajectories, and close the bootstrap argument for the spatial support of W, Z, A.

10.1 Upper bound of the trajectories

Lemma 10.1 *Let Φ denote either $\Phi_W^{y_0}$, $\Phi_Z^{y_0}$ or $\Phi_A^{y_0}$. For any $y_0\in\mathcal{X}_0$, we have that*

$$\begin{aligned} |\Phi_1(s)| &\le \frac{3}{2}\varepsilon^{\frac{1}{2}}\mathrm{e}^{\frac{3}{2}s}, \\ |\Phi_2(s)| &\le \frac{3}{2}\varepsilon^{\frac{1}{6}}\mathrm{e}^{\frac{s}{2}}. \end{aligned} \tag{10.2}$$

Proof We first deal with the case $\Phi=\Phi_W^{y_0}$. Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{-\frac{3}{2}s}\Phi_1(s)) &= \mathrm{e}^{-\frac{3}{2}s}g_W\circ\Phi, \\ \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{-\frac{s}{2}}\Phi_2(s)) &= \mathrm{e}^{-\frac{s}{2}}h_W\circ\Phi, \\ \Phi(-\log\varepsilon) &= y_0. \end{aligned} \tag{10.3}$$

Then the estimates are direct consequences of $|g_W| \le e^{\frac{s}{2}}$ and $|h_W| \le e^{\frac{s}{2}}$. We omit the detail, which is the same as that in [13]. The estimates for Φ_Z and Φ_A are similar.

Now we close the bootstrap bound for spatial support. We attempt to show that

$$\mathrm{supp}(DW, DZ, DA) \subset \frac{7}{8}\mathcal{X}(s) = \Big\{|y_1| \le \frac{7}{4}\varepsilon^{\frac{1}{2}}e^{\frac{3}{2}s},\ |y_2| \le \frac{7}{4}\varepsilon^{\frac{1}{6}}e^{\frac{s}{2}}\Big\}. \tag{10.4}$$

Since $\mathrm{supp}_x(D_xN, D_xT) \subset \frac{3}{4}\mathcal{X}(s) = \Big\{|x_1| \le \frac{3}{2}\varepsilon^{\frac{1}{2}}, |x_2| \le \frac{3}{2}\varepsilon^{\frac{1}{6}}\Big\}$, in $\Big(\frac{3}{4}\mathcal{X}(s)\Big)^c$, there hold

$$\begin{cases} g_W = \beta_\tau JW + \beta_\tau e^{\frac{s}{2}}\Big[-\dfrac{\partial_t f}{1+f_{x_1}} + J(\kappa + \beta_2 Z + 2\beta_1 V_1)\Big] \\ g_Z = \beta_2\beta_\tau JW + \beta_\tau e^{\frac{s}{2}}\Big[-\dfrac{\partial_t f}{1+f_{x_1}} + J(\beta_2\kappa + Z + 2\beta_1 V_1)\Big] \\ g_A = \beta_1\beta_\tau JW + \beta_\tau e^{\frac{s}{2}}\Big[-\dfrac{\partial_t f}{1+f_{x_1}} + J(\beta_1\kappa + \beta_1 Z + 2\beta_1 V_1)\Big], \end{cases} \tag{10.5}$$

$$h_W = h_Z = h_A = 2\beta_1\beta_\tau e^{-\frac{s}{2}}(V_2 + A), \tag{10.6}$$

$$\begin{cases} F_W = -2\beta_3\beta_\tau S\partial_2 A + \beta_\tau e^{-\frac{s}{2}}Q_{12}A, \\ F_Z = 2\beta_3\beta_\tau S\partial_2 A + \beta_\tau e^{-\frac{s}{2}}Q_{12}A, \\ F_A = -2\beta_3\beta_\tau S\partial_2 S - \beta_\tau e^{-\frac{s}{2}}Q_{12}U\cdot N. \end{cases} \tag{10.7}$$

We also define

$$\begin{cases} W_\infty(t) = \Big[\dfrac{\kappa_0}{2}(n_1+1) - \kappa\Big]e^{\frac{s}{2}}, \\ Z_\infty(t) = \dfrac{\kappa_0}{2}(n_1 - 1), \\ A_\infty(t) = -\dfrac{\kappa_0}{2}n_2, \\ S_\infty(t) = \dfrac{e^{-\frac{s}{2}}W_\infty + \kappa - Z_\infty}{2} = \dfrac{\kappa_0}{2}. \end{cases} \tag{10.8}$$

Then $W - W_\infty$, $Z - Z_\infty$, $A - A_\infty$ satisfy transport-type equations:

$$\begin{aligned} \Big(\partial_s - \frac{1}{2}\Big)(W - W_\infty) + \mathcal{V}_W\cdot\nabla(W - W_\infty) &= F_{W-W_\infty}, \\ \partial_s(Z - Z_\infty) + \mathcal{V}_Z\cdot\nabla(Z - Z_\infty) &= F_{Z-Z_\infty}, \\ \partial_s(A - A_\infty) + \mathcal{V}_A\cdot\nabla(A - A_\infty) &= F_{A-A_\infty}. \end{aligned} \tag{10.9}$$

where

$$\begin{aligned} F_{W-W_\infty} &= -\beta_3\beta_\tau e^{-\frac{s}{2}}(W - W_\infty)\partial_2 A + \beta_3\beta_\tau(Z - Z_\infty)\partial_2 A \\ &\quad + \beta_\tau e^{-\frac{s}{2}}Q_{12}(A - A_\infty) - 2\beta_3\beta_\tau S_\infty\partial_2 A, \\ F_{Z-Z_\infty} &= \beta_3\beta_\tau e^{-s}(W - W_\infty)\partial_2 A - \beta_3\beta_\tau e^{-\frac{s}{2}}(Z - Z_\infty)\partial_2 A \\ &\quad + 2\beta_3\beta_\tau e^{-\frac{s}{2}}S_\infty\partial_2 A + \beta_\tau e^{-s}Q_{12}(A - A_\infty), \\ F_{A-A_\infty} &= -\beta_3\beta_\tau e^{-s}(W - W_\infty)\partial_2 S + \beta_3\beta_\tau e^{-\frac{s}{2}}(Z - Z_\infty)\partial_2 S - 2\beta_3\beta_\tau e^{-\frac{s}{2}}S_\infty\partial_2 S \\ &\quad - \beta_\tau e^{-\frac{3}{2}s}Q_{12}(W - W_\infty) - \beta_\tau e^{-s}Q_{12}(Z - Z_\infty). \end{aligned} \tag{10.10}$$

For $y_0 \notin \frac{7}{8}\mathcal{X}(s)$, let $M' > |y_0|$ be a large enough constant. Define

$$Q_{\rm big} = \{|y_1| \le M', |y_2| \le M'\}, \quad Q_{\rm small}(s) = \{|y_1| \le {\rm e}^{\frac{3}{2}s}\mu_1(s), |y_2| \le {\rm e}^{\frac{s}{2}}\mu_2(s)\}, \tag{10.11}$$

where

$$\begin{cases} \mu_1(s) = \dfrac{3+\varepsilon}{2}\varepsilon^{\frac{1}{2}} - 2CM^{\frac{1}{4}}{\rm e}^{-s}, \\ \mu_2(s) = \dfrac{3+\varepsilon}{2}\varepsilon^{\frac{1}{6}} - 2CM^{\frac{1}{4}}{\rm e}^{-s}. \end{cases} \tag{10.12}$$

One can verify that $\frac{3}{4}\mathcal{X}(s) \subset Q_{\rm small} \subset \frac{7}{8}\mathcal{X}(s) \subset Q_{\rm big}$ if we take ε small enough and M' large enough. Define

$$E(y,s) = \frac{1}{2}({\rm e}^{-s}(W - W_\infty)^2 + (Z - Z_\infty) + 2(A - A_\infty)^2), \tag{10.13}$$

then we have

$$\frac{\rm d}{{\rm d}s}\int_{Q_{\rm big}\setminus Q_{\rm small}} E \le C\int_{Q_{\rm big}\setminus Q_{\rm small}} E. \tag{10.14}$$

From the initial condition, we can see that when $s = -\log\varepsilon$, $\int_{Q_{\rm big}\setminus Q_{\rm small}} E = 0$, thus

$$\int_{Q_{\rm big}\setminus Q_{\rm small}} E \equiv 0$$

at any time according to Gronwall's inequality. This tells us as long as $y_0 \notin \frac{7}{8}\mathcal{X}(s)$, $W(y_0,s) = W_\infty$, $Z(y_0,s) = Z_\infty$, $A(y_0,s) = A_\infty$ and finishes the proof of (IB-S).

10.2 Lower bounds for lagrangian trajectories

Lemma 10.2 *Suppose $|y_0| \ge l$, $s_0 \ge -\log\varepsilon$. Then we have*

$$|\Phi_W^{y_0}(s)| \ge |y_0|{\rm e}^{\frac{s-s0}{5}} \qquad \text{for all } s \ge s_0. \tag{10.15}$$

Proof It suffices to prove that $y \cdot \mathcal{V}_W \ge \frac{1}{5}|y|^2$. Note that by definition of $\mathcal{V}_W$, we can see that

$$y \cdot \mathcal{V}_W(y) \ge \frac{1}{2}|y|^2 + y_1^2 - \beta_\tau|y_1 JW| - |y_1 G_W| - |y_2 h_W|. \tag{10.16}$$

We split the estimate of W into two cases: $|y| \le L$ and $|y| > L$. If $|y| \le L$, by (B-$\widetilde{W}$-1) and (2.44) we have

$$\begin{aligned} |W(y)| &\le |W(y_1,y_2) - W(0,y_2)| + |W(0,y_2) - \overline{W}(0,y_2)| + \underbrace{|\overline{W}(0,y_2)|}_{=0} \\ &\le (1+\varepsilon^{\frac{1}{12}})|y_1| + \varepsilon^{\frac{1}{13}}|y_2|. \end{aligned} \tag{10.17}$$

If $|y| > L$, from the bootstrap assumption of W we have

$$|W(y)| \le (1+\varepsilon^{\frac{1}{20}})\eta^{\frac{1}{6}}(y) \le (1+\varepsilon^{\frac{1}{20}})^2|y|. \tag{10.18}$$

Then appealing to (9.1) and (9.3) we have the desired result.

