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Equivariant Tautological Integrals on Flag Varieties*

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Abstract The author apples the Atiyah-Bott-Berline-Vergne formula to the equivariant tautological integrals over flag varieties of types A, B, C, D, and recovers the formulas expressing the integrals as iterated residues at infinity, which were first obtained by Zielenkiewicz using symplectic reduction.

Keywords Localization, Flag variety, Residue
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1 Introduction

Let $\mathbb{P}^n$ be the complex projective space of dimension n and $x \in H^2(\mathbb{P}^n, \mathbb{Z})$ be the hyperplane class. Then $H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1}\rangle$. A general element in $H^*(\mathbb{P}^n, \mathbb{Z})$ can be written as $Q(x)$ for some polynomial Q with coefficients in $\mathbb{Z}$, and we have

$$\int_{\mathbb{P}^n} Q(x) = \mathrm{Res}_{z=0} \frac{Q(z)}{z^{n+1}} \mathrm{d}z. \tag{1.1}$$

Note that there is a natural action of the torus $T = (\mathbb{C}^*)^{n+1}$ on $\mathbb{P}^n$:

$$T \times \mathbb{P}^n \longrightarrow \mathbb{P}^n, \quad ((t_0, \cdots, t_n), [x_0 : \cdots : x_n]) \quad \longmapsto [t_0x_0 : \cdots : t_nx_n]. \tag{1.2}$$

We can also consider integrals of T-equivariant cohomology classes. Let H be the equivariant hyperplane class, $\mathfrak{t}$ be the Lie algebra of T and $\alpha_i = 2\pi\sqrt{-1}u_i$, $i = 1, \cdots, n+1$ be the weights of T defined by $\alpha_i(X_1, \cdots, X_{n+1}) = X_i, \forall (X_1, \cdots, X_{n+1}) \in \mathfrak{t}$. Then for any polynomial Q we have

$$\int_{\mathbb{P}^n} Q(H) = \mathrm{Res}_{z=\infty} \frac{-Q(z)}{\prod\limits_{1\le i\le n+1} (u_i + z)} \mathrm{d}z, \tag{1.3}$$

where $\mathrm{Res}_{z=\infty}$ is the residue at infinity (see [2, 8]). We call formulas like (1.1) nonequivariant residue formulas and call formulas like (1.3) residue formulas. By taking nonequivariant limit, we can see that (1.1) is a consequence of (1.3).

It is natural to ask for the similar formulas for general flag varieties. In [2], for a special class of type A flag varieties (the full flag of k-dimensional subspaces of $\mathrm{Sym}^{\le k}\mathbb{C}^n$), Bérczi and

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Szenes wrote the equivariant integrals as iterated residues at infinity. The main idea of [2] is to apply the Atiyah-Bott-Berline-Vergne formula to express the equivariant integrals as sums over fixed points and then play combinatorics. In this paper, we follow the idea of Bérczi and Szenes to derive the equivariant residue formulas for generalized flag varieties of types A, B, C, D.

Since the formulas share a lot of similarities, we only present here the formula in the type C case and refer the reader to later sections for other formulas. To state our result, we introduce some necessary notations. Let S be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$. For any increasing sequence of positive integers $0 = a_0 < a_1 < \cdots < a_s = k \leq n$, set $r_i = a_i - a_{i-1}$. Let $Fl^S = Fl^S(a_1, \cdots, a_s; 2n)$ be the variety parametrizing isotropic flags $E_1 \subset \cdots \subset E_s \subset \mathbb{C}^{2n}$, where $\dim E_i = a_i$. Then Fl^S can be identified with $Sp(2n;\mathbb{C})/P_{a_1,\cdots,a_s}$, where $P_{a_1,\cdots,a_s}$ is a parabolic subgroup. The maximal torus T of $Sp(2n;\mathbb{C})$ acts canonically on Fl^S. This action lifts canonically to the i -th tautological vector bundle which we denote by $\mathcal{E}_i$. Let $c_j^T\big(\big(\frac{\mathcal{E}_p}{\mathcal{E}_{p-1}}\big)^*\big)$ be the j-th equivariant Chern class of $\big(\frac{\mathcal{E}_p}{\mathcal{E}_{p-1}}\big)^*$ $(p = 1, \cdots, s; j = 1, \cdots, r_p)$. For any polynomial $Q(x_1, \cdots, x_k)$, which is invariant with respect to any permutation of $x_{a_i+1}, \cdots, x_{a_{i+1}}$ for any $i \in \{0, \cdots, s-1\}$, there exists a polynomial $\widetilde{Q}$ such that $Q(x_1, \cdots, x_k) = \widetilde{Q}(e_1^1, \cdots, e_1^{r_1}, \cdots, e_s^1, \cdots, e_s^{r_s})$, where $e_i^j = e_i^j(x_{a_{i-1}+1}, \cdots, x_{a_i})$ is the j-th elementary symmetric polynomial. Letting

$$I_C(Q) = \int_{Fl^S} \widetilde{Q}(c_1^T\Big(\Big(\frac{\mathcal{E}_1}{\mathcal{E}_0}\Big)^*\Big), \cdots, c_{r_1}^T\Big(\Big(\frac{\mathcal{E}_1}{\mathcal{E}_0}\Big)^*\Big), \cdots, c_1^T\Big(\Big(\frac{\mathcal{E}_s}{\mathcal{E}_{s-1}}\Big)^*\Big), \cdots, c_{r_s}^T\Big(\Big(\frac{\mathcal{E}_s}{\mathcal{E}_{s-1}})^*\Big)\Big),$$

then we prove the following theorem.

Theorem 6.1

$$\begin{aligned} I_C(Q) = & \frac{(-1)^k}{r_1! \cdots r_s!} \mathrm{Res}_\infty \times \\ & \frac{Q(z_1, \cdots, z_k) \prod\limits_{i>j} (z_i^2 - z_j^2) \prod\limits_{m=1}^{s} \prod\limits_{a_{m-1}+1 \leq i < j \leq a_m} (z_i - z_j)}{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{k} (z_j - u_i)(z_j + u_i)} \mathrm{d}z_1 \cdots \mathrm{d}z_k. \end{aligned} \tag{1.4}$$

Note that the equivariant Chern classes $c_i^T\big(\big(\frac{\mathcal{E}_j}{\mathcal{E}_{j-1}}\big)^*\big)$ $(i = 1, \cdots, r_j; j = 1, \cdots, s-1)$ generate the equivariant cohomolgy ring. Hence in principle we can compute the integral of any equivariant cohomology class over Fl^S via the above formula.

In the case of classical Grassmannians and maximal isotropic Grassmannians, the equivariant residue formulas were also proved by Zielenkiewicz in [13] using the same idea. Zielenkiewicz also suggested using JeffreyCKirwan nonabelian localization and symplectic reduction to obtain the residue formulas. Using that method, Zielenkiewicz obtained the equivariant residue formulas for flag varieties of types A,B,C,D in [14]. Later in [9] Weber and Zielenkiewicz also proved the residue formulas for the exceptional group G_2. In [7], Darondeau and Pragacz expressed the images of the nonequivariant cohomology classes under Gysin homomorphisms in terms of residues and Segre classes for flag bundles of types A, B, C, D. In [10–11] , the author also proved the nonequivariant residue formulas for flag varieties of type A using the ring structure of the cohomology ring. Note that the formulas in [7] (in the case that the base manifold is a single point) differs from the formulas obtained in [11, 14].

Our formulas coincide with those obtained in [14], except that in [14], the factor 2^k is missing in the type D case. In [14], the proofs of the formulas are based on the embeddings of the various flag varieties into the type A flag varieties. Our approach does not rely on such an embedding, and we believe that we can obtain the corresponding formulas for the exceptional groups along this line. As an application of the nonequivariant formulas, we also reprove the formulas (in the case that the base manifold is a single point) obtained in [7].

This paper is organized as follows. In Section 2, we fix some notations and collect the basic facts about flag varieties. In Section 3, as a warm up, we prove the type A case. In Section 4, we prove the type D case. In Section 5, we prove the type B case. In Section 6, we prove the type C case. In Section 7, we reprove the formulas (in the case that the base manifold is a single point) obtained in [7].

2 Preliminaries

In this section, we fix some notations and recall some basic facts about flag varieties.

2.1 Notations

Throughout this paper, we use $[n]$ to denote the set $\{1, \cdots, n\}$, and set $[0] = \emptyset$. We use $M_{n\times k}(\mathbb{C})$ to denote the set of $n\times k$ matrices over $\mathbb{C}$. For any sequence of increasing positive integers $0 = a_0 < a_1 < \cdots < a_s < n$, we set $r_i = a_i - a_{i-1}$ and $k = a_s$. We will consider the varieties parametrizing flags $E_1 \subset \cdots \subset E_s \subset \mathbb{C}^n$, satisfying $\dim E_i = a_i$ and certain additional conditions. In each case, we use $\mathcal{E}_i$ to denote the i-th universal vector bundle of rank a_i over the flag variety. We will see that there is a natural action of a torus T on the flag variety, which lifts to $\mathcal{E}_i$ canonically. By tautological integrals, we mean the integrals of the cohomology classes written as polynomials of the (equivariant) Chern classes of the bundles $\big(\frac{\mathcal{E}_i}{\mathcal{E}_{i-1}}\big)^*$ $(i = 1, \cdots, s)$. In general, such a class can be obtained in the following way. Let $Q(x_1, \cdots, x_k)$ be a polynomial which is invariant with respect to the $S_{r_1}\times\cdots\times S_{r_s}$ action. In other words, for any $i = 0, \cdots, s-1$, Q is invariant with respect to any permutation of $x_{a_i+1}, \cdots, x_{a_{i+1}}$. There exists a polynomial $\widetilde{Q}$ such that $Q(x_1, \cdots, x_k) = \widetilde{Q}(e_1^1, \cdots, e_1^{r_1}, \cdots, e_s^1, \cdots, e_s^{r_s})$ where e_i^j is the j-th elementary symmetric polynomial of $x_{a_{i-1}+1}, \cdots, x_{a_i}$. Then $\widetilde{Q}\big(c_1^T\big(\big(\frac{\mathcal{E}_1}{\mathcal{E}_0}\big)^*\big), \cdots, c_{r_1}^T\big(\big(\frac{\mathcal{E}_1}{\mathcal{E}_0}\big)^*\big), \cdots, c_{r_s}^T\big(\big(\frac{\mathcal{E}_s}{\mathcal{E}_{s-1}}\big)^*\big)\big)$ is such a class, where c_i^T is the i-th equivariant Chern class with respect to the action of T.

