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Approximations to Isentropic Planar Magneto-Hydrodynamics Equations by Relaxed Euler-Type Systems*

Yachun LI[1] Zhaoyang SHANG[2] Chenmu WANG[3] Liang ZHAO[4]

Abstract In this paper, the authors consider an approximation to the isentropic planar Magneto-hydrodynamics (MHD for short) equations by a kind of relaxed Euler-type system. The approximation is based on the generalization of the Maxwell law for non-Newtonian fluids together with the Maxwell correction for the Ampère law, hence the approximate system becomes a first-order quasilinear symmetrizable hyperbolic systems with partial dissipation. They establish the global-in-time smooth solutions to the approximate Euler-type equations in a small neighbourhood of constant equilibrium states and obtain the global-in-time convergence towards the isentropic planar MHD equations. In addition, they also establish the global-in-time error estimates of the limit based on stream function techniques and energy estimates for error variables.

Keywords Planar MHD equations, Relaxation limits, Global convergence, Stream function

2000 MR Subject Classification 35B25, 35L45, 35K45, 76W05

1 Introduction

Approximations of second-order parabolic equations by first-order hyperbolic equations have a long time history which dates back to the studies of Maxwell [24] in 1860s. Since then, there are a lot of studies concerning this topic. These approximations have not only the mathematical sense but also physical interpretations. The idea of these approximations can be explained by the following Cattaneo law for the heat equation $\partial_t\theta - \Delta\theta = 0$, which can be derived by the first law of thermodynamics $\partial_t\theta + \operatorname{div} q = 0$ together with the Fourier law of heat conduction

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[1]School of Mathematical Sciences, CMA-Shanghai, MOE-LSC and SHL-MAC, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: ycli@sjtu.edu.cn
[2]School of Finance, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China; School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: shangzhaoyang@sjtu.edu.cn
[3]School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; School of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China. E-mail: chenmuwang@sjtu.edu.cn
[4]Mathematical Modelling & Data Analytics Center, Oxford Suzhou Centre for Advanced Research, Suzhou 215123, Jiangsu, China. E-mail: liang.zhao@oxford-oscar.cn
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$q = -\nabla\theta$. The Cattaneo law perturbs Fourier law as

$$\varepsilon\partial_t q + q = -\nabla\theta,$$

then the heat equation becomes

$$\begin{cases} \partial_t\theta + \operatorname{div} q = 0, \\ \varepsilon\partial_t q + q = -\nabla\theta, \end{cases} \tag{1.1}$$

which is a symmetric first order hyperbolic system with partial dissipation for q. Taking $\varepsilon \to 0$, we recover the classical heat equation. The parameter $\varepsilon > 0$ is usually called the relaxation time and the limit $\varepsilon \to 0$ is the relaxation limit in the sense that the Fourier law is regarded as the stationary state of the system and zero relaxation means the recovery to the equilibrium. The hyperbolic structure of the relaxed system (1.1) is more physical since it avoids the major paradox of the heat equation that the heat waves are with infinite propagation speed. We refer the reader to [2–3] for the Cattaneo law for heat conduction, to [1, 25, 30, 32, 41] for approximation of the incompressible Navier-Stokes with Oldroyd-type derivatives describing non-Newtonian fluids, and to [6, 31] for the approximation of the Timoshenko-Fourier system by the Timoshenko-Cattaneo system.

The main purpose of the present paper is to approximate the planar Magneto-hydrodynamic (MHD for short) equations with relaxed hyperbolic systems. We start with the three-dimensional compressible MHD equations of the form (see [4, 16]),

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) = (\nabla\times\mathbf{B})\times\mathbf{B} + \operatorname{div}\Pi, \\ \partial_t\mathbf{B} - \nabla\times(\mathbf{u}\times\mathbf{B}) = -\nabla\times(\nu\nabla\times\mathbf{B}), \quad \operatorname{div}\mathbf{B} = 0, \end{cases} \tag{1.2}$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}^3$ is the spatial variable and $t \geq 0$ is the time variable. The unknown variables ρ is the density, $\mathbf{u} \in \mathbb{R}^3$ is the velocity and $\mathbf{B} \in \mathbb{R}^3$ is the magnetic field. The pressure $p(\rho)$ is sufficiently smooth and strictly increasing for all $\rho > 0$. The viscous stress tensor Π is given by

$$\Pi = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) + \lambda'(\operatorname{div}\mathbf{u})\mathbb{I}_3, \tag{1.3}$$

where $\nabla\mathbf{u}^{\mathrm{T}}$ is the transpose of the matrix $\nabla\mathbf{u}$, $\mathbb{I}_3$ is the 3×3 identity matrix and the viscosity coefficients μ and λ' satisfy

$$\mu > 0, \quad 2\mu + 3\lambda' \geq 0.$$

The parameter $\nu > 0$ is the magnetic diffusion coefficient.

Now we consider the planar MHD equation by assuming that the fluid moves in the $\mathbf{x}_1$ direction and is uniform in the transverse direction $(\mathbf{x}_2, \mathbf{x}_3)$. Denote $x = \mathbf{x}_1$, then

$$\rho = \rho(t,x), \quad \mathbf{u} = (u, \mathbf{w}^{\mathrm{T}})^{\mathrm{T}}(t,x), \quad \mathbf{B} = (b, \mathbf{b}^{\mathrm{T}})^{\mathrm{T}}(t,x), \tag{1.4}$$

where $\mathbf{w} = (u_2, u_3)^{\mathrm{T}}$ is the transverse velocity and $\mathbf{b} = (b_2, b_3)^{\mathrm{T}}$ is the transverse magnetic field. The first components u and b are the longitudinal velocity and longitudinal magnetic field, respectively. Since $\partial_x b = 0$ for all t, we may let $b = 1$ without loss of generality. After direct calculations, (1.2) can be reduced to

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 + p(\rho) + \frac{1}{2}|\mathbf{b}|^2\Big) = \partial_x\left(\lambda \partial_x u\right), \\ \partial_t(\rho \mathbf{w}) + \partial_x(\rho u \mathbf{w} - \mathbf{b}) = \partial_x(\mu \partial_x \mathbf{w}), \\ \partial_t \mathbf{b} + \partial_x(u\mathbf{b} - \mathbf{w}) = \partial_x(\nu \partial_x \mathbf{b}) \end{cases} \tag{1.5}$$

with $\lambda = \lambda^{'} + 2\mu > 0$. In addition, if we introduce $v = \rho^{-1}$, define the Lagrangian variables (y, t') by

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} = v\frac{\partial}{\partial x},$$

and still denote t' by t in the following, (1.5) in Euler coordinates (x, t) can be transformed into the following planar MHD equations in Lagrangian coordinates (y, t)

$$\begin{cases} \partial_t v - \partial_y u = 0, \\ \partial_t u + \partial_y\Big(p(v) + \frac{1}{2}|\mathbf{b}|^2\Big) = \partial_y\Big(\frac{\lambda \partial_y u}{v}\Big), \\ \partial_t \mathbf{w} - \partial_y \mathbf{b} = \partial_y\Big(\frac{\mu \partial_y \mathbf{w}}{v}\Big), \\ \partial_t(v\mathbf{b}) - \partial_y \mathbf{w} = \partial_y\Big(\frac{\nu \partial_y \mathbf{b}}{v}\Big), \end{cases} \tag{1.6}$$

where p becomes a function of v with $p'(v) < 0$.

There are rich literatures on the global well-posedness of the planar MHD equations (1.5) as well as (1.6). When the initial density is strictly positive, the MHD equations are of mixed hyperbolic-parabolic type in the sense of Shizuta-Kawashima [13, 34], so that the global existence of smooth solutions near constant equilibrium is guaranteed. For other related well-posedness results, we refer to [18, 22, 26, 37–38, 40]. For the results of non-isentropic planar MHD equations, we refer to [5, 7, 14, 17, 21, 33, 36, 39] and the references cited therein.

We now introduce the approximate system. Let

$$\tau = \frac{\sqrt{\lambda}\partial_y u}{v}, \quad \mathbf{S} = \frac{\sqrt{\mu}\partial_y \mathbf{w}}{v}, \quad \mathbf{J} = \frac{\sqrt{\nu}\partial_y \mathbf{b}}{v}. \tag{1.7}$$

We approximate the perturbed form of (1.7) by introducing the following constitution laws

$$\varepsilon_1^2(\partial_t \tau + u\partial_y \tau) + \tau = \frac{\sqrt{\lambda}\partial_y u}{v}, \tag{1.8}$$

$$\varepsilon_2^2(\partial_t \mathbf{S} + u\partial_y \mathbf{S}) + \mathbf{S} = \frac{\sqrt{\mu}\partial_y \mathbf{w}}{v}, \tag{1.9}$$

$$\varepsilon_3^2(\partial_t \mathbf{J} + u\partial_y \mathbf{J}) + \mathbf{J} = \frac{\sqrt{\nu}\partial_y \mathbf{b}}{v}, \tag{1.10}$$

where $\varepsilon_i > 0$ $(i = 1, 2, 3)$ are relaxation times. Denote $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^{\mathrm{T}}$ and $\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}$. Combining (1.6)–(1.10), we establish the relaxed Euler-type approximations of the planar MHD equations as follows

$$\begin{cases} \partial_t v - \partial_y u = 0, \\ \partial_t u + \partial_y \Big(p(v) + \dfrac{1}{2}|\mathbf{b}|^2 \Big) = \sqrt{\lambda}\partial_y \tau, \\ \partial_t \mathbf{w} - \partial_y \mathbf{b} = \sqrt{\mu}\partial_y \mathbf{S}, \\ \partial_t \mathbf{b} + \dfrac{\partial_y u}{v}\mathbf{b} - \dfrac{1}{v}\partial_y \mathbf{w} = \dfrac{\sqrt{\nu}\partial_y \mathbf{J}}{v}, \\ \varepsilon_1^2(\partial_t \tau + u\partial_y \tau) + \tau = \dfrac{\sqrt{\lambda}\partial_y u}{v}, \\ \varepsilon_2^2(\partial_t \mathbf{S} + u\partial_y \mathbf{S}) + \mathbf{S} = \dfrac{\sqrt{\mu}\partial_y \mathbf{w}}{v}, \\ \varepsilon_3^2(\partial_t \mathbf{J} + u\partial_y \mathbf{J}) + \mathbf{J} = \dfrac{\sqrt{\nu}\partial_y \mathbf{b}}{v}, \end{cases} \tag{1.11}$$

which is a first-order quasilinear hyperbolic system with initial data

$$(v, u, \mathbf{w}, \mathbf{b}, \tau, \mathbf{S}, \mathbf{J})|_{t=0} = (v_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \tau_0, \mathbf{S}_0, \mathbf{J}_0)(x). \tag{1.12}$$

The variables $\tau, \mathbf{S}$ and $\mathbf{J}$ are dissipative variables because there are damping terms in their corresponding equations. Formally letting $\varepsilon \to 0$ recovers the planar MHD equations (1.6).

It is necessary to show that the approximations (1.8)–(1.10) are physical. Constitutive laws (1.8)–(1.9) are approximations of the non-Newtonian fluids. In [24], Maxwell combined Newton's law of viscosity with Hooke's law of elasticity, and proposed a modification to the constitutive law of stress tensor (1.3) as follows

$$\varepsilon^2 \partial_t \Pi + \Pi = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) + \lambda'(\operatorname{div} \mathbf{u})\mathbb{I}_3.$$

However, the above law is not invariant under the following Galilean transformation

$$t' = t, \quad \mathbf{x}' = \mathbf{x} - Vt, \quad \Pi' = \Pi, \quad \mathbf{u}' = \mathbf{u} - V, \quad \forall\, V \in \mathbb{R}^3,$$

which may lead to paradoxical evolution of particles in a moving frame. To overcome it, the material derivative should be considered. The model then reads

$$\varepsilon^2 \left(\partial_t \Pi + (\mathbf{u} \cdot \nabla)\Pi) \right) + \Pi = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) + \lambda'(\operatorname{div} \mathbf{u})\mathbb{I}_3, \tag{1.13}$$

which reduces to approximations (1.8)–(1.9) for our case. The constitutive law (1.10) is actually the combination of the Ampère law together with the Maxwell correction of the Ampère law. Indeed, by the Ampère law with Maxwell correction, we have

$$\varepsilon \partial_t E - \nabla \times B = \mathbf{J},$$

where E is the electric field and $\mathbf{J}$ is the current density. We combine the above with the Ohm's law yielding

$$\varepsilon\partial_t\mathbf{J}+\mathbf{J}=\nabla\times B+\text{l.o.t},$$

which reduces to approximation (1.10) in our case. For more details, we refer to [1, 25, 30, 32] and references cited therein for more details.

System (1.11) can be regarded as symmetrizable hyperbolic (see Section 2 below) and consequently the local existence of smooth solutions is guaranteed by classical theories (see [12, 15, 23]). The global existence of smooth solutions and its convergence to the classical isentropic planar MHD equations (1.6) remain open as far as we know. However, for the isentropic Navier-Stokes equations with revised Maxwell constitutive laws, which is also called hyperbolic Navier-Stokes equations, there are rich literatures. In 2014, Yong [41] obtained the local existence and the local convergence to the classical isentropic Navier-Stokes equations under condition $\mathrm{tr}(\Pi)=0$, where $\mathrm{tr}(\Pi)$ means the trace of matrix Π. In 2021, Peng [27] constructed approximate systems with vector variables instead of tensor variables by using Hurwitz-Radon matrices in both compressible and incompressible cases, and established the uniform estimates with respect to ε_1 and ε_2 of the global smooth solutions near constant equilibrium state and the global-in-time convergence of the systems towards classical isentropic Navier-Stokes equations. He also obtained similar results for the isentropic Navier-Stokes equations with Maxwell constitutive law without condition $\mathrm{tr}(\Pi)=0$. For the results of non-isentropic Navier-Stokes equations with related Maxwell and Cattaneo constitutive law, we refer to [8–11, 30] and the references cited therein.