Lemma 10.3 *Let* Φ *denote either* $\Phi_Z^{y_0}$ *or* $\Phi_A^{y_0}$. *If*

$$\kappa_0 \geq \frac{3}{1-\max(\beta_1,\beta_2)}. \tag{10.19}$$

then for any $0 \leq \sigma_1 < \frac{1}{2}$ *and* $2\sigma_1 < \sigma_2$, *we have the bound*

$$\int_{-\log\varepsilon}^{\infty} \mathrm{e}^{\sigma_1 s'}(1+|\Phi_1(s')|)^{-\sigma_2}\mathrm{d}s' \leq C(\sigma_1,\sigma_2). \tag{10.20}$$

Proof The proof is the same as that in [13].

Lemma 10.4 *Let* Φ^{y_0} *denote either* $\Phi_Z^{y_0}$ *or* $\Phi_A^{y_0}$, *then*

$$\sup_{y_0\in\mathcal{X}_0}\int_{-\log\varepsilon}^{\infty} |\partial_1 W|\circ\Phi^{y_0}(s')\mathrm{d}s' \lesssim 1. \tag{10.21}$$

Proof Using Lemma 10.3 and the bootstrap assumption of $\partial_1 W$, we can deduce the above inequality.

11 Closure of Bootstrap Argument for $\partial_1 A$

Since the vorticity is purely transported by u, the bootstrap of $\partial_1 A$ is easy to close from the bound of the vorticity and bootstrap assumptions, in no need of the evolution equation of $\partial_1 A$.

Lemma 11.1 (Relating A and Ω) *We have the following identity*

$$\begin{aligned} J\mathrm{e}^{\frac{3}{2}s}\partial_1 A =& -(\alpha S)^{\frac{1}{\alpha}}\Omega - T_2\mathrm{e}^{\frac{3}{2}s}\partial_2\Big(\frac{\mathrm{e}^{\frac{3}{2}s}W+\kappa+Z}{2}\Big)\\ &- N_2\mathrm{e}^{-\frac{3}{2}s}\partial_1 A + U\cdot(N_2\partial_{x_2}T - T_2\partial_{x_2}N + J\partial_{x_1}T). \end{aligned}\tag{11.1}$$

Proof Note that curl $\mathring{u} = \partial_T\mathring{u}\cdot N - \partial_T\mathring{u}\cdot T$. We compute each term as follows:

$$\begin{aligned} \partial_T\mathring{u} &= T_1\partial_{\widetilde{x}_1}\mathring{u} + T_2\partial_{\widetilde{x}_2}\mathring{u} = T_1\frac{1}{1+f_{x_1}}\partial_{x_1}\mathring{u} + T_2\Big(-\frac{f_{x_2}}{1+f_{x_1}}\partial_{x_1}\mathring{u} + \partial_{x_2}\mathring{u}\Big)\\ &= \frac{f_{x_2}}{\sqrt{1+f_{x_2}^2}}\frac{1}{1+f_{x_1}}\partial_{x_1}\mathring{u} - \frac{f_{x_2}}{\sqrt{1+f_{x_2}^2}}\frac{1}{1+f_{x_1}}\partial_{x_1}\mathring{u} + \frac{\partial_{x_2}\mathring{u}}{\sqrt{1+f_{x_2}^2}} = T_2\partial_{x_2}\mathring{u}, \end{aligned}\tag{11.2}$$

$$\begin{aligned} \partial_N\mathring{u} &= N_1\partial_{\widetilde{x}_1}\mathring{u} + N_2\partial_{\widetilde{x}_2}\mathring{u} = \frac{1}{\sqrt{1+f_{x_2}^2}}\frac{1}{1+f_{x_1}}\partial_{x_1}\mathring{u} - \frac{f_{x_2}}{\sqrt{1+f_{x_1}^2}}\Big(-\frac{f_{x_2}}{1+f_{x_1}}\partial_{x_1}\mathring{u} + \partial_{x_2}\mathring{u}\Big)\\ &= \frac{\sqrt{1+f_{x_2}^2}}{1+f_{x_1}}\frac{1}{1+f_{x_1}}\partial_{x_1}\mathring{u} - \frac{f_{x_2}}{\sqrt{1+f_{x_2}^2}}\partial_{x_1}\mathring{u} = J\partial_{x_1}\mathring{u} + N_2\partial_{x_2}\mathring{u}. \end{aligned}\tag{11.3}$$

Thus, we have

$$\begin{aligned} \text{curl }\mathring{u} &= T_2\partial_{x_2}\mathring{u}\cdot N - (J\partial_{x_1}\mathring{u} + N_2\partial_{x_2}\mathring{u})\cdot T\\ &= T_2\partial_{x_2}(\mathring{u}\cdot N) - T_2\mathring{u}\partial_{x_2}N - J\partial_{x_1}(\mathring{u}\cdot T) + J\mathring{u}\partial_{x_1}T - N_2\partial_{x_2}(\mathring{u}\cdot T) + N_2\mathring{u}\cdot\partial_{x_2}T \end{aligned}$$

$$
\begin{aligned}
&= T_2\partial_{x_2}\Big(\frac{w+z}{2}\Big) - T_2\mathring{u}\partial_{x_2}N - J\partial_{x_1}a + J\mathring{u}\partial_{x_1}T - N_2\partial_{x_2}a + N_2\mathring{u}\partial_{x_2}T \\
&= T_2\partial_{x_2}\Big(\frac{w+z}{2}\Big) - J\partial_{x_1}a - N_2\partial_{x_2}a + \mathring{u}\cdot(N_2\partial_{x_2}T - T_2\partial_{x_2}N + J\partial_{x_1}T). \qquad (11.4)
\end{aligned}
$$

On the other hand, curl $\mathring{u} = \widetilde{\rho}\widetilde{\zeta} = (\alpha S)^{\frac{1}{\alpha}}\Omega$, thus we get the desired result.

With the help of this identity, we have

$$
\begin{aligned}
\mathrm{e}^{\frac{3}{2}s}|\partial_1 A| &\lesssim \kappa_0^{\frac{1}{\alpha}} + \mathrm{e}^{\frac{s}{2}}(\mathrm{e}^{-\frac{s}{2}} + M\varepsilon^{\frac{1}{2}}\mathrm{e}^{-\frac{s}{2}}) + \varepsilon^{\frac{1}{2}}\mathrm{e}^{\frac{s}{2}}M\varepsilon^{-\frac{1}{2}}\mathrm{e}^{\frac{s}{2}} + M^{\frac{1}{4}}(\varepsilon^{\frac{1}{2}}M^2\varepsilon - M^2\varepsilon + M^2\varepsilon^{\frac{2}{3}}) \\
&\le \frac{1}{2}M. \qquad (11.5)
\end{aligned}
$$

This improves the bootstrap bound for $\partial_1 A$.

12 Closure of Bootstrap Argument for Z and A

In this section we improve the bootstrap bound of Z and A.

Lemma 12.1 (Close Z bootstrap) *For the Riemann variable Z, we have the improved bootstrap bound:*

$$
\begin{aligned}
|Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M\varepsilon, \\
\mathrm{e}^{\frac{3}{2}s}|\partial_1 Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M^{\frac{1}{2}}, \\
\mathrm{e}^{\frac{s}{2}}|\partial_2 Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M\varepsilon^{\frac{1}{2}}, \\
\mathrm{e}^{\frac{3}{2}s}|\partial_{11} Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M^{\frac{1}{2}}, \\
\mathrm{e}^{\frac{3}{2}s}|\partial_{12} Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M, \\
\mathrm{e}^{s}|\partial_{22} Z\circ\Phi_Z^{y_0}(s)| &\le \frac{1}{2}M.
\end{aligned} \qquad (12.1)
$$

Proof Since $\mathrm{e}^{\mu s}\partial^\gamma Z$ obeys

$$
\partial_s(\mathrm{e}^{\mu S}\partial^\gamma Z) + D_Z^{(\gamma,\mu)}(\mathrm{e}^{\mu s}\partial^\gamma Z) + (\mathcal{V}_Z\cdot\nabla)(\mathrm{e}^{\mu s}\partial^\gamma Z) = \mathrm{e}^{\mu s}F_Z^{(\gamma)}, \qquad (12.2)
$$

by Gronwall's inequality we can see that

$$
\begin{aligned}
\mathrm{e}^{\mu s}|\partial^\gamma Z\circ\Phi_Z^{y_0}(s)| &\lesssim \varepsilon^{-\mu}|\partial^\gamma Z(y_0,-\log\varepsilon)|\exp\Big(-\int_{-\log\varepsilon}^{s} D_Z^{(\gamma,\mu)}\circ\Phi_Z^{y_0}(s')\mathrm{d}s'\Big) \\
&\quad + \int_{-\log\varepsilon}^{s}\mathrm{e}^{\mu s'}|F_Z^{(\gamma)}\circ\Phi_Z^{y_0}(s')|\exp\Big(-\int_{s'}^{s} D_Z^{(\gamma,\mu)}\circ\Phi_Z^{y_0}(s'')\mathrm{d}s''\Big)\mathrm{d}s', \qquad (12.3)
\end{aligned}
$$

where

$$
D_Z^{(\gamma,\mu)} = D_Z^{(\gamma)} - \mu = \frac{3}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \beta_2\beta_\tau\gamma_1 J\partial_1 W - \mu. \qquad (12.4)
$$

If we require that $\frac{3}{2}\gamma_1 + \frac{1}{2}\gamma_2 \ge \mu$, then we have

$$
D_Z^{(\gamma,\mu)} \le \beta_2\beta_\tau\gamma_1|J\partial_1 W| \overset{|\gamma|\le 2}{\le} 2|\partial_1 W|. \qquad (12.5)
$$

Thus the damping term is bound by

$$\begin{aligned}&\exp\Big(-\int_{s'}^{s} D_Z^{(\gamma,\mu)}\circ\Phi_Z^{y_0}(s'')\mathrm{d}s''\Big)\\&\lesssim \mathrm{e}^{-(\frac{3\gamma_1+\gamma_2}{2}-\mu)(s-s')}\exp\Big(\int_{s'}^{s}2|\partial_1 W|\circ\Phi_Z^{y_0}(s'')\mathrm{d}s''\Big)\\&\overset{(10.21)}{\lesssim}\ \mathrm{e}^{-(\frac{3\gamma_1+\gamma_2}{2}-\mu)(s-s')}.\end{aligned}\tag{12.6}$$

And finally we have

$$\begin{aligned}&\mathrm{e}^{\mu s}|\partial^\gamma Z\circ\Phi_Z^{y_0}(s)|\\&\lesssim\ \varepsilon^{-\mu}|\partial^\gamma Z(y_0,-\log\varepsilon)|+\int_{-\log\varepsilon}^{s}\mathrm{e}^{\mu s'}|F_Z^{(\gamma)}\circ\Phi_Z^{y_0}(s')|\mathrm{e}^{-(\frac{3\gamma_1+\gamma_2}{2}-\mu)(s-s')}\mathrm{d}s'.\end{aligned}\tag{12.7}$$

Next, for different multi-index γ, we choose different μ in the above inequality.