Next we introduce the notion of iterated residues (see [2, 8]). Let $w_1, \cdots, w_N$ be affine forms on $\mathbb{C}^n$, i.e., there exist constants $a_i^j, i = 1, \cdots, N; j = 0, \cdots, n$, such that $w_i = a_i^0 + \sum\limits_{j=1}^{n} a_i^j z_j$. Then for any holomorphic function $h(\mathbf{z})$, where $\mathbf{z} = (z_1, \cdots, z_n)$, the iterated residue at infinity of $\frac{h(\mathbf{z})}{\prod\limits_{i=1}^{N} w_i}\mathrm{d}\mathbf{z}$ is defined as

$$\mathrm{Res}_{z_1=\infty}\cdots\mathrm{Res}_{z_n=\infty}\frac{h(\mathbf{z})}{\prod\limits_{i=1}^{N} w_i}\mathrm{d}\mathbf{z}=\Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n\int_{M_R}\frac{h(\mathbf{z})}{\prod\limits_{i=1}^{N} w_i}\mathrm{d}\mathbf{z}, \tag{2.1}$$

where the integral is taken over the real torus $M_R=\{(z_1,\cdots,z_n)\mid |z_i|=R_i, i=1,\cdots,n\}$, $1\ll R_1\ll\cdots\ll R_n$ and the orientation of M_R is chosen such that $\mathrm{Res}_{z_1=\infty}\cdots\mathrm{Res}_{z_n=\infty}\frac{1}{z_1\cdots z_n}\mathrm{d}\mathbf{z}=(-1)^n$. For simplicity we use Res_∞ or $\mathrm{Res}_{\mathbf{z}=\infty}$ to denote $\mathrm{Res}_{z_1=\infty}\cdots\mathrm{Res}_{z_n=\infty}$. In practical computation, we can evaluate the iterated residue at infinity by iterated integrals

$$\mathrm{Res}_\infty\frac{h(\mathbf{z})}{\prod\limits_{i=1}^{N} w_i}\mathrm{d}\mathbf{z}=\Big(-\frac{1}{2\pi\sqrt{-1}}\int_{|z_1|=R_1}\cdots\Big(-\frac{1}{2\pi\sqrt{-1}}\int_{|z_n|=R_n}\frac{h(\mathbf{z})}{\prod\limits_{i=1}^{N} w_i}\mathrm{d}z_n\Big)\cdots\mathrm{d}z_1\Big), \tag{2.2}$$

where each integral is a contour integral of a function of one complex variable along a counterclockwise circle in the complex plane, which can be evaluated by the residue theorem. Note that in general the iterated residue depends on the order of the variables z_i. For example, let $\omega=\frac{1}{(z_1+z_2)z_1}\mathrm{d}z_1\mathrm{d}z_2$. We find that

$$-\frac{1}{2\pi\sqrt{-1}}\int_{|z_1|=R_1}\Big(-\frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=R_2}\frac{1}{(z_1+z_2)z_1}\mathrm{d}z_2\Big)\mathrm{d}z_1=1,\quad R_1\ll R_2, \tag{2.3}$$

while at the same time,

$$-\frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=R_2}\Big(-\frac{1}{2\pi\sqrt{-1}}\int_{|z_1|=R_1}\frac{1}{(z_1+z_2)z_1}\mathrm{d}z_1\Big)\mathrm{d}z_2=0,\quad R_2\ll R_1. \tag{2.4}$$

We remark that the iterated residues in this article, however, do not depend on the order of the variables, since we only consider residues of forms like

$$\frac{h(\mathbf{z})}{g_1(z_1)\cdots g_n(z_n)}\mathrm{d}\mathbf{z}, \tag{2.5}$$

where the g_i's are holomorphic functions in one variable.

2.2 Flag varieties and homogeneous vector bundles

Let G be a connected complex semisimple Lie group. Fix a maximal torus T in G. Let $\mathfrak{t}$ and $\mathfrak{g}$ be the Lie algebras of T and G, respectively. Let $R(G)$ be the set of roots of G with respect to T. By fixing an ordering on $\mathfrak{t}^*$, $R(G)$ can be written as the disjoint union of the set of positive roots $R^+(G)$ and the set of negative roots $R^-(G)$. Let $\Pi=\{\alpha_1,\cdots,\alpha_n\}$ be the simple roots with respect to this ordering. Then each positive root α can be written as $\alpha=\sum\limits_{i=1}^{n}m_i\alpha_i$ with $m_i\in\mathbb{Z}_{\geq 0}$, $\forall 1\leq i\leq n$. We have the root-space decomposition

$$\mathfrak{g}=\mathfrak{t}\oplus\bigoplus_{\alpha\in R(G)}\mathfrak{g}_\alpha, \tag{2.6}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_t(X) = \alpha(t)X, \forall t \in \mathfrak{t}\}$.

Let B be the Borel subgroup corresponding to the Lie subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus\limits_{\alpha \in R^-(G)} \mathfrak{g}_\alpha$. Then any parabolic subgroup containing B can be written in the form P_I, where $I \subset \{1, 2, \cdots, n\}$, and P_I is the parabolic subgroup corresponding to the following Lie algebra

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^-(G)} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in R_I^+} \mathfrak{g}_\alpha,$$

where $R_I^+ = \big\{ \sum\limits_{i=1}^{n} m_i \alpha_i \in R^+(G) \mid m_i = 0 \text{ if } i \in I \big\}$.

Let $X = G/P_I$. Then X is a smooth projective variety which is called a generalized flag variety. Recall that the restriction of every complex representation V of G to T is diagonalizable. Hence, as a T-representation $V \cong \oplus_\lambda V_\lambda$, where $\lambda \in \mathfrak{t}^*$ and V_λ is the subspace of V on which T acts by $\exp(t)(v) = \exp(\lambda(t)) \cdot v$, $\forall t \in \mathfrak{t}$. These λ's are called the weights of V. Let K be a compact real form of G. By Mostow theorem, K acts transitively on G/P_I. It is easy to see that $H = K \cap P_I$ is a connected Lie subgroup of K, which is determined by its Lie algebra. In this way, we represent the flag variety G/P_I as the quotient of two compact Lie groups of the same rank $G/P_I = K/H$ (see [1] for details).

Let $S = K \cap T$ be the common maximal torus in K and H. Denote the Lie algebra of S by $\mathfrak{s}$. Then $\Gamma = \{v \in \mathfrak{s} \mid \exp(v) = e\}$ is a lattice in $\mathfrak{s}$ which is isomorphic to $\mathbb{Z}^n$. The linear forms on $\mathfrak{s}$ which are integral on Γ can be identified with $H^1(S, \mathbb{Z})$, and we have $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \cong H^1(S, \mathbb{Z})$. Under the transgression map (see [3]), each integral linear form on $\mathfrak{s}$ corresponds to an element in $H^2(K/S, \mathbb{Z})$. In this paper, we use the same symbol to denote the integral linear form and the corresponding element in $H^2(K/S; \mathbb{Z})$. Since S is a compact real form of T, any weight λ of T when restricted to $\mathfrak{s}$ takes the form $2\pi\sqrt{-1}u$, where u is an integral linear form on $\mathfrak{s}$. Hence $\frac{\lambda}{2\pi\sqrt{-1}}$ represents an element in $H^2(K/S; \mathbb{Z})$.

Let $\{u_1, \cdots, u_n\}$ be a basis of $H^1(S, \mathbb{Z})$. Then the cohomology ring of the classifying space of S is $H^*(B_S, \mathbb{Z}) = \mathbb{Z}[u_1, \cdots, u_n]$ (see [3, Theorem 19.1]). Denote the Weyl groups of K and H by W_K and W_H, respectively. Then W_H can be viewed as a subgroup of W_K. $H^*(B_H, \mathbb{Z})$ can be regarded as the subring $\mathbb{Z}[u_1, \cdots, u_n]^{W_H}$ of $\mathbb{Z}[u_1, \cdots, u_n]$, which consists of W_H-invariant polynomials (see [3, Proposition 27.1]). Let $\langle \mathbb{Z}[u_1, \cdots, u_n]_+^{W_K} \rangle$ be the ideal in $\mathbb{Z}[u_1, \cdots, u_n]^{W_H}$ generated by homogeneous W_K-invariant polynomials of positive degrees. Then we have

$$H^*(K/H, \mathbb{Z}) \cong \mathbb{Z}[u_1, \cdots, u_n]^{W_H} / \langle \mathbb{Z}[u_1, \cdots, u_n]_+^{W_K} \rangle. \tag{2.7}$$

Note that the inverse of the above isomorphism is given by transgression (see [3, Theorem 22.2]): For any $f(u_1, \cdots, u_n) \in \mathbb{Z}[u_1, \cdots, u_n]^{W_H}$, we regard $u_i, i = 1, \cdots, l$ as elements of $H^2(K/S, \mathbb{Z})$ via transgression, then $f(u_1, \cdots, u_n)$ is an element in $H^*(K/S, \mathbb{Z})$. The fact that $f(u_1, \cdots, u_n)$ is invariant under W_H implies that $f(u_1, \cdots, u_n)$ descends to an element in $H^*(K/H, \mathbb{Z})$.

Let (V, φ) be a complex representation of P_I of dimension l and $\lambda_1, \cdots, \lambda_l$ be the weights of V as a representation of T. Let $\mathcal{V} = G \times_{P_I, \varphi} V$ be the corresponding homogeneous vector bundle over X (see [6]). We can rewrite $\mathcal{V}$ in the compact picture

$$\mathcal{V} = K \times_{H,\varphi|_H} V. \tag{2.8}$$

By [5, Thereom 10.3], the Chern class of $\mathcal{V}$ is $c(\mathcal{V}) = \prod\limits_{i=1}^{l} \big(1 + \frac{1}{2\pi\sqrt{-1}}\lambda_i\big)$. The expression $\prod\limits_{i=1}^{l} \big(1 + \frac{1}{2\pi\sqrt{-1}}\lambda_i\big)$ is W_H-invariant since W_H acts on $\{\lambda_1, \cdots, \lambda_l\}$. In particular, if we take $V = \mathfrak{g}/\mathfrak{p}$ and let φ be the adjoint representation, $G \times_{P_I,\varphi} \mathfrak{g}/\mathfrak{p}$ is the holomorphic tangent bundle on G/P_I, which we denote by $\mathcal{T}_X$. We have

$$c(\mathcal{T}_X) = \prod_{\alpha\in R^+(G)\setminus R_I^+} \Big(1 + \frac{1}{2\pi\sqrt{-1}}\alpha\Big). \tag{2.9}$$

2.3 Torus actions on flag varieties

The compact torus S acts canonically on K/H by sending $(t, \bar{k}) \in S \times K/H$ to $\overline{tk}$. The fixed point set X^S can be determined completely (see [4]).

Proposition 2.1 (see [4])

$$X^S = \{wH \in K/H \mid w \in W_K\}. \tag{2.10}$$

In particular, X^S is bijective to W_K/W_H.

For any homogeneous vector bundle $\mathcal{V} = K \times_H V$, S acts on $\mathcal{V}$ by sending $(t,(k,v)) \in S \times \mathcal{V}$ to $(tk, v) \in \mathcal{V}$. Under these actions, the projection $\mathcal{V} \to K/H$ is S-equivariant. For any $kH \in K/H$, let $\mathcal{V}|_{kH}$ be the fiber of $\mathcal{V}$ over kH. Then for any fixed point $wH \in X^S$, S acts on $\mathcal{V}|_{wH}$. Suppose that the weights of V as a representation of T are $\lambda_1, \cdots, \lambda_n$. Then the weight of $\mathcal{V}|_{wH}$ can be calculated as follows. Since $\mathcal{V}|_{wH} = \{w\} \times_H V$, for any $t \in S, (w, v) \in \{w\} \times_H V$ we have $t(w, v) = (tw, v) = (ww^{-1}tw, v) = (w, w^{-1}twv) \in \{w\} \times_H V$. Hence one can see that the weights of $\mathcal{V}|_{wH}$ are $w\lambda_1, \cdots, w\lambda_n$.