The main purposes of this paper is to prove the global existence of smooth solutions to Cauchy problem (1.11)–(1.12) near constant equilibrium states and establish the global-in-time convergence rates towards the classical isentropic planar MHD equations. The existence of the global-in-time smooth solutions is based on the uniform estimates of the local-in-time smooth solutions with respect to time and small parameters $\vec{\varepsilon}=(\varepsilon_1,\varepsilon_2,\varepsilon_3)^{\mathrm{T}}$ together with the classical bootstrap arguments. The global-in-time convergence of solutions under the limit $\varepsilon\to 0$ is based on the uniform estimates and some compactness arguments. Remark that our system can not be included in the studies of [19, 28–29, 42] in that the structure of our limiting system is different from those of systems in these mentioned results. More precisely, our limiting system is a mixed hyperbolic-parabolic type in the sense of Shizuta-Kawashima rather than a parabolic system and violates the condition (e.g., [19, (**A3**)) needed for deriving a parabolic limiting system. It is worth mentioning that the choice of $\varepsilon_1,\varepsilon_2$ and ε_3 can be made independently, hence our results indeed include some partial limits result, for example if we let $\varepsilon_3\to 0$ with ε_1 and ε_2 fixed, we can get the MHD equations for non-Newtonian fluids.

The major difficulty of this paper is to obtain the global-in-time convergence rates, i.e., to establish the global-in-time error estimates between the smooth solution to the relaxed time system and those to the planar MHD limiting system. The proof is based on the stream function techniques together with the energy estimates of the error variables. We first use energy methods to obtain directly the error estimates for dissipative variables $\tau,\mathbf{S},\mathbf{J}$. For the non-dissipative variables $v, u, \mathbf{w}, \mathbf{b}$, we have to adopt the stream function techniques to establish the error estimates. It is worth emphasizing that our treatments are different from those in [19–20, 42] in that the energy estimates of the error variables for dissipative variables $\tau, \mathbf{S}, \mathbf{J}$ can not be decoupled from the stream function estimates for non-dissipative variables. Consequently, careful combinations of estimates are needed.

This paper is organized as follows. In Section 2, we introduce preliminaries and state our main results. In Section 3, we establish the uniform estimate of smooth solutions near the equilibrium state with respect to the time and small parameters $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^{\mathrm{T}}$ and prove the convergence of the relaxed system towards the planar MHD system. Section 4 is devoted to the proof of the global-in-time convergence rate.

2 Preliminaries and Main Results

2.1 Notations and inequalities

For later purpose, we introduce the following notations. We denote $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_s$ the generic norms of $L^2 \overset{\text{def}}{=} L^2(\mathbb{K})$, $L^\infty \overset{\text{def}}{=} L^\infty(\mathbb{K})$ and $H^s \overset{\text{def}}{=} H^s(\mathbb{K})$, respectively, with $\mathbb{K} = \mathbb{R}$ for Cauchy problem and $\mathbb{K} = \mathbb{T}$ for periodic problem. Moreover, $\langle \cdot, \cdot \rangle$ stands for the inner product of L^2. In the following, we require that $s \geq 2$ is a positive integer and $C > 0$ is a generic constant independent of $\varepsilon_i (i = 1, 2, 3)$ and any time.

Next, we introduce the Moser-type calculus inequalities, which will be frequently used in later proof.

Lemma 2.1 (Moser-type calculus inequalities, see [23]) *Let $s \geq 2$ be an integer and $1 \leq l \leq s$. Then it holds*

$$\|\partial_y^l(uv) - u\partial_y^l v\| \leq C\|\partial_y u\|_{s-1}\|v\|_{l-1}, \quad \|\partial_y^l(uv)\| \leq C\|u\|_s\|v\|_l.$$

For periodic problems, we need to introduce the following notation. For a given scalar or vector function, we denote its mean value over the torus as

$$\mathcal{M}(g(t,x))(t) = \int_{\mathbb{T}} g(t,x)\mathrm{d}x.$$

2.2 Symmetrizable hyperbolicity

System (1.11) can be rewritten into the following

$$D_0(\varepsilon)\partial_t W + A(W)\partial_y W = -Q(W), \tag{2.1}$$

where

$$W = (v-1, u, \mathbf{w}^{\mathrm{T}}, \mathbf{b}^{\mathrm{T}}, \tau, \mathbf{S}^{\mathrm{T}}, \mathbf{J}^{\mathrm{T}})^{\mathrm{T}},\ D_0(\varepsilon) = \mathbf{diag}(\mathbb{I}_6, \varepsilon_1^2, \varepsilon_2^2\mathbb{I}_2, \varepsilon_3^2\mathbb{I}_2),\ Q(W) = (0_{1\times 6}, \tau, \mathbf{S}^{\mathrm{T}}, \mathbf{J}^{\mathrm{T}})^{\mathrm{T}},$$

where $\mathbb{I}_d$ is the $d \times d$ unit matrix, T is the transpose of a vector or a matrix and

$$A(W) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ p'(v) & 0 & 0 & \mathbf{b}^{\mathrm{T}} & -\sqrt{\lambda} & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_2 & 0 & -\sqrt{\mu}\mathbb{I}_2 & 0 \\ 0 & \dfrac{\mathbf{b}}{v} & -\dfrac{1}{v}\mathbb{I}_2 & 0 & 0 & 0 & -\dfrac{\sqrt{\nu}}{v}\mathbb{I}_2 \\ 0 & -\dfrac{\sqrt{\lambda}}{v} & 0 & 0 & \varepsilon_1^2 u & 0 & 0 \\ 0 & 0 & -\dfrac{\sqrt{\mu}}{v}\mathbb{I}_2 & 0 & 0 & \varepsilon_2^2 u\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & -\dfrac{\sqrt{\nu}}{v}\mathbb{I}_2 & 0 & 0 & \varepsilon_3^2 u\mathbb{I}_2 \end{pmatrix}.$$

Now if we introduce the following symmetrizer

$$A_0(W) = \mathrm{diag}(-p'(v), 1, \mathbb{I}_2, v\mathbb{I}_2, v, v\mathbb{I}_2, v\mathbb{I}_2),$$

such that

$$\widetilde{A}_0(W) := A_0(W)D_0(\varepsilon) = \mathrm{diag}(-p'(v), 1, \mathbb{I}_2, v\mathbb{I}_2, \varepsilon_1^2 v, \varepsilon_2^2 v\mathbb{I}_2, \varepsilon_3^2 v\mathbb{I}_2)$$

and

$$\widetilde{A}(W) := A_0(W)A(W) = \begin{pmatrix} 0 & p'(v) & 0 & 0 & 0 & 0 & 0 \\ p'(v) & 0 & 0 & \mathbf{b}^{\mathrm{T}} & -\sqrt{\lambda} & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{I}_2 & 0 & -\sqrt{\mu}\mathbb{I}_2 & 0 \\ 0 & \mathbf{b} & -\mathbb{I}_2 & 0 & 0 & 0 & -\sqrt{\nu}\mathbb{I}_2 \\ 0 & -\sqrt{\lambda} & 0 & 0 & \varepsilon_1^2 vu & 0 & 0 \\ 0 & 0 & -\sqrt{\mu}\mathbb{I}_2 & 0 & 0 & \varepsilon_2^2 vu\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & -\sqrt{\nu}\mathbb{I}_2 & 0 & 0 & \varepsilon_3^2 vu\mathbb{I}_2 \end{pmatrix}.$$

It is clear that $A_0(W)$ is symmetric and positive definite and $\widetilde{A}(W)$ is symmetric, so that the relaxed system (1.11) is a first-order quasi-linear symmetrizable hyperbolic system, to which the local existence of smooth solutions is guaranteed by classical theories (see [12, 15, 23]).

For later purpose, we need to split the giant matrix $A(W)$ into several partitioned matrices. Let $W = (W_1^{\mathrm{T}}, W_2^{\mathrm{T}})^{\mathrm{T}}$ with

$$W_1 = (v-1, u, \mathbf{w}^{\mathrm{T}}, \mathbf{b}^{\mathrm{T}})^{\mathrm{T}} \quad \text{and} \quad W_2 = (\tau, \mathbf{S}^{\mathrm{T}}, \mathbf{J}^{\mathrm{T}})^{\mathrm{T}}. \tag{2.2}$$

We call W_1 non-dissipative variables and W_2 dissipative variables in that it admits damping structures in $Q(W)$. In addition, we introduce the partitioned matrices defined by

$$A_0(W) := \mathrm{diag}(A_0^{11}(W), A_0^{22}(W)), \quad A(W) := \begin{pmatrix} A^{11}(W) & A^{12}(W) \\ A^{21}(W) & A^{22}(W) \end{pmatrix}$$

with

$$A_0^{11}(W) = \mathrm{diag}\left(-p'(v), 1, \mathbb{I}_2, v\mathbb{I}_2\right), \quad A_0^{22}(W) = \mathrm{diag}\left(v, v\mathbb{I}_2, v\mathbb{I}_2\right),$$

$$A^{11}(W) := \begin{pmatrix} 0 & -1 & 0 & 0 \\ p'(v) & 0 & 0 & \mathbf{b}^{\mathrm{T}} \\ 0 & 0 & 0 & -\mathbb{I}_2 \\ 0 & \dfrac{\mathbf{b}}{v} & -\dfrac{1}{v}\mathbb{I}_2 & 0 \end{pmatrix}, \quad A^{12}(W) := \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{\lambda} & 0 & 0 \\ 0 & -\sqrt{\mu}\mathbb{I}_2 & 0 \\ 0 & 0 & -\dfrac{\sqrt{\nu}}{v}\mathbb{I}_2 \end{pmatrix}$$

and

$$A^{21}(W) := \begin{pmatrix} 0 & -\frac{\sqrt{\lambda}}{v} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{\mu}}{v}\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{\nu}}{v}\mathbb{I}_2 \end{pmatrix}, \quad A^{22}(W) := \begin{pmatrix} \varepsilon_1^2 u & 0 & 0 \\ 0 & \varepsilon_2^2 u\mathbb{I}_2 & 0 \\ 0 & 0 & \varepsilon_3^2 u\mathbb{I}_2 \end{pmatrix}.$$

Clearly, $\widetilde{A}(W) = A_0(W)A(W) = \begin{pmatrix} \widetilde{A}^{11}(W) & \widetilde{A}^{12}(W) \\ \widetilde{A}^{21}(W) & \widetilde{A}^{22}(W) \end{pmatrix}$ with

$$\widetilde{A}^{11}(W) := A_0^{11}(W)A^{11}(W), \quad \widetilde{A}^{12}(W) = \widetilde{A}^{21}(W)^{\mathrm{T}} = A_0^{11}(W)A^{12}(W)$$

and

$$\widetilde{A}^{22}(W) = A_0^{22}(W)A^{22}(W).$$

2.3 Main results

The main results of this paper are as follows.

Theorem 2.1 (Global existence and uniform estimates) *Let $s \geq 2$ be an integer and $(v_0-1, u_0, \mathbf{w}_0, \mathbf{b}_0, \tau_0, \boldsymbol{S}_0, \mathbf{J}_0) \in H^s$. Then there exist two positive constants δ and C independent of ε, such that if*

$$\|v_0 - 1\|_s + \|u_0\|_s + \|\mathbf{w}_0\|_s + \|\mathbf{b}_0\|_s + \varepsilon_1\|\tau_0\|_s + \varepsilon_2\|\boldsymbol{S}_0\|_s + \varepsilon_3\|\mathbf{J}_0\|_s < \delta,$$

then for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, 1]$, the Cauchy problem (1.11)–(1.12) *admits a unique global smooth solution $(v, u, \mathbf{w}, \mathbf{b}, \tau, \boldsymbol{S}, \mathbf{J})$ satisfying*

$$\begin{aligned} &\|W_1(t)\|_s^2 + \varepsilon_1^2\|\tau(t)\|_s^2 + \varepsilon_2^2\|\boldsymbol{S}(t)\|_s^2 + \varepsilon_3^2\|\mathbf{J}(t)\|_s^2 + \int_0^t (\|\partial_y W_1(t')\|_{s-1}^2 + \|W_2(t')\|_s^2)\mathrm{d}t' \\ &\leq C(\|v_0 - 1\|_s^2 + \|u_0\|_s^2 + \|\mathbf{w}_0\|_s^2 + \|\mathbf{b}_0\|_s^2 + \varepsilon_1^2\|\tau_0\|_s^2 + \varepsilon_2^2\|\boldsymbol{S}_0\|_s^2 + \varepsilon_3^2\|\mathbf{J}_0\|_s^2), \end{aligned} \tag{2.3}$$

where W_1 and W_2 are defined in (2.2).

Theorem 2.2 (The relaxation limits) *Let $(v, u, \mathbf{w}, \mathbf{b}, \tau, \boldsymbol{S}, \mathbf{J})$ be the global solution obtained in Theorem theorem* 2.1. *If there exist functions $(\overline{v}_0, \overline{u}_0, \overline{\mathbf{w}}_0, \overline{\mathbf{b}}_0) \in H^s$ satisfying*

$$(v_0 - 1, u_0, \mathbf{w}_0, \mathbf{b}_0) \rightharpoonup (\overline{v}_0 - 1, \overline{u}_0, \overline{\mathbf{w}}_0, \overline{\mathbf{b}}_0) \quad \textit{weakly in } H^s,$$

then there exist functions $(\overline{v} - 1, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}) \in L^\infty(\mathbb{R}^+; H^s)$ and $(\overline{\tau}, \overline{\boldsymbol{S}}, \overline{\mathbf{J}}) \in L^2(\mathbb{R}^+; H^s)$, such that as $\varepsilon \to 0$, up to subsequences,

$$(v - 1, u, \mathbf{w}, \mathbf{b}) \rightharpoonup (\overline{v} - 1, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}) \quad \textit{weakly} - * \textit{ in } L^\infty(\mathbb{R}^+; H^s), \tag{2.4}$$

$$(\tau, \boldsymbol{S}, \mathbf{J}) \rightharpoonup (\overline{\tau}, \overline{\boldsymbol{S}}, \overline{\mathbf{J}}) \quad \textit{weakly in } L^2(\mathbb{R}^+; H^s), \tag{2.5}$$

where

$$\overline{\tau} = \frac{\sqrt{\lambda}\partial_y \overline{u}}{\overline{v}}, \quad \overline{\boldsymbol{S}} = \frac{\sqrt{\mu}\partial_y \overline{\mathbf{w}}}{\overline{v}}, \quad \overline{\mathbf{J}} = \frac{\sqrt{\nu}\partial_y \overline{\mathbf{b}}}{\overline{v}}$$

and $(\overline{v}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}})$ is the solution to the planar MHD system in Lagrangian coordinate (1.6).