Case 1 $\gamma=(0,0)$. We set $\mu=0$. From (3.38) and (9.15), we have

$$|Z\circ\Phi_Z^{y_0}(s)|\lesssim\varepsilon+\int_{-\log\varepsilon}^{s}\mathrm{e}^{-s'}\mathrm{d}s'\lesssim\varepsilon\le\frac{1}{2}M\varepsilon.\tag{12.8}$$

Case 2 $\gamma=(1,0)$. We set $\mu=\frac{3}{2}$. Also from (3.38) and (9.15), we have

$$\begin{aligned}\mathrm{e}^{\frac{3}{2}s}|Z\circ\Phi_Z^{y_0}(s)|&\lesssim\varepsilon^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}}+\int_{-\log\varepsilon}^{s}\mathrm{e}^{-\frac{3}{2}}\mathrm{e}^{\frac{3}{2}}\eta^{-\frac{1}{6}+\frac{2}{3(k-2)}}\circ\Phi_Z^{y_0}(s')\mathrm{d}s'\\&\lesssim 1+\int_{-\log\varepsilon}^{s}(1+|\Phi_1(s')|^2)^{-\frac{1}{6}+\frac{2}{3(k-2)}}\mathrm{d}s'\overset{(10.20)}{\lesssim}1\le\frac{1}{2}M^{\frac{1}{2}}.\end{aligned}\tag{12.9}$$

Case 3 $\gamma=(2,0)$. We set $\mu=\frac{3}{2}$ and deduce that

$$\begin{aligned}&\mathrm{e}^{\frac{3}{2}s}|\partial_{11}Z\circ\Phi_Z^{y_0}(s)|\\&\lesssim\varepsilon^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}}+\int_{-\log\varepsilon}^{s}\mathrm{e}^{\frac{3}{2}s'}\mathrm{e}^{-\frac{3}{2}s'}(1+M\eta^{-\frac{1}{3}}\circ\Phi(s'))\mathrm{e}^{-\frac{3}{2}(s-s')}\mathrm{d}s'\\&\lesssim 1+M\int_{-\log\varepsilon}^{s}\mathrm{e}^{-\frac{1}{8}(s-s')}(1+|\Phi_1(s')|)^{-\frac{2}{3}}\mathrm{d}s'\\&\overset{(10.20)}{\lesssim}\ 1+M\mathrm{e}^{-\frac{s}{8}}\le\frac{1}{2}M^{\frac{1}{2}}.\end{aligned}\tag{12.10}$$

Case 4 $\gamma=(1,1)$. We set $\mu=\frac{3}{2}$ to see that

$$\begin{aligned}&\mathrm{e}^{\frac{3}{2}s}|\partial_{12}Z\circ\Phi_Z^{y_0}(s)|\\&\lesssim\varepsilon^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}}+\int_{-\log\varepsilon}^{s}\mathrm{e}^{\frac{3}{2}s'}\mathrm{e}^{-\frac{3}{2}s1'}(M^{\frac{1}{2}}+M^2\eta^{-\frac{1}{3}}\circ\Phi(s'))\mathrm{e}^{-\frac{1}{2}(s-s')}\mathrm{d}s'\\&\lesssim 1+M^{\frac{1}{2}}+M^2\int_{-\log\varepsilon}^{s}\mathrm{e}^{-\frac{1}{8}(s-s')}(1+|\Phi_1(s')|)^{-\frac{2}{3}}\mathrm{d}s'\\&\overset{(10.20)}{\lesssim}\ M^{\frac{1}{2}}+M^2\mathrm{e}^{-\frac{s}{8}}\le\frac{1}{2}M.\end{aligned}\tag{12.11}$$

Case 5 $\gamma=(0,2)$. We set $\mu=1$ to obtain

$$\begin{aligned}\mathrm{e}^{\frac{3}{2}s}|\partial_{22}Z\circ\Phi_Z^{y_0}(s)| &\lesssim \varepsilon^{-\frac{1}{2}}\varepsilon+\int_{-\log\varepsilon}^{s}\mathrm{e}^{s'}M^{\frac{1}{4}}\mathrm{e}^{-(\frac{3}{2}-\frac{1}{k-3})s'}\mathrm{d}s'\\ &\lesssim \varepsilon^{\frac{1}{2}}+M^{\frac{1}{4}}\varepsilon^{\frac{1}{2}-\frac{1}{k-3}}\le\frac{1}{2}M.\end{aligned}\tag{12.12}$$

Next we close the bootstrap argument of A by proving (IB-A).

Lemma 12.2 (Close A bootstrap) *For the Riemann variable A, we have the improved bootstrap bound:*

$$\begin{aligned}|A\circ\Phi_A^{y_0}(s)| &\le \frac{1}{2}M\varepsilon,\\ \mathrm{e}^{\frac{s}{2}}|\partial_2 A\circ\Phi_A^{y_0}(s)| &\le \frac{1}{2}M\varepsilon^{\frac{1}{2}},\\ \mathrm{e}^{s}|\partial_{22}A\circ\Phi_A^{y_0}(s)| &\le \frac{1}{2}M.\end{aligned}\tag{12.13}$$

Proof As in the closure of Z bootstrap, if $\mu=\frac{3\gamma_1+\gamma_2}{2}$, we have

$$\mathrm{e}^{\mu s}|\partial^\gamma A\circ\Phi_A^{y_0}(s)|\lesssim\ \varepsilon^{-\mu}|\partial^\gamma A(y_0,-\log\varepsilon)|+\int_{-\log\varepsilon}^{s}\mathrm{e}^{\mu s'}|F_A^{(\gamma)}\circ\Phi_A^{y_0}(s')|\mathrm{d}s'.\tag{12.14}$$

For different multi-index γ, we choose different values of μ in the above inequality.

Case 1 $\gamma=(0,0)$. We set $\mu=0$. From (3.39) and (9.16), we have

$$|A\circ\Phi_A^{y_0}(s)|\lesssim\varepsilon+\int_{-\log\varepsilon}^{s}M^{\frac{1}{4}}\mathrm{e}^{-s'}\mathrm{d}s'\lesssim M^{\frac{1}{4}}\varepsilon\le\frac{1}{2}M\varepsilon.\tag{12.15}$$

Case 2 $\gamma=(0,1)$. We set $\mu=\frac{1}{2}$ and deduce that

$$\mathrm{e}^{\frac{s}{2}}|\partial_2 A\circ\Phi_A^{y_0}(s)|\lesssim\varepsilon^{-\frac{1}{2}}\varepsilon+\int_{-\log\varepsilon}^{s}\mathrm{e}^{\frac{s'}{2}}M^{\frac{1}{4}}\mathrm{e}^{-s'}\mathrm{d}s'\lesssim M^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}\le\frac{1}{2}M\varepsilon^{\frac{1}{2}}.\tag{12.16}$$

Case 3 $\gamma=(0,2)$. We set $\mu=1$ and deduce that

$$\begin{aligned}e^{s}|\partial_{22}A\circ\Phi_A^{y_0}(s)| &\lesssim \varepsilon^{-1}\varepsilon+\int_{-\log\varepsilon}^{s}\mathrm{e}^{s'}\mathrm{e}^{-\left(1-\frac{3}{k-2}\right)s'}\eta^{-\frac{1}{6}}\circ\Phi_A^{y_0}(s')\mathrm{d}s'\\ &\lesssim 1+M^{\frac{1}{4}}\int_{-\log\varepsilon}^{s}\mathrm{e}^{\frac{2}{k-2}s'}(1+|\Phi_1(s)|)^{-\frac{1}{3}}\mathrm{d}s'\overset{(10.20)}{\lesssim}1\le\frac{1}{2}M.\end{aligned}\tag{12.17}$$

13 Closure of Bootstrap Argument for W and $\widetilde{W}$

In this section, we prove the improved bootstrap bounds (IB-W), (IB-$\widetilde{W}^0$), (IB-$\widetilde{W}$-1), (IB-$\widetilde{W}$-2) and (IB-$\widetilde{W}$-3) for W and $\widetilde{W}$.