For any homogeneous symmetric polynomial $Q(x_1, \cdots, x_l)$, $l = \dim V$, it can be written as a polynomial in elementary symmetric polynomials, say $\widetilde{Q}(e_1, \cdots, e_l)$. Let $c_i^S(\mathcal{V})$, $i = 1, \cdots, l$, be the equivariant Chern classes of V. By Atiyah-Bott-Berline-Vergne formula, we can express $\int_X \widetilde{Q}(c_1^S(\mathcal{V}), \cdots, c_l^S(\mathcal{V}))$ as the sums of contributions at fixed points

$$\int_X \widetilde{Q}(c_1^S(\mathcal{V}), \cdots, c_l^S(\mathcal{V})) = (2\pi\sqrt{-1})^{\dim X} \sum_{w\in W_K/W_H} \frac{Q(wu_1, \cdots, wu_l)}{\prod\limits_{\alpha\in R^+(G)\setminus R_I^+} w\alpha}, \tag{2.11}$$

where $u_i = \frac{1}{2\pi\sqrt{-1}}\lambda_i$, $i = 1, \cdots, l$.

In the following sections, in order to use (2.11) we need to determine in each case the following datum:

(1) The fixed point set;

(2) the complementary roots $R^+(G) \setminus R_I^+$;

(3) the weights of the tautological bundles at fixed points.

3 Partial Flag Varieties of Type A

To illustrate our method, we include here the simplest case: The type A flag varieties. Now, let $Fl = Fl(a_1, \cdots, a_s; n)$ be the variety of type A parametrizing flags

$$E_1 \subset \cdots \subset E_s \subset \mathbb{C}^n, \tag{3.1}$$

satisfying $\dim E_i = a_i$. $SL(n;\mathbb{C})$ acts transitively on the set of such flags, and Fl can be identified with $SL(n;\mathbb{C})/P$ for some parabolic subgroup P. The torus $T = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by

$$(t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n) := (t_1 z_1, \cdots, t_n z_n), \tag{3.2}$$

which induces a T-action on Fl canonically. Note that unlike the type B,C,D cases, the torus we use here is not a maximal torus of $SL(n;\mathbb{C})$ ($SL(n;\mathbb{C})$ has rank $n-1$). In order to use the Atiyah-Bott-Berline-Vergne formula to compute $\int_{Fl} \widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*), \cdots, c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*))$, we need to determine the weights of the fibers of the tangent bundle and the bundles $(\mathcal{E}_i/\mathcal{E}_{i-1})^*$ at the fixed points. We refer the reader to [12] for an elementary analysis and we only state the results. The fixed point set of this action is indexed by the set $\mathcal{I} = \{(I_1, \cdots, I_s) \mid I_i \subset \{1, \cdots, k\}, |I_i| = r_i, I_i \cap I_j = \varnothing, \forall i \neq j\}$. For any $(I_1, \cdots, I_s) \in \mathcal{I}$, the corresponding fixed point $P_{I_1, \cdots, I_s}$ is the following flag:

$$\operatorname{span}\{e_i \mid i \in I_1\} \subset \operatorname{span}\{e_i \mid i \in I_1 \cup I_2\} \subset \cdots \subset \operatorname{span}\{e_i \mid i \in I_1 \cup \cdots \cup I_s\} \subset \mathbb{C}^n. \tag{3.3}$$

Let $\mathfrak{t}$ be the Lie algebra of T. Let $\lambda_1, \cdots, \lambda_n$ be the standard weights of T, i.e., $\lambda_i(X) = X_i$, $\forall X = (X_1, \cdots, X_n) \in \mathfrak{t}$. Then the weights of the tangent space of Fl at $P_{I_1, \cdots, I_s}$ are

$$\bigcup_{p=1}^{s} \{\lambda_i - \lambda_j \mid j \in I_p, i \notin I_1 \cup \cdots \cup I_p\}. \tag{3.4}$$

The weights of $(\mathcal{E}_i/\mathcal{E}_{i-1})^*|_{P_{I_1, \cdots, I_s}}$ are

$$\{-\lambda_j \mid j \in I_i\}. \tag{3.5}$$

Denote $u_i = \frac{1}{2\pi\sqrt{-1}}\lambda_i$, $i = 1, \cdots, n$. For any $I = \{i_1, \cdots, i_p\} \subset [n]$ $(i_1 < \cdots < i_p)$, denote $u_I = (u_{i_1}, \cdots, u_{i_p})$. Using the Atiyah-Bott-Berline-Vergne formula and combining (3.4)–(3.5), we have

$$\int_{Fl} \widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*), \cdots, c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*)) = \sum_{(I_1, \cdots, I_s) \in \mathcal{I}} C_{I_1, \cdots, I_s}, \tag{3.6}$$

where

$$C_{I_1, \cdots, I_s} = \frac{Q(-u_{I_1}, \cdots, -u_{I_s})}{\prod\limits_{p=1}^{s} \prod\limits_{\substack{j \in I_p \\ i \notin I_1 \cup \cdots \cup I_p}} (u_i - u_j)}. \tag{3.7}$$

We can rewrite $C_{I_1,\cdots,I_s}$ as

$$\frac{Q(-u_{I_1},\cdots,-u_{I_s})\prod\limits_{\substack{i,j\in I_1\cup\cdots\cup I_s\\ i\neq j}}(u_i-u_j)}{\prod\limits_{j\in I_1\cup\cdots\cup I_s}\prod\limits_{\substack{1\leq i\leq n\\ i\neq j}}(u_i-u_j)\prod\limits_{p=1}^{s}\prod\limits_{j\in I_p}\prod\limits_{i\in I_{p+1}\cup\cdots\cup I_s}(u_i-u_j)},\tag{3.8}$$

and we recognize it as the residue:

$$\mathrm{Res}_{z=(-u_{I_1},\cdots,-u_{I_s})}\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\leq i,j\leq k\\ i\neq j}}(z_i-z_j)}{\prod\limits_{1\leq j\leq k}\prod\limits_{1\leq i\leq n}(u_i+z_j)\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\leq i\leq a_p\\ a_p+1\leq j\leq k}}(z_i-z_j)}\mathrm{d}\mathbf{z}.\tag{3.9}$$

Hence, we have

$$\begin{aligned}&\sum_{(I_1,\cdots,I_s)\in\mathcal{I}}C_{I_1,\cdots,I_s}\\ &=\sum_{(I_1,\cdots,I_s)\in\mathcal{I}}\mathrm{Res}_{z=(-u_{I_1},\cdots,-u_{I_s})}\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\leq i,j\leq k\\ i\neq j}}(z_i-z_j)}{\prod\limits_{1\leq j\leq k}\prod\limits_{1\leq i\leq n}(u_i+z_j)\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\leq i\leq a_p\\ a_p+1\leq j\leq k}}(z_i-z_j)}\mathrm{d}\mathbf{z}\\ &=\mathrm{Res}_\infty\frac{(-1)^k}{r_1!\cdots r_s!}\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\leq i,j\leq k\\ i\neq j}}(z_i-z_j)}{\prod\limits_{1\leq j\leq k}\prod\limits_{1\leq i\leq n}(u_i+z_j)\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\leq i\leq a_p\\ a_p+1\leq j\leq k}}(z_i-z_j)}\mathrm{d}\mathbf{z},\end{aligned}$$

where the last equality follows from the residue theorem and the factor $r_1!\cdots r_s!$ comes from the $S_{r_1}\times\cdots\times S_{r_s}$-invariance of the function

$$\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\leq i,j\leq k\\ i\neq j}}(z_i-z_j)}{\prod\limits_{1\leq j\leq k}\prod\limits_{1\leq i\leq n}(u_i+z_j)\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\leq i\leq a_p\\ a_p+1\leq j\leq k}}(z_i-z_j)}.$$

We have proved the following theorem which was originally proved by Zielenkiewicz in [14, Theorem 4.19].

Theorem 3.1 *Let* $I_A(Q)=\int_{Fl(a_1,\cdots,a_s;n)}\widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*),\cdots,c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*))$*. Then we have*

$$I_A(Q)=\mathrm{Res}_\infty\frac{(-1)^k}{r_1!\cdots r_s!}\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\leq i,j\leq k\\ i\neq j}}(z_i-z_j)}{\prod\limits_{1\leq j\leq k}\prod\limits_{1\leq i\leq n}(u_i+z_j)\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\leq i\leq a_p\\ a_p+1\leq j\leq k}}(z_i-z_j)}\mathrm{d}\mathbf{z}.\tag{3.10}$$

Taking the nonequivariant limit, we have the folloming corollary

Corollary 3.1 *Let* $u_{a_{p-1}+1}, \cdots, u_{a_p}$ *be the Chern roots of* $(\mathcal{E}_p/\mathcal{E}_{p-1})^*$. *Then*

$$\int_{Fl(a_1,\cdots,a_s;n)} Q(u_1,\cdots,u_k) = \mathrm{Res}_{\mathbf{z}=0}\frac{1}{r_1!\cdots r_s!}\frac{Q(z_1,\cdots,z_k)\prod\limits_{\substack{1\le i,j\le k\\ i\ne j}}(z_i-z_j)}{(z_1\cdots z_k)^n\prod\limits_{p=1}^{s}\prod\limits_{\substack{a_{p-1}+1\le i\le a_p\\ a_p+1\le j\le k}}(z_i-z_j)}\mathrm{d}\mathbf{z}. \tag{3.11}$$

The formula (3.11) coincides with the formula obtained in [11]. In [11], it was proved by using the structure of the cohomology ring.

4 Flag Varieties of Type D

4.1 The flag varieties of type D

Consider the vector space $\mathbb{C}^{2n}$, and fix a nondegenerate symmetric bilinear form $\mathfrak{S}$ on it. In this section, we consider flags in $\mathbb{C}^{2n}$:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s \subset \mathbb{C}^{2n}, \tag{4.1}$$

satisfying $\dim E_i = a_i$ and $\mathfrak{S}|_{E_i\times E_i} = 0$, $\forall 1 \le i \le s$. We denote by $Fl^O = Fl^O(a_1,\cdots,a_s;2n)$ the variety parametrizing the isotropic flags (4.1). Fix a basis $e_1,\cdots,e_n,f_1,\cdots,f_n$ of $\mathbb{C}^{2n}$ such that $\mathfrak{S}(e_i,e_j)=0=\mathfrak{S}(f_i,f_j), \mathfrak{S}(e_i,f_j)=\delta_{ij}$. With respect to this basis, $\mathfrak{S}$ has the following matrix representation:

$$\begin{pmatrix} 0 & I_n \\ I_n & 0, \end{pmatrix},$$

where I_n is the $n\times n$ identity matrix. Let $SO(2n;\mathbb{C})$ be the group of linear isomorphisms of $\mathbb{C}^{2n}$ preserving $\mathfrak{S}$ with determinant 1. Then $SO(2n;\mathbb{C})$ acts on Fl^O in a natural way, and we denote by $P_{a_1,\cdots,a_s}$ the isotropy group at the following flag:

$$\mathrm{span}\{f_1\cdots,f_{a_1}\} \subset \mathrm{span}\{f_1\cdots,f_{a_2}\} \subset \cdots \subset \mathrm{span}\{f_1\cdots,f_{a_s}\} \subset \mathbb{C}^{2n}. \tag{4.2}$$

With respect to the basis $e_1,\cdots,e_n,f_1,\cdots,f_n$, matrices in $P_{a_1,\cdots,a_s}$ take the following form:

$$\left(\begin{array}{c:cc} & \overbrace{0}^{a_s} & \overbrace{B_1}^{n-a_s} \\ A & 0 & B_2 \\ \hdashline C & U & V \\ & 0 & W \end{array}\right)\begin{array}{l} \}a_s \\ \}n-a_s \\ \}a_s \\ \}n-a_s \end{array} \quad , \tag{4.3}$$

where U is a block upper-triangular matrix

$$\begin{pmatrix} U_1 & * & * \\ & \ddots & * \\ & & U_s \end{pmatrix} \tag{4.4}$$

with $U_i \in M_{r_i \times r_i}(\mathbb{C})$, where $r_i = a_i - a_{i-1}$. When $a_s < n$, the action of $SO(n)$ is transitive and we can identify $Fl^O(a_1, \cdots, a_s; 2n)$ with $SO(2n; \mathbb{C})/P_{a_1,\cdots,a_s}$. When $a_s = n$, $Fl^O(a_1, \cdots, a_s; 2n)$ has two isomorphic connected components, and $SO(n)$ acts transitively on each of the components. In the following, we assume that $a_s < n$. However, the results also hold for the case $a_s = n$, and the reader can check that the proof is exactly the same. In the case $a_s = n$, although (4.20) only contains the fixed points lying on one of the connected components, the whole fixed point set is again indexed by (4.23) and is given by the formula (4.24).