Theorem 2.3 (Global convergence rates) *Under the conditions in Theorems* 2.1–2.2*, let* $(v, u, \mathbf{w}, \mathbf{b}, \tau, \boldsymbol{S}, \mathbf{J})$ *be the unique smooth solution to* (1.11)–(1.12) *and* $(\overline{v}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}})$ *be the unique solution to* (1.6)*. Then there exists a constant* $\delta > 0$*, independent of* ε*, such that if*

$$\|v_0 - 1\|_s + \|u_0\|_s + \|\mathbf{w}_0\|_s + \|\mathbf{b}_0\|_s + \varepsilon_1\|\tau_0\|_s + \varepsilon_2\|\boldsymbol{S}_0\|_s + \varepsilon_3\|\mathbf{J}_0\|_s < \delta,$$

and for any given positive constants α *and* C_1 *independent of* ε *satisfying*

$$\|(v_0 - \overline{v}_0, u_0 - \overline{u}_0, \mathbf{w}_0 - \overline{\mathbf{w}}_0, \mathbf{b}_0 - \overline{\mathbf{b}}_0)\|_{s-1} + \|(\varepsilon_1\tau_0, \varepsilon_2\boldsymbol{S}_0, \varepsilon_3\mathbf{J}_0)\|_{s-1} < C_1\varepsilon^{\alpha},$$

then for all $\varepsilon \in (0, 1]$*, there exists a positive constant* C_2 *independent of* ε*, such that*

$$\begin{aligned}&\|(v - \overline{v}, u - \overline{u}, \mathbf{w} - \overline{\mathbf{w}}, \mathbf{b} - \overline{\mathbf{b}}, \varepsilon_1(\tau - \overline{\tau}), \varepsilon_2(\boldsymbol{S} - \overline{\boldsymbol{S}}), \varepsilon_3(\mathbf{J} - \overline{\mathbf{J}}))(t)\|_{s-1}^2\\ &+ \int_0^t \|(v - \overline{v}, u - \overline{u}, \mathbf{w} - \overline{\mathbf{w}}, \mathbf{b} - \overline{\mathbf{b}}, \tau - \overline{\tau}, \boldsymbol{S} - \overline{\boldsymbol{S}}, \mathbf{J} - \overline{\mathbf{J}})(t')\|_{s-1}^2 \mathrm{d}t' \le C_2\varepsilon^{2\alpha_1},\end{aligned}$$

where $\alpha_1 = \min(1, \alpha)$*.*

3 Global Existence and Convergence

In this section, we establish the global uniform estimate of the smooth solutions with respect to small parameters. Let $T > 0$ and $(v, u, \mathbf{w}, \mathbf{b}, \tau, \mathbf{S}, \mathbf{J})$ be the unique local smooth solution defined on the time interval $[0, T]$. Recall the definitions of W_1 and W_2 in (2.2). We introduce the total energy as

$$\mathscr{E}(t) = \|W_1(t)\|_s^2 + \varepsilon_1^2\|\tau(t)\|_s^2 + \varepsilon_2^2\|\mathbf{S}(t)\|_s^2 + \varepsilon_3^2\|\mathbf{J}(t)\|_s^2,$$

which we assume to be sufficiently small for all $0 \le t \le T$. We also introduce the dissipative energy

$$\mathscr{D}(t) = \|\partial_y W_1(t)\|_{s-1}^2 + \|W_2(t)\|_s^2.$$

In this section, we tend to establish estimates of the following type

$$\mathscr{E}(t) + \int_0^t \mathscr{D}(t')\mathrm{d}t' \le \mathscr{E}(0).$$

The smallness of $\mathscr{E}(t)$ leads to the fact that

$$\frac{1}{2} \le v \le \frac{3}{2}, \quad p'(v) \le -p_1 \tag{3.1}$$

for $p_1 > 0$ a positive constant.

We first give the L^2-estimate.

Lemma 3.1 (L^2-estimate) *It holds*

$$\begin{aligned}&\|v(t) - 1\|^2 + \|u(t)\|^2 + \|\mathbf{w}(t)\|^2 + \|\mathbf{b}(t)\|^2 + \varepsilon_1^2\|\tau(t)\|^2 + \varepsilon_2^2\|\boldsymbol{S}(t)\|^2 + \varepsilon_3^2\|\mathbf{J}(t)\|^2\\ &+ \int_0^t (\|\tau(t')\|^2 + \|\boldsymbol{S}(t')\|^2 + \|\mathbf{J}(t')\|^2)\mathrm{d}t'\\ &\le C\int_0^t \mathscr{E}(t')^{\frac{1}{2}}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0).\end{aligned} \tag{3.2}$$

Proof Let $\eta(W)$ be denoted by

$$\eta(W)=-P(v)+\frac{1}{2}u^2+\frac{1}{2}|\mathbf{w}|^2+\frac{1}{2}v|\mathbf{b}|^2+\frac{1}{2}\varepsilon_1^2v\tau^2+\frac{1}{2}\varepsilon_2^2v|\mathbf{S}|^2+\frac{1}{2}\varepsilon_3^2v|\mathbf{J}|^2,$$

where $P'(v)=p(v)$. Then according to the system (1.11), direct calculations give

$$\begin{aligned}\partial_t\eta(W)&=-p(v)\partial_tv+u\partial_tu+\mathbf{w}\cdot\partial_t\mathbf{w}+\frac{1}{2}\partial_tv|\mathbf{b}|^2+v\mathbf{b}\cdot\partial_t\mathbf{b}+\frac{1}{2}\varepsilon_1^2\tau^2\partial_tv+\varepsilon_1^2v\tau\partial_t\tau\\&\quad+\frac{1}{2}\varepsilon_2^2\partial_tv|\mathbf{S}|^2+\varepsilon_2^2v\mathbf{S}\cdot\partial_t\mathbf{S}+\frac{1}{2}\varepsilon_3^2\partial_tv|\mathbf{J}|^2+\varepsilon_3^2v\mathbf{J}\cdot\partial_t\mathbf{J}\\&=-p(v)\partial_yu-u\partial_y(p(v))-\frac{1}{2}u\partial_y(|\mathbf{b}|^2)+\sqrt{\lambda}u\partial_y\tau+\mathbf{w}\cdot\partial_y\mathbf{b}+\sqrt{\mu}\mathbf{w}\cdot\partial_y\mathbf{S}\\&\quad+\frac{1}{2}\partial_yu|\mathbf{b}|^2-\partial_yu|\mathbf{b}|^2+\mathbf{b}\cdot\partial_y\mathbf{w}+\sqrt{\nu}\mathbf{b}\cdot\partial_y\mathbf{J}+\frac{1}{2}\varepsilon_1^2\tau^2\partial_yu\\&\quad-\varepsilon_1^2vu\tau\partial_y\tau-v\tau^2+\sqrt{\lambda}\tau\partial_yu+\frac{1}{2}\varepsilon_2^2\partial_yu|\mathbf{S}|^2-\varepsilon_2^2vu\mathbf{S}\cdot\partial_y\mathbf{S}-v|\mathbf{S}|^2\\&\quad+\sqrt{\mu}\mathbf{S}\cdot\partial_y\mathbf{w}+\frac{1}{2}\varepsilon_3^2\partial_yu|\mathbf{J}|^2-\varepsilon_3^2vu\mathbf{J}\cdot\partial_y\mathbf{J}-v|\mathbf{J}|^2+\sqrt{\nu}\partial_y\mathbf{b}\cdot\mathbf{J}.\end{aligned}$$

Then, if we denote

$$\psi(W)=-p(v)u-\frac{1}{2}u|\mathbf{b}|^2+\sqrt{\lambda}u\tau+\mathbf{w}\cdot\mathbf{b}+\sqrt{\mu}\mathbf{w}\cdot\mathbf{S}+\sqrt{\nu}\mathbf{b}\cdot\mathbf{J}$$

and the remaining terms

$$R(W)=\frac{1}{2}\varepsilon_1^2\tau^2\partial_yu-\varepsilon_1^2vu\tau\partial_y\tau+\frac{1}{2}\varepsilon_2^2\partial_yu|\mathbf{S}|^2-\varepsilon_2^2vu\mathbf{S}\cdot\partial_y\mathbf{S}+\frac{1}{2}\varepsilon_3^2\partial_yu|\mathbf{J}|^2-\varepsilon_3^2vu\mathbf{J}\cdot\partial_y\mathbf{J},$$

we obtain

$$\partial_t\eta(W)+v\tau^2+v|\mathbf{S}|^2+v|\mathbf{J}|^2=\partial_y\psi(W)+R(W).$$

Since v is close to 1, so that by the Taylor expansion of $p(v)$ at $v=1$, we obtain

$$\begin{aligned}\partial_tP(v)=\partial_t(P(v)-P(1))&=\partial_t\Big(p(1)v+\frac{1}{2}p'(\widetilde{v})(v-1)^2\Big)\\&=\partial_y(p(1)u)+\partial_t\Big(\frac{1}{2}p'(\widetilde{v})(v-1)^2\Big),\end{aligned}$$

where $\widetilde{v}$ is between v and 1. This yields

$$\partial_t\widetilde{\eta}(W)+\partial_y(-p(1)u-\psi(W))+v\tau^2+v|\mathbf{S}|^2+v|\mathbf{J}|^2=R(W),\tag{3.3}$$

where by (3.1), there exists a constant $c_0>0$, such that

$$\begin{aligned}\widetilde{\eta}(W)&\overset{\text{def}}{=}\eta(W)+P(v)-\frac{1}{2}p'(\widetilde{v})(v-1)^2\\&\geq c_0(|v-1|^2+u^2+|\mathbf{w}|^2+|\mathbf{b}|^2+\varepsilon_1^2\tau^2+\varepsilon_2^2|\mathbf{S}|^2+\varepsilon_3^2|\mathbf{J}|^2).\end{aligned}$$

Noticing that by the Cauchy-Schwarz inequality and the Moser-type calculus inequalities,

$$\Big|\int_{\mathbb{K}}R(W)\mathrm{d}y\Big|\leq C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t)$$

integrating (3.3) over $[0,t]\times\mathbb{K}$ with $t\in(0,T]$ ends the proof.

We then have the following higher order estimates.

Lemma 3.2 (Higher order estimates) *It holds*

$$\mathscr{E}(t)+\int_0^t \|W_2(t')\|_s^2\mathrm{d}t' \le C\int_0^t \mathscr{E}(t')^{\frac{1}{2}}\mathscr{D}(t')\mathrm{d}t'+C\mathscr{E}(0). \tag{3.4}$$

Proof For all integers l with $1\le l\le s$, applying ∂_y^l to both sides of (2.1), making the inner product of the resulting equation with $2A_0(W)\partial_y^l W$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \widetilde{A}_0(W)\partial_y^l W,\partial_y^l W\rangle &= \langle \partial_t\widetilde{A}_0(W)\partial_y^l W,\partial_y^l W\rangle+\langle \partial_y\widetilde{A}(W)\partial_y^l W,\partial_y^l W\rangle\\ &\quad -\langle 2A_0(W)\partial_y^l W,\partial_y^l Q(W)\rangle+\langle 2A_0(W)\partial_y^l W,J^l\rangle,\end{aligned} \tag{3.5}$$

where the commutator J^l is defined as

$$J^l=-\partial_y^l(A(W)\partial_y W)+A(W)\partial_y^{l+1}W.$$

First, noticing that

$$\|\partial_t v\|_\infty\le C\|u\|_s,$$

then from the definitions of the $D_0(\varepsilon)$, $A_0(W)$ and $\widetilde{A}_0(W)$, we get

$$|\langle \partial_t\widetilde{A}_0(W)\partial_y^l W,\partial_y^l W\rangle|\le C\|\partial_t v\|_\infty\|\partial_y^l W\|^2\le C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t).$$

Next, noticing that

$$\|\partial_y\widetilde{A}(W)\|_\infty\le C\|\partial_y W_1\|_\infty\le C\|\partial_y W_1\|_{s-1},$$

then

$$|\langle \partial_y\widetilde{A}(W)\partial_y^l W,\partial_y^l W\rangle|\le C\|\partial_y\widetilde{A}(W)\|_\infty\|\partial_y^l W\|^2\le C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t).$$

Afterwards, according to the definition of $Q(W)$, we conclude from (3.1) that

$$\langle 2A_0(W)\partial_y^l W,\partial_y^l Q(W)\rangle=\langle 2v\partial_y^l\tau,\partial_y^l\tau\rangle+\langle 2v\partial_y^l\mathbf{S},\partial_y^l\mathbf{S}\rangle+\langle 2v\partial_y^l\mathbf{J},\partial_y^l\mathbf{J}\rangle\ge\|\partial_y^l W_2\|^2.$$

Finally, we estimate the term in (3.5) containing commutator J^l. By the Moser-type calculus inequalities, we have

$$\|J^l\|\le C\|\partial_y A(W)\|_{s-1}\|\partial_y W\|_{l-1}\le C\|\partial_y W_1\|_{s-1}\|\partial_y W\|_{l-1},$$

hence

$$|\langle 2A_0(W)\partial_y^l W,J^l\rangle|\le C\|\partial_y W_1\|_{s-1}\|\partial_y W\|_{s-1}\|\partial_y^l W\|\le C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t).$$

Combining all these estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A_0(W)D_0(\varepsilon)\partial_y^l W,\partial_y^l W\rangle+\|\partial_y^l\tau\|^2+\|\partial_y^l\mathbf{S}\|^2+\|\partial_y^l\mathbf{J}\|^2\le C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t). \tag{3.6}$$

Notice that there exists a constant $c_1>0$, such that

$$c_1\|\partial_y^l V\|^2\le\langle A_0(W)D_0(\varepsilon)\partial_y^l W,\partial_y^l W\rangle\le C\|\partial_y^l V\|^2,$$

where

$$V=(v,u,\mathbf{w}^{\mathrm{T}},\mathbf{b}^{\mathrm{T}},\varepsilon_1\tau,\varepsilon_2\mathbf{S}^{\mathrm{T}},\varepsilon_3\mathbf{J}^{\mathrm{T}})^{\mathrm{T}}.$$

Integrating (3.6) over $[0,t]$ with $t\in(0,T]$, summing for all $1\le l\le s$ and combining (3.2) yield (3.4).