13.1 Closure of bootstrap argument for high order derivatives of $\widetilde{W}$

As stated in (2.53), $\partial^\gamma\widetilde{W}$ satisfies the equation

$$\partial_s\partial^\gamma\widetilde{W}+D_{\widetilde{W}}^{(\gamma)}\partial^\gamma\widetilde{W}+(\mathcal{V}_W\cdot\nabla)\partial^\gamma\widetilde{W}=\widetilde{F}_W^{(\gamma)},\tag{13.1}$$

where the damping term has a lower bound according to (5.10), (2.43) and (5.3):

$$
\begin{aligned}
D_{\widetilde{W}}^{(\gamma)} &= \frac{3\gamma_1+\gamma_2-1}{2} + \beta_\tau J(\partial_1\overline{W}+\gamma_1\partial_1 W) \\
&\geq \frac{3}{2}+\gamma_1-(1+\varepsilon^{\frac{1}{2}})(1+\gamma_1(1+\varepsilon^{\frac{1}{12}})) \geq \frac{3}{2}-1+\gamma_1-\gamma_1-C\varepsilon^{\frac{1}{12}} \geq \frac{1}{3}.
\end{aligned} \tag{13.2}
$$

From the equation of $\partial^\gamma \widetilde{W}$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}s}|\partial^\gamma\widetilde{W}\circ\Phi_W^{y_0}| + (D_{\widetilde{W}}^{(\gamma)}\circ\Phi_W^{y_0})|\partial^\gamma\widetilde{W}\circ\Phi_W^{y_0}| \leq |\widetilde{F}_W^{(\gamma)}\circ\Phi_W^{y_0}|. \tag{13.3}
$$

If $|\gamma|=4$ and $|y|\leq l$, from (9.17) and (13.2), we have

$$
\mathrm{e}^{\frac{s}{3}}|\partial^\gamma\widetilde{W}\circ\Phi_W^{y_0}(s)| \leq \varepsilon^{-\frac{1}{3}}\varepsilon^{\frac{1}{8}} + C\mathrm{e}^{\frac{s}{3}}(\varepsilon^{\frac{1}{7}}+\varepsilon^{\frac{1}{10}}(\log M)^{\gamma_2-1}). \tag{13.4}
$$

Thus for $|\gamma|=4$ and $|y|\leq l$, we have

$$
|\partial^\gamma\widetilde{W}\circ\Phi_W^{y_0}(s)| \leq \frac{1}{4}\varepsilon^{\frac{1}{10}}(\log M)^{\gamma_2}. \tag{13.5}
$$

Now we consider the case $|\gamma|=3$, $y=0$. Letting $y=0$ in (2.53), we have

$$
\begin{aligned}
|\partial_s\partial^\gamma\widetilde{W}^0| &= |\widetilde{F}_W^{(\gamma),0} - G_W^0\partial_1\partial^\gamma\widetilde{W}^0 - h_W^0\partial_2\partial^\gamma\widetilde{W}^0 - (1-\beta_\tau)(1+\gamma_1)\partial^\gamma\widetilde{W}^0| \\
&\lesssim \mathrm{e}^{-(\frac{1}{2}-\frac{1}{k-3})s} + M\mathrm{e}^{-s}\varepsilon^{\frac{1}{10}}(\log M)^4 + M\mathrm{e}^{-s}\varepsilon^{\frac{1}{4}} \lesssim \mathrm{e}^{-(\frac{1}{2}-\frac{1}{k-3})s}.
\end{aligned} \tag{13.6}
$$

Thus from (3.35), one can see that

$$
|\partial^\gamma\widetilde{W}^0(s)| \leq |\partial^\gamma\widetilde{W}(-\log\varepsilon)| + C\mathrm{e}^{-(\frac{1}{2}-\frac{1}{k-3})s} \leq \frac{1}{10}\varepsilon^{\frac{1}{4}}. \tag{13.7}
$$

Next, we consider the case $|\gamma|\leq 3$, $|y|\leq l$. For $|\gamma|=3$, by (13.5) and (13.7), we have

$$
|\partial^\gamma\widetilde{W}| \leq \varepsilon^{\frac{1}{4}} + \frac{1}{2}\varepsilon^{\frac{1}{10}}(\log M)^{\gamma_2+1}|y| \leq \frac{1}{2}(\log M)^4\varepsilon^{\frac{1}{10}}|y| + \frac{1}{2}M\varepsilon^{\frac{1}{4}}. \tag{13.8}
$$

Now by induction and $\partial^\gamma\widetilde{W}^0=0$ for $|\gamma|\leq 2$, we can close the bootstrap argument of $\partial^\gamma\widetilde{W}$ as in the case $|\gamma|=3$.

13.2 A general discusion of weighted estimates

In order to close the bootstrap argument for W and $\widetilde{W}$ fully, let us consider the evolution of $q=\eta^\mu R$, where R satisfies a transport-type equation:

$$
\partial_s R + D_R R + \mathcal{V}_W\cdot R = F_R. \tag{13.9}
$$

We assume $|\mu|\leq\frac{1}{2}$. One can deduce that q satisfies

$$
\partial_s q + D_q q - \mathcal{V}_W\cdot\nabla q = \eta^\mu F_R, \tag{13.10}
$$

where

$$
D_q = D_R - \mu\eta^{-1}\mathcal{V}_W\cdot\nabla q = D_R - 3\mu + 3\mu\eta^{-1} - 2\mu\underbrace{\eta^{-1}(y_1 g_W + 3y_2^5 h_W)}_{D_\eta}. \tag{13.11}
$$

By (5.10), (9.1), (9.3) and the bootstrap assumption for W, one can see that $|D_\eta| \le 3\eta^{-\frac{1}{3}}$. Thus $D_q \ge D_R - 3\mu + 3\mu\eta^{-1} - 6|\mu|\eta^{-\frac{1}{3}}$.

By composing q with the trajectory of $\mathcal{V}_W$, we have

$$\begin{aligned}|q \circ \Phi_W^{y_0}(s)| \le{}& |q(y_0, s_0)| \exp\Big(-\int_{s_0}^{s} D_q \circ \Phi_W^{y_0}(s')\mathrm{d}s'\Big)\\ &+ \int_{s_0}^{s} |F_q^{(\gamma)} \circ \Phi_W^{y_0}(s')| \exp\Big(-\int_{s'}^{s} D_q \circ \Phi_W^{y_0}(s'')\mathrm{d}s''\Big)\mathrm{d}s',\end{aligned} \tag{13.12}$$

where (y_0, s_0) is the starting position and starting time of the trajectory. Note that s_0 need not to be $-\log\varepsilon$.

If $|y_0| \ge l$, we have that

$$\begin{aligned}2\mu\int_{s'}^{s} D_\eta \circ \Phi_W^{y_0}(s'')\mathrm{d}s'' &\overset{|\mu|\le\frac{1}{2}}{\le} \int_{s_0}^{s} 3\eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s')\mathrm{d}s'\\ &\le 3\cdot 2^{\frac{1}{3}}\int_{s_0}^{\infty}(1 + l^2 \mathrm{e}^{\frac{2}{5}(s'-s_0)})^{-\frac{1}{3}}\mathrm{d}s' \le -30\log l.\end{aligned} \tag{13.13}$$

Consequently, we can bound q by

$$\begin{aligned}&|q \circ \Phi_W^{y_0}|\\ \le{}& l^{-30}|q(y_0, s_0)| \exp\Big[-\int_{s_0}^{s}(D_R - 3\mu + 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s')\mathrm{d}s'\Big]\\ &+ l^{-30}\int_{s_0}^{s} |F_q^{(\gamma)} \circ \Phi_W^{y_0}(s')| \exp\Big[-\int_{s'}^{s}(D_R - 3\mu + 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'')\mathrm{d}s''\Big]\mathrm{d}s'.\end{aligned} \tag{13.14}$$

We remark that as long as $|y_0| \ge l$ and $p > 0$, one can verify that

$$\int_{s_0}^{\infty} \eta^{-p} \circ \Phi_W^{y_0}(s)\mathrm{d}s \lesssim_p -\log l \tag{13.15}$$

If $|y_0| \ge L$, we have another inequality

$$\begin{aligned}2\mu\int_{s'}^{s} D_\eta \circ \Phi_W^{y_0}(s'')\mathrm{d}s'' &\overset{|\mu|\le\frac{1}{2}}{\le} \int_{s_0}^{s} 3\eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s')\mathrm{d}s'\\ &\le 3\cdot 2^{\frac{1}{3}}\int_{s_0}^{\infty}(1 + L^2 \mathrm{e}^{\frac{2}{5}(s'-s_0)})^{-\frac{1}{3}}\mathrm{d}s' \le CL^{-\frac{20}{3}}.\end{aligned} \tag{13.16}$$

In this case, q is bounded by

$$\begin{aligned}|q \circ \Phi_W^{y_0}| \le{}& \mathrm{e}^{\varepsilon}|q(y_0, s_0)| \exp\Big[-\int_{s_0}^{s}(D_R - 3\mu + 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s')\mathrm{d}s'\Big]\\ &+ \mathrm{e}^{\varepsilon}\int_{s_0}^{s} |F_q^{(\gamma)} \circ \Phi_W^{y_0}(s')| \exp\Big[-\int_{s'}^{s}(D_R - 3\mu + 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'')\mathrm{d}s''\Big]\mathrm{d}s'.\end{aligned} \tag{13.17}$$

13.3 Closure of bootstrap argument for $\widetilde{W}$

For different multi-index γ, we choose different values of μ, and we will use (13.14) or (13.17), depending on the location of y. We establish the estimates case by case.