Denote the Lie algebras of $SO(2n; \mathbb{C})$ and $P_{a_1,\cdots,a_s}$ by $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Let T be the maximal torus in $SO(2n; \mathbb{C})$ which consists of diagonal matrices in the following form

$$\mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}), \quad t_i \in \mathbb{C}^*,\ i = 1, \cdots, n,$$

and let $\mathfrak{t}$ be its Lie algebra. Choose an ordering on $\mathfrak{t}^*$ such that the simple roots of $SO(2n; \mathbb{C})$ are

$$\lambda_1 - \lambda_2, \cdots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n,$$

where $\lambda_i(X) = X_i$, $\forall X = \mathrm{diag}(X_1, \cdots, X_n, -X_1, \cdots, -X_n) \in \mathfrak{t}$, $1 \le i \le n$. By a direct computation, one can check that $P_{a_1,\cdots,a_s}$ is exactly the parabolic subgroup associated to the roots

$$\lambda_{a_1} - \lambda_{a_1+1}, \lambda_{a_2} - \lambda_{a_2+1}, \cdots, \lambda_{a_s} - \lambda_{a_s+1} \tag{4.5}$$

(when $a_s = n-1$, $P_{a_1,\cdots,a_s}$ is the parabolic subgroup associated to $\lambda_{a_1} - \lambda_{a_1+1}, \lambda_{a_2} - \lambda_{a_2+1}, \cdots,$ $\lambda_{a_{s-1}} - \lambda_{a_{s-1}+1}, \lambda_{a_s} - \lambda_{a_s+1}, \lambda_{n-1} + \lambda_n$; when $a_s = n$, $P_{a_1,\cdots,a_s}$ is the parabolic subgroup associated to $\lambda_{a_1} - \lambda_{a_1+1}, \lambda_{a_2} - \lambda_{a_2+1}, \cdots, \lambda_{a_{s-1}} - \lambda_{a_{s-1}+1}, \lambda_{n-1} + \lambda_n$), and $\mathfrak{p}$ consists of the matrices of the following form

$$\begin{array}{rc} & \begin{array}{cccc} \overbrace{\quad}^{a_s} & \overbrace{\quad}^{n-a_s} & \overbrace{\quad}^{a_s} & \overbrace{\quad}^{n-a_s} \end{array} \\ \begin{array}{r} a_s\{ \\ n-a_s\{ \\ a_s\{ \\ n-a_s\{ \end{array} & \left(\begin{array}{cc:cc} A & 0 & 0 & 0 \\ B & C & 0 & G \\ \hdashline D & E & -A^t & -B^t \\ -E^t & F & 0 & -C^t \end{array}\right) \end{array}, \tag{4.6}$$

where D, F, G are skew-symmetric matrices, B, C, E are arbitrary matrices whose sizes are indicated in (4.6) and A is a block lower triangular matrix

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ * & & \ddots \\ * & * & A_s \end{pmatrix} \tag{4.7}$$

with $A_i \in M_{r_i \times r_i}(\mathbb{C})$. Hence the weights of the adjoint representation of $P_{a_1,\cdots,a_s}$ on $\mathfrak{g}/\mathfrak{p}$ are

$$\begin{aligned} &\{\lambda_i + \lambda_j \mid 1 \le i < j \le a_s\} \\ \cup&\{\lambda_i \pm \lambda_j \mid 1 \le i \le a_s, a_s + 1 \le j \le n\} \\ \cup&\{\lambda_i - \lambda_j \mid a_{m-1} + 1 \le i \le a_m, a_m + 1 \le j \le a_s, m = 1, \cdots, s-1\}. \end{aligned} \tag{4.8}$$

By (4.8), we have

$$\dim Fl^O(a_1, \cdots, a_s; 2n) = \binom{a_s}{2} + 2a_s(n - a_s) + \sum_{1 \le i < j \le s} r_i r_j. \tag{4.9}$$

Let $\widetilde{\iota} : Fl^O(a_1, \cdots, a_s; 2n) \to Gr(a_i; 2n)$ be the canonical map sending $0 \subset E_1 \subset \cdots \subset E_s \subset \mathbb{C}^{2n}$ to E_i. $\widetilde{\iota}$ fits into the following commutative diagram

$$\begin{tikzcd} Fl^O(a_1, \cdots, a_s; 2n) \arrow[r, "\widetilde{\iota}"] \arrow[d] & Gr(a_i; 2n) \arrow[d] \\ SO(2n; \mathbb{C})/P_{a_1, \cdots, a_s} \arrow[r, "\iota"] & SL(2n; \mathbb{C})/P_{a_i} \end{tikzcd} , \tag{4.10}$$

where ι is induced by the canonical inclusion of $SO(2n; \mathbb{C})$ into $SL(2n; \mathbb{C})$ and P_{a_i} is the subgroup of $SL(2n; \mathbb{C})$ consisting of the matrices of the following form

$$\left(\begin{array}{c:cc} \multirow{2}{*}{A} & \overbrace{0}^{a_i} & \overbrace{B_1}^{n-a_i} \\ & 0 & B_2 \\ \hdashline \multirow{2}{*}{C} & U & V \\ & 0 & W \end{array}\right)\!\!\begin{array}{l} \}a_i \\ \}n-a_i \\ \}a_i \\ \}n-a_i \end{array} \quad . \tag{4.11}$$

One can check that the universal vector bundle over $Gr(a_i; 2n)$ can be identified with the following homogeneous vector bundle

$$SL(2n; \mathbb{C}) \times_{P_{a_i}, \rho_i} \mathbb{C}^{a_i}, \tag{4.12}$$

where ρ_i is the representation of P_{a_i} given by

$$\rho_i(g)(v) = U \cdot v \tag{4.13}$$

for any element $g \in P_{a_i}$ in the form (4.11) and any element $v \in \mathbb{C}^{a_i}$.

Since the universal bundle $\mathcal{E}_i$ over $Fl^O(a_1, \cdots, a_s; 2n)$ is just the pullback of $\mathcal{E}$ under $\widetilde{\iota}$, $\mathcal{E}_i/\mathcal{E}_{i-1}$ can be identified with the following homogeneous vector bundle

$$\mathcal{E}_i/\mathcal{E}_{i-1} = SO(2n; \mathbb{C}) \times_{P_{a_1, \cdots, a_s}, \pi_i} \mathbb{C}^{r_i}, \tag{4.14}$$

where π_i is the representation of $P_{a_1, \cdots, a_s}$ given by

$$\pi_i(g)(v) = U_i \cdot v, \quad \forall v \in \mathbb{C}^{r_i} \tag{4.15}$$

for any element $g \in P_{a_1, \cdots, a_s}$ in the form (4.3).

By (4.14), the weights of $(\mathcal{E}_i/\mathcal{E}_{i-1})|_{p_0}$ are

$$-\lambda_{a_{i-1}+1}, \cdots, -\lambda_{a_i}, \tag{4.16}$$

and the Chern roots of $\mathcal{E}_i/\mathcal{E}_{i-1}$ are

$$-u_{a_{i-1}+1}, \cdots, -u_{a_i}, \tag{4.17}$$

where p_0 is the flag represented by (4.2) and

$$u_i = \frac{1}{2\pi\sqrt{-1}} \lambda_i, \ i = 1, \cdots, n.$$

Let K be a compact real form of $SO(2n;\mathbb{C})$, $H = K \cap P_{a_1,\cdots,a_s}$ and $S = K \cap T$ be the common maximal torus of K and H. Let W_K and W_H be the Weyl groups of K and H with respect to S. For any root α, denote the simple reflection with respect to α by s_α. Then

$$W_K = \langle s_{\lambda_1-\lambda_2}, \cdots, s_{\lambda_{n-1}-\lambda_n}, s_{\lambda_{n-1}+\lambda_n} \rangle, \tag{4.18}$$

$$W_H = \langle s_{\lambda_i-\lambda_{i+1}}, s_{\lambda_{n-1}+\lambda_n} \mid i \in [n-1] \setminus \{a_1, \cdots, a_s\} \rangle. \tag{4.19}$$

By a direct computation, W_K is generated by all permutations of $\lambda_1, \cdots, \lambda_n$ and even sign changes. Comparing (4.9) with (4.18), it follows immediately that W_H is generated by

$$\begin{aligned} &\text{permutations of } \{\lambda_{a_{i-1}+1}, \cdots, \lambda_{a_i}\}, \quad i = 1, \cdots, s, \\ &\text{permutations of } \{\lambda_{a_s+1}, \cdots, \lambda_n\}, \\ &\text{even sign changes of } \lambda_{a_s+1}, \cdots, \lambda_n. \end{aligned}$$

Any polynomial $Q(u_1, \cdots, u_k)$ which is symmetric with respect to the canonical $S_{r_1} \times \cdots \times S_{r_s}$ action is W_H-invariant, and therefore represents a cohomology class on $Fl^O(a_1, \cdots, a_s; 2n)$.

4.2 Torus action on $Fl^O(a_1, \cdots, a_s; 2n)$

As is discussed in Section 2, the fixed point set of the action of S on $Fl^O(a_1, \cdots, a_s; 2n)$ is

$$\{wH \in K/H \mid w \in W_K\}, \tag{4.20}$$

which is bijective to W_K/W_H. To determine the flag corresponding to wH, we recall the following notation.

Definition 4.1 *A $k \times k$ matrix M is called a permutation matrix if there exists a permutation $\sigma \in S_k$ such that the i-th row of M is $\varepsilon_{\sigma(i)}$, where*

$$\boldsymbol{\varepsilon}_i = (0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \cdots, 0) \ (i = 1, \cdots, k).$$

Let Per_n be the set of $n \times n$ permutation matrices. One can easily check that for any element g in $\mathrm{Per}_{2n} \cap \left\{ \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix} \right) \mid A, B \in M_{n\times n}(\mathbb{C})1 \right\}$, g preserves $\mathfrak{S}$ (here we identify g with the linear isomorphism that it represents with respect to the basis $e_1, \cdots, e_n, f_1, \cdots, f_n$). Note that $g^T g = I_{2n}$, hence if we further require $\det g = 1$ (i.e., g represents an even permutation), g must lie in $SO(2n;\mathbb{C})$. We can characterize the condition $\det g = 1$ in the following way.