Next, we obtain the dissipative estimates for $\partial_y W_1$. In the following, we denote $\kappa>0$ a sufficiently small positive constant, of which the value is determined in (3.11).

Lemma 3.3 (Dissipative estimates for $\partial_y u,\partial_y\mathbf{w}$ and $\partial_y\mathbf{b}$) *It holds*

$$\int_0^t\|\partial_y(u,\mathbf{w},\mathbf{b})(t')\|_{s-1}^2\mathrm{d}t'\le C\kappa\int_0^t\|\partial_yW_1(t')\|_{s-1}^2\mathrm{d}t'+C\int_0^t\mathscr{E}(t')^{\frac12}\mathscr{D}(t')\mathrm{d}t'+C\mathscr{E}(0).\quad(3.7)$$

Proof Let $Z=(\sqrt{\lambda}u,\sqrt{\mu}\mathbf{w}^{\mathrm{T}},\sqrt{\nu}\mathbf{b}^{\mathrm{T}})^{\mathrm{T}}$. The last three equations in (1.11) can be rewritten into the following

$$D_1(\varepsilon)v(\partial_tW_2+u\partial_yW_2)+vW_2=\partial_yZ,\quad\text{where } D_1(\varepsilon)=\mathrm{diag}(\varepsilon_1^2,\varepsilon_2^2\mathbb{I}_2,\varepsilon_3^2\mathbb{I}_2).$$

Let m be an integer with $0\le m\le s-1$. Applying ∂_y^m to the above equation, and taking inner product with $\partial_y^{m+1}Z$ in L^2, we have after certain integration by parts,

$$\begin{aligned}\|\partial_y^{m+1}Z\|^2&=\frac{\mathrm{d}}{\mathrm{d}t}\langle D_1(\varepsilon)\partial_y^m(vW_2),\partial_y^{m+1}Z\rangle-\langle D_1(\varepsilon)\partial_y^m(\partial_yuW_2),\partial_y^{m+1}Z\rangle\\&\quad+\langle D_1(\varepsilon)\partial_y^{m+1}(vW_2),\partial_y^m\partial_tZ\rangle+\langle D_1(\varepsilon)\partial_y^m(vu\partial_yW_2),\partial_y^{m+1}Z\rangle\\&\quad+\langle\partial_y^m(vW_2),\partial_y^{m+1}Z\rangle.\end{aligned}$$

By the Moser-type calculus inequalities, we have

$$|\langle D_1(\varepsilon)\partial_y^m(\partial_yuW_2),\partial_y^{m+1}Z\rangle|\le C\mathscr{E}(t)^{\frac12}\mathscr{D}(t),$$

and further by using the Young inequality,

$$|\langle D_1(\varepsilon)\partial_y^m(vu\partial_yW_2),\partial_y^{m+1}Z\rangle+\langle\partial_y^m(vW_2),\partial_y^{m+1}Z\rangle|\le\frac12\|\partial_y^{m+1}Z\|^2+C\|W_2\|_s^2.$$

By using system (1.11), we obtain that

$$\|\partial_y^m\partial_tZ\|\le C\|\partial_yW\|_{s-1},$$

then we have

$$|\langle D_1(\varepsilon)\partial_y^{m+1}(vW_2),\partial_y^m\partial_tZ\rangle|\le\kappa\|\partial_yW\|_{s-1}^2+C\|W_2\|_s^2.$$

Combining all these estimates, we obtain

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\langle D_1(\varepsilon)\partial_y^m(vW_2),\partial_y^{m+1}Z\rangle+\frac12\|\partial_y^{m+1}Z\|^2\\&\le C\mathscr{E}(t)^{\frac12}\mathscr{D}(t)+C\|W_2\|_s^2+\kappa\|\partial_yW\|_{s-1}^2.\end{aligned}\quad(3.8)$$

Noticing that

$$|\langle D_1(\varepsilon)\partial_y^m(vW_2),\partial_y^{m+1}Z\rangle|\le C\mathscr{E}(t),\quad\forall\ t\ge0,$$

integrating (3.8) over $[0,t]$ for any $t\in(0,T]$, summing the resulting equation for all $0\le m\le s-1$ and using (3.4) yield (3.7).

Lemma 3.4 (Dissipative estimates for $\partial_y v$) *It holds*

$$\int_0^t \|\partial_y v(t')\|_{s-1}\mathrm{d}t' \le C\kappa \int_0^t \|\partial_y W_1(t')\|_{s-1}^2\mathrm{d}t' + C\int_0^t \mathscr{E}(t')^{\frac{1}{2}}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0). \tag{3.9}$$

Proof Let m be an integer with $0 \le m \le s-1$. Applying ∂_y^m to the second equation in (1.11), and taking inner product with $\partial_y^{m+1}v$ in L^2, we have

$$\begin{aligned}\langle -p'(v)\partial_y^{m+1}v, \partial_y^{m+1}v\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_y^m u, \partial_y^{m+1}v\rangle + \langle \partial_y^{m+1}u, \partial_y^m\partial_t v\rangle + \langle \partial_y^m(\mathbf{b}\cdot\partial_y\mathbf{b}), \partial_y^{m+1}v\rangle \\ &\quad - \langle \sqrt{\lambda}\partial_y^{m+1}\tau, \partial_y^{m+1}v\rangle + \langle \partial_y^m(p'(v)\partial_y v) - p'(v)\partial_y^{m+1}v, \partial_y^{m+1}v\rangle.\end{aligned}$$

It is clear that by (3.1),

$$\langle -p'(v)\partial_y^{m+1}v, \partial_y^{m+1}v\rangle \ge p_1\|\partial_y^{m+1}v\|^2.$$

From the Moser-type calculus inequalities, we have

$$|\langle \partial_y^m(\mathbf{b}\cdot\partial_y\mathbf{b}), \partial_y^{m+1}v\rangle + \langle \partial_y^m(p'(v)\partial_y v) - p'(v)\partial_y^{m+1}v, \partial_y^{m+1}v\rangle| \le C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t),$$

and further by using (1.11) and the Young inequality,

$$|\langle \partial_y^{m+1}u, \partial_y^m\partial_t v\rangle + \langle \sqrt{\lambda}\partial_y^{m+1}\tau, \partial_y^{m+1}v\rangle| \le C\|\partial_y u\|_{s-1}^2 + \frac{p_1}{2}\|\partial_y^{m+1}v\|^2 + C\|\tau\|_s^2.$$

These estimates imply that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_y^m u, \partial_y^{m+1}v\rangle + \frac{p_1}{2}\|\partial_y^{m+1}v\|^2 \le C\|\partial_y u\|_{s-1}^2 + C\|\tau\|_s^2 + C\mathscr{E}(t)^{\frac{1}{2}}\mathscr{D}(t). \tag{3.10}$$

Noticing that

$$|\langle \partial_y^m u, \partial_y^{m+1}v\rangle| \le C\mathscr{E}(t), \quad \forall\, t \ge 0,$$

integrating (3.10) over $[0,t]$ for any $t \in (0,T]$, summing the resulting equation for all $0 \le m \le s-1$ and using (3.4) yield

$$\begin{aligned}&\int_0^t \|\partial_y v(t')\|_{s-1}\mathrm{d}t' \\ &\le C\kappa\int_0^t \|\partial_y W_1(t')\|_{s-1}^2\mathrm{d}t' + C\int_0^t \mathscr{E}(t')^{\frac{1}{2}}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0) + C\int_0^t \|\partial_y u(t')\|_{s-1}^2\mathrm{d}t',\end{aligned}$$

which yields (3.9) by noticing (3.7).

Proof of Theorem 2.1 Combining (3.7) and (3.9), we conclude that there exists a positive constant $c_2 > 0$, such that

$$\int_0^t \|\partial_y W_1(t')\|_{s-1}\mathrm{d}t' \le c_2\kappa\int_0^t \|\partial_y W_1(t')\|_{s-1}^2\mathrm{d}t' + C\int_0^t \mathscr{E}(t')^{\frac{1}{2}}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0).$$

We then choose κ small enough such that

$$c_2\kappa < 1, \tag{3.11}$$

then
$$\int_0^t \|\partial_y W_1(t')\|_{s-1}\mathrm{d}t' \le C\int_0^t \mathscr{E}(t')^{\frac12}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0).$$
Combining the above estimate with (3.4), we have
$$\mathscr{E}(t) + \int_0^t \mathscr{D}(t')\mathrm{d}t' \le C\int_0^t \mathscr{E}(t')^{\frac12}\mathscr{D}(t')\mathrm{d}t' + C\mathscr{E}(0),$$
this yields (2.3) by noticing that $\mathscr{E}(t)$ is sufficiently small. By the bootstrap principle, it also implies the global-in-time existence of solution to system (1.11).

Proof of Theorem 2.2 The uniform estimate (2.3) implies that for any $\varepsilon \in (0,1]$, sequences $\{(v^\varepsilon - 1, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{b}^\varepsilon)\}_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}^+; H^s)$ and sequences $\{(\tau^\varepsilon, \mathbf{S}^\varepsilon, \mathbf{J}^\varepsilon)\}_{\varepsilon>0}$ are bounded in $L^2(\mathbb{R}^+; H^s)$. It follows that there exist functions $(\overline{v}-1, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}) \in L^\infty(\mathbb{R}^+; H^s)$ and $(\overline{\tau}, \overline{\mathbf{S}}, \overline{\mathbf{J}}) \in L^2(\mathbb{R}^+; H^s)$, such that (2.4)–(2.5) hold. In addition, as $\varepsilon \to 0$,
$$\begin{aligned}
&\varepsilon_1^2(\partial_t\tau^\varepsilon + u^\varepsilon\partial_y\tau^\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{K}),\\
&\varepsilon_2^2(\partial_t\mathbf{S}^\varepsilon + u^\varepsilon\partial_y\mathbf{S}^\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{K}),\\
&\varepsilon_3^2(\partial_t\mathbf{J}^\varepsilon + u^\varepsilon\partial_y\mathbf{J}^\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{K}).
\end{aligned}$$

Moreover, from the first four equations in (1.11), it is easy to see that $\{\partial_t v^\varepsilon\}_{\varepsilon>0}$, $\{\partial_t u^\varepsilon\}_{\varepsilon>0}$, $\{\partial_t \mathbf{w}^\varepsilon\}_{\varepsilon>0}$ and $\{\partial_t \mathbf{b}^\varepsilon\}_{\varepsilon}$ are bounded in $L^2(\mathbb{R}^+; H^{s-1})$. Hence, by a classical compactness theorem (see [35]), for all $T>0$, $\{v^\varepsilon\}_{\varepsilon>0}$, $\{u^\varepsilon\}_{\varepsilon>0}$, $\{\mathbf{w}^\varepsilon\}_{\varepsilon>0}$ and $\{\mathbf{b}^\varepsilon\}_{\varepsilon>0}$ are relatively compact in $C([0,T]; H^{s-1}_{\mathrm{loc}}(\mathbb{K}))$. As a consequence, as $\varepsilon \to 0$, up to subsequences,
$$(v^\varepsilon - 1, u^\varepsilon, \mathbf{w}^\varepsilon, \mathbf{b}^\varepsilon) \to (\overline{v}-1, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}), \quad \text{strongly in } C([0,T]; H^{s-1}_{\mathrm{loc}}(\mathbb{K})).$$

This is sufficient to pass the limit $\varepsilon \to 0$ in (1.11) in the sense of distributions to obtain that
$$\begin{cases}
\partial_t\overline{v} - \partial_y\overline{u} = 0,\\
\partial_t\overline{u} + \partial_y\Big(p(\overline{v}) + \dfrac12|\overline{\mathbf{b}}|^2\Big) = \sqrt{\lambda}\partial_y\overline{\tau},\\
\partial_t\overline{\mathbf{w}} - \partial_y\overline{\mathbf{b}} = \sqrt{\mu}\partial_y\overline{\mathbf{S}},\\
\partial_t\overline{\mathbf{b}} + \dfrac{\partial_y\overline{u}}{\overline{v}}\overline{\mathbf{b}} - \dfrac{1}{\overline{v}}\partial_y\overline{\mathbf{w}} = \dfrac{\sqrt{\nu}}{\overline{v}}\partial_y\overline{\mathbf{J}}
\end{cases} \tag{3.12}$$
with
$$\overline{\tau} = \frac{\sqrt{\lambda}\partial_y\overline{u}}{\overline{v}}, \quad \overline{\mathbf{S}} = \frac{\sqrt{\mu}\partial_y\overline{\mathbf{w}}}{\overline{v}}, \quad \overline{\mathbf{J}} = \frac{\sqrt{\nu}\partial_y\overline{\mathbf{b}}}{\overline{v}}. \tag{3.13}$$
Substituting (3.13) into (3.12) ends the proof.