Case 1 $|\gamma|=0$, $l\le|y|\le L$. In this case we set $\mu=-\frac{1}{6}$. Thus we have $q=\eta^{-\frac{1}{6}}\widetilde{W}$ and $D_R-3\mu+3\mu\eta^{-1}=-\frac{1}{2}\eta^{-1}+\beta_\tau J\partial_1\overline{W}$. We estimate the damping term and the forcing term.

$$\begin{aligned}&-\int_{s'}^{s}(\beta_\tau J\partial_1\overline{W}-\frac{1}{2}\eta^{-1})\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\\&\le(1+\varepsilon^{\frac{1}{2}})\int_{s_0}^{s}|\partial_1\overline{W}\circ\Phi_W^{y_0}(s'')|\mathrm{d}s''+\frac{1}{2}\int_{s_0}^{s}\eta^{-1}\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\\&\le 2\int_{s_0}^{s}\eta^{-\frac{1}{3}}\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\le-20\log l,\end{aligned}\tag{13.18}$$

$$\int_{s_0}^{s}|(\eta^{-\frac{1}{6}}\widetilde{F}_W)\circ\Phi_W^{y_0}(s')|\mathrm{d}s'\lesssim\int_{s_0}^{s}M\varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{3}}\circ\Phi_W^{y_0}(s')\mathrm{d}s'\le-\varepsilon^{\frac{1}{8}}\log l.\tag{13.19}$$

According to Lemma 10.2, it is possible to require that either $|y_0|=l$ or $s_0=-\log\varepsilon$, thus we can use the initial condition or bootstrap assumptions to bound $|q(y_0,s_0)|$. From (13.14), (3.35) and (B-$\widetilde{W}$-1), we have

$$\begin{aligned}|\eta^{-\frac{1}{6}}\widetilde{W}\circ\Phi_W^{y_0}(s)|&\le l^{-30}|\widetilde{W}(y_0,s_0)|\eta^{-\frac{1}{6}}(y_0)l^{-20}+l^{-30}l^{-20}(-\varepsilon^{\frac{1}{8}})\log l\\&\le l^{-50}\eta^{-\frac{1}{6}}(y_0)\max(\varepsilon^{\frac{1}{10}}\eta^{-\frac{1}{6}}(y_0),2(\log M)^4\varepsilon^{\frac{1}{10}}l^4)\\&\quad-l^{-50}\varepsilon^{\frac{1}{8}}\log l\le\frac{1}{2}\varepsilon^{\frac{1}{11}}.\end{aligned}\tag{13.20}$$

Case 2 $\gamma=(1,0)$, $l\le|y|\le L$. Let $\mu=\frac{1}{3}$. Then we have $D_R-3\mu+3\mu\eta^{-1}\ge\beta_\tau J(\partial_1\overline{W}+\partial_1 W)$, and

$$-\int_{s'}^{s}(D_R-3\mu+3\mu\eta^{-1})\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\le 4\int_{s_0}^{\infty}\eta^{-\frac{1}{3}}\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\le-40\log l,\tag{13.21}$$

$$\int_{s_0}^{s}|F_q\circ\Phi_W^{y_0}(s')|\mathrm{d}s'\lesssim\varepsilon^{\frac{1}{11}}\int_{s_0}^{s}(\eta^{\frac{1}{3}}\eta^{-\frac{1}{2}})\circ\Phi_W^{y_0}(s')\mathrm{d}s'\lesssim-\varepsilon^{\frac{1}{11}}\log l.\tag{13.22}$$

Now we can bound q by

$$\begin{aligned}|\eta^{\frac{1}{3}}\widetilde{W}\circ\Phi_W^{y_0}(s)|&\le l^{-30}|\widetilde{W}(y_0,s_0)|\eta^{\frac{1}{3}}(y_0)l^{-40}+l^{-30}l^{-40}(-\varepsilon^{\frac{1}{11}})\log l\\&\le l^{-70}\eta^{\frac{1}{3}}(y_0)\max(\varepsilon^{\frac{1}{11}}\eta^{-\frac{1}{3}}(y_0),2(\log M)^4\varepsilon^{\frac{1}{10}}l^3)-l^{-70}\varepsilon^{\frac{1}{11}}\log l\\&\le\frac{1}{2}\varepsilon^{\frac{1}{12}}.\end{aligned}\tag{13.23}$$

Case 3 $\gamma=(0,1)$, $l\le|y|\le L$. Let $\mu=0$. Then we have $D_R-3\mu+3\mu\eta^{-1}=\beta_\tau J\partial_1\overline{W}$, and $|F_q|\lesssim\varepsilon^{\frac{1}{12}}\eta^{-\frac{1}{3}}$. The rest is almost the same as Case 2.

13.4 Closure of bootstrap for W

Similarly, for different γ we choose different values of μ, and we will use (13.14) or (13.17), depending on the location of y.

Case 1 $|\gamma|=2$, $|y|\ge l$. Now let $R=\partial^\gamma W$, and

$$\mu=\begin{cases}\frac{1}{3}, & \gamma=(2,0),(1,1),\\ \frac{1}{6}, & \gamma=(0,2).\end{cases}\tag{13.24}$$

The damping term becomes

$$3\mu - D_R = \begin{cases} -\gamma_1 + \dfrac{1}{2} - \beta_\tau(1+\gamma_1 \mathbb{1}_{\gamma_1\geq 2})J\partial_1 W, & \gamma_1 \geq 1, \\ -\beta_\tau J\partial_1 W, & \gamma_1 = 0. \end{cases} \tag{13.25}$$

When $\gamma_1 = 0$, we have

$$\int_{s'}^{s} (3\mu - D_R)\circ \Phi_W^{y_0}(s'')\mathrm{d}s'' \leq 2\int_{s_0}^{\infty} |\partial_1 W|\circ \Phi_W^{y_0}(s'')\mathrm{d}s'' \leq -20\log l, \tag{13.26}$$

and the forcing term is bound by

$$\int_{s_0}^{s} |\eta^{\frac{1}{6}} F_W^{(0,2)}|\circ \Phi_W^{y_0}(s')\mathrm{d}s' \lesssim M^{\frac{2}{3}} \int_{s_0}^{s} (\eta^{\frac{1}{6}}\eta^{-\frac{1}{3}+\frac{1}{3(k-3)}})\circ \Phi_W^{y_0}(s')\mathrm{d}s' \leq -M^{\frac{5}{6}}\log l. \tag{13.27}$$

Thus, we have that

$$\begin{aligned} &\big|\eta^{\frac{1}{6}}\partial_{22}W\circ \Phi_W^{y_0}(s)\big| \\ &\leq l^{-30}\eta^{\frac{1}{6}}(y_0)|\partial_{22}W(y_0,s_0)|l^{-20} - l^{-30}M^{\frac{5}{6}}\log l \\ &\leq l^{-50}\eta^{\frac{1}{6}}(y_0)\max\Big(\eta^{-\frac{1}{6}}(y_0), \frac{6}{7}\eta^{-\frac{1}{6}}(y_0) + 2(\log M)^4\varepsilon^{\frac{1}{10}}l^2\eta^{-\frac{1}{6}}(y_0)\|\eta^{\frac{1}{6}}\|_{L^\infty(|y|\leq l)}\Big) \\ &\quad - l^{-50}M^{\frac{5}{6}}\log l \\ &\overset{\varepsilon\text{ small}}{\leq} -2l^{-50}M^{\frac{5}{6}}\log l \overset{M\text{ large}}{\leq} \frac{1}{2}M. \end{aligned} \tag{13.28}$$

When $\gamma_1 > 0$, we have that

$$\begin{aligned} &\exp\Big(\int_{s'}^{s}(3\mu - D_R)\circ\Phi_W^{y_0}(s'')\mathrm{d}s''\Big) \\ &\leq \exp\Big\{3\int_{s'}^{s}|\partial_1 W|\circ \Phi_W^{y_0}(s'')\mathrm{d}s'' + \int_{s'}^{s}\Big(\frac{1}{2}-1\Big)\mathrm{d}s''\Big\} \\ &\leq \exp\Big\{4\int_{s'}^{s}\eta^{-\frac{1}{3}}\circ \Phi_W^{y_0}(s'')\mathrm{d}s'' - \frac{1}{2}(s-s')\Big\} \leq l^{-80}\mathrm{e}^{-\frac{1}{2}(s-s')}, \end{aligned} \tag{13.29}$$

and $|F_q| = \big|\eta^{\frac{1}{3}}F_W^{(\gamma)}\big| \lesssim \eta^{\frac{1}{3}}M^{\frac{1}{3}\gamma_2}\eta^{-\frac{1}{3}} \leq M^{\frac{1}{3}\gamma_2+\frac{1}{6}}$. Thus, we have the bound for $\partial^\gamma W$:

$$\begin{aligned} &\big|\eta^{\frac{1}{3}}\partial^\gamma W\big|\circ\Phi_W^{y_0}(s) \\ &\leq l^{-20}\eta^{\frac{1}{3}}(y_0)|\partial^\gamma W(y_0,s_0)|l^{-80}\mathrm{e}^{-\frac{1}{2}(s-s_0)} + L^{-20}\int_{s_0}^{s} M^{\frac{1}{3}\gamma_2+\frac{1}{6}}l^{-80}\mathrm{e}^{-\frac{1}{2}(s-s')}\mathrm{d}s' \\ &\leq l^{-100}\eta^{\frac{1}{3}}(y_0)\max(\eta^{-\frac{1}{3}}(y_0), C\eta^{-\frac{1}{2}}(y_0) + 2(\log M)^4\varepsilon^{\frac{1}{10}}l^2\eta^{-\frac{1}{3}}(y_0)\|\eta^{\frac{1}{3}}\|_{L^\infty(|y|\leq l)})\mathrm{e}^{-\frac{1}{2}(s-s_0)} \\ &\quad + l^{-101}M^{\frac{1}{3}\gamma_2+\frac{1}{6}} \\ &\leq l^{-100}\max(1, C+3(\log M)^4\varepsilon^{\frac{1}{10}}l^2) + l^{-101}M^{\frac{1}{3}\gamma_2+\frac{1}{6}} \\ &\leq M^{\frac{1+\gamma_2}{3}}\underbrace{(CM^{-\frac{1}{3}} + l^{-101}M^{-\frac{1}{6}})}_{<\frac{1}{2}\text{ when }M\text{ large}} \leq \frac{1}{2}M^{\frac{1+\gamma_2}{3}}. \end{aligned} \tag{13.30}$$