Note that any g permutes $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$, and in turn determines a function $\phi_g : \{1, 2, \cdots, n\} \to \{\pm 1, \pm 2, \cdots, \pm n\}$ such that

$$g f_i = \begin{cases} f_{\phi_g(i)}, & \text{if } \phi_g(i) > 0, \\ e_{-\phi_g(i)}, & \text{if } \phi_g(i) < 0. \end{cases} \tag{4.21}$$

Then $\det g = 1$ if and only if

$$\sharp\{i \mid \phi_g(i) < 0, i \in [n]\} \text{ is even.} \tag{4.22}$$

Any $g \in \mathrm{Per}_{2n} \cap \left\{ \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix} \right) \mid A, B \in M_{n\times n}(\mathbb{C}) \right\}$ satisfying (4.22) lies in the normalizer of the maximal torus and in turn represents a element w in the Weyl group. This establishes a one-to-one correspondence between the set of elements in $\mathrm{Per}_{2n} \cap \left\{ \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix} \right) \mid A, B \in M_{n\times n}(\mathbb{C}) \right\}$ satisfying (4.22) and the Weyl group W_K. And under this correspondence, the fixed point wH is the flag obtained by acting g on the flag (4.2).

By the above description of the fixed point wH, geometrically, the fixed points can be indexed by the set

$$\mathcal{F} = \{(I_1, \cdots, I_s; J_1, \cdots, J_s) \mid I_i, J_i \subset \{1, \cdots, n\}, |I_i \cup J_i| = r_i, I_i \cap J_j = \emptyset, \forall i, j\}. \tag{4.23}$$

For simplicity, we use $(I; J)$ to denote $(I_1, \cdots, I_s; J_1, \cdots, J_s)$. For any $(I; J) \in \mathcal{F}$, the corresponding fixed point $P_{I;J}$ is the following flag

$$\mathrm{span}\left\{e_i, f_j \middle| \begin{array}{c} i \in I^1, \\ j \in J^1 \end{array} \right\} \subset \mathrm{span}\left\{e_i, f_j \middle| \begin{array}{c} i \in I^2, \\ j \in J^2 \end{array} \right\} \subset \cdots \subset \mathrm{span}\left\{e_i, f_j \middle| \begin{array}{c} i \in I^s \\ j \in J^s \end{array} \right\}, \tag{4.24}$$

where for any $(I; J) \in \mathcal{F}$ we denote $I^p = \bigcup\limits_{1\le m\le p} I_m$ and $J^p = \bigcup\limits_{1\le m\le p} J_m$, $p = 1, \cdots, s$.

Using this notation, the flag (4.2) is $P_{(\emptyset,\cdots,\emptyset);(\{1,\cdots,a_1\},\cdots,\{a_{s-1}+1,\cdots,a_s\})}$. Under the identification of λ_i with i, the Weyl group W_K acts on the set $\{\pm 1, \cdots, \pm n\}$, and for any $w \in W_K$ the fixed point wH is exactly $P_{I;J}$ such that the image of $[a_i]$ under w is $w([a_i]) = (-I^i) \cup J^i$, $i = 1, \cdots, s$. Since the weights of the torus action on the tangent space at wH is obtained by applying w to the weights of the tangent space at H. The weights of the tangent space at $P_{I,J}$ are

$$\begin{array}{l} \{\widetilde{\lambda}_i + \widetilde{\lambda}_j \mid i, j \in I^s \cup J^s, i < j\} \\ \cup\{\widetilde{\lambda}_i + \lambda_j \mid i \in I^s \cup J^s, j \notin I^s \cup J^s\} \cup \{\widetilde{\lambda}_i - \lambda_j \mid i \in I^s \cup J^s, j \notin I^s \cup J^s\} \\ \cup\{\widetilde{\lambda}_i - \widetilde{\lambda}_j \mid i \in I_m \cup J_m, j \in (I^s \cup J^s) \setminus (I^m \cup J^m),\ m = 1, \cdots, s\}, \end{array} \tag{4.25}$$

where

$$\widetilde{\lambda}_m = \begin{cases} \lambda_m, & \text{if } m \in J^s, \\ -\lambda_m, & \text{otherwise.} \end{cases}$$

Similarly, the weights of $\mathcal{E}_i/\mathcal{E}_{i-1}$ at $P_{I;J}$ are

$$\{-\widetilde{\lambda}_j \mid j \in I_i \cup J_i\}. \tag{4.26}$$

For any $(S_{r_1} \times \cdots \times S_{r_s})$-invariant polynomial, $Q(x_1, \cdots, x_k)$, consider the following integral

$$I_D(Q) = \int_{Fl^O(a_1,\cdots,a_s;2n)} \widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*), \cdots, c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*)). \tag{4.27}$$

By the Atiyah-Bott-Berline-Vergne formula, the above integral can be expressed by the following summation:

$$\sum_{(I;J)\in\mathcal{F}} \frac{Q(\widetilde{u}_{I_1\cup J_1}, \cdots, \widetilde{u}_{I_s\cup J_s})}{\prod\limits_{\substack{i,j\in I^s\cup J^s \\ i<j}} (\widetilde{u}_i + \widetilde{u}_j) \prod\limits_{\substack{i\in I^s\cup J^s, \\ j\notin I^s\cup J^s}} (u_i^2 - u_j^2) \prod\limits_{m=1}^{s} \prod\limits_{\substack{i\in I_m\cup J_m, \\ j\in(I^s\cup J^s)\setminus(I^m\cup J^m)}} (\widetilde{u}_i - \widetilde{u}_j)}, \tag{4.28}$$

where for any $i = 1, \cdots, s$, if $I_i \cup J_i = \{j_1, \cdots, j_{r_i}\}$ $(j_1 < \cdots < j_r)$, we use $\tilde{u}_{I_i \cup J_i}$ to denote $(\widetilde{u}_{j_1}, \cdots, \widetilde{u}_{j_{r_i}})$, where

$$\widetilde{u}_j = \begin{cases} u_j, & \text{if } j \in J_i, \\ -u_j, & \text{otherwise.} \end{cases}$$

To avoid nested subscripts, we simply denote a_s by k. In the following, we give a second proof of the following theorem which was originally proved by Zielenkiewicz in [14, Theorem 4.25] using symplectic reduction.

Theorem 4.1

$$\begin{aligned} & I_D(Q) \\ & = (-1)^k \frac{2^k}{r_1! \cdots r_s!} \operatorname{Res}_\infty \frac{Q(z_1, \cdots, z_k) \prod\limits_{i>j} (z_i^2 - z_j^2) \prod\limits_{i=1}^{k} z_i \prod\limits_{\substack{1 \le m \le s \\ a_{m-1} < i < j \le a_m}} (z_i - z_j)}{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{k} (z_j - u_i)(z_j + u_i)} \mathrm{d}\mathbf{z}. \end{aligned} \tag{4.29}$$

To prove the theorem, we make some preparations. Let

$$f(z) = \frac{Q(z_1, \cdots, z_k) \prod\limits_{i>j} (z_i^2 - z_j^2) \prod\limits_{i=1}^{k} z_i \prod\limits_{m=1}^{s} \prod\limits_{a_{m-1} < i < j \le a_m} (z_i - z_j)}{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{k} (z_j - u_i)(z_j + u_i)}. \tag{4.30}$$

It is obvious that the set of poles of f that contributes nonzero residues can be identified with

$$\mathcal{P} = \{(\varepsilon, \upsilon) \mid \varepsilon : [k] \to \{-1, 1\}, \upsilon : [k] \to [n], \upsilon \text{ injective}\}, \tag{4.31}$$

where any $(\varepsilon, \upsilon) \in \mathcal{P}$ corresponds to the pole $(z_1, \cdots, z_k) = (\varepsilon(1) u_{\upsilon(1)}, \cdots, \varepsilon(k) u_{\upsilon(k)})$. There is a canonical projection

$$\pi : \mathcal{P} \to \mathcal{F} \tag{4.32}$$

which sends (ε, υ) to $(I, J) = (I_1, \cdots, I_s; J_1, \cdots, J_s) \in \mathcal{F}$, where

$$I_m = \upsilon([a_m] \setminus [a_{m-1}] \cap \epsilon^{-1}(-1)), \quad J_m = \upsilon([a_m] \setminus [a_{m-1}] \cap \varepsilon^{-1}(1)), \quad m = 1, \cdots, s. \tag{4.33}$$

The symmetric group S_k acts on $\mathcal{P}$ via acting on $[k]$, and its subgroup $S_{r_1} \times \cdots \times S_{r_s}$ acts transitively on each fiber of π. In particular, $|\mathcal{P}| = r_1! \cdots r_s! |\mathcal{F}|$. A key observation that will be used in the proof of Theorem 4.1 is that the residues of f are the same at poles in a single fiber of π. This follows from the fact that f is $S_{r_1} \times \cdots \times S_{r_s}$-invariant. To see this, we rewrite f as follows

$$f(z_1, \cdots, z_k) = \frac{Q(z_1, \cdots, z_k) \prod\limits_{i \ne j} (z_i - z_j) \prod\limits_{i<j} (z_i + z_j) \prod\limits_{i=1}^{k} z_i}{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{k} (z_j - u_i)(z_j + u_i) \prod\limits_{m=1}^{s} \prod\limits_{\substack{a_{m-1}+1 \le i \le a_m, \\ a_m + 1 \le j \le k}} (z_i - z_j)}. \tag{4.34}$$

Then the $S_{r_1} \times \cdots \times S_{r_s}$-invariance of f follows from the $S_{r_1} \times \cdots \times S_{r_s}$-invariance of

$$\prod_{m=1}^{s} \prod_{\substack{a_{m-1}+1 \le i \le a_m, \\ a_m+1 \le j \le k}} (z_i - z_j),$$

which is obvious since

$$\prod_{\substack{a_{m-1}+1 \le i \le a_m, \\ a_m+1 \le j \le k}} (z_i - z_j) is S_{r_m} \times S_{r_{m+1}+\cdots+r_s} - invariant.$$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1 By the residue theorem, $(-1)^k \mathrm{Res}_\infty f(z)\mathrm{d}\mathbf{z} = \sum\limits_{p \in \mathcal{P}} \mathrm{Res}_p f(z)\mathrm{d}\mathbf{z}$. Using the $S_{r_1} \times \cdots \times S_{r_s}$-invariance of f, we have

$$\sum_{p \in \mathcal{P}} \mathrm{Res}_p f(z)\mathrm{d}\mathbf{z} = \sum_{(I,J) \in \mathcal{F}} r_1! \cdots r_s! \mathrm{Res}_{z_{I,J}} f(z)\mathrm{d}\mathbf{z}, \tag{4.35}$$

where $z_{I,J}$ is a pole of f such that $\pi(z_{I,J}) = (I, J) \in \mathcal{F}$. We can compute $\mathrm{Res}_{z_{I,J}} f(z)\mathrm{d}\mathbf{z}$ directly

$$\begin{aligned}
&\mathrm{Res}_{z_{I,J}} f(z)\mathrm{d}\mathbf{z} \\
=&\mathrm{Res}_{z_{I,J}} \frac{Q(z_1, \cdots, z_k) \prod\limits_{i \ne j}(z_i - z_j) \prod\limits_{i<j}(z_i + z_j) \prod\limits_{i=1}^{k} z_i}{\prod\limits_{i=1}^{n} \prod\limits_{j=1}^{k} (z_j - u_i)(z_j + u_i) \prod\limits_{m=1}^{s} \prod\limits_{\substack{a_{m-1}+1 \le i \le a_m, \\ a_m+1 \le j \le k}} (z_i - z_j)} \mathrm{d}\mathbf{z} \\
=&\frac{Q(\widetilde{u}_{I_1 \cup J_1}, \cdots, \widetilde{u}_{I_s \cup J_s}) \prod\limits_{\substack{i,j \in I^s \cup J^s \\ i \ne j}} (\widetilde{u}_i - \widetilde{u}_j) \prod\limits_{\substack{i,j \in I^s \cup J^s \\ i<j}} (\widetilde{u}_i + \widetilde{u}_j) \prod\limits_{i \in I^s \cup J^s} (\widetilde{u}_i)}{\prod\limits_{j \in I^s \cup J^s} \prod\limits_{\substack{i=1 \\ i \ne j}}^{n} (\widetilde{u}_j - u_i)(\widetilde{u}_j + u_i) \cdot 2^k \prod\limits_{i \in I^s \cup J^s} (\widetilde{u}_i) \prod\limits_{m=1}^{s} \prod\limits_{\substack{i \in I_m \cup J_m, \\ j \in (I^s \cup J^s) \backslash (I^m \cup J^m)}} (\widetilde{u}_i - \widetilde{u}_j)} \\
=&\frac{Q(\widetilde{u}_{I_1 \cup J_1}, \cdots, \widetilde{u}_{I_s \cup J_s})}{2^k \prod\limits_{\substack{i,j \in I^s \cup J^s \\ i>j}} (\widetilde{u}_i + \widetilde{u}_j) \prod\limits_{\substack{i \notin I^s \cup J^s \\ j \in I^s \cup J^s}} (u_j^2 - u_i^2) \prod\limits_{m=1}^{s} \prod\limits_{\substack{i \in I_m \cup J_m, \\ j \in (I^s \cup J^s) \backslash (I^m \cup J^m)}} (\widetilde{u}_i - \widetilde{u}_j)}.
\end{aligned}$$

Substituting the last expression into (4.35) and comparing with (4.28)–(4.29) follows.