4 Global Convergence Rates

In this section, we tend to establish the global-in-time convergence rate problems between the relaxed system and the original planar MHD system in periodic domains. Let $x \in \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/[-\pi,\pi]$ be a torus over $\mathbb{R}$. For simplicity, we denote $\overline{W} = (\overline{W}_1, \overline{W}_2)^{\mathrm{T}}$ with
$$\overline{W}_1 = (\overline{v}-1, \overline{u}, \overline{\mathbf{w}}^{\mathrm{T}}, \overline{\mathbf{b}}^{\mathrm{T}})^{\mathrm{T}}, \quad \overline{W}_2 = (\overline{\tau}, \overline{\mathbf{S}}^{\mathrm{T}}, \overline{\mathbf{J}}^{\mathrm{T}})^{\mathrm{T}}$$

and

$$U = W - \overline{W}, \quad U_1 = W_1 - \overline{W}_1, \quad U_2 = W_2 - \overline{W}_2.$$

In addition, we denote

$$\mathcal{E}(t) = \|U_1(t)\|_{s-1}^2 + \varepsilon_1^2\|\tau(t) - \overline{\tau}(t)\|_{s-1}^2 + \varepsilon_2^2\|\mathbf{S}(t) - \overline{\mathbf{S}}(t)\|_{s-1}^2 + \varepsilon_3^2\|\mathbf{J}(t) - \overline{\mathbf{J}}(t)\|_{s-1}^2$$

and

$$\mathcal{D}(t) = \|U(t)\|_{s-1}^2.$$

In this section, we will establish the estimates of the following type

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(t')\mathrm{d}t' \le C\varepsilon^{2\alpha_1},$$

where α_1 is defined in Theorem 2.3. We recall here the total energy

$$\mathscr{E}(t) = \|v(t) - 1\|_s^2 + \|u(t)\|_s^2 + \|\mathbf{w}(t)\|_s^2 + \|\mathbf{b}(t)\|_s^2 + \varepsilon_1^2\|\tau(t)\|_s^2 + \varepsilon_2^2\|\mathbf{S}(t)\|_s^2 + \varepsilon_3^2\|\mathbf{J}(t)\|_s^2$$

as well as the dissipative energy

$$\mathscr{D}(t) = \|\partial_y v(t)\|_{s-1}^2 + \|\partial_y u(t)\|_{s-1}^2 + \|\partial_y \mathbf{w}(t)\|_{s-1}^2 + \|\partial_y \mathbf{b}(t)\|_{s-1}^2 + \|\tau(t)\|_s^2 + \|\mathbf{S}(t)\|_s^2 + \|\mathbf{J}(t)\|_s^2.$$

For convenience, we also denote

$$\overline{\mathscr{E}}(t) = \|\overline{v}(t) - 1\|_s^2 + \|\overline{u}(t)\|_s^2 + \|\overline{\mathbf{w}}(t)\|_s^2 + \|\overline{\mathbf{b}}(t)\|_s^2$$

as well as

$$\overline{\mathscr{D}}(t) = \|\partial_y \overline{W}_1(t)\|_s^2 + \|\overline{W}_2(t)\|_s^2 + \|\partial_t \overline{W}_2(t)\|_{s-2}.$$

We first give an estimate on the limiting system, which is a direct consequence of the weak convergence of solutions together with the lower semi-continuity of norms.

Lemma 4.1 *Let $(\overline{v}_0, \overline{u}_0, \overline{\mathbf{w}}_0, \overline{\mathbf{b}}_0)$ be the weak limit of $(v_0, u_0, \mathbf{w}_0, \mathbf{b}_0)$ in H^s. Then the limiting solutions $(\overline{v}, \overline{u}, \overline{\mathbf{w}}, \overline{\mathbf{b}}, \overline{\tau}, \overline{\boldsymbol{S}}, \overline{\mathbf{J}})$ satisfy*

$$\overline{\mathscr{E}}(t) + \int_0^t \overline{\mathscr{D}}(t')\mathrm{d}t' \le C\|(\overline{v}_0 - 1, \overline{u}_0, \overline{\mathbf{w}}_0, \overline{\mathbf{b}}_0)\|_s^2.$$

4.1 Error estimates of dissipative variables

The formal limit of (2.1) is the following

$$D_0(0)\partial_t \overline{W} + A(\overline{W})\partial_y \overline{W} = -Q(\overline{W}). \tag{4.1}$$

Subtracting the above equation from (2.1), we have

$$D_0(\varepsilon)\partial_t U + A(W)\partial_y U + Q(W) - Q(\overline{W}) = -f, \tag{4.2}$$

where

$$f = (D_0(\varepsilon) - D_0(0))\partial_t \overline{W} + (A(W) - A(\overline{W}))\partial_y \overline{W}.$$

For integers $l \le s-1$, applying ∂_y^l to both sides of (4.2), and making the inner product of the resulting equation with $2A_0(W)\partial_y^l U$ in L^2 yield that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle D_0(\varepsilon)A_0(W)\partial_y^l U, \partial_y^l U\rangle + \langle 2A_0(W)\partial_y^l U, \partial_y^l (Q(W)-Q(\overline{W}))\rangle \\
&= \langle (\partial_t \widetilde{A}_0(W) + \partial_y \widetilde{A}(W))\partial_y^l U, \partial_y^l U\rangle - \langle 2A_0(W)\partial_y^l U, \partial_y^l f\rangle - \langle 2A_0(W)\partial_y^l U, \partial_y^l G^l\rangle \\
&\stackrel{\text{def}}{=} K_1^l + K_2^l + K_3^l
\end{aligned} \tag{4.3}$$

with the natural correspondence of K_1^l, K_2^l and K_3^l, and the commutator is defined as

$$G^l := \partial_y^l (A(W)\partial_y U) - A(W)\partial_y^{l+1} U.$$

We then treat the terms K_1^l, K_2^l and K_3^l one by one in a series of lemmas as follows.

Lemma 4.2 (Estimates of K_1^l) *It holds*

$$|K_1^l| \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}(t).$$

Proof Notice that $\|\partial_t v\|_\infty$ is bounded by $\|u\|_s$, then it is clear that

$$|\langle \partial_t \widetilde{A}_0(W)\partial_y^l U, \partial_y^l U\rangle| \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}(t).$$

Similarly,

$$|\langle \partial_y \widetilde{A}(W)\partial_y^l U, \partial_y^l U\rangle| \le \|\partial_y W_1\|_\infty \|\partial_y^l U\|^2 \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}(t).$$

This ends the proof.

Lemma 4.3 (Estimates of K_2^l) *It holds*

$$|K_2^l| \le C\varepsilon^2(\mathscr{D}(t) + \overline{\mathscr{D}}(t)) + C\overline{\mathscr{E}}(t)^{\frac{1}{2}}\mathcal{D}(t). \tag{4.4}$$

Proof We first treat the term in f containing $(D_0(\varepsilon)-D_0(0))\partial_t\overline{W}$. Recall that

$$(D_0(\varepsilon)-D_0(0))\partial_t\overline{W} = (0_{1\times 6}, \varepsilon_1^2\partial_t\overline{\tau}, \varepsilon_2^2\partial_t\overline{\mathbf{S}}^{\mathrm{T}}, \varepsilon_3^2\partial_t\overline{\mathbf{J}}^{\mathrm{T}})^{\mathrm{T}}.$$

Consequently, we deduce that for $l=0$,

$$|\langle 2A_0(W)U, (D_0(\varepsilon)-D_0(0))\partial_t\overline{W}\rangle| \le C\varepsilon^2\overline{\mathscr{D}}(t) + C\varepsilon^2\mathscr{D}(t),$$

and for $1 \le l \le s-1$,

$$\begin{aligned}
&|\langle 2A_0(W)\partial_y^l U, (D_0(\varepsilon)-D_0(0))\partial_t\partial_y^l\overline{W}\rangle| \\
&\le |\langle 2\partial_y A_0(W)\partial_y^l U, (D_0(\varepsilon)-D_0(0))\partial_t\partial_y^{l-1}\overline{W}\rangle| \\
&\quad + |\langle 2A_0(W)\partial_y^{l+1} U, (D_0(\varepsilon)-D_0(0))\partial_t\partial_y^{l-1}\overline{W}\rangle| \\
&\le C\varepsilon^2\|\partial_y v\|_\infty\|U\|_{s-1}\|\partial_t\overline{W}_2\|_{s-2} + C\varepsilon^2\|\partial_y U\|_{s-1}\|\partial_t\overline{W}_2\|_{s-2} \\
&\le C\varepsilon^2(\mathscr{D}(t) + \overline{\mathscr{D}}(t)).
\end{aligned}$$

For the term containing $(A(W)-A(\overline{W}))\partial_y\overline{W}$, by the Moser-type calculus inequalities, we deduce

$$\|\partial_y^l((A(W)-A(\overline{W}))\partial_y\overline{W})\| \le C\|(A(W)-A(\overline{W})\|_{s-1}\|\partial_y\overline{W}\|_{s-1}.$$

Notice that for $B=A^{11}(W), A^{12}(W)$ and $A^{21}(W)$, we have

$$\|(B(W)-B(\overline{W})\|_{s-1} \le C\|U\|_{s-1} \le C\mathcal{D}(t)^{\frac{1}{2}},$$

as a result, we obtain that

$$\begin{aligned}
&|\langle 2A_0(W)\partial_y^l U, \partial_y^l((A(W)-A(\overline{W}))\partial_y\overline{W})\rangle| \\
&\le C\overline{\mathscr{E}}(t)^{\frac{1}{2}}\mathcal{D}(t) + |\langle 2v\partial_y^l U_2, \partial_y^l(A^{22}(W)-A^{22}(\overline{W})\partial_y\overline{W}_2)\rangle| \\
&\le C\overline{\mathscr{E}}(t)^{\frac{1}{2}}\mathcal{D}(t) + C\varepsilon^2(\mathscr{D}(t)+\overline{\mathscr{D}}(t)).
\end{aligned}$$

Combining all these estimates yields (4.4).

Lemma 4.4 (Estimates of K_3^l) *It holds*

$$|K_3^l| \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}(t).$$

Proof The classical Moser-type calculus inequalities yield

$$\|G^l\| \le C\|\nabla A(W)\|_{s-1}\|U\|_{s-1} \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}^{\frac{1}{2}}(t).$$

This implies that

$$|K_3^l| \le C\|G^l\|\|\partial_y^l U\| \le C\mathscr{E}(t)^{\frac{1}{2}}\mathcal{D}(t),$$

which ends the proof.

Combining all these estimates, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle D_0(\varepsilon)A_0(W)\partial_y^l U, \partial_y^l U\rangle + \langle 2A_0(W)\partial_y^l U, \partial_y^l(Q(W)-Q(\overline{W}))\rangle \\
&\le C(\mathscr{E}(t)^{\frac{1}{2}}+\overline{\mathscr{E}}(t)^{\frac{1}{2}})\mathcal{D}(t) + C\varepsilon^2(\mathscr{D}(t)+\overline{\mathscr{D}}(t)).
\end{aligned} \tag{4.5}$$

Notice that there exists a constant $C_1>0$, such that

$$C_1\mathcal{E}(t) \le \langle D_0(\varepsilon)A_0(W)\partial_y^l U, \partial_y^l U\rangle \le C\mathcal{E}(t),$$

and there exists a $C_2>0$, such that

$$\langle 2A_0(W)\partial_y^l U, \partial_y^l(Q(W)-Q(\overline{W}))\rangle \ge C_2\|\partial_y^l U_2\|^2.$$

Integrating (4.5) over $[0,t]$ and summing up for all $l\le s-1$ yield

$$\mathcal{E}(t) + \int_0^t \|U_2(t')\|_{s-1}^2\mathrm{d}t' \le C\int_0^t (\mathscr{E}(t')^{\frac{1}{2}}+\overline{\mathscr{E}}(t')^{\frac{1}{2}})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}. \tag{4.6}$$

4.2 Error estimates for non-dissipative variables

We now use the stream function to study the dissipative estimates for $W_1 - \overline{W}_1$. We first introduce the method of constructing stream functions over 1-D torus, the idea of which is initialized in [42].

Lemma 4.5 (Existence of stream function over 1-D torus) *Consider the following conservation law over* 1*-D torus*

$$\partial_t z + \partial_x h = 0, \quad x \in \mathbb{T}$$

with $\mathcal{M}(z) = 0$. *Then there exists a stream function* Φ *satisfying*

$$\begin{cases} \partial_t \Phi = -h + \dfrac{1}{2\pi}\mathcal{M}(h), \\ \partial_x \Phi = z, \\ \mathcal{M}(\Phi) = 0. \end{cases}$$

Proof Let Φ be the unique solution to the following problem

$$\partial_x \Phi = z, \quad \mathcal{M}(\Phi) = 0 \quad \text{for } t \geq 0.$$

To this end, applying time derivative to both sides of the equation above yields

$$\partial_x(\partial_t \Phi) = \partial_t z = -\partial_x h,$$

which yields that there exists a function $\kappa(t)$ independent of x, such that

$$\partial_t \Phi = -h + \kappa(t).$$

Noticing that $\mathcal{M}(\partial_t \Phi) = 0$, we deduce that

$$\kappa(t) = \frac{1}{2\pi}\mathcal{M}(h),$$

which ends the proof.