Case 2 $|\gamma| = 0$ and $|y| \geq L$. Let $\mu = -\frac{1}{6}$. Now we have $3\mu - D_R - 3\mu\eta^{-1} = \frac{1}{2}\eta^{-1}$ and $F_q = \eta^{-\frac{1}{6}}(F_W - \mathrm{e}^{-\frac{s}{2}}\beta_\tau\dot\kappa)$. And we bound the damping term and the forcing term by

$$\int_{s'}^{s}\frac{1}{2}\eta^{-1}\circ\Phi_W^{y_0}(s'')\mathrm{d}s'' \leq \int_{s_0}^{\infty}(1+L^2\mathrm{e}^{\frac{2}{5}(s''-s_0)})^{-1}\mathrm{d}s''$$

$$\leq L^{-2}\int_{s_0}^{\infty} \mathrm{e}^{-\frac{2}{5}(s''-s_0)}\mathrm{d}s'' \leq L^{-1} = \varepsilon^{\frac{1}{10}}, \tag{13.31}$$

$$\begin{aligned}\int_{s_0}^{s} |F_q \circ \Phi_W^{y_0}(s')|\mathrm{d}s' &\lesssim \int_{s_0}^{s} (\mathrm{e}^{-\frac{s'}{2}} + M\mathrm{e}^{-\frac{s'}{2}})\eta^{-\frac{1}{6}} \circ \Phi_W^{y_0}(s')\mathrm{d}s' \\ &\lesssim M\int_{s_0}^{s} \mathrm{e}^{-\frac{s'}{2}}\mathrm{d}s' \leq \varepsilon^{\frac{1}{3}}.\end{aligned} \tag{13.32}$$

Thus, we have that

$$\begin{aligned}&|\eta^{-\frac{1}{6}}W| \circ \Phi_W^{y_0}(s) \\ &\leq \mathrm{e}^{\varepsilon}\eta^{-\frac{1}{6}}(y_0)|W(y_0,s_0)|\mathrm{e}^{\varepsilon^{\frac{1}{10}}} + \mathrm{e}^{\varepsilon}\varepsilon^{\frac{1}{3}}\mathrm{e}^{\varepsilon^{\frac{1}{10}}} \\ &\leq \mathrm{e}^{\varepsilon}\mathrm{e}^{\varepsilon^{\frac{1}{10}}}\eta^{-\frac{1}{6}}(y_0)\max(\eta^{\frac{1}{6}}(y_0)(1+\varepsilon^{\frac{1}{11}}), \eta^{\frac{1}{6}}(y_0) + \varepsilon^{\frac{1}{12}}\eta^{-\frac{1}{3}}(y_0)) + \mathrm{e}^{\varepsilon}\varepsilon^{\frac{1}{3}}\mathrm{e}^{\varepsilon^{\frac{1}{10}}} \\ &\leq 1 + \varepsilon^{\frac{1}{19}}.\end{aligned} \tag{13.33}$$

Case 3 $\gamma = (1,0)$ and $|y| \geq L$. In this case, we can see that $q = \eta^{\frac{1}{3}}\partial_1 W$, $3\mu - D_R - 3\mu\eta^{-1} = -\beta_\tau J\partial_1 W - \eta^{-1} \leq -\beta_\tau J\partial_1 W$, and

$$\begin{aligned}\int_{s'}^{s} (3\mu - D_R - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'')\mathrm{d}s'' &\leq 2\int_{s'}^{s} |\partial_1 W| \circ \Phi_W^{y_0}(s'')\mathrm{d}s'' \\ &\lesssim \int_{s_0}^{\infty} (1 + L^2\mathrm{e}^{\frac{2}{5}(s''-s_0)})^{-\frac{1}{3}}\mathrm{d}s'' \lesssim L^{-\frac{2}{3}} \leq \varepsilon.\end{aligned} \tag{13.34}$$

The forcing term is bound by

$$\begin{aligned}\int_{s_0}^{s} |F_q \circ \Phi_W^{y_0}(s')|\mathrm{d}s' &\lesssim \int_{s_0}^{s} \varepsilon^{\frac{1}{4}}|\eta^{\frac{1}{3}}\eta^{-\frac{1}{2}+\frac{2}{3(k-2)}}| \circ \Phi_W^{y_0}(s')\mathrm{d}s' \\ &\lesssim \varepsilon^{\frac{1}{4}}\int_{s_0}^{s} \eta^{-\frac{1}{12}} \circ \Phi_W^{y_0}(s')\mathrm{d}s' \lesssim \varepsilon^{\frac{1}{12}}.\end{aligned} \tag{13.35}$$

Thus we have the bound

$$\begin{aligned}&|\eta^{\frac{1}{3}}\partial_1 W| \circ \Phi_W^{y_0}(s) \\ &\leq \mathrm{e}^{\varepsilon}\eta^{\frac{1}{3}}(y_0)|\partial_1 W(y_0,s_0)|\mathrm{e}^{\varepsilon} + C\mathrm{e}^{\varepsilon}\varepsilon^{\frac{1}{4}}\mathrm{e}^{\varepsilon} \\ &\leq \mathrm{e}^{\varepsilon}\mathrm{e}^{2\varepsilon}\eta^{\frac{1}{3}}(y_0)\max(\eta^{-\frac{1}{3}}(y_0)(1+\varepsilon^{\frac{1}{112}}), \eta^{-\frac{1}{3}}(y_0) + \varepsilon^{\frac{1}{12}}\eta^{-\frac{1}{3}}(y_0)) + C\mathrm{e}^{2\varepsilon}\varepsilon^{\frac{1}{4}} \\ &\leq 1 + \varepsilon^{\frac{1}{13}}.\end{aligned} \tag{13.36}$$

Case 4 $\gamma = (0,1)$ and $|y| \geq L$. Let $\mu = 0$. We have $q = R = \partial_2 W$, and $3\mu - D_R - 3\mu\eta^{-1} = -\beta_\tau J\partial_1 W$. Thus we have the bound for damping term

$$\int_{s'}^{s} (3\mu - D_R - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'')\mathrm{d}s'' \leq \varepsilon. \tag{13.37}$$

The forcing term is bound by

$$\int_{s_0}^{s} |F_q \circ \Phi_W^{y_0}(s')|\mathrm{d}s' \lesssim \int_{s_0}^{s} M^2\varepsilon^{\frac{1}{6}}\eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s')\mathrm{d}s' \leq \varepsilon^{\frac{1}{8}}. \tag{13.38}$$

Finally we have that

$$|\partial_2 W| \circ \Phi_W^{y_0}(s) \leq \mathrm{e}^{\varepsilon}|\partial_2 W(y_0,s_0)|\mathrm{e}^{\varepsilon} + \mathrm{e}^{\varepsilon}\varepsilon^{\frac{1}{8}}\mathrm{e}^{\varepsilon} \leq \mathrm{e}^{2\varepsilon}\max\Big(\frac{3}{4}, \frac{2}{3} + \varepsilon^{\frac{1}{13}}\Big) + \mathrm{e}^{2\varepsilon}\varepsilon^{\frac{1}{8}} \leq \frac{5}{6}. \tag{13.39}$$

14 Proof of the Main Theorem

In this section we prove the main theorem, discuss the Hölder regularity of w and deduce a lower bound of the vorticity.

Proof of the main theorem The local well-posedness of (u,σ) in physical variables implies the local well-posedness of $(W,Z,A,\kappa,\tau,\xi,n,\phi)$ in self-similar variables, and the global existence of $(W,Z,A,\kappa,\tau,\xi,n,\phi)$ in self-similar variables is obtained via the bootstrap bound.

Now we prove the solution has the desired blow-up behavior. From the bootstrap assumptions and $\tau(t)-t=\int_t^{T_*}(1-\dot\tau(t'))\mathrm{d}t'$ we can see that $c(T_*-t)\le\tau-t=\mathrm{e}^{-s}\le C(T_*-t)$. Since $R(t)\in SO(2)$, using (5.10) and (5.23), we have that

$$|[(R(t)N)\cdot\nabla_{\mathrm{x}}]u|=|N\cdot\nabla_{\widetilde{x}}\widetilde{u}|=\Big|\Big(\frac{\sqrt{1+f_{x_2}^2}}{1+f_{x_1}}\partial_{x_1}-\frac{f_{x_2}}{\sqrt{1+f_{x_2}^2}}\partial_{x_2}\Big)\mathring{u}\Big|$$
$$\le(1+\varepsilon^{\frac{2}{3}})(1+\varepsilon^{\frac{3}{4}})\mathrm{e}^s+\varepsilon\le\frac{1+\varepsilon^{\frac{1}{2}}}{T_*-t}. \tag{14.1}$$

Similarly, we can see that the directional derivative of u along the shock front is bounded:

$$|[(R(t)T)\cdot\nabla_{\mathrm{x}}]u|=|T\cdot\nabla_{\widetilde{x}}\widetilde{u}|=\Big|\frac{1}{\sqrt{1+f_{x_2}^2}}\partial_{x_2}\mathring{u}\Big|\le1+\varepsilon^{\frac{1}{2}}. \tag{14.2}$$

In a same way, we can prove that $|[(R(t)N)\cdot\nabla_{\mathrm{x}}]\sigma|\le\frac{1+\varepsilon^{\frac{1}{2}}}{T_*-t}$ and $|[(R(t)T)\cdot\nabla_{\mathrm{x}}]\sigma|\le1+\varepsilon^{\frac{1}{2}}$. Consequently, we have that

$$|\nabla_{\mathrm{x}}u(t)|\le|[(R(t)N)\cdot\nabla_{\mathrm{x}}]u|+|[(R(t)T)\cdot\nabla_{\mathrm{x}}]u|\le\frac{1+2\varepsilon^{\frac{1}{2}}}{T_*-t}, \tag{14.3}$$

$$|\nabla_{\mathrm{x}}\sigma(t)|\le|[(R(t)N)\cdot\nabla_{\mathrm{x}}]\sigma|+|[(R(t)T)\cdot\nabla_{\mathrm{x}}]\sigma|\le\frac{1+2\varepsilon^{\frac{1}{2}}}{T_*-t}. \tag{14.4}$$

From the bootstrap assumptions $|\dot\xi|\le M^{\frac{1}{4}}$ and $|\dot n_2|\le M^2\varepsilon^{\frac{1}{2}}$, we know that both ξ and n have limits as $t\to T_*$.