Taking $\lim\limits_{u \to 0}$ in (4.29), we have the following corollary.

Corollary 4.1 *Let $u_{a_{p-1}+1}, \cdots, u_{a_p}$ be the Chern roots of $(\mathcal{E}_p/\mathcal{E}_{p-1})^*$, then*

$$\begin{aligned}
&\int_{Fl^O(a_1, \cdots, a_s; 2n)} Q(u_1, \cdots, u_k) \\
&= \frac{2^k}{r_1! \cdots r_s!} \mathrm{Res}_{z=0} \frac{Q(z_1, \cdots, z_k) \prod\limits_{i>j}(z_i^2 - z_j^2) \prod\limits_{\substack{1 \le m \le s \\ a_{m-1}<i<j \le a_m}} (z_i - z_j)}{(z_1 \cdots z_k)^{2n-1}} \mathrm{d}\mathbf{z}.
\end{aligned} \tag{4.36}$$

5 Flag Varieties of Type B

5.1 The flag varieties of type B

Consider the vector space $\mathbb{C}^{2n+1}$, and let $\mathfrak{S}$ be a nondegenerate symmetric bilinear form on it. We denote by $Fl^O = Fl^O(a_1, \cdots, a_s; 2n+1)$ the flag variety parametrizing the isotropic flags

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s \subset \mathbb{C}^{2n+1} \tag{5.1}$$

satisfying $\dim_{\mathbb{C}} E_i = a_i$, $i = 1, \cdots, s$. Unlike the type D case, $Fl^O(a_1, \cdots, a_s; 2n+1)$ is always connected.

Fix a basis $g, e_1, \cdots, e_n, f_1, \cdots, f_n$ of $\mathbb{C}^{2n+1}$ such that $\mathfrak{S}(e_i, e_j) = \mathfrak{S}(f_i, f_j) = \mathfrak{S}(g, f_i) = \mathfrak{S}(g, e_i) = 0, \mathfrak{S}(e_i, f_j) = \delta_{ij}$, $\mathfrak{S}(g, g) = 1$. With respect to this basis, $\mathfrak{S}$ has the following matrix representation:

$$\left(\begin{array}{c:cc} 1 & & \\ \hdashline & & I \\ & I & \end{array}\right), \tag{5.2}$$

where I is the $n \times n$ identity matrix. Let $SO(2n+1; \mathbb{C})$ be the group of linear isomorphisms of $\mathbb{C}^{2n+1}$ preserving $\mathfrak{S}$. $SO(2n+1; \mathbb{C})$ acts on $Fl^O(a_1, \cdots, a_s; 2n+1)$ transitively, and we denote by $P_{a_1, \cdots, a_s}$ the isotropy group at the following flag

$$\text{span}\{f_1 \cdots, f_{a_1}\} \subset \text{span}\{f_1 \cdots, f_{a_2}\} \subset \cdots \subset \text{span}\{f_1 \cdots, f_{a_s}\} \subset \mathbb{C}^{2n}. \tag{5.3}$$

With respect to the basis $g, e_1, \cdots, e_n, f_1, \cdots, f_n$, matrices in $P_{a_1, \cdots, a_s}$ take the following form

$$\begin{array}{l} \left(\begin{array}{c:c:cc} c & \alpha & 0 & \beta' \\ \hdashline \gamma^t & \multirow{2}{*}{A} & 0 & B_1 \\ & & 0 & B_2 \\ \hdashline \delta^t & \multirow{2}{*}{C} & U & V \\ & & \underbrace{0}_{a_s} & \underbrace{W}_{n-a_s} \end{array}\right) \begin{array}{l} \\ \}a_s \\ \}n-a_s \\ \}a_s \\ \}n-a_s \end{array} \end{array}, \tag{5.4}$$

where $\alpha, \beta', \gamma, \delta$ are row vectors and U is a block upper-triangular matrix

$$\begin{pmatrix} U_1 & * & * \\ & \ddots & * \\ & & U_s \end{pmatrix} \tag{5.5}$$

with $U_i \in M_{r_i \times r_i}(\mathbb{C})$. In this way, we identify $Fl^O(a_1, \cdots, a_s; 2n+1)$ with $SO(2n+1; \mathbb{C})/P_{a_1, \cdots, a_s}$.

Denote the Lie algebras of $SO(2n+1;\mathbb{C})$ and $P_{a_1,\cdots,a_s}$ by $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Then $\mathfrak{g}$ consists of the matrices of the following form

$$\left(\begin{array}{c:c:c} 0 & \alpha & \beta \\ \hdashline -\beta^t & A & C \\ \hdashline -\alpha^t & D & -A^t \end{array}\right), \tag{5.6}$$

where C and D are skew-symmetric matrices.

Let T be the maximal torus in $SO(2n+1;\mathbb{C})$ consisting of the matrices of the following form

$$\mathrm{diag}(1, t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}), \quad t_i \in \mathbb{C}^*,\ i = 1, \cdots, n$$

and let $\mathfrak{t}$ be its Lie algebra. Choose an ordering on $\mathfrak{t}^*$ such that the simple roots of $SO(2n+1;\mathbb{C})$ are

$$\lambda_1 - \lambda_2, \cdots, \lambda_{n-1} - \lambda_n, \lambda_n, \tag{5.7}$$

where $\lambda_i(X) = X_i$, $\forall X = \mathrm{diag}(0, X_1, \cdots, X_n, -X_1, \cdots, -X_n) \in \mathfrak{t}$, $1 \le i \le n$. Let η_j be the j-th simple root in (5.7). By a direct computation, one can show that the parabolic subalgebra associated to η_j consists of the matrices of the following form

$$\begin{array}{r} \\ j\{ \\ n-j\{ \\ j\{ \\ n-j\{ \end{array}\left(\begin{array}{c:cc:cc} 0 & \overbrace{\alpha_1}^{j} & \overbrace{\alpha_2}^{n-j} & \overbrace{0}^{j} & \overbrace{\beta}^{n-j} \\ \hdashline 0 & A & 0 & 0 & 0 \\ -\beta^t & B & C & 0 & G \\ \hdashline -\alpha_1^t & D & E & -A^t & -B^t \\ -\alpha_2^t & -E^t & F & 0 & -C^t \end{array}\right), \tag{5.8}$$

where A, B, C, E are arbitrary matrices of given sizes, D, F, G are skew-symmetric matrices and $\alpha_1, \alpha_2, \beta$ are row vectors. Hence $P_{a_1,\cdots,a_s}$ is exactly the parabolic subgroup associated to $\eta_{a_1}, \cdots, \eta_{a_s}$, and its Lie algebra $\mathfrak{p}$ consists of the matrices of the following form

$$\begin{array}{r} \\ a_s\{ \\ n-a_s\{ \\ a_s\{ \\ n-a_s\{ \end{array}\left(\begin{array}{c:cc:cc} 0 & \overbrace{\alpha_1}^{a_s} & \overbrace{\alpha_2}^{n-a_s} & \overbrace{0}^{a_s} & \overbrace{\beta}^{n-a_s} \\ \hdashline 0 & A & 0 & 0 & 0 \\ -\beta^t & B & C & 0 & G \\ \hdashline -\alpha_1^t & D & E & -A^t & -B^t \\ -\alpha_2^t & -E^t & F & 0 & -C^t \end{array}\right), \tag{5.9}$$

where D, F, G are skew-symmetric matrices, B, C, E are arbitrary matrices whose sizes are indicated in (5.9) and A is a block lower triangular matrix

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ * & & \ddots \\ * & * & A_s \end{pmatrix} \tag{5.10}$$

with $A_i \in M_{r_i \times r_i}(\mathbb{C})$. By (5.9) the weights of the adjoint representation of $P_{a_1,\cdots,a_s}$ on $\mathfrak{g}/\mathfrak{p}$ are:

$$\begin{aligned} &\{\lambda_i + \lambda_j \mid 1 \le i < j \le a_s\} \\ \cup &\{\lambda_i \pm \lambda_j \mid 1 \le i \le a_s, a_s + 1 \le j \le n\} \\ \cup &\{\lambda_i - \lambda_j \mid a_{m-1} + 1 \le i \le a_m, a_m + 1 \le j \le a_s, m = 1, \cdots, s-1\} \\ \cup &\{\lambda_i \mid 1 \le i \le a_s\}. \end{aligned} \tag{5.11}$$

By (5.11), we have

$$\dim Fl^O(a_1,\cdots,a_s;2n+1)=\binom{a_s}{2}+2a_s(n-a_s)+\sum_{1\le i<j\le s}r_ir_j+a_s. \tag{5.12}$$

The quotients of the universal bundles can be written as homogeneous vector bundles in the following way. Let π_i be the representation of $P_{a_1,\cdots,a_s}$ on $\mathbb{C}^{r_i}$ given by

$$\pi_i(g)(v)=U_i\cdot v,\quad \forall v\in\mathbb{C}^{r_i} \tag{5.13}$$

for any element $g\in P_{a_1,\cdots,a_s}$ in the form (5.4). Then we have

$$\mathcal{E}_i/\mathcal{E}_{i-1}=SO(2n;\mathbb{C})\times_{P_{a_1,\cdots,a_s},\pi_i}\mathbb{C}^{r_i}. \tag{5.14}$$

Hence the Chern roots of $\mathcal{E}_i/\mathcal{E}_{i-1}$ are $-u_{a_{i-1}+1},\cdots,-u_{a_i}$, where

$$u_i=\frac{1}{2\pi\sqrt{-1}}\lambda_i.$$

Let K be a compact real form of $SO(2n+1;\mathbb{C})$, $H=K\cap P_{a_1,\cdots,a_s}$ and $S=K\cap T$ be the common maximal torus of K and H. Let W_K and W_H be the Weyl groups of K and H with respect to S. By a direct computation, W_K is generated by all permutations of $\lambda_1,\cdots,\lambda_n$ and all sign changes. W_H is generated by permutations of $\{\lambda_{a_{i-1}+1},\cdots,\lambda_{a_i}\}, i=1,\cdots,s$, permutations of $\{\lambda_{a_s+1},\cdots,\lambda_n\}$ and all sign changes of $\lambda_{a_s+1},\cdots,\lambda_n$.