Noticing that the fourth equation in (1.11) is equivalent to the last equation in (1.6) by using the first equation in (1.11), we construct the error equation for $\mathbf{b} - \overline{\mathbf{b}}$ with

$$\partial_t(v\mathbf{b} - \overline{v}\overline{\mathbf{b}}) - \partial_y[(\mathbf{w} - \overline{\mathbf{w}}) + \sqrt{\nu}(\mathbf{J} - \overline{\mathbf{J}})] = 0,$$

In addition, we subtract (3.12)–(3.13) excluding the equation for $\overline{\mathbf{b}}$ from system (1.11) excluding the equation for $\mathbf{b}$, which leads to the error system

$$\begin{cases} \partial_t(v - \overline{v}) - \partial_y(u - \overline{u}) = 0, \\ \partial_t(u - \overline{u}) + \partial_y\Big[p(v) - p(\overline{v}) + \dfrac{1}{2}|\mathbf{b}|^2 - \dfrac{1}{2}|\overline{\mathbf{b}}|^2 - \sqrt{\lambda}(\tau - \overline{\tau})\Big] = 0, \\ \partial_t(\mathbf{w} - \overline{\mathbf{w}}) - \partial_y[(\mathbf{b} - \overline{\mathbf{b}}) - \sqrt{\mu}(\mathbf{S} - \overline{\mathbf{S}})] = 0, \\ \partial_t(v\mathbf{b} - \overline{v}\overline{\mathbf{b}}) - \partial_y[(\mathbf{w} - \overline{\mathbf{w}}) - \sqrt{\nu}(\mathbf{J} - \overline{\mathbf{J}})] = 0, \\ \varepsilon_1^2 \partial_t \tau + \varepsilon_1^2 u \partial_y \tau + (\tau - \overline{\tau}) = \sqrt{\lambda}\Big(\dfrac{\partial_y u}{v} - \dfrac{\partial_y \overline{u}}{\overline{v}}\Big), \\ \varepsilon_2^2 \partial_t \mathbf{S} + \varepsilon_2^2 u \partial_y \mathbf{S} + (\mathbf{S} - \overline{\mathbf{S}}) = \sqrt{\mu}\Big(\dfrac{\partial_y \mathbf{w}}{v} - \dfrac{\partial_y \overline{\mathbf{w}}}{\overline{v}}\Big), \\ \varepsilon_3^2 \partial_t \mathbf{J} + \varepsilon_3^2 u \partial_y \mathbf{J} + (\mathbf{J} - \overline{\mathbf{J}}) = \sqrt{\nu}\Big(\dfrac{\partial_y \mathbf{b}}{v} - \dfrac{\partial_y \overline{\mathbf{b}}}{\overline{v}}\Big). \end{cases} \tag{4.7}$$

Noticing that the first four equations of the above system are conservative. We immediately have

$$\mathcal{M}((v-\overline{v},u-\overline{u},\mathbf{w}-\overline{\mathbf{w}},v\mathbf{b}-\overline{v}\overline{\mathbf{b}}))=\mathcal{M}((v_0-\overline{v}_0,u_0-\overline{u}_0,\mathbf{w}_0-\overline{\mathbf{w}}_0,v_0\mathbf{b}_0-\overline{v}_0\overline{\mathbf{b}}_0)).$$

Without loss of generality, we may assume that $\mathcal{M}((v_0-\overline{v}_0,u_0-\overline{u}_0,\mathbf{w}_0-\overline{\mathbf{w}}_0,v_0\mathbf{b}_0-\overline{v}_0\overline{\mathbf{b}}_0)^{\mathrm{T}})=0$. Otherwise, we may introduce new variables removing the average from the variables. According to Lemma 4.5, we have immediately the following existence of four stream functions ϕ_1, ϕ_2, ϕ_3 and ϕ_4, in which the conditions for ϕ_1 are

$$\partial_t\phi_1=u-\overline{u},\quad \partial_y\phi_1=v-\overline{v},\quad \mathcal{M}(\phi_1)=0.$$

The conditions for ϕ_2 are

$$\begin{cases}\partial_t\phi_2=-\Big[p(v)-p(\overline{v})+\dfrac12|\mathbf{b}|^2-\dfrac12|\overline{\mathbf{b}}|^2-\sqrt{\lambda}(\tau-\overline{\tau})\Big]\\ \qquad\qquad +\dfrac{1}{2\pi}\mathcal{M}\Big(p(v)-p(\overline{v})+\dfrac12|\mathbf{b}|^2-\dfrac12|\overline{\mathbf{b}}|^2-\sqrt{\lambda}(\tau-\overline{\tau})\Big),\\ \partial_y\phi_2=u-\overline{u},\\ \mathcal{M}(\phi_2)=0.\end{cases}$$

The conditions for ϕ_3 are

$$\partial_t\phi_3=(\mathbf{b}-\overline{\mathbf{b}})-\sqrt{\mu}(\mathbf{S}-\overline{\mathbf{S}})+\mathcal{M}(\sqrt{\mu}(\mathbf{S}-\overline{\mathbf{S}}))-\mathcal{M}(\mathbf{b}-\overline{\mathbf{b}}),\quad \partial_y\phi_3=\mathbf{w}-\overline{\mathbf{w}},\quad \mathcal{M}(\phi_3)=0.$$

The conditions for ϕ_4 are

$$\partial_t\phi_4=(\mathbf{w}-\overline{\mathbf{w}})-\sqrt{\nu}(\mathbf{J}-\overline{\mathbf{J}})+\sqrt{\nu}\mathcal{M}(\mathbf{J}-\overline{\mathbf{J}}),\quad \partial_y\phi_4=v\mathbf{b}-\overline{v}\overline{\mathbf{b}},\quad \mathcal{M}(\phi_4)=0.$$

In the following, we denote $\mu>0$ a sufficiently small constant to be determined in (4.21).

Lemma 4.6 (Dissipative estimates for $u-\overline{u}$) *It holds*

$$\int_0^T\|u-\overline{u}\|_{s-1}^2\mathrm{d}t\le C\int_0^T(\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'+\mu\int_0^T\mathcal{D}(t')\mathrm{d}t'+C\varepsilon^{2\alpha_1}. \tag{4.8}$$

Proof Letting $0\le l\le s-1$, applying ∂_y^l to the fifth equation in (4.7), we have

$$\varepsilon_1^2\partial_t\partial_y^l\tau+\varepsilon_1^2\partial_y^l(u\partial_y\tau)+\partial_y^l(\tau-\overline{\tau})-\sqrt{\lambda}\partial_y^l\Big(\frac{\partial_y u}{v}-\frac{\partial_y\overline{u}}{\overline{v}}\Big)=0.$$

Taking the inner product of this equality with $\partial_y^l\phi_2$ in L^2, and integrating over $[0,T]$, we get

$$\begin{aligned}0&=\int_0^T\Big\langle\partial_y^l\phi_2,\varepsilon_1^2\partial_t\partial_y^l\tau+\varepsilon_1^2\partial_y^l(u\partial_y\tau)+\partial_y^l(\tau-\overline{\tau})-\sqrt{\lambda}\partial_y^l\Big(\frac{\partial_y u}{v}-\frac{\partial_y\overline{u}}{\overline{v}}\Big)\Big\rangle\mathrm{d}t\\ &:=L_{\phi_2}^1+L_{\phi_2}^2+L_{\phi_2}^3+L_{\phi_2}^4\end{aligned}$$

with the natural correspondence of $L_{\phi_2}^1,L_{\phi_2}^2,L_{\phi_2}^3$ and $L_{\phi_2}^4$, of which the terms are treated one by one as follows. For $L_{\phi_2}^1$, we have

$$L_{\phi_2}^1=\varepsilon_1^2\int_0^T\langle\partial_y^l\phi_2,\partial_t\partial_y^l\tau\rangle\mathrm{d}t$$

$$= \varepsilon_1^2 \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \langle \partial_y^l \phi_2, \partial_y^l \tau \rangle \mathrm{d}t - \varepsilon_1^2 \int_0^T \langle \partial_t \partial_y^l \phi_2, \partial_y^l \tau \rangle \mathrm{d}t.$$

For all $\varepsilon \in (0,1]$, we use (2.3) to obtain

$$\begin{aligned}
\Big|\varepsilon_1^2 \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \langle \partial_y^l \phi_2, \partial_y^l \tau \rangle \mathrm{d}t\Big| &= \varepsilon_1^2 |\langle \partial_y^l \phi_2(T), \partial_y^l \tau(T)\rangle - \langle \partial_y^l \phi_2(0), \partial_y^l \tau(0)\rangle| \\
&\le C\|\partial_y^l \phi_2(T)\|^2 + C\|\partial_y^l \phi_2(0)\|^2 + C\varepsilon_1^2 \\
&\le C\|\partial_y^l (u-\overline{u})(T)\|^2 + C\varepsilon^{2\alpha_1} \\
&\le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1},
\end{aligned}$$

in which we have used (4.6) and the Poincaré inequality. In addition,

$$\begin{aligned}
&\varepsilon_1^2 \Big|\int_0^T \langle \partial_t \partial_y^l \phi_2, \partial_y^l \tau \rangle\Big| \\
&\le \varepsilon_1^2 \int_0^T \Big|\Big\langle \partial_y^l(-p(v)+p(\overline{v}) - \frac12|\mathbf{b}|^2 + \frac12|\overline{\mathbf{b}}|^2 + \sqrt{\lambda}(\tau-\overline{\tau})), \partial_y^l \tau\Big\rangle\Big|\mathrm{d}t \\
&\quad + C\varepsilon_1^2 \int_0^T \Big|\Big\langle \mathcal{M}(-p(v)+p(\overline{v}) - \frac12|\mathbf{b}|^2 + \frac12|\overline{\mathbf{b}}|^2 + \sqrt{\lambda}(\tau-\overline{\tau})), \partial_y^l \tau\Big\rangle\Big|\mathrm{d}t \\
&\le \mu \int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon_1^2.
\end{aligned}$$

These estimates imply that

$$|L_{\phi_2}^1| \le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}. \tag{4.9}$$

For $L_{\phi_2}^2$, by using Poincaré inequality, we have

$$\begin{aligned}
|L_{\phi_2}^2| &= \Big|\int_0^T \langle \partial_y^l \phi_2, \varepsilon_1^2 \partial_y^l (u\partial_y \tau)\rangle \mathrm{d}t\Big| \\
&\le C\varepsilon_1^2 \int_0^T \|\partial_y^l (u-\overline{u})\| \|u\|_{s-1} \|\partial_y \tau\|_{s-1} \mathrm{d}t \\
&\le \mu \int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon_1^2.
\end{aligned} \tag{4.10}$$

For $L_{\phi_2}^3$, we have

$$\begin{aligned}
|L_{\phi_2}^3| &= \Big|\int_0^T \langle \partial_y^l \phi_2, \partial_y^l (\tau-\overline{\tau})\rangle \mathrm{d}t\Big| \\
&\le \int_0^T \|\partial_y^l (u-\overline{u})\| \|\partial_y^l (\tau-\overline{\tau})\| \mathrm{d}t \\
&\le \mu \int_0^T \|\partial_y^l (u-\overline{u})\|^2 \mathrm{d}t + C\int_0^T \|\partial_y^l (\tau-\overline{\tau})\|^2 \mathrm{d}t \\
&\le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}.
\end{aligned} \tag{4.11}$$

For $L^4_{\phi_2}$, we treat as follows. Notice that

$$\frac{\partial_y u}{v}-\frac{\partial_y\overline{u}}{\overline{v}}=\frac{\partial_y u-\partial_y\overline{u}}{v}-\partial_y\overline{u}\frac{v-\overline{v}}{v\overline{v}}=\partial_y\Big(\frac{u-\overline{u}}{v}\Big)+\frac{\partial_y v(u-\overline{u})}{v^2}-\partial_y\overline{u}\frac{v-\overline{v}}{v\overline{v}}.$$

Based on this, we have

$$\begin{aligned}&\Big\langle\partial_y^l\phi_2,\partial_y^{l+1}\Big(\frac{u-\overline{u}}{v}\Big)\Big\rangle\\&=-\Big\langle\partial_y^l(u-\overline{u}),\partial_y^l\Big(\frac{u-\overline{u}}{v}\Big)\Big\rangle\\&=-\Big\langle\partial_y^l(u-\overline{u}),\frac{\partial_y^l(u-\overline{u})}{v}\Big\rangle-\Big\langle\partial_y^l(u-\overline{u}),\partial_y^l\Big(\frac{u-\overline{u}}{v}\Big)-\frac{\partial_y^l(u-\overline{u})}{v}\Big\rangle,\end{aligned}$$

where

$$\Big\langle\partial_y^l(u-\overline{u}),\frac{\partial_y^l(u-\overline{u})}{v}\Big\rangle\geq\frac{2}{3}\|\partial_y^l(u-\overline{u})\|^2,$$

and by the Moser-type calculus inequalities,

$$\Big|\Big\langle\partial_y^l(u-\overline{u}),\partial_y^l\Big(\frac{u-\overline{u}}{v}\Big)-\frac{\partial_y^l(u-\overline{u})}{v}\Big\rangle\Big|\leq C\|\partial_y v\|_{s-1}\|u-\overline{u}\|_{s-1}^2\leq C\int_0^T\mathscr{E}(t')^{\frac12}\mathcal{D}(t')\mathrm{d}t'.$$

In addition, by the Moser-type calculus inequalities,

$$\Big|\Big\langle\partial_y^l\phi_2,\partial_y^l\Big(\frac{\partial_y v(u-\overline{u})}{v^2}-\partial_y\overline{u}\frac{v-\overline{v}}{v\overline{v}}\Big)\Big\rangle\Big|\leq C\int_0^T(\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.$$

These imply

$$L^4_{\phi_2}\geq\frac{2}{3}\int_0^T\|\partial_y^l(u-\overline{u})\|^2\mathrm{d}t-C\int_0^T(\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.\tag{4.12}$$

Combining all these estimates and summing for all $l\leq s-1$ yield (4.8).