Next, by the definition of n and N, and the coordinate transformations, we have $n(t)=R(t)N(0,t)$. Furthermore, we can see that

$$|[(R(t)N)\cdot\nabla_{\mathrm{x}}]u(\xi(t),t)|=\Big|\Big(\frac{\sqrt{1+f_{x_2}^2}}{1+f_{x_1}}\partial_{x_1}-\frac{f_{x_2}}{\sqrt{1+f_{x_2}^2}}\partial_{x_2}\Big)\mathring{u}(0,t)\Big|$$
$$=\Big|\frac{-\mathrm{e}^s+\partial_{x_1}z(0,t)}{2}\widetilde{\mathrm{e}}_1+\partial_{x_1}a(0,t)\widetilde{\mathrm{e}}_2\Big|\ge\Big(\frac{1}{2}-\varepsilon^{\frac{1}{2}}\Big)\mathrm{e}^s. \tag{14.5}$$

Similarly, we have

$$|[(R(t)N)\cdot\nabla_{\mathrm{x}}]\sigma(\xi(t),t)|=|\partial_{x_1}\mathring{\sigma}(0,t)|=\Big|\frac{-\mathrm{e}^s-\partial_{x_1}z(0,t)}{2}\Big|\ge\Big(\frac{1}{2}-\varepsilon^{\frac{1}{2}}\Big)\mathrm{e}^s. \tag{14.6}$$

Thus, we can conclude that $\|\nabla_{\mathrm{x}}u\|_{L^\infty}\ge|[(R(t)N)\cdot\nabla_{\mathrm{x}}]u(\xi(t),t)|\ge\frac{c}{T_*-t}$, and $\|\nabla_{\mathrm{x}}\sigma\|_{L^\infty}\ge|[(R(t)N)\cdot\nabla_{\mathrm{x}}]\sigma(\xi(t),t)|\ge\frac{c}{T_*-t}$.

Next, we prove (3.25). It suffices to prove that $\|\partial_{x_1}w\|_{L^\infty(B_x^C(0,\delta))}\le C(\delta)$. From (IB-$W$), we have that

$$\begin{aligned}\|\partial_{x_1}w\|_{L^\infty(B_x^C(0,\delta))}&\le(1+\varepsilon^{\frac{1}{13}})\mathrm{e}^s\Big\|\frac{1}{(1+y_1^2+y_2^6)^{\frac13}}\Big\|_{L_y^\infty(\{\mathrm{e}^{-3s}y_1^2+\mathrm{e}^{-s}y_2^2\le\delta^2\}^c)}\\&\le 2\delta^{-2}(1+\varepsilon^{\frac{1}{13}})\frac{\mathrm{e}^s}{(1+\mathrm{e}^{3s})^{\frac13}}\le 3\delta^{-2}.\end{aligned}\tag{14.7}$$

Now we have completed the proof of the main shock formation result and (3.23)–(3.30). The Hölder bound is left to the next subsection.

14.1 Hölder regularity for w

We now prove that Riemann invariant w posseses a uniform $\frac13$-Hölder bound up to the blow-up time.

Proposition 14.1 *For the Riemann variable w, we have that $w\in L^\infty([-\varepsilon,T_*);C^{\frac13})$.*

Proof The proof of this proposition is the same as that in [13], and for the reader's convenience we outline the proof here.

Using the bootstrap assumptions we directly compute the $C^{\frac13}$ norm

$$\begin{aligned}&\frac{|w(x_1,x_2,t)-w(x_1',x_2',t)|}{|x-x'|^{\frac13}}\\&=\frac{\mathrm{e}^{-\frac{s}{2}}|W(y,s)-W(y',s)|}{[\mathrm{e}^{-3s}(y_1-y_1')^2+\mathrm{e}^{-s}(y_2-y_2')^2]^{\frac16}}\\&\le\frac{|W(y_1,y_2,s)-W(y_1',y_2,s)|}{|y_1-y_1'|^{\frac13}}+\mathrm{e}^{-\frac{s}{3}}\frac{|W(y_1',y_2,s)-W(y_1',y_2',s)|}{|y_2-y_2'|^{\frac13}}\\&\lesssim\frac{\int_{y_1'}^{y_1}(1+z^2)^{-\frac13}\mathrm{d}z}{|y_1-y_1'|^{\frac13}}+\mathrm{e}^{-\frac{s}{3}}|y_2-y_2'|^{\frac23}\overset{y\in\mathcal{X}(s)}{\lesssim}1.\end{aligned}\tag{14.8}$$

Now we have proved that w is uniformly Hölder-$\frac13$ continuous with respect to x. And one can check that the transformation $\widetilde{x}\mapsto x$, $\mathrm{x}\mapsto\widetilde{x}$ do not affect the Hölder-$\frac13$ continuity of w.

14.2 Discussion of the vorticity

From (2.9), we know that in $\widetilde{x}$-coordinate, the specific vorticity $\widetilde{\zeta}$ is purely transported by $\widetilde{u}+\widetilde{v}$. From (5.19), (5.23) and the estimate (5.10) of $|f|$, we can deduce that $|\widetilde{u}+\widetilde{v}|\lesssim M^{\frac14}$ on $\{|\widetilde{x}_1|\le 10\varepsilon^{\frac12},|\widetilde{x}_2|\le 10\varepsilon^{\frac16}\}\supset B_{\tilde{x}}(0,\varepsilon^{\frac34})$. Note that $|T_*-t_0|=|T_*+\varepsilon|\lesssim\varepsilon$. Hence if $\widetilde{\zeta}(\widetilde{x},t_0)\ge c_0$ for some $c_0>0$ on $B_{\tilde{x}}(0,\varepsilon^{\frac34})$, then $\widetilde{\zeta}(\widetilde{x},t)\ge\frac{c_0}{2}$ on $B_{\tilde{x}}\Big(0,\frac{\varepsilon^{\frac34}}{2}\Big)$, upon taking ε to be sufficiently small.

From the bootstrap assumptions and (8.8) we have that

$$\Big|S-\frac{\kappa_0}{2}\Big|\lesssim|\kappa-\kappa_0|+\mathrm{e}^{-\frac{s}{2}}|W|+|Z|\lesssim M\varepsilon+\varepsilon^{\frac16}\lesssim\varepsilon^{\frac16}.$$

Thus the sound speed $\widetilde{\sigma}\ge\frac{\kappa_0}{4}$, and $|\widetilde{\omega}|=|\widetilde{\zeta}||\widetilde{\rho}|=|\zeta|(\alpha|\sigma|)^{\frac1\alpha}\ge\frac{c_0}{2}\cdot\left(\frac{\alpha\kappa_0}{4}\right)^{\frac1\alpha}$ on $B_{\tilde{x}}\Big(0,\frac{\varepsilon^{\frac34}}{2}\Big)$.

The initial conditions stated in subsection 3.1 can not rule out the possibility that $\widetilde{\zeta}(\widetilde{x}, t_0)$ have a positive lower bound on $B_{\widetilde{x}}\left(0, \varepsilon^{\frac{3}{4}}\right)$, thus there do exist solutions satisfying the listed initial condition and present non-zero vorticity at the blow-up point.

Data availability statement Data sharing is not applicable to this article as no new data were created or analyzed in this study.

A Toy model of 1D Burgers profile

Consider the following Cauchy problem for the 1D Burgers equation:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = u_0(x) := -x\mathrm{e}^{-x^2}. \end{cases} \tag{A.1}$$

It is well-known for the Burgers equation that the blow-up time is

$$T = -\frac{1}{\inf\limits_{x\in\mathbb{R}} \partial_x u_0} = 1,$$

the blow-up point is $(x,t) = (0,1)$, and

$$\|\partial_x u(\cdot,t)\|_{L^\infty} \le \frac{1}{1-t}. \tag{A.2}$$

Now we claim that $\frac{1}{\sqrt{1-t}}u((1-t)^{\frac{3}{2}}y, t)$ converges uniformly to a profile (a fixed stationary function) $\overline{U}(y)$ on any compact set as $t \to 1$. This fact characterizes the blow-up behavior of u. We can formally write this fact as

$$u(x,t) \sim (1-t)^{\frac{1}{2}}\overline{U}((1-t)^{-\frac{3}{2}}x) \quad \text{as } t \to 1. \tag{A.3}$$

To closely investigate this fact, we use the "self-similar transformation" $y = (1-t)^{-\frac{3}{2}}x$. y is a "zoom-in" version of x in the sense that any compact set of y corresponds to a set of x that converging to 0. Thus in y-coordinate we can observe the behavior of u near the blow-up point in detail as $t \to 1$.

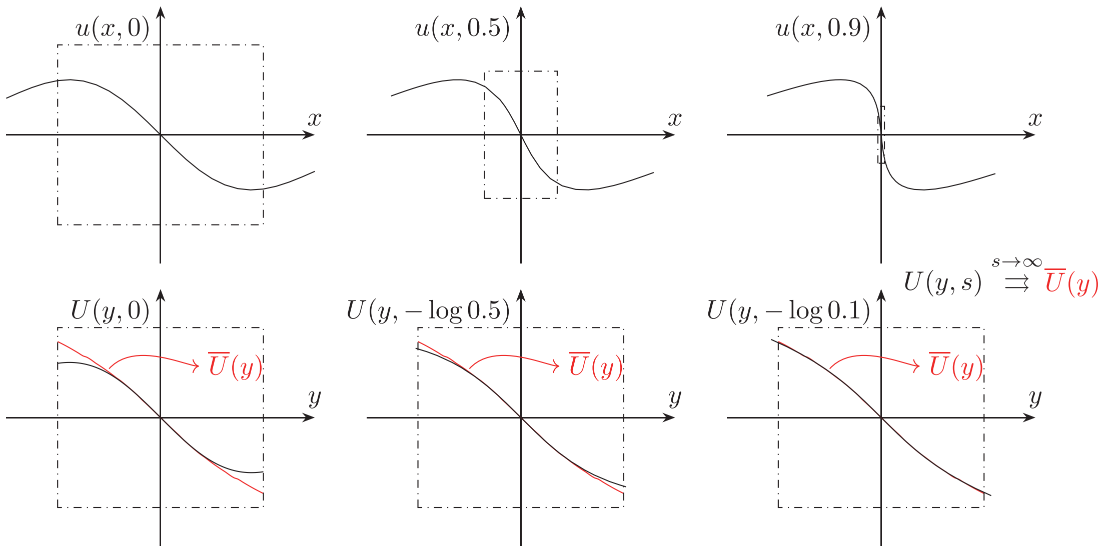
For the sake of convenience we introduce the self-similar time $s = -\log(1-t)$, thus $1-t = \mathrm{e}^{-s}$. This "self-similar time" has the advantage that $t \to 1$ is equivalent to $s \to \infty$. Now the self-similar transformation becomes $y = \mathrm{e}^{\frac{3}{2}s}x$ and we can rewrite $\frac{1}{\sqrt{1-t}}u((1-t)^{\frac{3}{2}}y, t)$ in the self-similar coordinate as

$$U(y,s) := \frac{1}{\sqrt{1-t}}u((1-t)^{\frac{3}{2}}y, t) = \mathrm{e}^{\frac{s}{2}}u(\mathrm{e}^{-\frac{3}{2}s}y, 1-\mathrm{e}^{-s}). \tag{A.4}$$