5.2 Torus action on $Fl^O(a_1,\cdots,a_s;2n+1)$

As in Section 4, the fixed points of the torus action are also indexed by the set $\mathcal{F}$. For any $(I;J)\in\mathcal{F}$, the corresponding fixed point $P_{I;J}$ is the following flag

$$\operatorname{span}\left\{e_i,f_j\middle|\begin{array}{c}i\in I^1,\\ j\in J^1\end{array}\right\}\subset\operatorname{span}\left\{e_i,f_j\middle|\begin{array}{c}i\in I^2,\\ j\in J^2\end{array}\right\}\subset\cdots\subset\operatorname{span}\left\{e_i,f_j\middle|\begin{array}{c}i\in I^s\\ j\in J^s\end{array}\right\}. \tag{5.15}$$

The weights of the tangent space at $P_{I;J}$ are

$$\begin{aligned}&\{\widetilde{\lambda}_i+\widetilde{\lambda}_j\mid i,j\in I^s\cup J^s,i<j\}\\ \cup&\{\widetilde{\lambda}_i+\lambda_j\mid i\in I^s\cup J^s,j\notin I^s\cup J^s\}\cup\{\widetilde{\lambda}_i-\lambda_j\mid i\in I^s\cup J^s,j\notin I^s\cup J^s\}\\ \cup&\{\widetilde{\lambda}_i-\widetilde{\lambda}_j|i\in I_m\cup J_m,j\in(I^s\cup J^s)\setminus(I^m\cup J^m),m=1,\cdots,s\}\\ \cup&\{\widetilde{\lambda}_i\mid i\in I^s\cup J^s\}\end{aligned} \tag{5.16}$$

and the weights of $\mathcal{E}_i/\mathcal{E}_{i-1}$ at $P_{I;J}$ are

$$\{-\widetilde{\lambda}_j\mid j\in I_i\cup J_i\}. \tag{5.17}$$

For any $(S_{r_1}\times\cdots\times S_{r_s})$-invariant polynomial $Q(x_1,\cdots,x_k)$, we again use Atiyah-Bott-Berline-Vergne formula to express

$$\begin{aligned}I_B(Q)=\int_{Fl^O}&\widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*),\cdots,c_{r_1}^T((\mathcal{E}_1/\mathcal{E}_0)^*),\cdots,\\ &c_1^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*),\cdots,c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*)),\end{aligned} \tag{5.18}$$

as sums over fixed points

$$\sum_{(I;J)\in\mathcal{F}} \frac{Q(\widetilde{u}_i; i\in I^s\cup J^s)}{\prod\limits_{\substack{i,j\in I^s\cup J^s,\\ i<j}}(\widetilde{u}_i+\widetilde{u}_j)\prod\limits_{\substack{i\in I^s\cup J^s,\\ j\notin I^s\cup J^s}}(u_i^2-u_j^2)\prod\limits_{m=1}^{s}\prod\limits_{\substack{i\in I_m\cup J_m,\\ j\in(I^s\cup J^s)\setminus(I^m\cup J^m)}}(\widetilde{u}_i-\widetilde{u}_j)\prod\limits_{i\in I^s\cup J^s}(\widetilde{u}_i)}. \tag{5.19}$$

To avoid nested subscript, we again denote a_s by k. By modifying the proof of Theorem 4.1 slightly, one can prove (see [14, Theorem 4.26]) the following theorem.

Theeorem 5.1

$$\begin{aligned} & I_B(Q) \\ & = (-1)^k\frac{2^k}{r_1!\cdots r_s!}\mathrm{Res}_\infty \frac{Q(z_1,\cdots,z_k)\prod\limits_{i>j}(z_i^2-z_j^2)\prod\limits_{m=1}^{s}\prod\limits_{a_{m-1}+1\le i<j\le a_m}(z_i-z_j)}{\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{k}(z_j-u_i)(z_j+u_i)}\mathrm{d}\mathbf{z}. \end{aligned} \tag{5.20}$$

Taking $\lim\limits_{u\to 0}$ in (5.20), we have the following corollary.

Corollary 5.1 *Let $u_{a_{p-1}+1},\cdots,u_{a_p}$ be the Chern roots of $(\mathcal{E}_p/\mathcal{E}_{p-1})^*$, Then*

$$\begin{aligned} & \int_{Fl^O(a_1,\cdots,a_s;2n)} Q(u_1,\cdots,u_k) \\ & = \frac{2^k}{r_1!\cdots r_s!}\mathrm{Res}_{z=0}\frac{Q(z_1,\cdots,z_k)\prod\limits_{i>j}(z_i^2-z_j^2)\prod\limits_{\substack{1\le m\le s\\ a_{m-1}<i<j\le a_m}}(z_i-z_j)}{(z_1\cdots z_k)^{2n}}\mathrm{d}\mathbf{z}. \end{aligned} \tag{5.21}$$

6 Flag Varieties of Type C

6.1 The flag varieties of type C

Consider the vector space $\mathbb{C}^{2n}$, and fix a nondegenerate skew-symmetric bilinear form $\mathfrak{S}$ on it. Denote by $Fl^S = Fl^S(a_1,\cdots,a_s;2n)$ the flag variety parametrizing the isotropic flags

$$0 = E_0 \subset E_1 \subset\cdots\subset E_s \subset \mathbb{C}^{2n}, \tag{6.1}$$

satisfying $\dim_{\mathbb{C}} E_i = a_i$, $i=1,\cdots,s$. $Fl^S(a_1,\cdots,a_s;2n)$ is always connected.

Fix a basis $e_1,\cdots,e_n,f_1,\cdots,f_n$ of $\mathbb{C}^{2n}$ such that $\mathfrak{S}(e_i,e_j)=0=\mathfrak{S}(f_i,f_j), \mathfrak{S}(e_i,f_j)=\delta_{ij}$. With respect to this basis, $\mathfrak{S}$ has the following matrix representation

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $n\times n$ identity matrix. Let $Sp(2n;\mathbb{C})$ be the group of linear isomorphisms of $\mathbb{C}^{2n}$ preserving $\mathfrak{S}$. $Sp(2n;\mathbb{C})$ acts on $Fl^S(a_1,\cdots,a_s;2n)$ transitively, and we denote by $P_{a_1,\cdots,a_s}$ the isotropy group at the following flag

$$\mathrm{span}\{f_1\cdots,f_{a_1}\}\subset \mathrm{span}\{f_1\cdots,f_{a_2}\}\subset\cdots\subset\mathrm{span}\{f_1\cdots,f_{a_s}\}\subset\mathbb{C}^{2n}. \tag{6.2}$$

With respect to the basis $e_1, \cdots, e_n, f_1, \cdots, f_n$, matrices in $P_{a_1,\cdots,a_s}$ take the following form

$$\begin{array}{cl}
\begin{array}{ccc} & \overbrace{\quad}^{a_s} & \overbrace{\quad}^{n-a_s} \end{array} & \\
\left(\begin{array}{c:cc} A & 0 & B_1 \\ & 0 & B_2 \\ \hdashline C & U & V \\ & 0 & W \end{array}\right) & \begin{array}{l} \}a_s \\ \}n-a_s \\ \}a_s \\ \}n-a_s \end{array}
\end{array}, \tag{6.3}$$

where U is a block upper-triangular matrix

$$\begin{pmatrix} U_1 & * & * \\ & \ddots & * \\ & & U_s \end{pmatrix} \tag{6.4}$$

with $U_i \in M_{r_i \times r_i}(\mathbb{C})$. In this way, we identify $Fl^S(a_1, \cdots, a_s; 2n)$ with $Sp(2n;\mathbb{C})/P_{a_1,\cdots,a_s}$.

Denote the Lie algebras of $Sp(2n;\mathbb{C})$ and $P_{a_1,\cdots,a_s}$ by $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Then $\mathfrak{g}$ consists of matrices of the following form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \tag{6.5}$$

where B and C are symmetric matrices.

Let T be the maximal torus in $Sp(2n;\mathbb{C})$ consisting of matrices of the following forms

$$\mathrm{diag}(t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}), \quad t_i \in \mathbb{C}^*,\ i = 1, \cdots, n,$$

and let $\mathfrak{t}$ be its Lie algebra. Choose an ordering on $\mathfrak{t}^*$ such that the simple roots of $Sp(2n;\mathbb{C})$ are

$$\lambda_1 - \lambda_2, \cdots, \lambda_{n-1} - \lambda_n, 2\lambda_n, \tag{6.6}$$

where $\lambda_i(X) = X_i$, $\forall X = \mathrm{diag}(X_1, \cdots, X_n, -X_1, \cdots, -X_n) \in \mathfrak{t}$, $1 \le i \le n$. Let η_j be the j-th simple root in (6.6). By a direct computation, one can show that the parabolic subalgebra associated to η_j consists of the matrices of the following form

$$\begin{array}{rc}
 & \begin{array}{cccc} \overbrace{\ }^{j} & \overbrace{\quad}^{n-j} & \overbrace{\ }^{j} & \overbrace{\quad}^{n-j} \end{array} \\
\begin{array}{r} j\{ \\ n-j\{ \\ j\{ \\ n-j\{ \end{array} & \left(\begin{array}{cc:cc} A & 0 & 0 & 0 \\ B & C & 0 & G \\ \hdashline D & E & -A^t & -B^t \\ E^t & F & 0 & -C^t \end{array}\right)
\end{array}, \tag{6.7}$$

where A, B, C, E are arbitrary matrices of given sizes and D, F, G are symmetric matrices. Hence $P_{a_1,\cdots,a_s}$ is exactly the parabolic subgroup associated to $\eta_{a_1}, \cdots, \eta_{a_s}$, and its Lie algebra $\mathfrak{p}$ consists of the matrices of the following form

$$\begin{array}{rc}
 & \begin{array}{cccc} \overbrace{\ }^{a_s} & \overbrace{\quad}^{n-a_s} & \overbrace{\ }^{a_s} & \overbrace{\quad}^{n-a_s} \end{array} \\
\begin{array}{r} a_s\{ \\ n-a_s\{ \\ a_s\{ \\ n-a_s\{ \end{array} & \left(\begin{array}{cc:cc} A & 0 & 0 & 0 \\ B & C & 0 & G \\ \hdashline D & E & -A^t & -B^t \\ E^t & F & 0 & -C^t \end{array}\right)
\end{array}, \tag{6.8}$$

where D, F, G are symmetric matrices, B, C, E are arbitrary matrices whose sizes are indicated in (6.8) and A is a block lower triangular matrix

$$\begin{pmatrix} A_1 & & \\ * & \ddots & \\ * & * & A_s \end{pmatrix} \tag{6.9}$$

with $A_i \in M_{r_i \times r_i}(\mathbb{C})$. By (6.8) the weights of the adjoint representation of $P_{a_1,\cdots,a_s}$ on $\mathfrak{g}/\mathfrak{p}$ are

$$\begin{aligned} &\{2\lambda_i \mid 1 \le i \le a_s\} \cup \{\lambda_i + \lambda_j | 1 \le i < j \le a_s\} \\ \cup &\{\lambda_i \pm \lambda_j \mid 1 \le i \le a_s, a_s + 1 \le j \le n\} \\ \cup &\{\lambda_i - \lambda_j \mid a_{m-1} + 1 \le i \le a_m, a_m + 1 \le j \le a_s, m = 1, \cdots, s-1\}. \end{aligned} \tag{6.10}$$

By (6.10), we have

$$\dim Fl^S(a_1, \cdots, a_s; 2n) = \binom{a_s}{2} + 2a_s(n - a_s) + \sum_{1 \le i < j \le s} r_i r_j + a_s. \tag{6.11}$$

By a similar argument as in Section 4, the Chern roots of $\mathcal{E}_i/\mathcal{E}_{i-1}$ are $-u_{a_{i-1}+1}, \cdots, -u_{a_i}$, where $u_i = \frac{1}{2\pi\sqrt{-1}}\lambda_i$.