Lemma 4.7 (Dissipative estimates for $\mathbf{w}-\overline{\mathbf{w}}$) *It holds*

$$\int_0^T\|\mathbf{w}-\overline{\mathbf{w}}\|_{s-1}^2\mathrm{d}t\leq C\int_0^T(\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'+\mu\int_0^T\mathcal{D}(t')\mathrm{d}t'+C\varepsilon^{2\alpha_1}.\tag{4.13}$$

Proof Letting $0\leq l\leq s-1$, applying ∂_y^l to the sixth equation in (4.7), we have

$$\varepsilon_2^2\partial_y^l\partial_t\mathbf{S}+\varepsilon_2^2\partial_y^l(u\partial_y\mathbf{S})+\partial_y^l(\mathbf{S}-\overline{\mathbf{S}})-\sqrt{\mu}\partial_y^l\Big(\frac{\partial_y\mathbf{w}}{v}-\frac{\partial_y\overline{\mathbf{w}}}{\overline{v}}\Big)=0.$$

Taking the inner product of this equality with $\partial_y^l\phi_3$ in L^2, and integrating over $[0,T]$, we get

$$\begin{aligned}0=&\int_0^T\Big\langle\partial_y^l\phi_3,\varepsilon_2^2\partial_y^l\partial_t\mathbf{S}+\varepsilon_2^2\partial_y^l(u\partial_y\mathbf{S})+\partial_y^l(\mathbf{S}-\overline{\mathbf{S}})-\sqrt{\mu}\partial_y^l\Big(\frac{\partial_y\mathbf{w}}{v}-\frac{\partial_y\overline{\mathbf{w}}}{\overline{v}}\Big)\Big\rangle\mathrm{d}t\\:=&L^1_{\phi_3}+L^2_{\phi_3}+L^3_{\phi_3}+L^4_{\phi_3}\end{aligned}\tag{4.14}$$

with the natural correspondence of $L^1_{\phi_3},L^2_{\phi_3},L^3_{\phi_3}$ and $L^4_{\phi_3}$, of which the treatments are similar to those in the above lemma. In fact, similar to (4.9)–(4.12), we have

$$|L^1_{\phi_3}|\leq C\int_0^T(\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'+\mu\int_0^T\mathcal{D}(t')\mathrm{d}t'+C\varepsilon^{2\alpha_1},$$

$$
\begin{aligned}
|L_{\phi_3}^2| &\le \mu \int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon_2^2,\\
|L_{\phi_3}^3| &\le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1},\\
L_{\phi_3}^4 &\ge \frac{2}{3}\int_0^T \|\partial_y^l(\mathbf{w}-\overline{\mathbf{w}})\|^2\mathrm{d}t - C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.
\end{aligned}
$$

These estimates are enough for us to obtain (4.13).

Lemma 4.8 (Dissipative estimates for $\mathbf{b}-\overline{\mathbf{b}}$) *It holds*

$$
\int_0^T \|\mathbf{b}-\overline{\mathbf{b}}\|_{s-1}^2\mathrm{d}t \le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}. \tag{4.15}
$$

Proof Letting $0 \le l \le s-1$, applying ∂_y^l to the last equation in (4.7), we have

$$
\varepsilon_3^2\partial_y^l\partial_t\mathbf{J} + \varepsilon_3^2\partial_y^l(u\partial_y\mathbf{J}) + \partial_y^l(\mathbf{J}-\overline{\mathbf{J}}) - \sqrt{\nu}\partial_y^l\Big(\frac{\partial_y\mathbf{b}}{v} - \frac{\partial_y\overline{\mathbf{b}}}{\overline{v}}\Big) = 0.
$$

Taking the inner product of this equality with $\partial_y^l\phi_4$ in L^2, and integrating over $[0,T]$, we get

$$
\begin{aligned}
0 &= \int_0^T \Big\langle \partial_y^l\phi_4, \varepsilon_3^2\partial_y^l\partial_t\mathbf{J} + \varepsilon_3^2\partial_y^l(u\partial_y\mathbf{J}) + \partial_y^l(\mathbf{J}-\overline{\mathbf{J}}) - \sqrt{\nu}\partial_y^l\Big(\frac{\partial_y\mathbf{b}}{v} - \frac{\partial_y\overline{\mathbf{b}}}{\overline{v}}\Big)\Big\rangle\mathrm{d}t\\
&:= L_{\phi_4}^1 + L_{\phi_4}^2 + L_{\phi_4}^3 + L_{\phi_4}^4
\end{aligned} \tag{4.16}
$$

with the natural correspondence of $L_{\phi_4}^1, L_{\phi_4}^2, L_{\phi_4}^3$ and $L_{\phi_4}^4$, which are treated term by term as follows. For $L_{\phi_4}^1$, we have

$$
\begin{aligned}
L_{\phi_4}^1 &= \varepsilon_3^2\int_0^T \langle\partial_y^l\phi_4, \partial_y^l\partial_t\mathbf{J}\rangle\mathrm{d}t\\
&= \varepsilon_3^2\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_y^l\phi_4, \partial_y^l\mathbf{J}\rangle\mathrm{d}t - \varepsilon_3^2\int_0^T \langle\partial_y^l\partial_t\phi_4, \partial_y^l\mathbf{J}\rangle\mathrm{d}t,
\end{aligned}
$$

in which similarly

$$
\begin{aligned}
\Big|\varepsilon_3^2\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_y^l\phi_4, \partial_y^l\mathbf{J}\rangle\mathrm{d}t\Big| &= \varepsilon_3^2|\langle\partial_y^l\phi_4(T), \partial_y^l\mathbf{J}(T)\rangle - \langle\partial_y^l\phi_4(0), \partial_y^l\mathbf{J}(0)\rangle|\\
&\le C\|\partial_y^l(v\mathbf{b} - \overline{v}\overline{\mathbf{b}})(T)\|^2 + C\varepsilon^{2\alpha_1}\\
&\le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_3^2\Big|\int_0^T \langle\partial_y^l\partial_t\phi_4, \partial_y^l\mathbf{J}\rangle\mathrm{d}t\Big| &\le \varepsilon_3^2\int_0^T |\langle\partial_y^l((\mathbf{w}-\overline{\mathbf{w}}) - \sqrt{\nu}(\mathbf{J}-\overline{\mathbf{J}}) + \sqrt{\nu}\mathcal{M}(\mathbf{J}-\overline{\mathbf{J}})), \partial_y^l\mathbf{J}\rangle|\mathrm{d}t\\
&\le \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon_3^2.
\end{aligned}
$$

These estimates imply that

$$
|L_{\phi_4}^1| \le C\int_0^T (\mathscr{E}(t')^{\frac12} + \overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}.
$$

For $L^2_{\phi_4}$, similarly to (4.10), we have

$$\begin{aligned}|L^2_{\phi_4}| &\le C\varepsilon_3^2\int_0^T \|\partial_y^l(v\mathbf{b}-\overline{v}\overline{\mathbf{b}})\|\|\partial_y^l(u\partial_y\mathbf{J})\|\mathrm{d}t\\ &\le C\varepsilon_3^2\int_0^T \|\partial_y^l((v-\overline{v})\mathbf{b})+\partial_y^l(\overline{v}(\mathbf{b}-\overline{\mathbf{b}}))\|\|\partial_y^l(u\partial_y\mathbf{J})\|\mathrm{d}t\\ &\le \mu\int_0^T \mathcal{D}(t')\mathrm{d}t'+C\varepsilon_3^2.\end{aligned}$$

Similarly to (4.11), for $L^3_{\phi_4}$, we have

$$\begin{aligned}|L^3_{\phi_4}| &\le C\int_0^T \|\partial_y^l(v\mathbf{b}-\overline{v}\overline{\mathbf{b}})\|\|\partial_y^l(\mathbf{J}-\overline{\mathbf{J}})\|\mathrm{d}t\\ &\le C\int_0^T \|\partial_y^l((v-\overline{v})\mathbf{b})+\partial_y^l(\overline{v}(\mathbf{b}-\overline{\mathbf{b}}))\|\|\partial_y^l(\mathbf{J}-\overline{\mathbf{J}})\|\mathrm{d}t\\ &\le \mu\int_0^T \|\partial_y^l U_1\|^2\mathrm{d}t+C\int_0^T \|\partial_y^l(\mathbf{J}-\overline{\mathbf{J}})\|^2\mathrm{d}t\\ &\le C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'+\mu\int_0^T \mathcal{D}(t')\mathrm{d}t'+C\varepsilon^{2\alpha_1}.\end{aligned}$$

For $L^4_{\phi_4}$, the treatments are similar to that for (4.12). We first have

$$\frac{\partial_y\mathbf{b}}{v}-\frac{\partial_y\overline{\mathbf{b}}}{\overline{v}}=\frac{\partial_y\mathbf{b}-\partial_y\overline{\mathbf{b}}}{v}-\partial_y\overline{\mathbf{b}}\frac{v-\overline{v}}{v\overline{v}}=\partial_y\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)+\frac{\partial_y v(\mathbf{b}-\overline{\mathbf{b}})}{v^2}-\partial_y\overline{\mathbf{b}}\frac{v-\overline{v}}{v\overline{v}}.$$

Notice also

$$v\mathbf{b}-\overline{v}\overline{\mathbf{b}}=(v-\overline{v})\mathbf{b}+\overline{v}(\mathbf{b}-\overline{\mathbf{b}}).$$

We have

$$\begin{aligned}L^4_{\phi_4} &= -\sqrt{\nu}\int_0^T \Big\langle\partial_y^l\phi_4,\partial_y^{l+1}\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)\Big\rangle\mathrm{d}t\\ &\quad-\sqrt{\nu}\int_0^T \Big\langle\partial_y^l\phi_4,\partial_y^l\Big(\frac{\partial_y v(\mathbf{b}-\overline{\mathbf{b}})}{v^2}-\partial_y\overline{\mathbf{b}}\frac{v-\overline{v}}{v\overline{v}}\Big)\Big\rangle\mathrm{d}t\\ &=\sqrt{\nu}\int_0^T \Big\langle\partial_y^l((v-\overline{v})\mathbf{b}+\overline{v}(\mathbf{b}-\overline{\mathbf{b}})),\partial_y^l\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)\Big\rangle\mathrm{d}t\\ &\quad-\sqrt{\nu}\int_0^T \Big\langle\partial_y^l\phi_4,\partial_y^l\Big(\frac{\partial_y v(\mathbf{b}-\overline{\mathbf{b}})}{v^2}-\partial_y\overline{\mathbf{b}}\frac{v-\overline{v}}{v\overline{v}}\Big)\Big\rangle\mathrm{d}t\\ &=\sqrt{\nu}\int_0^T \Big\langle\partial_y^l((v-\overline{v})\mathbf{b}),\partial_y^l\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)\Big\rangle\mathrm{d}t\\ &\quad+\sqrt{\nu}\int_0^T \Big\langle\overline{v}\partial_y^l((\mathbf{b}-\overline{\mathbf{b}})),\frac{1}{v}\partial_y^l(\mathbf{b}-\overline{\mathbf{b}})\Big\rangle\mathrm{d}t\\ &\quad+\sqrt{\nu}\int_0^T \Big\langle\overline{v}\partial_y^l((\mathbf{b}-\overline{\mathbf{b}})),\partial_y^l\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)-\frac{1}{v}\partial_y^l(\mathbf{b}-\overline{\mathbf{b}})\Big\rangle\mathrm{d}t\\ &\quad+\sqrt{\nu}\int_0^T \Big\langle\partial_y^l(\overline{v}(\mathbf{b}-\overline{\mathbf{b}}))-\overline{v}\partial_y^l((\mathbf{b}-\overline{\mathbf{b}})),\partial_y^l\Big(\frac{\mathbf{b}-\overline{\mathbf{b}}}{v}\Big)\Big\rangle\mathrm{d}t\\ &\quad-\sqrt{\nu}\int_0^T \Big\langle\partial_y^l\phi_4,\partial_y^l\Big(\frac{\partial_y v(\mathbf{b}-\overline{\mathbf{b}})}{v^2}-\partial_y\overline{\mathbf{b}}\frac{v-\overline{v}}{v\overline{v}}\Big)\Big\rangle\mathrm{d}t\end{aligned}$$

$$= L^{4,1}_{\phi_4} + L^{4,2}_{\phi_4} + L^{4,3}_{\phi_4} + L^{4,4}_{\phi_4} + L^{4,5}_{\phi_4}$$

with the natural correspondence. Making use of the Moser-type calculus inequalities, it is clear that

$$|L^{4,1}_{\phi_4}| + |L^{4,3}_{\phi_4}| + |L^{4,4}_{\phi_4}| \le C \int_0^T \mathscr{E}(t')^{\frac{1}{2}} \mathcal{D}(t')\mathrm{d}t'$$

and

$$\begin{aligned} |L^{4,5}_{\phi_4}| &\le C \int_0^T \|\partial_y^l (v\mathbf{b} - \overline{v}\overline{\mathbf{b}})\| \Big\| \partial_y^l \Big(\frac{\partial_y v(\mathbf{b} - \overline{\mathbf{b}})}{v^2} - \partial_y \overline{\mathbf{b}} \frac{v-\overline{v}}{v\overline{v}} \Big) \Big\| \mathrm{d}t \\ &\le C \int_0^T (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}}) \mathcal{D}(t')\mathrm{d}t'. \end{aligned} \tag{4.17}$$

In addition, we have for $L^{4,2}_{\phi_4}$,

$$L^{4,2}_{\phi_4} \ge \frac{1}{3} \|\partial_y^l (\mathbf{b} - \overline{\mathbf{b}})\|^2.$$

These estimates imply

$$L^4_{\phi_4} \ge \frac{1}{3} \int_0^T \|\partial_y^l (\mathbf{b} - \overline{\mathbf{b}})\|^2 \mathrm{d}t - C \int_0^T (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}}) \mathcal{D}(t')\mathrm{d}t'.$$

Combining all these estimates and summing for all $l \le s-1$ yield (4.15).