In this coordinate, the proposition we claimed becomes

$$\boxed{U(y,s) \overset{s\to\infty}{\rightrightarrows} \overline{U}(y) \quad y \in K \text{ for all compact } K.} \tag{A.5}$$

The following figures show the graphs of U and how U converges to $\overline{U}$:



We now prove the convergence. Firstly, from (A.2) and the self-similar transformation we have that

$$\|\partial_y U(\cdot, s)\|_{L^\infty} \le 1. \tag{A.6}$$

From chain rule we can deduce that $U(s, y)$ satisfies

$$\begin{cases} \Big(\partial_s - \dfrac{1}{2}\Big) U + \Big(\dfrac{3}{2} y + U\Big)\partial_y U = 0, \\ U(y, 0) = U_0(y) = u_0(y) = -y\mathrm{e}^{-y^2}. \end{cases} \tag{A.7}$$

Ignoring the time-dependent term in the above equation, we have

$$-\frac{1}{2} W + \Big(\frac{3}{2} y + W\Big)\partial_y W = 0, \tag{A.8}$$

which is called the self-similar Burgers equation. Using ODE techniques we can find a first integral of this equation: $y = -W_C(y) - CW_C(y)^3$. If we impose the constraint $W_C'''(0) = 6 = u_0'''(0)$, then C must be 1. We select $\overline{U}$ to be the function that implicitly determined by the identity $y = -\overline{U}(y) - \overline{U}(y)^3$, the solution of this cubic equation is

$$\overline{U}(y) = \Big(-\frac{y}{2} + \Big(\frac{1}{27} + \frac{y^2}{4}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{3}} - \Big(\frac{y}{2} + \Big(\frac{1}{27} + \frac{y^2}{4}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{3}}. \tag{A.9}$$

One can verify that $\overline{U}(0) = U_0(0)$, $\overline{U}'(0) = U_0'(0)$, $\overline{U}''(0) = U_0''(0)$, and $\overline{U}'''(0) = U_0'''(0)$. Thus we can check by the above explicit expression of $\overline{U}$ that

$$|\overline{U}(y) - U_0(y)| = |\overline{U}(y) + y\mathrm{e}^{-y^2}| \le My^4 \tag{A.10}$$

holds for some $M > 0$ and $-1 \le \overline{U}_y \le 0$.

Now we are ready to prove (A.5). Define $\widetilde{U}(y)=U(y)-\overline{U}(y)$. Then subtracting (A.8) from (A.7), we have

$$\begin{cases}\partial_s\widetilde{U}-\widetilde{D}\widetilde{U}+\Big(\dfrac{3}{2}y+U\Big)\widetilde{U}_y=0,\\ \widetilde{D}=\dfrac{1}{2}-\overline{U}_y,\\ \widetilde{U}(y,0)=\widetilde{U}_0(y):=u_0(y)-\overline{U}(y).\end{cases}\tag{A.11}$$

Notice that (A.7) is a transport equation. We define its Lagrange trajectories by

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}s}\Phi_{y_0}(s)=\Big(\dfrac{3}{2}y+U\Big)\circ\Phi_{y_0}(s),\\ \Phi_{y_0}(0)=y_0.\end{cases}\tag{A.12}$$

From (A.6) we have $|U(y)|\le|y|$, and $\Big(\frac{3}{2}y+U\Big)\cdot y\ge\frac{1}{2}y^2$. Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\Phi_{y_0}(s)|^2=\Phi_{y_0}(s)\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{y_0}(s)=\Big[\Big(\frac{3}{2}y+U\Big)\cdot y\Big]\circ\Phi_{y_0}(s)\ge\frac{1}{2}|\Phi_{y_0}(s)|^2.\tag{A.13}$$

If $\Phi_{y_0}(s)=y$, from the above inequality we have $\mathrm{e}^{-s}|y|^2\ge|y_0|^2$. Rewriting (A.11) along the Lagrange trajectories, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{U}\circ\Phi_{y_0}(s)=\Big(\frac{1}{2}-\overline{U}_y\Big)\circ\Phi_{y_0}(s)\cdot\widetilde{U}\circ\Phi_{y_0}(s).\tag{A.14}$$

From $-1\le\overline{U}_y\le0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|\widetilde{U}\circ\Phi_{y_0}(s)|\le\frac{3}{2}|\widetilde{U}\circ\Phi_{y_0}(s)|.\tag{A.15}$$

Thus we can conclude that

$$\begin{aligned}|\widetilde{U}(y,s)|&=|\widetilde{U}\circ\Phi_{y_0}(s)|\\ &\overset{\text{(A.15)}}{\le}\mathrm{e}^{\frac{3}{2}s}|\widetilde{U}\circ\Phi_{y_0}(0)|\\ &=\mathrm{e}^{\frac{3}{2}s}|\widetilde{U}(y_0,0)|\\ &\le\mathrm{e}^{\frac{3}{2}s}My_0^4\\ &\overset{\text{(A.10)}}{\le}M\mathrm{e}^{-\frac{s}{2}}y^4.\end{aligned}\tag{A.16}$$

From this inequality we know that $\widetilde{U}$ converge to 0 uniformly on any compact set, or equivalently it holds that $U\rightrightarrows\overline{U}$ on any compact set.

Though we prove the convergence in the case of a specific initial datum, the proof can be modified to apply to almost all initial data. In fact, take any $u_0\in C_c^\infty$, there exists a point $x_0\in\mathbb{R}$ and an integer $k\ge1$, such that $u_0'(x_0)=\inf\limits_{x\in\mathbb{R}}u_0'(x)$, and $u_0^{(j)}(x_0)=0$ holds for $2\le j\le2k$, while $u_0^{(2k+1)}(x_0)>0$. In this case, a rescaled version of the solution u will eventually converge to a solution $\overline{U}$ of the self-similar Burgers equation $-\frac{1}{2k}\overline{U}(y)+\Big[\Big(1+\frac{1}{2k}\Big)y+\overline{U}(y)\Big]\overline{U}_y(y)=0$. In this sense, the self-similar Burgers equation plays a universal role in the blow-up of the Burgers equation.

B Interpolation

Here we state the interpolation inequalities that are used in this paper.

Lemma B.1 (Gagliardo-Nirenberg inequalities) *Suppose that* $1 \leq q, r \leq \infty$, $1 \leq p < \infty$, $j < m$ *are non-negative integers,* $\theta \in [0,1]$, *and they satisfy the relations*

$$\frac{1}{p} = \frac{j}{n} + \theta\Big(\frac{1}{r} - \frac{m}{n}\Big) + \frac{1-\theta}{q}, \qquad \frac{j}{m} \leq \theta \leq 1. \tag{B.1}$$

Then $\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^{\theta}_{L^r(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}$ *holds for any* $u \in L^q(\mathbb{R}^n)$ *such that* $D^m u \in L^r(\mathbb{R}^n)$, *with two exceptional cases:*

(1) *If* $j = 0$, $q = \infty$ *and* $rm < n$, *then an additional assumption is needed: Either* u *tends to* 0 *at infinity, or* $u \in L^s(\mathbb{R}^n)$ *for some finite value of* s.

(2) *If* $r > 1$ *and* $m - j - \frac{n}{r}$ *is a non-negative integer, then the additional assumption* $\frac{j}{m} \leq \theta < 1$ *is needed.*

A frequently used special case in this paper is that

$$\|D^j \varphi\|_{L^{\frac{2m}{j}}(\mathbb{R}^n)} \lesssim \|\varphi\|^{\frac{j}{m}}_{\dot{H}^m(\mathbb{R}^n)}\|\varphi\|^{1-\frac{j}{m}}_{L^\infty(\mathbb{R}^n)} \tag{B.2}$$

holds for any $u \in \dot{H}^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Lemma B.2 *Suppose that* $k \geq 4$, $0 \leq l \leq k-3$ *are integers,* $q \in (4, 2(k-1)]$, *then*

$$\big\|D^{2+l}\phi D^{k-1-l}\varphi\big\|_{L^2(\mathbb{R}^2)} \lesssim_{k,q} \|D^k\phi\|^a_{L^2(\mathbb{R}^2)}\|D^2\phi\|^{1-a}_{L^q(\mathbb{R}^2)}\|D^k\varphi\|^b_{L^2(\mathbb{R}^2)}\|D^2\varphi\|^{1-b}_{L^q(\mathbb{R}^2)} \tag{B.3}$$

holds for any $\phi, \varphi \in \dot{H}^k(\mathbb{R}^2) \cap \dot{W}^{2,q}(\mathbb{R}^2)$, *where* a, b *are given by*

$$a = \frac{\frac{1}{q} - \frac{1}{p} + \frac{l}{1}}{\frac{k}{2} + \frac{1}{q} - \frac{3}{2}}, \quad b = \frac{\frac{1}{q} - \frac{1}{2} + \frac{1}{p} + \frac{k-3-l}{2}}{\frac{k}{2} + \frac{1}{q} - \frac{3}{2}}, \tag{B.4}$$

and

$$p = \frac{2q(k-3)}{(q-3)l + 2(k-3)}.$$

Moreover, we have that

$$a + b = 1 - \frac{\frac{1}{2} - \frac{1}{q}}{\frac{k-3}{2} + \frac{1}{q}} \in (0,1)$$

is independent of l.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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