Let K, H, S, W_K, W_H be as in the previous cases. By a direct computation, W_K is generated by all permutations of $\lambda_1, \cdots, \lambda_n$ and all sign changes. W_H is generated by

$$\begin{gathered} \text{permutations of } \{\lambda_{a_{i-1}+1}, \cdots, \lambda_{a_i}\}, \quad i = 1, \cdots, s, \\ \text{permutations of } \{\lambda_{a_s+1}, \cdots, \lambda_n\}, \\ \text{sign changes of } \lambda_{a_s+1}, \cdots, \lambda_n. \end{gathered}$$

6.2 Torus action on $Fl^S(a_1, \cdots, a_s; 2n)$

As in Section 4, the fixed points are also indexed by the set $\mathcal{F}$. For any $(I, J) \in \mathcal{F}$, the corresponding fixed point $P_{I,J}$ is the following flag

$$\operatorname{span}\left\{e_i, f_j \middle| \begin{array}{l} i \in I^1, \\ j \in J^1 \end{array}\right\} \subset \operatorname{span}\left\{e_i, f_j \middle| \begin{array}{l} i \in I^2, \\ j \in J^2 \end{array}\right\} \subset \cdots \subset \operatorname{span}\left\{e_i, f_j \middle| \begin{array}{l} i \in I^s \\ j \in J^s \end{array}\right\}. \tag{6.12}$$

The weights of the tangent space at $P_{I,J}$ are

$$\begin{aligned} &\{\widetilde{\lambda}_i \mid i \in I^s \cup J^s\} \cup \{\widetilde{\lambda}_i + \widetilde{\lambda}_j \mid i, j \in I^s \cup J^s, i < j\} \\ \cup &\{\widetilde{\lambda}_i + \lambda_j \mid i \in I^s \cup J^s, j \notin I^s \cup J^s\} \cup \{\widetilde{\lambda}_i - \lambda_j \mid i \in I^s \cup J^s, j \notin I^s \cup J^s\} \\ \cup &\{\widetilde{\lambda}_i - \widetilde{\lambda}_j \mid i \in I_m \cup J_m, j \in (I^s \cup J^s) \setminus (I^m \cup J^m), m = 1, \cdots, s\}, \end{aligned} \tag{6.13}$$

where

$$\widetilde{\lambda}_m = \begin{cases} \lambda_m, & \text{if } m \in J^s, \\ -\lambda_m, & \text{otherwise.} \end{cases}$$

Similarly, the weights of $\mathcal{E}_i/\mathcal{E}_{i-1}$ at $P_{I,J}$ are

$$\{-\widetilde{\lambda}_j \mid j \in I_i \cup J_i\}. \tag{6.14}$$

For any $(S_{r_1}\times\cdots\times S_{r_s})$-invariant polynomial $Q(x_1,\cdots,x_k)$, by the Atiyah-Bott-Berline-Vergne formula, the following integral

$$\begin{aligned}&I_C(Q)\\&=\int_{Fl^S}\widetilde{Q}(c_1^T((\mathcal{E}_1/\mathcal{E}_0)^*),\cdots,c_{r_1}^T((\mathcal{E}_1/\mathcal{E}_0)^*),\cdots,c_1^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*),\cdots,c_{r_s}^T((\mathcal{E}_s/\mathcal{E}_{s-1})^*))\end{aligned}\tag{6.15}$$

can be expressed as the sums over fixed points

$$\begin{aligned}&\sum_{(I,J)\in\mathcal{F}}\\&\times\frac{Q(\widetilde{u}_i;i\in I^s\cup J^s)}{\prod\limits_{\substack{i,j\in I^s\cup J^s,\\ i<j}}(\widetilde{u}_i+\widetilde{u}_j)\prod\limits_{\substack{i\in I^s\cup J^s,\\ j\notin I^s\cup J^s}}(u_i^2-u_j^2)\prod\limits_{m=1}^{s}\prod\limits_{\substack{i\in I_m\cup J_m,\\ j\in(I^s\cup J^s)\backslash(I^m\cup J^m)}}(\widetilde{u}_i-\widetilde{u}_j)\prod\limits_{i\in I^s\cup J^s}(\widetilde{2u_i})}.\end{aligned}\tag{6.16}$$

Note that this expression differs from (5.19) only in the factor 2^k, where as before $k=a_s$. Hence by dividing 2^k in Theorem 5.1 and Corollary 5.1, we immediately obtain the corresponding formulas for $Fl^S(a_1,\cdots,a_s;2n)$ (see [14, Theorem 4.22]).

Theorem 6.1

$$I_C(Q)=\frac{(-1)^k}{r_1!\cdots r_s!}\mathrm{Res}_\infty\frac{Q(z_1,\cdots,z_k)\prod\limits_{i>j}(z_i^2-z_j^2)\prod\limits_{m=1}^{s}\prod\limits_{a_{m-1}+1\le i<j\le a_m}(z_i-z_j)}{\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{k}(z_j-u_i)(z_j+u_i)}\mathrm{d}\mathbf{z}.\tag{6.17}$$

Corollary 6.1 *Let $u_{a_{p-1}+1},\cdots,u_{a_p}$ be the Chern roots of $(\mathcal{E}_p/\mathcal{E}_{p-1})^*$. Then*

$$\begin{aligned}&\int_{Fl^S(a_1,\cdots,a_s;2n)}Q(u_1,\cdots,u_k)\\&=\frac{1}{r_1!\cdots r_s!}\mathrm{Res}_{z=0}\frac{Q(z_1,\cdots,z_k)\prod\limits_{i>j}(z_i^2-z_j^2)\prod\limits_{\substack{1\le m\le s\\ a_{m-1}<i<j\le a_m}}(z_i-z_j)}{(z_1\cdots z_k)^{2n}}\mathrm{d}\mathbf{z}.\end{aligned}\tag{6.18}$$

We remark that the absense of 2^k in these two formulas is caused by the presence of the weights $2\lambda_i, i=1,\cdots,k$ in the adjoint representation.

7 Reprove Darondeau-Pragacz's Formulas

In this section, we use our nonequivariant formulas (4.36), (5.21) and (6.18) to reprove the formulas (in the case that the base manifold is a single point) in [7]. To state the formulas in [7], for any monomial m and any laurent polynomial f, we denote by $[m](f)$ the coefficient of m in f.

The formula for flag varieties of types B and D in [7, Theorem 3.1] is

$$\int_{Fl^O(a_1,\cdots,a_s;m)}Q(u_1,\cdots,u_k)=2^k[z_1^{e_1}\cdots z_k^{e_k}]\Big(Q(z_1,\cdots,z_k)\prod_{1\le i<j\le k}(z_i^2-z_j^2)\Big),\tag{7.1}$$

where $m = 2n$ or $2n+1$, $e_j = m-1-i$ and i is determined by the equation $j = k - a_l + i$ with $l \in \{1, \cdots, s\}$ and $i \in \{1, \cdots, r_l\}$.

The formula for flag varieties of type C in [7, Theorem 2.1] is

$$\int_{Fl^S(a_1,\cdots,a_s;2n)} Q(u_1,\cdots,u_k) = [z_1^{e_1}\cdots z_k^{e_k}]\Big(Q(z_1,\cdots,z_k)\prod_{1\le i<j\le k}(z_i^2-z_j^2)\Big), \tag{7.2}$$

where $e_j = 2n-i$ and i is determined by the equation $j = k-a_l+i$ with $l \in \{1,\cdots,s\}$ and $i \in \{1,\cdots,r_l\}$.

We only give the proof for the Grassmannian $OG(k;2n) = Fl^O(k;2n)$ since the other cases are essentially the same. In the case of $OG(k;2n)$, the formula (7.1) takes the following form

$$\begin{aligned}\int_{OG(k;2n)} Q(u_1,\cdots,u_k) &= 2^k[z_1^{2n-2}z_2^{2n-3}\cdots z_k^{2n-1-k}]\Big(Q(z_1,\cdots,z_k)\prod_{i<j}(z_i^2-z_j^2)\Big)\\ &= 2^k[z_1^{-1}z_2^{-2}\cdots z_k^{-k}]\Big(\frac{Q(z_1,\cdots,z_k)\prod_{i<j}(z_i^2-z_j^2)}{(z_1\cdots z_k)^{2n-1}}\Big).\end{aligned} \tag{7.3}$$

However, by (4.36), we have

$$\begin{aligned}\int_{OG(k;2n)} Q(u_1,\cdots,u_k) &= \frac{2^k}{k!}\mathrm{Res}_{z=0}\frac{Q(z_1,\cdots,z_k)\prod_{i>j}(z_i^2-z_j^2)\prod_{i<j}(z_i-z_j)}{(z_1\cdots z_k)^{2n-1}}\\ &= \frac{2^k}{k!}\mathrm{Res}_{z=0}\frac{Q(z_1,\cdots,z_k)\prod_{i<j}(z_i^2-z_j^2)}{(z_1\cdots z_k)^{2n-1}}\prod_{i>j}(z_i-z_j).\end{aligned} \tag{7.4}$$

We are going to show the right hand sides of (7.3) and (7.4) are equal. Since the right hand sides of (7.3) and (7.4) both vanish unless $\deg(Q) = \dim OG(k;2n) = \binom{k}{2} + 2k(n-k)$, we assume $\deg(Q) = \binom{k}{2}+2k(n-k)$. Let $f(z_1,\cdots,z_k) = \frac{Q(z_1,\cdots,z_k)\prod_{i<j}(z_i^2-z_j^2)}{(z_1\cdots z_k)^{2n-1}}$. By

$$\prod_{i>j}(z_i-z_j) = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & z_k & \cdots & z_k^{k-1}\end{vmatrix},$$

we see that

$$f(z_{\sigma(1)},\cdots,z_{\sigma(k)}) = (-1)^{\mathrm{sgn}(\sigma)}f(z_1,\cdots,z_k). \tag{7.5}$$

In particular, for any monomial $z_1^{p_1}\cdots z_k^{p_k}$, we have

$$[z_{\sigma(1)}^{p_1}\cdots z_{\sigma(k)}^{p_k}](f) = (-1)^{\mathrm{sgn}(\sigma)}[z_1^{p_1}\cdots z_k^{p_k}](f). \tag{7.6}$$

Also note that

$$\prod_{i>j}(z_i-z_j) = \sum_{\sigma\in S_k}(-1)^{\mathrm{sgn}(\sigma)}z_{\sigma(1)}^0 z_{\sigma(2)}^1\cdots z_{\sigma(k)}^{k-1}. \tag{7.7}$$

Hence the right hand side of (7.4) equals

$$\frac{2^k}{k!}\sum_{\sigma\in S_k}(-1)^{\mathrm{sgn}(\sigma)}[z_{\sigma(1)}^{-1}z_{\sigma(2)}^{-2}\cdots z_{\sigma(k)}^{-k}](f), \tag{7.8}$$

which by (7.6) equals

$$\frac{2^k}{k!}\sum_{\sigma\in S_k}[z_1^{-1}z_2^{-2}\cdots z_k^{-k}](f)=2^k[z_1^{-1}z_2^{-2}\cdots z_k^{-k}](f). \tag{7.9}$$

The proof is completed.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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