Lemma 4.9 (Dissipative estimates for $v - \overline{v}$) *It holds*

$$\int_0^T \|v-\overline{v}\|^2_{s-1} \mathrm{d}t \le C \int_0^T (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}}) \mathcal{D}(t')\mathrm{d}t' + \mu \int_0^T \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}. \tag{4.18}$$

Proof Letting $0 \le l \le s-1$, applying ∂_y^l to the second equation in (4.7), taking the inner product of the resulting equation with $\partial_y^l \phi_1$ in L^2, and integrating over $[0,T]$, we get

$$\begin{aligned} 0 &= \int_0^T \Big\langle \partial_y^l \phi_1, \partial_t \partial_y^l (u-\overline{u}) + \partial_y^{l+1}(p(v) - p(\overline{v})) + \frac{1}{2}\partial_y^{l+1}(|\mathbf{b}|^2 - |\overline{\mathbf{b}}|^2) - \sqrt{\lambda}\partial_y^{l+1}(\tau - \overline{\tau}) \Big\rangle \mathrm{d}t \\ &:= L^1_{\phi_1} + L^2_{\phi_1} + L^3_{\phi_1} + L^4_{\phi_1} \end{aligned} \tag{4.19}$$

with the natural correspondence of $L^1_{\phi_1}, L^2_{\phi_1}, L^3_{\phi_1}$ and $L^4_{\phi_1}$, which are treated term by term as follows. For $L^1_{\phi_1}$, we have that

$$L^1_{\phi_1} = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \langle \partial_y^l \phi_1, \partial_y^l (u-\overline{u}) \rangle \mathrm{d}t - \int_0^T \langle \partial_t \partial_y^l \phi_1, \partial_y^l (u-\overline{u}) \rangle \mathrm{d}t,$$

in which

$$\begin{aligned} \Big| \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \langle \partial_y^l \phi_1, \partial_y^l (u-\overline{u}) \rangle \mathrm{d}t \Big| &\le C\|\partial_y^l (v-\overline{v})(T)\|^2 + C\|\partial_y^l (u-\overline{u})(T)\|^2 + C\varepsilon^{2\alpha_1} \\ &\le C \int_0^T (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}}) \mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}, \end{aligned}$$

and by using (4.8),

$$\begin{aligned}
&\Big|\int_0^T \langle \partial_t\partial_y^l\phi_1, \partial_y^l(u-\overline{u})\rangle \mathrm{d}t\Big| \\
&\le \mu\int_0^T \|\partial_y^l(v-\overline{v})\|^2\mathrm{d}t + C\int_0^T \|\partial_y^l(u-\overline{u})\|^2\mathrm{d}t \\
&\le \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}. \qquad (4.20)
\end{aligned}$$

These imply

$$|L^1_{\phi_1}| \le \mu\int_0^T \mathcal{D}(t')\mathrm{d}t' + C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}.$$

For $L^2_{\phi_1}$, we first notice that

$$p(v)-p(\overline{v}) = \int_0^1 p'(\widetilde{v})(v-\overline{v})\mathrm{d}s,$$

where $\widetilde{v} = v + s(v-\overline{v})$ is between v and $\overline{v}$. Then we have

$$\begin{aligned}
L^2_{\phi_1} &= -\int_0^T \langle \partial_y^{l+1}\phi_1, \partial_y^l(p(v)-p(\overline{v}))\rangle \mathrm{d}t \\
&= -\int_0^T \Big\langle \partial_y^l(v-\overline{v}), \int_0^1 p'(\widetilde{v})\partial_y^l(v-\overline{v})\mathrm{d}s\Big\rangle \mathrm{d}t \\
&\quad -\int_0^T \Big\langle \partial_y^l(v-\overline{v}), \partial_y^l\Big(\int_0^1 p'(\widetilde{v})(v-\overline{v})\mathrm{d}s\Big) - \int_0^1 p'(\widetilde{v})\partial_y^l(v-\overline{v})\mathrm{d}s\Big\rangle \mathrm{d}t,
\end{aligned}$$

in which

$$-\int_0^T \Big\langle \partial_y^l(v-\overline{v}), \int_0^1 p'(\widetilde{v})\partial_y^l(v-\overline{v})\mathrm{d}s\Big\rangle \mathrm{d}t \ge p_1\int_0^T \|\partial_y^l(v-\overline{v})\|^2\mathrm{d}t$$

and

$$\begin{aligned}
&\Big|\int_0^T \Big\langle \partial_y^l(v-\overline{v}), \partial_y^l\Big(\int_0^1 p'(\widetilde{v})(v-\overline{v})\mathrm{d}s\Big) - \int_0^1 p'(\widetilde{v})\partial_y^l(v-\overline{v})\mathrm{d}s\Big\rangle \mathrm{d}t\Big| \\
&\le C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.
\end{aligned}$$

These imply

$$L^2_{\phi_1} \ge p_1\int_0^T \|\partial_y^l(v-\overline{v})\|^2\mathrm{d}t - C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.$$

For $L^3_{\phi_1}$, we have by using the conditions of the stream function,

$$|L^3_{\phi_1}| = \Big|\int_0^T \langle \partial_y^l(v-\overline{v}), \frac12\partial_y^l((\mathbf{b}+\overline{\mathbf{b}})\cdot(\mathbf{b}-\overline{\mathbf{b}}))\rangle \mathrm{d}t\Big| \le C\int_0^T (\mathscr{E}(t')^{\frac12}+\overline{\mathscr{E}}(t')^{\frac12})\mathcal{D}(t')\mathrm{d}t'.$$

Now the last term is treated as follows

$$|L^4_{\phi_1}| = \Big|\int_0^T \langle \partial_y^l(v-\overline{v}), \partial_y^l(\tau-\overline{\tau})\rangle \mathrm{d}t\Big|$$

$$\leq \mu \int_0^T \|v-\overline{v}\|_{s-1}^2 \mathrm{d}t + C\int_0^T \|\tau-\overline{\tau}\|_{s-1}^2 \mathrm{d}t$$
$$\leq \mu \int_0^T \mathcal{D}(t')\mathrm{d}t' + C\int_0^T (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1}.$$

Combining all these estimates and summing for all $l \leq s-1$ yield (4.18).

Proof of Theorem 2.3 Combining all these lemmas and (4.6), we conclude that there exists a constant $C_3 > 0$, such that

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(t')\mathrm{d}t' \leq C\int_0^t (\mathscr{E}(t')^{\frac{1}{2}} + \overline{\mathscr{E}}(t')^{\frac{1}{2}})\mathcal{D}(t')\mathrm{d}t' + C\varepsilon^{2\alpha_1} + C_3\mu \int_0^T \mathcal{D}(t')\mathrm{d}t'.$$

Notice that $\mathscr{E}(t)$ and $\overline{\mathscr{E}}(t)$ are both sufficiently small for all $t \geq 0$. Then there exists a constant $C_4 > 0$, such that

$$\mathcal{E}(t) + C_4\int_0^t \mathcal{D}(t')\mathrm{d}t' \leq C\varepsilon^{2\alpha_1} + C_3\mu \int_0^T \mathcal{D}(t')\mathrm{d}t'.$$

We then choose $\mu > 0$ small enough such that

$$C_3\mu < C_4. \tag{4.21}$$

This ends the proof.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Bresch, D. and Prange, C., Newtonian limit for weakly viscoelastic fluid flows, *SIAM J. Math. Anal.*, **46**(2), 2014, 1116–1159.

[2] Cattaneo, C., Sulla conduzione del calore, *Atti Sem. Mat. Fis. Univ. Modena*, **3**, 1949, 83–101.

[3] Cattaneo, C., Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée, *C. R. Acad. Sci. Paris*, **247**, 1958, 431–433.

[4] Chen, F., Introduction to Plasma Physics and Controlled Fusion, **1**, PlenumPress, New York, 1984.

[5] Chen, G.-Q. and Wang, D., Global solutions of nonlinear magnetohydrodynamics with large initial data, *J. Differential Equations*, **182**(2), 2002, 344–376.

[6] Fernández Sare, H. D. and Racke, R., On the stability of damped Timoshenko systems: Cattaneo versus Fourier law, *Arch. Ration. Mech. Anal.*, **194**(1), 2009, 221–251.

[7] Hu, Y. and Ju, Q., Global large solutions of magnetohydrodynamics with temperature-dependent heat conductivity, *Z. Angew. Math. Phys.*, **66**(3), 2015, 865–889.

[8] Hu, Y. and Racke, R., Compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Hyperbolic Differ. Equ.*, **13**(2), 2016, 233–247.

[9] Hu, Y. and Racke, R., Compressible Navier-Stokes equations with revised Maxwell's law, *J. Math. Fluid Mech.*, **19**(1), 2017, 77–90.

[10] Hu, Y. and Racke, R., Hyperbolic compressible Navier-Stokes equations, *J. Differential Equations*, **269**(4), 2020, 3196–3220.

[11] Hu, Y., Racke, R. and Wang, N., Formation of singularities for one-dimensional relaxed compressible Navier-Stokes equations, *J. Differential Equations*, **327**, 2022, 145–165.

[12] Kato, T., The Cauchy problem for quasi–linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, **58**(3), 1975, 181–205.

[13] Kawashima, S., *Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics*, PhD thesis, Kyoto Uninversity, 1983.

[14] Kawashima, S. and Okada, M., Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Proc. Japan Acad. Ser. A Math. Sci.*, **58**(9), 1982, 384–387.

[15] Lax, P. D., Hyperbolic systems of conservation laws and the mathematical theory of shock waves, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, **11**, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973.

[16] Li, T. and Qin, T., *Physics and Partial Differential Equations,* **1**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Higher Education Press, Beijing, 2012.

[17] Li, W., Shang, Z. and Tang, F., Global existence of large solutions to the planar magnetohydrodynamic equations with zero magnetic diffusivity, *J. Math. Anal. Appl.*, **496**(1), 2021, 31 pp.

[18] Li, Y., Global well-posedness to the one-dimensional model for planar non-resistive magnetohydrodynamics with large data and vacuum, *J. Math. Anal. Appl.*, **462**(2), 2018, 1342–1356.

[19] Li, Y., Peng, Y.-J. and Zhao, L., Convergence rate from hyperbolic systems of balance laws to parabolic systems, *Appl. Anal.*, **100**(5), 2021, 1079–1095.

[20] Li, Y., Peng, Y.-J. and Zhao, L., Convergence rates in zero-relaxation limits for Euler-Maxwell and Euler-Poisson systems, *J. Math. Pures Appl.* (9), **154**, 2021, 185–211.

[21] Li, Y. and Shang, Z., Global large solutions to planar magnetohydrodynamics equations with temperature-dependent coefficients, *J. Hyperbolic Differ. Equ.*, **16**(3), 2019, 443–493.

[22] Liu, X. and Qin, Y., Global solutions to isentropic planar magnetohydrodynamic equations with density-dependent viscosity, *Math. Methods Appl. Sci.*, **41**(12), 2018, 4448–4464.

[23] Majda, A., Compressible fluid flow and systems of conservation laws in several space variables, **53**, *Applied Mathematical Sciences*, Springer-Verlag, New York, 1984.

[24] Maxwell, J. C., IV., on the dynamical theory of gases, *Philosophical Transactions of the Royal Society of London*, **157**, 1867, 49–88.

[25] Molinet, L. and Talhouk, R., Newtonian limit for weakly viscoelastic fluid flows of Oldroyd type, *SIAM J. Math. Anal.*, **39**(5), 2008, 1577–1594.

[26] Ou, Y., Shi, P. and Wittwer, P., Large time behaviors of strong solutions to magnetohydrodynamic equations with free boundary and degenerate viscosity, *J. Math. Phys.*, **59**(8), 2018, 081510, 34 pp.

[27] Peng, Y.-J., Relaxed Euler systems and convergence to Navier-Stokes equations, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **38**(2), 2021, 369–401.

[28] Peng, Y.-J. and Wasiolek, V., Parabolic limit with differential constraints of first–order quasilinear hyperbolic systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33**(4), 2016, 1103–1130.

[29] Peng, Y.-J. and Wasiolek, V., Uniform global existence and parabolic limit for partially dissipative hyperbolic systems, *J. Differential Equations*, **260**(9), 2016, 7059–7092.

[30] Peng, Y.-J. and Zhao, L., Global convergence to compressible full Navier-Stokes equations by approximation with Oldroyd-type constitutive laws, *J. Math. Fluid Mech.*, **24**(2), 2022, 17 pp.

[31] Said-Houari, B. and Kasimov, A., Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same, *J. Differential Equations*, **255**(4), 2013, 611–632.

[32] Saut, J.-C., Some remarks on the limit of viscoelastic fluids as the relaxation time tends to zero, Trends in applications of pure mathematics to mechanics (Bad Honnef, 1985), **249**, Lecture Notes in Phys., Springer-Verlog, Berlin, 1986, 364–369.

[33] Shang, Z., Global large solutions to the Cauchy problem of planar magnetohydrodynamics equations with temperature-dependent coefficients, *J. Dyn. Control Syst.*, **28**(1), 2022, 163–205.

[34] Shizuta, Y. and Kawashima, S., Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.*, **14**(2), 1985, 249–275.

[35] Simon, J., Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* (4), **146**, 1987, 65–96.

[36] Smagulov, S. S., Durmagambetov, A. A. and Iskenderova, D. A., Cauchy problems for equations of magnetogasdynamics, *Differentsial' nye Uravneniya*, **29**(2), 1993, 337–348, 367 pp.

[37] Tao, Q., Yang, Y. and Gao, J., A free boundary problem for planar compressible Hall-magnetohydrodynamic equations, *Z. Angew. Math. Phys.*, **69**(1), 2018, 22 pp.

[38] Tao, Q., Yang, Y. and Yao, Z.-A., Global existence and exponential stability of solutions for planar compressible Hall-magnetohydrodynamic equations, *J. Differential Equations*, **263**(7), 2017, 3788–3831.

[39] Wang, D., Large solutions to the initial-boundary value problem for planar magnetohydrodynamics, *SIAM J. Appl. Math.*, **63**(4), 2003, 1424–1441.

[40] Ye, X. and Zhang, J., On the behavior of boundary layers of one-dimensional isentropic planar MHD equations with vanishing shear viscosity limit, *J. Differential Equations*, **260**(4), 2016, 3927–3961.

[41] Yong, W.-A., Newtonian limit of Maxwell fluid flows, *Arch. Ration. Mech. Anal.*, **214**(3), 2014, 913–922.

[42] Zhao, L. and Xi, S., Convergence rate from systems of balance laws to isotropic parabolic systems, a periodic case, *Asymptot. Anal.*, **124**, 2021, 163–198.